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CRANFIELD

THE USE OF A SIMPLE COMPOSITE ELEMENT TO DESCRIBE THE CREEP PROPERTIES OF FIBRE REINFORCED COMPOSITES

by

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DEPARTMENT OF MATERIALS

The use of a simple composite element to describe the creep properties of fibre reinforced composites

- by -

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SUMMARY

The stress-strain relationship for a composite material is dependent on both the geometry and the stress-strain relationships of the component phases.

This note describes a technique by which the stress-strain relationship can be calculated for any fibre reinforced composite where the matrix has linear viscoelastic properties and the fibres are linearly elastic. The distribution of fibres within the composite is assumed to be macroscopically homogeneous but the distribution of fibre orientation can take any configurations. The problem is solved initially for the case where both phases are linearly elastic. A simple composite element from which a composite can be built up is defined and the stress-strain relationship for this element is calculated using variational methods. By summing these elements assuming either uniform stress or uniform strain throughout the composite, upper and lower bounds to the stiffness matrix of the composite are obtained. Using the correspondence principle these bounds for the purely elastic case are transformed to give the bounds for the viscoelastic case.

The theoretical answers obtained using this method are compared with those obtained using a more simple model for the mode of combination of the two phases.

Contents

Nomenclature

C_{ijkl}	-	elastic stiffnesses	Tensor notation where i,j,k,l are integers taking the values 1,2,3.
S_{ijkl}	-	elastic compliances	
σ_{ij}	-	stress	
ϵ_{ij}	-	strain (tensor)	
$C_{q,r}$	-	elastic stiffnesses	Matrix notation where q,r are integers taking the values 1,2,3,4,5,6.
S_{qr}	-	elastic compliances	
σ_q	-	stress	
ϵ_q	-	strain (engineering)	
E	-	Young's modulus	
ν	-	Poisson's ratio	
G	-	shear modulus	
K	-	bulk modulus	
v	-	volume fraction	
a_{ij}	-	direction cosine; i,j take values 1,2,3.	
α_i, β_i	-	Fourier coefficients; i are integers.	
$f(\theta)$	-	distribution of fibre orientation	
t	-	time	
p	-	transformed variable	
S_i	-	magnitudes of discrete retardation spectra	
γ_i	-	retardation times	
$\underset{\sim}{C}$	-	stiffness matrix	
$\underset{\sim}{S}$	-	compliance matrix	

suffixes

f	-	fibrous phase
m	-	matrix phase

superfixes

U	-	upper limit of compliance
L	-	lower limit of compliance

A tilde, ~, below a letter denotes a matrix

A circumflex, ∧, above a letter denotes a Laplace transform.

Introduction

The stiffness of a composite material depends both on the geometry of the structure of the material and on the stiffness of the component phases. The composite materials under consideration here are fibre reinforced linear viscoelastic materials and consequently any analysis of their stress-strain characteristics must take the geometry and orientation of the fibrous phase into account as well as the time-dependence of the matrix.

Cox (1952) has analysed a mat of ideal fibres, assuming that these fibres have no flexural stiffness and that in consequence they can only transmit loads in tension. He characterises the orientation of the fibres in the mat by a distribution function. This represents the number of fibres at a given angle to a specified direction in a unit width perpendicular to their axial direction. The assumptions made by Cox seem valid in the context of a mat with no means of interconnection between the fibres. Using this analysis Arridge (1963) has combined an ideal fibrous mat with an elastic matrix by assuming that the strains in the two phases are equal. These principles of Cox and Arridge have been extended to allow for the matrix material being linearily viscoelastic by Dootson (1968) who has obtained Volterra integral equations relating the creep compliance of a composite to the geometry and stiffness of the two phases. These equations have been solved using several techniques (see Mikhlin (1964)) and the calculated compliances compared with the experimentally obtained compliances of several glass fibre and polyester resin systems.

In a composite material it seems likely that, due to the connection between the fibres, the fibres affect the stiffness of the whole other than axially. Bishop (1966) has tried to allow for this by introducing two hypothetical lateral fibres to act in conjunction with each fibre. While this artifice can be used empirically to improve predictions of the mechanical properties of the composite, it is not very satisfactory from a theoretical point of view.

In this note it is intended to use a more rigorous elastic analysis, based on the variational principles used by Hashin and Rosen (1964), to calculate the five elastic stiffnesses required to characterise a simple composite element. Summing these using a distribution function in the same way as Cox has done, the elastic solution for a fibre reinforced composite can be obtained.

Using the correspondence principle proved by Biot (1954), associating elastic and viscoelastic problems, this elastic solution can be used to yield the viscoelastic solution required. This technique is explained by Williams (1964) who suggests that the complicated transform inversion involved can be bypassed by an approximation method such as the collocation method proposed by Schapery (1962).

