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The Exact Flow Behind a Yawed Conical Shock *

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Flight Lieutenant G. Radhakrishnan, I.A.F.,
B.Sc.(Eng)., D.C.Ae., A.M.I.E.(Ind).

SUMMARY

The exact flow behind a yawed conical shock wave is investigated. A simple numerical method of solving the differential equations of motion behind the shock wave is evolved.

This method is applied to the case of the flow of a perfect gas behind a conical shock of semi-apex angle 30° yawed at 20° to a free stream of Mach number 10. The shape of the body which would produce such a shock wave is determined. The properties of the flow between the shock wave and the body surface are investigated particularly with respect to the variation of entropy and the streamline pattern.

The existence of a singular generator on the body surface in the plane of yaw and on the "leeward" side, at which the entropy is many-valued is brought out. It is found that, downstream of the shock, all stream lines curve round and tend to converge to this singular generator.

The body obtained by the present investigation is compared to the yawed circular cone which according to Stone's first order theory would produce the same shock wave dealt with in this particular case.

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It is convenient to combine equations (1), (2) and (3) to give

$$v \frac{\partial S}{\partial \psi} + \frac{\bar{w}}{\sin\psi} \frac{\partial S}{\partial \omega} = 0 \qquad (5)$$

Equation of Energy

$$C_p T + \frac{q^2}{2} = \text{constant} \qquad (6)$$

where $q^2 = u^2 + v^2 + w^2$

C_p = specific heat at constant pressure

Equation (5) is general for any conical flow and in fact defines the lines of constant entropy which correspond to the streamlines. If L is the streamline projection on the sphere r = constant,

$$\frac{dS}{dL} = \frac{\partial S}{\partial \psi} \frac{d\psi}{dL} + \frac{\partial S}{\partial \omega} \frac{d\omega}{dL} = 0.$$

Using equation (5) we have

$$\left(\frac{d\omega}{d\psi}\right)_L = \frac{w}{v \sin\psi} \qquad (7)$$

2.4. Conical flow without axial symmetry

In reference 9 Ferri has discussed in detail the properties of supersonic conical flow without axial symmetry and has shown that singularities must exist in any such flow. He considers a conical body placed in a free stream inclined to its axis and by physical reasoning shows that the entropy must be constant on the surface of the cone or must change in a discontinuous manner. It has been shown by Ferri that such a discontinuity occurs along the generator of the cone on the "leeward" meridian plane ($\omega = 0$) at which the entropy is many valued. The character of the flow is such that stream lines downstream of the shock cone curve round and converge to this singular generator. The entropy on the cone surface is equal to that on the "windward" meridian plane ($\omega = \pi$).

3. METHOD OF SOLUTION

3.1. Procedure for numerical solution

The differential equations of motion set out in paragraph 2.3 can be integrated step-by-step with respect to ψ making use of numerical differentiation to obtain derivatives with respect to ω. The method in brief, is as follows: Consider the circle of intersection ($\psi = \psi_w$) of the shock cone and sphere r = constant. Choose a large number of azimuthal stations around this circle. The quantities u, v, w, S and T and their derivatives with respect to ω are known on this circle from the shock wave equations. Substituting these values

Having obtained the body shape, the distribution of pressure on the surface could be found. The values of C_P, the pressure coefficient, at the various azimuthal stations around the body surface are compared with the values just behind the shock in Fig. 14.

The head lift and drag coefficient of the body as defined in Appendix B have been calculated.

The lift coefficient $C_L = 0.410$

The drag coefficient $C_D = 0.545$

4. DISCUSSION

4.1. Method of Solution.

As mentioned earlier, the numerical procedure was found to work in a very satisfactory manner up to $\psi = 26^\circ 30'$, when the solution was in the neighbourhood of the singular point in the "leeward" meridian plane $\omega = 0$. The main difficulty from this stage onwards was that the value of v tended towards zero and a subsequent change in sign (the change in v itself was quite regular throughout). This factor was highly critical since the evaluation of $\frac{\partial u}{\partial \psi}$, $\frac{\partial w}{\partial \psi}$ and $\frac{\partial S}{\partial \psi}$ involved division by v. This meant that whilst the value of v passed through zero and changed sign, it was possible to get large magnitudes of the above derivatives changing in sign quite rapidly. However, this was found to be highly critical only in the case of the evaluation of $\frac{\partial w}{\partial \psi}$. It was this feature which was responsible for the extreme care necessary to continue the solution beyond $\psi = 26^\circ 30'$ in the vicinity of $\omega = 0$.

