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On the Natural Frequencies of a Reinforced Circular Cylinder

-by-

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SUMMARY

This report is concerned with the calculation of the natural frequencies of an ideal structure somewhat representative of an aircraft fuselage. The results are based upon a simplified "shell" theory which permits proper allowance to be made for the shear stresses and the corresponding displacements. No use is made of the so-called "shear deflection", but it is shown in the Appendix that, for the special case considered, this approach would yield the same answer. Numerical results are given in §7 and comparison is made with both the usual beam theory results and the frequencies calculated on the assumption that flexibility in shear is of primary importance.

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CONTENTS

1. Statement of the Problem

It is well known that shear flexibility in a beam can affect the natural frequencies and modes of vibration. However the precise degree in which this effect enters into the determination of the frequencies of aircraft structures does not seem to be accurately known. This report attempts to throw light upon this question, by the analysis of an ideal case somewhat representative of an aircraft fuselage.

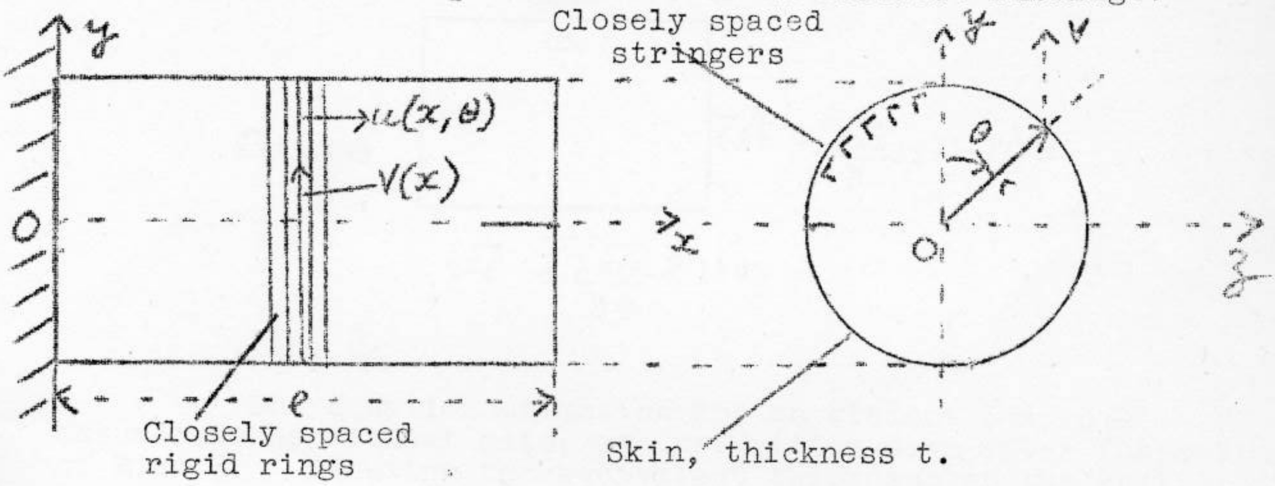

Fig. 1.

Consider a cantilever in the form of a reinforced circular cylindrical shell. (Fig.1.) The closely spaced rings, which for simplicity will be considered rigid in their planes, carry masses amounting to a uniform distribution of M per unit length. The problem to be solved here consists in the determination of the natural frequencies for flexural vibration parallel to a diametral plane which will be taken as xOy, (see Fig.1.) Comparison may then be made with the more usual calculation which neglects the shear flexibility.