Moduli of a representative composite element

In order to analyse the elastic behaviour of a composite material it is necessary to assume a mode of combination of the component phases. Arridge (1963) has assumed that a composite formed from a mat of continuous fibres embedded in a homogeneous isotropic matrix can be adequately represented by considering the two phases to undergo equal strains and to have no interaction with each other. After Cox (1952) he assumes that the fibres have no flexural stiffness and can consequently only transmit loads in tension. It would be expected that the errors incurred by these assumptions are small in calculations of the stiffness of the composite parallel to the fibre axis, as neither the interaction between the phases nor the flexural stiffness of the fibres will have much effect on this. Conversely, the shear stiffness and the stiffness normal to the fibre axis, calculated for the composite, would be expected to contain large errors.

To eliminate these errors it is necessary to consider both phases to be isotropic and homogeneous and to take the stress distribution in the two phases into account. However, to calculate the stress distribution for each configuration of fibres and applied stress field would be an extraordinarily lengthy process. As an alternative we can consider a composite of this kind as being formed from a number of representative composite elements. Each of these is composed of many, infinitely long, parallel fibres in a cylinder of the matrix material with its axis parallel to the fibre axes. The fibres are assumed to be placed randomly in this element and the element is assumed to be large enough to be macroscopically homogeneous. Both phases are assumed to be isotropic and homogeneous. This representation of a composite allows for interaction between the phases, and the fibres may be taken to contribute to the stiffness of the composite both in shear and in deformation normal to their axes in addition to their contribution to the stiffness parallel to their axes. The elastic constants of such an element may be calculated from the constants of the individual phases and the elastic constants of any fibre reinforced composite may be obtained by a suitable combination of these elements.

Hashin and Rosen (1964) have derived expressions for the macroscopic elastic moduli of composite materials where the reinforcement takes the form of parallel cylindrical fibres. They assume the composite material to be macroscopically homogeneous and that it can therefore be split into representative subregions of the type already described here. Their analysis takes the form of a variational method which calculates bounds for the moduli by the use of the theorems of minimum potential and complementary energy. For random fibre placement a geometric approximation is involved and thus the resulting bounds are only approximate. They show that in this case the bounds are coincident.

Hashin and Rosen define the axis of their element as the 1-axis with the 2- and 3-axes mutually perpendicular in the transverse plane. The first modulus calculated is defined as the plane strain bulk modulus and is

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CORRIGENDA

p. 5 Equations 1 to 5 on this page should be:-

$$\frac{1}{2}(C_{22}+C_{23}) = \frac{(3K_m+G_m)}{3} + \frac{v_f}{\frac{3}{(3K_f+G_f)-(3K_m+G_m)} + \frac{3v_m}{(3K_m+4G_m)}} \tag{1}$$

$$C_{44} = \frac{1}{2}(C_{22}-C_{23}) = G_m + \frac{v_f}{\frac{1}{(G_f-G_m)} + \frac{v_m(3K_m+7G_m)}{2G_m(3K_m+4G_m)}} \tag{2}$$

$$C_{66} = G_m \left\{\frac{G_f(1+v_f) + G_m v_m}{G_f v_m + G_m(1+v_f)}\right\} \tag{3}$$

$$C_{22} - \frac{2C_{12}^2}{C_{22}+C_{23}} = v_m E_m + v_f E_f + \frac{3v_m v_f \left(\frac{G_f}{(3K_f+G_f)} - \frac{G_m}{(3K_m+G_m)}\right)^2}{\left(\frac{v_m}{(3K_f+G_f)} + \frac{v_f}{(3K_m+G_m)} + \frac{1}{3G_m}\right)} \tag{4}$$

$$\frac{C_{12}}{C_{22}+C_{23}} = v_m \nu_m + v_f \nu_f + \frac{v_m v_f(\nu_f - \nu_m)\left(\frac{1}{(3K_m+G_m)} - \frac{1}{(3K_f+G_f)}\right)}{\left(\frac{v_m}{(3K_f+G_f)} + \frac{v_f}{(3K_m+G_m)} + \frac{1}{3G_m}\right)} \tag{5}$$

associated with the volume change due to a plane strain system in the 2,3 plane. In terms of the elastic stiffnesses of the element this modulus is $(c_{22}+c_{23})/2$ and so from Hashin and Rosen's analysis we obtain

$$\frac{1}{2}(C_{22}+C_{23}) = K_m + \frac{v_f}{\frac{1}{K_f-K_m} + \frac{v_m}{K_m+G_m}} \qquad (1)$$