As mentioned earlier, this highly critical region between $\omega = 0$ and (as it turned out) $\omega = 45^\circ$ was investigated separately using smaller values of $\Delta\psi$ than that used for the remainder of the azimuthal stations. Here it may be mentioned that the above stated difficulties encountered when $v \longrightarrow 0$ and changes sign, were avoided in the case of the "windward" side. This was because the surface of the solid body (as represented by the line of constant entropy of magnitude equal to that of the entropy on the "windward" plane $\omega = \pi$) was obtained before the critical region ($v \longrightarrow 0$) was reached. The solution was not carried any further because the behaviour of the flow inside the body surface was of no special interest in the present case.

4.2. Properties of the flow.

4.2.1. Velocity.

It is found that the variation of the velocity components is quite regular and exhibit no peculiarities. However, the variation in the values of u and particularly w in the vicinity of the singular point needs some consideration. Some difficulty was experienced in the finding of the numerical values of w and u in the region $60^\circ > \omega > 0^\circ$ for values of ψ smaller than $26^\circ 30'$. Although it appeared that the values of u and w behaved regularly in this region it was considered that accurate numerical values could only be obtained if smaller intervals of ω were used in the numerical method.

The component v is found to vary in a very regular manner. This is quite understandable since the evaluation of $\frac{\partial v}{\partial \psi}$ depends on $(v^2 - a^2)$ with $v^2 \ll a^2$.

4.2.2. Temperature.

The variation of temperature follows from the way in which the velocity changes. It is found that the variation in temperature throughout the field is quite regular.

4.2.3. Entropy and streamlines.

The distribution of entropy in the flow behind the shock cone is represented in Fig. 2, 7, 8. The projections of constant entropy lines (they correspond to streamlines) on the sphere r = constant are represented in Fig. 2. The location of the singular point, on the "leeward" meridian plane $\omega = 0$, at which the entropy is many valued is also indicated in the figure. It is found that the streamlines, after leaving the shock cone, curve round and converge to the singular point. The surface of the hypothetical solid body (corresponding to the constant entropy line having the same entropy as a plane $\omega = \pi$) which will produce the shock wave dealt with here is also indicated in the figure.

One feature in the pattern of the streamlines near the singular point may be pointed out. From equation (5) we have

$$v \frac{\partial S}{\partial \psi} + \frac{w}{\sin\psi} \frac{\partial S}{\partial \omega} = 0$$ from which we have as equ. (7)

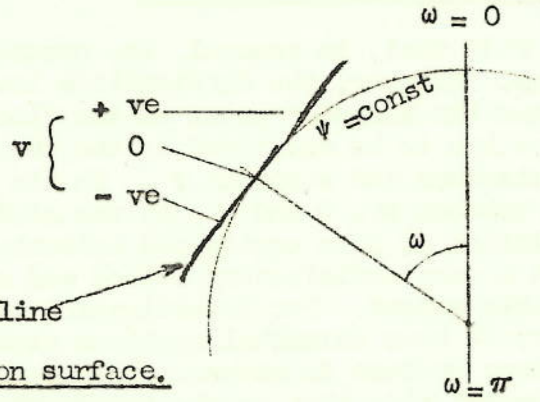
$$\left(\frac{d\omega}{d\psi}\right)_{\text{streamline}} = \frac{w}{v \sin\psi}$$

or more conveniently

$$\left(\frac{d\psi}{d\omega}\right)_{\text{streamline}} = \frac{v \sin\psi}{w}$$

Except in the meridian plane when v = 0 (and when w = 0) the equation is indeterminate, the above equation holds good generally. Hence, when $v \to 0$ and changes sign (but $w \neq 0$) the streamlines will tend to "flatten" out and become parallel to the line ψ = constant at v = 0 and then "curl up" when v becomes positive. This is illustrated in the accompanying diagram.

This happened in the case of a few streamlines on the "leeward" side.



4.2.4. Pressure distribution on surface.

The pressure distribution on the body was worked out and a comparison with the values on the downstream side of the shock wave is made in Fig.14. This indicates that there is an expansion in the flow between the shock wave and body except in a small region $\omega = 140$ to 180° on the "windward" side where a slight compression of the flow takes place.