2. Displacements and Strains

Since the rings are rigid their displacement parallel to Oy will be given by a function of x only, say V(x). This displacement will be imposed upon the skin-stringer shell, whose remaining longitudinal freedom may be represented by a displacement u parallel to Ox. "u" will be a function of both x and the polar angle θ. The direct strain e_{xx} parallel to Ox will be $\partial u/\partial x$. The shear strain $e_{x\theta}$ arises from both u and the tangential component V sin θ of V. (see Fig.2). We thus derive:-

$$e_{xx} = \frac{\partial u}{\partial x}\ ,\ e_{x\theta} = \frac{\partial u}{r\partial\theta} - \frac{dV}{dx}\sin\theta \qquad \text{.....(1)}$$

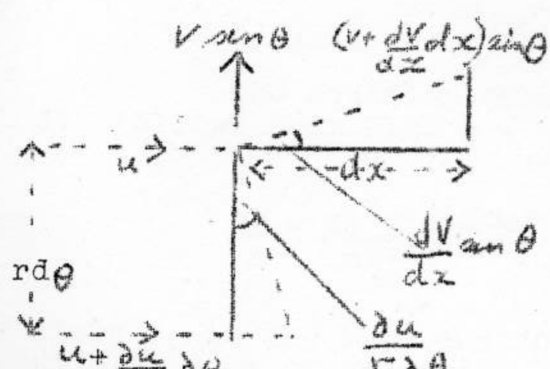

Fig. 2.

3. Stresses and Equations of Motion

The stresses follow from the strains in the usual way. Denoting the direct stress by $\widehat{xx}$, the shear stress by $\widehat{x\theta}$ and neglecting any hoop stresses in the skin we find

$$\widehat{xx} = E\, e_{xx} \quad , \quad \widehat{x\theta} = G\, e_{x\theta} \tag{2}$$

where E,G are the direct and shear moduli respectively.

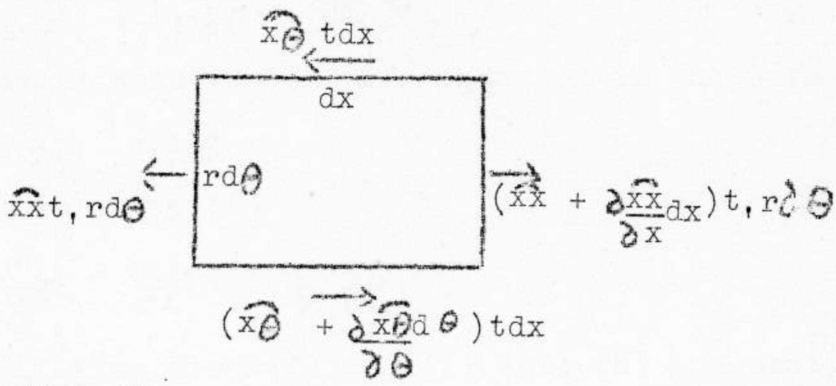

Fig. 3.

The equation of motion for an element (dx, rdθ) assumed to be without mass, can be written down after inspection of Fig. 3. Denoting the equivalent thickness of the shell for carrying direct stress $\widehat{xx}$ by t, (= stringer area per unit length of circumference plus skin thickness) we find:-

$$t, \frac{\partial \widehat{xx}}{\partial x} + t\,\frac{\partial \widehat{x\theta}}{r\,\partial \theta} = 0 \tag{3}$$

The total shear force Y acting across a section of our cylinder is given by,

$$Y = -\int_0^{2\pi} \widehat{x\theta}\, \sin.\theta . t . r d\theta \tag{4}$$

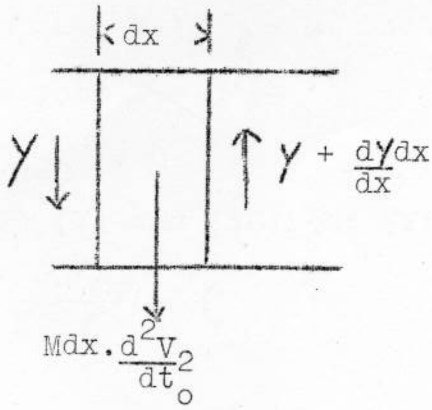

Fig. 4.