Similarly, considering the shear modulus associated with a pure shear strain in the 2,3 plane we obtain

$$C_{44} = \frac{1}{2}(C_{22}-C_{23}) = G_m + \frac{v_f}{\frac{1}{G_f-G_m} + \frac{v_m(K_m+2G_m)}{2G_m(K_m+G_m)}} \qquad (2)$$

From the modulus associated with a pure shear strain in either the 1,2 or 1,3 planes we obtain

$$C_{66} = G_m \left\{\frac{G_f(1+v_f) + G_m v_m}{G_f v_m + G_m(1+v_f)}\right\} \qquad (3)$$

If we consider the element to be subjected to a longitudinal stress only, then the longitudinal Young's modulus can be calculated as well as the associated Poisson's ratio. These two give the relations

$$C_{11} - \frac{2C_{12}^2}{C_{22}+C_{23}} = v_m E_m + v_f E_f + \frac{v_m v_f}{3} \frac{\left(\frac{G_f}{3K_f+G_f} - \frac{G_m}{3K_m+G_m}\right)^2}{\left(\frac{v_m}{3K_f+G_f} + \frac{v_f}{3K_m+G_m} + \frac{1}{3G_m}\right)} \qquad (4)$$

and

$$\frac{C_{12}}{C_{22}+C_{23}} = v_m \nu_m + v_f \nu_f + v_m v_f \frac{(\nu_f-\nu_m)\left(\frac{1}{3K_m+G_m} - \frac{1}{3K_f+G_f}\right)}{\left(\frac{v_m}{3K_f+G_f} + \frac{v_f}{3K_m+G_m} + \frac{1}{3G_m}\right)} \qquad (5)$$

respectively. From these five equations we can calculate the five elastic constants, C_{11}, C_{22}, C_{66}, C_{12}, C_{23}, needed to characterise this transversely isotropic element. These equations have been written in terms of G, K, E, ν, of which only two are required to describe each isotropic phase, in order to simplify the resulting expressions. As this element is transversely isotropic, the stress-strain relationship can be written as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{12} & C_{23} & C_{22} & & & \\ & & & (C_{22}-C_{23})/2 & & \\ & & & & C_{65} & \\ & & & & & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \qquad (6)$$

in matrix notation. It should be noted that in equation (4) the last term on the right hand side is small and can in general be ignored as, for the range of values expected of the variables, it does not exceed 1% of the total. With this term eliminated the 'law of mixtures' usually quoted for the longitudinal Young's modulus remains.

Extension to a complete composite

The elastic constants of the representative composite element relate the components of one second rank tensor (stress) to those of another (strain). They are therefore a fourth rank tensor and on transforming from one set of axes to another

$$C'_{ijkl} = a_{im} \cdot a_{jn} \cdot a_{ko} \cdot a_{lp} \cdot C_{mnop} \qquad (7)$$

as is described by Hearmon (1961). For a rotation through an angle θ from the 1-axis towards the 2-axis about the 3-axis the direction cosines, a_{ij}, are

$$a_{11} = m,\ a_{12} = n,\ a_{21} = -n,\ a_{22} = m$$
$$a_{33} = 1,\ a_{13} = a_{23} = a_{31} = a_{32} = 0 \qquad (8)$$

where $m = \cos\theta$ and $n = \sin\theta$. Consequently, if we rotate the composite element described through an angle θ about the 3-axis, then the stiffness matrix becomes

$$\underset{\sim}{C}(\theta) = \begin{bmatrix} C(\theta)_{11} & C(\theta)_{12} & C(\theta)_{13} & & & C(\theta)_{16} \\ C(\theta)_{12} & C(\theta)_{22} & C(\theta)_{23} & & & C(\theta)_{26} \\ C(\theta)_{13} & C(\theta)_{23} & C(\theta)_{33} & & & C(\theta)_{36} \\ & & & C(\theta)_{44} & C(\theta)_{45} & \\ & & & C(\theta)_{45} & C(\theta)_{55} & \\ C(\theta)_{16} & C(\theta)_{26} & C(\theta)_{36} & & & C(\theta)_{66} \end{bmatrix} \qquad (9)$$