4.2.5. Comparison with first order solution.

The body shape obtained by the present method is compared here with the first order yawed cone solution (Ref.2.6) in Fig. 15. It is observed that the body is smaller than the corresponding cone in the fist order solution. The body is not wholly circular; however, it is noted that it is mostly circular with a small hump on the "leeward" side. The smaller size of the body as noted in the case of the present solution might mean that in actual practice the assumption of the first order theory at comparatively large yaw with respect to shape of the shock cone may be valid but that it may be necessary to make a correction for the change in size of the shock cone.

The head lift and drag coefficients of the conical body (of non-circular cross section) obtained by the present method have been calculated using expressions defined in Appendix B.

Head lift coefficient $\underline{C_L = 0.410}$

and a Head drag coefficient $\underline{C_D = 0.545}$

These first order values were obtained only as a means of checking the orders of magnitude of C_L and C_D obtained for the body of the present solution. A direct comparison between the two sets of values cannot be considered to have any conclusive significance.

4.3. Method of numerical solution.

It is felt that, in general, the numerical investigation was satisfactory. However, the difficulties involved in carrying on the solution near the singular point on the "leeward" side have shown that extreme care has to be exercised in the choice of the interval between azimuthal stations and steps in ψ. In the present investigation the region between $\omega = 0$ and $\omega = 45$ was studied separately by carrying out the solution at five equispaced azimuthal stations. It is felt that this is not a very satisfactory method and could be improved upon to a considerable extent. For investigation of the flow in this region it is necessary to have azimuthal stations closer to each other than 15°. It may perhaps be best to choose a larger number of azimuthal stations on the "leeward" side than on the "windward" side. For future work it is suggested that azimuthal stations should be spaced at intervals of $\Delta\omega = 5^\circ$ from $\omega = 0$ to $\omega = 75$ and at intervals of 15° from $\omega = 75$ to $\omega = 180^\circ$.

5. CONCLUSIONS

It has been found that the numerical method adopted for the investigation of the exact flow behind a yawed conical shock is simple to use and produced reasonably satisfactory results. The accuracy of the method can be improved by choosing a smaller interval between azimuthal stations.

As a particular case, the flow behind a conical shock of semi-apex angle 30° inclined at 20° to a free stream of Mach Number 10 has been investigated and the shape of the conical body (of non-circular section) which would produce such a shock wave has been determined and compared with the yawed cone solution. In this case, it has been found that the shape departs from circular only to a small extent on the "leeward" side. More significantly, it is noted that the size of the body is smaller than that of the circular cone which according to Stone's first order theory (Ref. 2, 6) would produce the given shock wave.

The properties of the flow between the shock cone and the solid body surface have been determined and the pattern of the streamlines has been studied. The existence of a singular generator on the body surface in the "leeward" meridian plane $\omega = 0$, at which the entropy is many valued has been well brought out.

6. ACKNOWLEDGEMENTS.

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REFERENCES.

1.	Woods, B.A.	The supersonic flow around a yawed cone. College of Aeronautics Thesis, June 1956, (unpublished).
2.	Stone, A.H.	On the supersonic flow past a slightly yawing cone. Journal of Mathematics and Physics, Vol.27. 1948.
3.	Taylor G.I., and Maccoll J.W.	The air pressure on a cone moving at high speeds. Proc. Roy.Soc. Series A, Vol.139, 1933.
4.	Stone, A.H.	On supersonic flow past a slightly yawing cone II. Journal of Mathematics and Physics, Vol.30. 1951.
5.	Kopal, Z and staff of Computing Lab.	Tables of supersonic flow around cones. M.I.T. Report No. 1, 1947.
6.	"	Tables of supersonic flow around yawing cones. M.I.T. Report No. 3, 1947.
7.	"	Tables of supersonic flow around cone of large yaw. M.I.T. Report No. 5, 1949.
8.	Roberts, R.C., and Riley, J.D.	A guide to the use of M.I.T. cone tables. Journal Aero.Sc. Vol.21 May 1954.
9.	Ferri, A.	Supersonic Flow around circular cones at angles of attack. NACA Report 1045 (1951) T.N.2236 (1950).
10.	Bickley, W.G.	Formulae for numerical differentiation. Mathematical Gazette, Vo. 25 (1941), p.22.
11.	Ames Research Staff.	Equations, tables and charts for compressible flow. NACA Report 1135.(1953).
12.	Young, G.B.W., and Siska, C.P.	Supersonic flow around cones at large yaw. Journal Aero.Sc. Vol.19 No. 2. Feb. (1952).