The equation of motion for an element of the cylinder dx follows from Fig. 4:-

$$\frac{dY}{dx} = M\frac{d^2V}{dt_o^2} = -Mp^2V \tag{5}$$

where t_o is the time and "p" the angular frequency in a natural mode of vibration. All our variables vary with time as exp (ipt_o)

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4. The Differential Equation for V.

Substituting from (1) into (2) and from thence into (3) we find,

$$Et_1 \frac{\partial^2 u}{\partial x^2} + \frac{Gt}{r}\left(\frac{\partial^2 u}{r\partial\theta^2} - \frac{dV}{dx}\cos\theta\right) = 0 \qquad (6)$$

We seek a solution of this equation in the form

$$u = U(x).\ \cos\theta \qquad (7)$$

This yields by (6)

$$Et_1 \frac{d^2 U}{dx^2} - \frac{GtU}{r^2} - \frac{Gt}{r}\frac{dV}{dx} = 0 \qquad (8)$$

Substituting from (1) and (2) into (4) and using (7) we find,

$$Y = \pi G t r\left(\frac{U}{r} + \frac{dV}{dx}\right) \qquad (9)$$

Substituting from (9) into (5) we find:-

$$\pi G t r\left(\frac{1}{r}\frac{dU}{dx} + \frac{d^2V}{dx^2}\right) = -Mp^2V \qquad (10)$$

Eliminating U from (8) and (10) we obtain,

$$\frac{d^4V}{dx^4} + \frac{Mp^2}{\pi G t r}\frac{d^2V}{dx^2} - \frac{Mp^2V}{\pi E t_1 r^3} = 0 \qquad (11)$$

5. The Boundary Conditions

At the free end $x = \ell$ we must have,

$$(\widehat{xx})_{x=\ell} = 0, \quad (Y)_{x=\ell} = 0 \qquad (12)$$

while at the root $x = 0$:-

$$(u)_{x=0}, \quad (V)_{x=0} = 0 \qquad (13)$$

Using (1),(2),(7) and (10) the first of (12) gives,

$$\left(\frac{d^2V}{dx^2}\right)_{x=\ell} = -\frac{Mp^2}{\pi G t r}(V)_{x=\ell} \qquad (14)$$

/Equations ...

Equations (9),(8) and (10) transform the second of (12) to

$$\left(\frac{d^3V}{dx^3}\right)_{x=\ell} = - \frac{Mp^2}{\pi G t r} \left(\frac{dV}{dx}\right)_{x=\ell} \qquad (15)$$

Using (7),(8) and (10) the first of (13) becomes

$$\left(\frac{dV}{dx}\right)_{x=0} = - \frac{Et_1 r^2}{Gt\left(1 + \frac{Et_1 Mp^2 r}{\pi G^2 t^2}\right)} \left(\frac{d^3V}{dx^3}\right)_{x=0} \qquad (16)$$

For completeness the second of (13) is,

$$(V)_{x=0} = 0 \qquad (17)$$

Since (11) is of the fourth order we can satisfy (14),(15),(16) and (17) and obtain a frequency equation.

6. The Frequency Equation

Solving (11) and satisfying (14),(15),(16) and (17) we find after considerable transformation a frequency equation which is best expressed in terms of the parameters:-

$$\varpi = \frac{Mp^2 \ell^4}{\pi E t_1 r^3} \qquad (18)$$

$$\rho = \frac{Et_1}{Gt} \cdot \frac{r^2}{\ell^2} \qquad (19)$$

$$\left.\begin{aligned} \alpha_1 &= \left[\frac{\{\varpi(4 + \rho^2 \varpi)\}^{\frac{1}{2}} - \varpi \rho}{2}\right]^{\frac{1}{2}} \\ \alpha_2 &= \left[\frac{\{\varpi(4 + \rho^2 \varpi)\}^{\frac{1}{2}} + \varpi \rho}{2}\right]^{\frac{1}{2}} \end{aligned}\right\} \qquad (20)$$