where

$$
\begin{aligned}
C(\theta)_{11} &= C_{11}m^4 + C_{12}\,2m^2n^2 + C_{22}\,n^4 + C_{66}\,4m^2n^2 \\
C(\theta)_{12} &= C_{11}m^2n^2 + C_{12}(m^4+n^4) + C_{22}m^2n^2 - C_{66}\,4m^2n^2 \\
C(\theta)_{16} &= -C_{11}m^3n + C_{12}(m^3n - mn^3) + C_{22}mn^3 + C_{66}.2(m^3n - mn^3) \\
C(\theta)_{22} &= C_{11}n^4 + C_{12}\,2m^2n^2 + C_{22}m^4 + C_{66}\,4m^2n^2 \\
C(\theta)_{26} &= -C_{11}mn^3 + C_{12}(mn^3-m^3n) + C_{22}m^3n + C_{66}.2(mn^3-m^3n) \\
C(\theta)_{66} &= C_{11}m^2n^2 - C_{12}\,2m^2n^2 + C_{22}m^2n^2 + C_{66}(m^2-n^2)^2 \\
C(\theta)_{13} &= C_{12}m^2 + C_{23}n^2 \\
C(\theta)_{23} &= C_{12}n^2 + C_{23}m^2 \\
C(\theta)_{36} &= -C_{12}mn + C_{23}mn \\
C(\theta)_{44} &= (C_{22}-C_{23})m^2/2 + C_{66}n^2 \\
C(\theta)_{55} &= (C_{22}-C_{23})n^2/2 + C_{66}m^2 \\
C(\theta)_{45} &= (C_{22}-C_{23})mn/2 - C_{66}mn \\
C(\theta)_{33} &= C_{33}
\end{aligned}
\qquad (10)
$$

The stress-strain relationship for a representative composite element with its axis oriented at an angle θ to the 1-axis in the 1,2 plane is thus given by

$$\underset{\sim}{\sigma} = \underset{\sim}{C}(\theta).\underset{\sim}{\epsilon} \qquad (11)$$

Cox (1952) described the distribution of orientation of the axial directions of the fibres in a mat by a distribution function, $f(\theta)$, which represents the number of fibres at a given angle to a specified direction in a unit width perpendicular to their axial direction. Using $f(\theta)$ to describe the distribution of the axial directions of representative composite elements and assuming that the strains throughout the composite are uniform gives the stress-strain relationship for the composite as

$$\underset{\sim}{\sigma} = \int_0^{\pi} \underset{\sim}{C}(\theta).f(\theta)d\theta \;.\; \underset{\sim}{\epsilon} \qquad (12)$$

The alternative assumption that the stresses throughout the composite are uniform gives this relationship as

$$\underset{\sim}{\epsilon} = \int_0^{\pi} \underset{\sim}{C}^{-1}(\theta).f(\theta)d\theta \;.\; \underset{\sim}{\sigma} \qquad (13)$$

Consequently we can write the stiffness matrix of the composite as either

$$\underset{\sim}{C}^{U} = \int_0^{\pi} \underset{\sim}{C}(\theta).f(\theta)d\theta \qquad (14)$$

or

$$(\underset{\sim}{C}^{L})^{-1} = \int_{0}^{\pi} \underset{\sim}{C}^{-1}(\theta).f(\theta)d\theta \qquad (15)$$

depending on which assumption is made. These two assumptions should give upper and lower bounds to the stiffness matrix of the composite.

The distribution function, $f(\theta)$, is periodic with a period of π and it can consequently be written as a Fourier series

$$\begin{aligned} \pi f(\theta) &= 1 + \alpha_1\cos 2\theta + \alpha_2\cos 4\theta + \ldots. \\ &\quad + \beta_1\sin 2\theta + \beta_2\sin 4\theta + \ldots. \end{aligned} \qquad (16)$$

Since equation (10), representing the rotation of the element through an angle θ, is concerned with powers of trigonometrical functions no higher than the fourth, further terms do not effect the stiffness matrix of the composite. By expanding the powers of $\cos\theta$ and $\sin\theta$ to give $\underset{\sim}{C}(\theta)$ or $\underset{\sim}{C}^{-1}(\theta)$ in multiangular form and integrating we obtain the non-zero elements of $\underset{\sim}{C}^{U}$ as