APPENDIX A.

DETAILS OF SOLUTION.

A.1. Procedure for numerical solution.

The differential equations of motion (1) to (6) set out in paragraph 2.3 can be expressed in non-dimensional form by effecting the following substitutions. (Primes denote non-dimensional quantities).

$$u' = \frac{u}{V_1} \quad , \quad v' = \frac{v}{V_1} \quad , \quad w = \frac{w}{V_1}$$

$$S' = \frac{S}{C_p} \quad , \quad T' = \frac{T}{V_1^2/C_p} \qquad \text{where } V_1 = \text{free stream velocity}$$

and

$$a' = \frac{a}{V_1} \qquad a^2 = (a'\,V_1)^2 = \alpha\, R\, T' \,\frac{V_1^2}{C_p}$$

$$(a')^2 = (\gamma - 1)\, T'$$

using the above relations we have

$$v' \frac{\partial u'}{\partial \psi} + \frac{w'}{\sin\psi} \frac{\partial u'}{\partial \omega} - v^2 - w^2 = 0 \qquad (1)A$$

$$- u' \frac{\partial u'}{\partial \psi} - w' \frac{\partial w'}{\partial \psi} + \frac{w'}{\sin\psi} \frac{\partial v'}{\partial \omega} + u'\, v' - w' \cot\psi = T' \frac{\partial S'}{\partial \psi} \qquad (2)A$$

$$v' \operatorname{Sin}\psi \frac{\partial w'}{\partial \psi} - u' \frac{\partial u'}{\partial \omega} - v' \frac{\partial v'}{\partial \omega} + u'\, w' \operatorname{Sin}\psi + v'\, w' \cos\psi = T' \frac{\partial S'}{\partial \omega} \qquad (3)A$$

$$u' \left(v'^2 + w'^2 - 2\left[\alpha - 1\right] T'\right) - (\alpha - 1) T'\, v' \cot\psi + \frac{\partial v'}{\partial \psi}\left(v'^2 - \left[\alpha - 1\right] T'\right)$$

$$+ \frac{1}{\sin\psi} \frac{\partial w'}{\partial \omega}\left(w'^2 - \left[\alpha - 1\right] T'\right) + v'\, w' \left(\frac{\partial w'}{\partial \psi} + \frac{1}{\sin\psi} \frac{\partial v'}{\partial \omega}\right) = 0 \qquad (4)A$$

$$v' \frac{\partial S'}{\partial \psi} + \frac{w'}{\sin\psi} \frac{\partial S'}{\partial \omega} = 0 \qquad (5)A$$

$$T' + \frac{q'^2}{2} = \text{constant}$$

$$= T'_1 + \frac{V_1'^2}{2} = T'_1 + \tfrac{1}{2} \qquad (6)A$$

Hereafter these non-dimensional quantities will be used and the primes will be omitted.

Lift and Drag of Equivalent Cone.

The semi-apex angle of a circular cone that will in axi-symmetric flow at $M_1 = 10$, produce a conical shock wave of semi-apex and $\psi_W = 30^\circ$ is $\psi_S = 26.6^\circ$ (approx). This was obtained from chart 5 in Ref. 11.

Making use of the 1st order theory of Stone (Ref. 2) we have that when the cone is yawed with respect to the free stream (at an angle β) the shock will retain its size and shape but its axis will be inclined to the free stream at an angle (in general not equal to β). From part II of Ref. 6 we have that for $\psi_S = 26.6^\circ$, $M_1 = 10$ (by graphical interpolation) $\frac{\alpha}{\beta} = 1.046$.

$\therefore \beta = 19.1^\circ$. We have further that

$K_N = 0.628$ and $K_D = 0.167$ where K_N and K_D are coefficients of normal and drag forces defined according to wind co-ordinates in Ref. 6. The transformation to the more practical body co-ordinate system can be effected as follows. This method was pointed out by Young and Siska in Reference 12 who give the following formulae for the transformations.