The frequency equation is found to be:-

$$2 + (2 + \rho^2 \varpi) \cosh\alpha_1 \cos\alpha_2 - \varpi^{\frac{1}{2}} \rho \sinh\alpha_1 \sin\alpha_2 = 0 \qquad (21)$$

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In the case where shear flexibility is neglected (i.e. $\rho = 0$) equation (21) reduces to:-

$$\cosh \varpi^{\frac{1}{4}} \cos \varpi^{\frac{1}{4}} + 1 = 0 \qquad (22)$$

In the case where shear flexibility is paramount (i.e. $\rho \rightarrow \infty$) Equation (21) yields:-

$$\cos (\rho^{\frac{1}{2}} \varpi^{\frac{1}{2}}) = 0$$

or

$$\varpi^{\frac{1}{2}} = (2n + 1)\, \pi / 2\rho^{\frac{1}{2}} \qquad (23)$$

7. Numerical Results

If we take $E/_G = 2.5$ $t_{1/t} = 1.6$ $\ell/_{2r} = 5$ as fairly representative of fuselage construction we find $\rho = {}^1/25$. Adopting this value the comparison of frequencies which vary as $\varpi^{\frac{1}{2}}$ (see(18)), as given by equations (21),(22) and (23) is as follows:-

	No. of Mode	(1)	(2)	(3)	(4)
(A)	$\varpi^{\frac{1}{2}}$(Eq(21))*	3.226	14.57	31.38	48.27
(B)	$\varpi^{\frac{1}{2}}$(Eq(22))‡	3.516	22.03	61.70	120.82
(C)	$\varpi^{\frac{1}{2}}$(Eq(23))	7.854	23.56	39.27	54.98

Table of Values of $\varpi^{\frac{1}{2}} = p\ell^2 \left(\frac{M}{\pi E t_1 r^3}\right)^{\frac{1}{2}}$

Line (A) includes shear flexibility,
Line (B) disregards it, while
Line (C) treats it as paramount.

/8. Conclusions

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* I am indepted to Dr. S. Kirkby and the Computing Section of the Aerodynamics Department for these results.

‡ Taken from "Theory of Sound" (Rayleigh).

8. Conclusions

Inspection of the table of § 7. shows that, while the usual theory (line(B)) gives a fair approximation to the fundamental frequency, it is already seriously in error in estimating the first overtone. The assumption that shear flexibility is of primary importance (line(C)) begins to yield fair approximations at the third overtone. The conclusion is therefore reached, that some method of calculation more accurate than the usual theory of beam vibration, is required for the estimation of fuselage frequencies, if anything more than a rough estimate of the fundamental is required.

/Appendix....

APPENDIX

Note on the "Shear" Deflection

If we write

$$u = -r\,\frac{dV_b}{dx} \quad , \quad V_s = V - V_b$$

Equation (9) gives,

$$Y = \pi G t r\,\frac{dV_s}{dx} = \tfrac{1}{2}GA\,\frac{dV_s}{dx}\,, \quad \text{where } A = 2\pi r t$$

Remembering that the max. shear stress in a thin circular tube is equal to twice the mean, we see that V_s is the so-called "shear deflection".
Equations (8) and (9) yield,

$$Y = \pi E t_1 r^2\,\frac{d^2u}{dx^2} = -\,E.(\pi r^3 t_1)\,\frac{d^3V_b}{dx^3}$$

Since $\pi r^3 t_1$ is the moment of inertia of our section, we recognise a fundamental equation of beam theory and so can identify V_b with the "bending deflection". It follows that the results of this report may be obtained from the usual theory of beams combined with the notion of shear deflection. This is not really remarkable since equation (7) implies that "plane sections remain plane". However this agreement is only valid for the thin circular tube. The arguments of this Appendix and equation (7) upon which they are based are not valid for a tube of another section.

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