$$\begin{aligned}
C_{11}^{U} &= \{C_{11}(6+4\alpha_1+\alpha_2) + C_{12}.2(2-\alpha_2) + C_{22}(6-4\alpha_1+\alpha_2) + C_{66}.4(2.\alpha_2)\}/16 \\
C_{12}^{U} &= \{C_{11}(2-\alpha_2) + C_{12}.2(6+\alpha_2) + C_{22}(2-\alpha_2) - C_{66}.4(2-\alpha_2)\}/16 \\
C_{16}^{U} &= \{-C_{11}(2\beta_1+\beta_2) + C_{12}.2\beta_2 + C_{22}(2\beta_1-\beta_2) + C_{66}.4\beta_2\}/16 \\
C_{22}^{U} &= \{C_{11}(6-4\alpha_1+\alpha_2) + C_{12}.2(2-\alpha_2) + C_{22}(6+4\alpha_1+\alpha_2) + C_{66}.4(2-\alpha_2)\}/16 \\
C_{26}^{U} &= \{-C_{11}(2\beta_1-\beta_2) - C_{12}.2\beta_2 + C_{22}(2\beta_1+\beta_2) - C_{66}\,4\beta_2\}/16 \\
C_{66}^{U} &= \{C_{11}(2-\alpha_2) - C_{12}.2(2-\alpha_2) + C_{22}(2-\alpha_2) + C_{66}.4(2+\alpha_2)\}/16 \\
C_{13}^{U} &= \{C_{12}(2+\alpha_1) + C_{23}(2-\alpha_1)\}/4 \\
C_{23}^{U} &= \{C_{12}(2-\alpha_1) + C_{23}(2+\alpha_1)\}/4 \\
C_{36}^{U} &= \{C_{12}.\beta_1 + C_{23}\beta_1\}/4 \\
C_{44}^{U} &= \{(C_{22}-C_{23})(2+\alpha_1)/2 + C_{66}(2-\alpha_1)\}/4 \\
C_{55}^{U} &= \{(C_{22}-C_{23})(2-\alpha_1)/2 + C_{66}(2+\alpha_1)\}/4 \\
C_{45}^{U} &= \{(C_{22}-C_{23})\beta_1/2 - C_{66}\,\beta_1\}/4 \\
C_{33}^{U} &= C_{22}
\end{aligned} \qquad (17)$$

and, similarly, for the lower bound case

$$
\begin{aligned}
S^L_{11} &= \{S_{11}(6+4\alpha_1+\alpha_2) + S_{12}.2(2-\alpha_2) + S_{22}(6-4\alpha_1+\alpha_2) + S_{66}(2-\alpha_2)\}/16 \\
S^L_{12} &= \{S_{11}(2-\alpha_2) + S_{12}.2(6+\alpha_2) + S_{22}(2-\alpha_2) - S_{66}(2-\alpha_2)\}/16 \\
S^L_{16} &= \{- S_{11}\, 2(2\beta_1+\beta_2) + S_{12}.4\beta_2 + S_{22}\, 2(2\beta_1-\beta_2) + S_{66}\, 2\beta_2\}/16 \\
S^L_{22} &= \{S_{11}(6-4\alpha_1+\alpha_2) + S_{12}\, 2(2-\alpha_2) + S_{22}(6+4\alpha_1+\alpha_2) + S_{66}(2-\alpha_2)\}/16 \\
S^L_{26} &= \{- S_{11}\, 2(2\beta_1-\beta_2) - S_{12}\, 4\beta_2 + S_{22}\, 2(2\beta_1+\beta_2) - S_{66}\, 2\beta_2\}/16 \\
S^L_{66} &= \{S_{11}\, 4(2-\alpha_2) - S_{12}\, 8(2-\alpha_2) + S_{22}\, 4(2-\alpha_2) + S_{66}\, 4(2+\alpha_2)\}/16 \\
S^L_{13} &= \{S_{12}(2+\alpha_1) + S_{23}(2-\alpha_1)\}/4 \\
S^L_{23} &= \{S_{12}(2-\alpha_1) + S_{23}(2-\alpha_1)\}/4 \\
S^L_{36} &= \{S_{12}\, 2\beta_1 + S_{23}\, 2\beta_1\}/4 \\
S^L_{44} &= \{2(S_{22}-S_{23})(2+\alpha_1) + S_{66}(2-\alpha_1)\}/4 \\
S^L_{55} &= \{2(S_{22}-S_{23})(2-\alpha_1) + S_{66}(2+\alpha_1)\}/4 \\
S^L_{45} &= \{2(S_{22}-S_{23})\beta_1 - S_{66}\beta_1\}/4 \\
S^L_{33} &= S_{22}
\end{aligned}
\qquad (18)
$$

where

$$S_{11} = \frac{C_{22} + C_{23}}{C_{11}(C_{22}+C_{23}) - 2C_{12}^2}, \qquad S_{12} = \frac{C_{12}}{2C_{12}^2 - C_{11}(C_{22}+C_{23})}$$

$$S_{22} = \frac{C_{11}C_{22} - C_{12}^2}{C_{11}(C_{22}^2-C_{23}^2) - 2C_{12}^2(C_{22}-C_{23})},$$

$$S_{23} = \frac{C_{12}^2 - C_{11}C_{23}}{C_{11}(C_{22}^2 - C_{23}^2) - 2C_{12}^2(C_{22} - C_{23})}$$

$$S_{66} = \frac{1}{C_{66}}, \quad 2(S_{22}-S_{23}) = \frac{2}{(C_{22} - C_{23})} \qquad (19)$$

and $\underset{\sim}{S}^L$ repredents $(\underset{\sim}{C}^L)^{-1}$. By inverting $\underset{\sim}{S}^L$, obtained in equation (18), we thus obtain the lower bound for the stiffness matrix. We therefore know both the upper and lower bounds of the stiffness matrix of the composite material. The limits within which the behaviour of the composite must lie are therefore given by the two equations

$$\underset{\sim}{\sigma} = \underset{\sim}{C}^U \cdot \underset{\sim}{\epsilon} \qquad (20)$$

and

$$\underset{\sim}{\sigma} = \underset{\sim}{C}^L \cdot \underset{\sim}{\epsilon}$$