Normal Force Coefficient

$$C_N = \left(\frac{8\beta}{\pi}\right) K_n = \frac{8 \times .333}{\pi} \times .628 = \underline{0.533}$$

$\beta = 19.1^\circ = 0.333$ radians

$\cos\beta = 0.945$

$\sin\beta = 0.327$

Axial Force Coefficient

$$C_A = \frac{8}{\pi} K_D = \frac{8 \times .167}{\pi} = \underline{0.425}$$

$$C_L = C_N \cos\beta - C_A \sin\beta = .504 - .139 = \underline{0.365}$$

$$C_D = C_A \cos\beta + C_N \sin\beta = .402 + .174 = \underline{0.576}$$

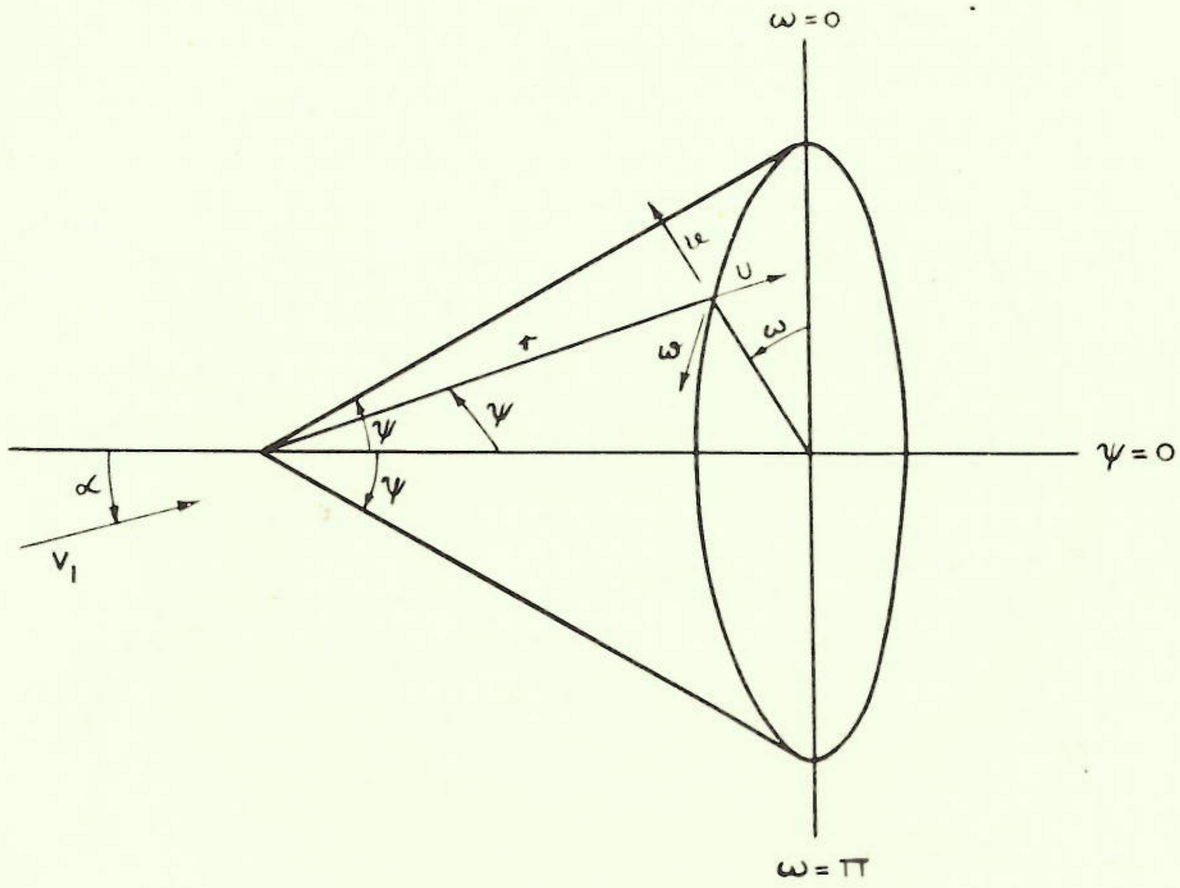

FIG. 1. THE COORDINATE SYSTEM AND NOMENCLATURE