Due to the conditions of stress and strain to which Hashin and Rosen assume their element to be subjected, these two bounds to the behaviour of the composite only coincide for a uniaxially reinforced composite.

Laplace transformation solution for the stress-strain relationship of a fibre reinforced linear viscoelastic composite

Biot (1954) has proved for the general anisotropic case that any viscoelastic problem can be associated with the corresponding problem where all the components are elastic.

Williams (1964) in his review on the structural analysis of viscoelastic materials, discusses this correspondence rule and the techniques used in its application. The method depends on transforming the equilibrium, compatibility, and boundary conditions with respect to time and thus obtaining a set of associated equations in the transform plane in terms of the transformed variable, p. Having solved these associated equations, the final step involves the inversion of the transformed solution back to real time.

Before obtaining this solution it is necessary to define the time-dependent behaviour of the matrix material in general terms. Dootson (1968) has discussed the general accuracy of two different methods of describing the creep compliance of a viscoelastic material. The first of these is of a simple power law relationship with time of the type suggested by Findley (1962)

$$\epsilon_m(t) = (a+bt^n).\sigma_m \tag{21}$$

This is often a good approximation but it is limited to a small range of shapes of creep curve. A more complicated approximation is that obtained by the use of a discrete spectrum for retardation times:

$$\epsilon_m(t) = \left\{S_o + \sum_{i=1}^{n} S_i(1-e^{-t/\gamma_i})\right\}\sigma_m \tag{22}$$

This approximation is capable of fitting a large range of creep curves to a high degree of accuracy and as it is a more general method it will be used here. In order to complete the description of the time-dependence of the isotropic matrix it is necessary to define the Poisson's ratio. Turner (1966) has suggested that the assumption that the Bulk modulus of the material remains constant often provides an acceptable approximation to the Poisson's ratio, and this approximation will be used here.

Equation (20) describes the upper and lower bounds of the behaviour of the elastic composite and so to obtain the solution for the viscoelastic composite we must replace all time-dependent variables by their Carson transforms. This gives the general relationship as

$$p.\hat{\underset{\sim}{\sigma}}(p) = p.\hat{\underset{\sim}{C}}(p) \cdot p\hat{\underset{\sim}{\epsilon}}(p) \tag{23}$$

where $\hat{\underset{\sim}{C}}(p)$ is known in terms of the transformed modulus, $\hat{E}_m(p)$, and the transformed Poisson's ratio, $\hat{\nu}_m(p)$, of the matrix material. These are given by

$$p\,\hat{E}_m(p) = \left\{S_o + \sum_{i=1}^{n} S_i\left(\frac{1}{1+p\gamma_i}\right)\right\}^{-1} \tag{24}$$

and

$$p.\hat{\nu}_m(p) = \frac{1}{2}\left\{1 - \frac{p}{3K_m}\hat{E}_m(p)\right\} \tag{25}$$

If we wish to calculate the strain response to a given stress input we must first invert the matrix of the transformed stiffnesses and then take the inverse transform of the resulting expression. To invert the transform exactly requires either the use of transform tables or of a formal inversion using

$$\underset{\sim}{\epsilon}(t) = \frac{1}{2\pi i}\oint \hat{\underset{\sim}{\epsilon}}(p).e^{pt}.dp \tag{26}$$

both of which are liable to be difficult in general.

Let us consider how we may invert the transform numerically for the particular case of the creep of the composite where the stress is applied as a step input. For this case equation (23) can be written in the form

$$p.\hat{\underset{\sim}{\epsilon}}(p) = \left(p\hat{\underset{\sim}{C}}(p)\right)^{-1}.\underset{\sim}{\sigma} \tag{27}$$

It has already been described how a series of exponential terms describes the creep compliance of the matrix accurately, and it seems reasonable to assume that the same form of approximation can be used to describe the creep behaviour of the composite. Thus we assume that

$$\underset{\sim}{\epsilon}(t) = \underset{\sim}{S}'(t) \cdot \underset{\sim}{\sigma} \tag{28}$$

where

$$S'_{qr} = \left\{S'_o + \sum_{i=1}^{n} S'_i(1-e^{-t/\gamma_i})\right\}_{qr} \tag{29}$$

describes the creep behaviour of the composite. Transforming this to the p plane we obtain

$$p\,\hat{\underset{\sim}{\varepsilon}}(p) = p.\hat{\underset{\sim}{S}}'(p)\,.\,\underset{\sim}{\sigma} \tag{30}$$

where

$$\hat{S}'_{qr} = \left\{ S'_o + \sum_{i=1}^{n} S_i{}' \left(\frac{1}{1+p\gamma_i} \right) \right\}_{qr} \tag{31}$$

Using either the collocation method suggested by Schapery (1962) or a linear regression technique we can calculate the values of $(S'_i)_{qr}$ that give the best fit of $\hat{S}'_{qr}$ to the elements of the inverted matrix of transformed stiffnesses. Consequently we can substitute these values of S'_i into equations (28) and (29) to give us the creeep behaviour of the composite.

Comparison of the elemental method with the simplified fibre method

In order to ascertain the merits of the elemental method for describing the time-dependence of fibre reinforced materials it is necessary to compare the results obtained from it with those obtained by some other method. Here the comparison will be made with the simplified fibre method, originally suggested by Cox (1952) and Arridge (1963) for the elastic case and extended to the time dependent case by Dootson (1968).

So that these methods can be compared it was necessary to write a computer program capable of using the method described in this note. The language in which the program was written is Algol and the program has been developed and run on the Cranfield Computing Centre's ICT 1905 computer. The flow diagram of the program showing the order of the steps used in the calculation of the bounds to the compliance is shown in Fig. 3.

The most direct comparison between the two methods can be obtained by considering the angular variation of compliance for a unidirectionally reinforced composite. As for this particular case there is both stress and strain compatibility between the elements the upper and lower bounds to the solution coincide. In Fig. 1 the angular variation of the compliance for an isophthalic polyester resin reinforced unidirectionally by 'E' glass having a volume fraction of 0.24, is shown as predicted by the two methods. Parallel to the direction of the fibres it can be seen that the two models yield the same result, both for the initial compliance and the time-dependent compliance which is represented here by the 1,000 min. curve. As the angle between the line of action of the applied stress and the fibre axis increases, the elemental model gives rise to a stiffer composite than does the simplified fibre model. This is due to the simplified fibre model assuming that the fibres only have stiffness along their axes while the elemental model assumes them to be isotropic.

It is interesting to note that the compliance predicted by the simplified fibre model exceeds that of the unreinforced resin at angles greater than 36° from the fibre axis. This is due to the fact that the model considers the fibres to have no stiffness normal to their axes and consequently to act as voids in this direction. The elemental model, as it allows fibre contribution normal to the fibre axis, is stiffer than the unreinforced resin.

Both models predict that the compliance is maximum at about 60° from the fibre axis. This is due to the stiffening effect of the Poisson's ratio of the fibres normal to their axis. For the fibres to be able to stiffen the composite in this way they must be capable of taking a compressive load. It is likely that in practice the fibres may tend to buckle under compressive loads even though they are embedded in a constraining medium and that this stiffening effect at 90° may be less noticeable.

The second set of calculations that have been made using the elemental model is for the case of a random distribution of fibres in the plane of the composite. This is to show the difference between the bounds predicted, assuming either stress or strain compatibility, when the fibres are not all parallel. The upper and lower bounds predicted by the elemental method for an isophthalic polyester resin reinforced with randomly orientated 'E' glass fibres are shown in Fig. 2. For this particular case the bounds differ by about 30% for low values of time and 50% for high values of time. These bounds are compared with the simplified fibre prediction which is equivalent to a lower bound of the compliance.

In conclusion to this comparison between the two methods it should be noted that without any experimental results to compare these predictions with, no absolute value can be placed on the merits of either method. The elemental model used in this note seems the more realistic and the Laplace transform method of solution is certainly superior to the Integral Equation Techniques used previously. To improve the model suggested here it would appear that it is necessary to decrease the distance apart of the bounds for the non-parallel fibre case by improving the stress and strain compatibility.

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Figure 1. Angular variation of compliance for a unidirectionally reinforced polyester resin. $v_f = 0{\cdot}24$

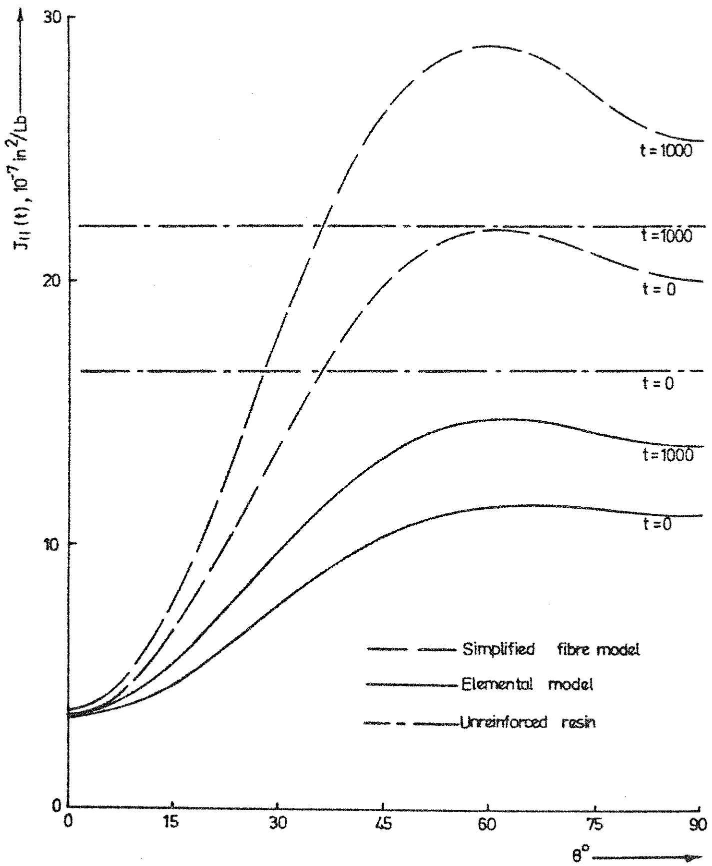

Figure 2. Tensile creep curves predicted for a randomly reinforced polyester resin. $v_f = 0{\cdot}215$.

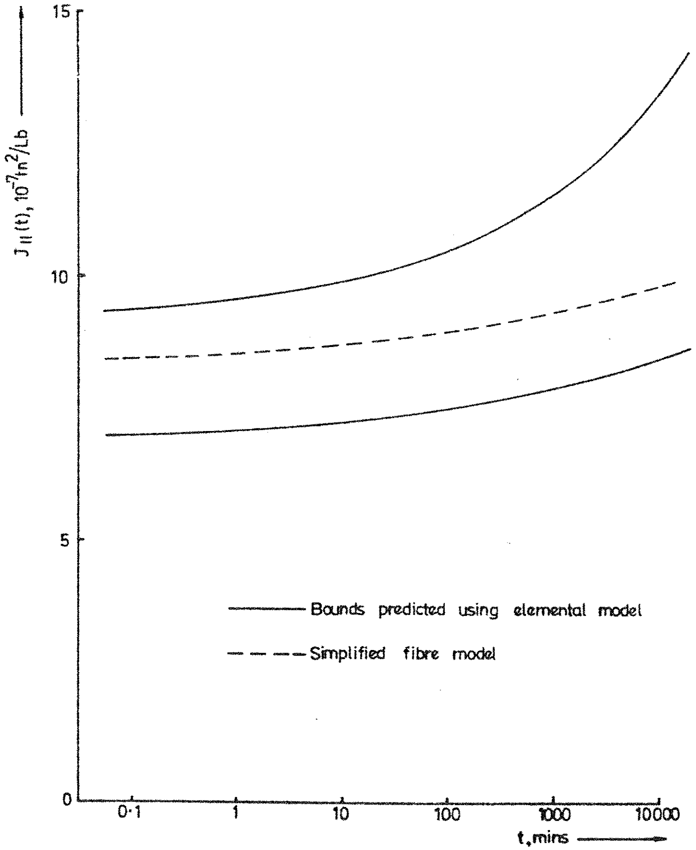

Figure 3 Flow diagram of the computer program

INPUT

N	
S[0:N] G[1:N]	} describes time dependence of matrix (eqn. 22)
A1,A2,B1,B2	describe fibre orientation (eqn. 16)
VF,VM	volume fractions of phases
EF,NUF	Young's modulus and Poisson's ratio of fibres
NUM	Instantaneous Poisson's ratio of matrix

↓ eqns. 24 and 25

P-multiplied Laplace Transforms of Young's modulus and Poisson's ratio of the matrix

↓ eqns. 1,2,3,4,5

P-multiplied Laplace Transform of the stiffness matrix of a composite element

- ↓ eqn. 17
  - Upper bound of P-multiplied Laplace Transform of the stiffness matrix of the composite
  - ↓ (inversion)
  - P-multiplied Laplace Transform of the compliance matrix of the composite for Upper bound stiffness
- ↓ eqn. 19 (inversion)
  - P-multiplied Laplace Transform of the compliance matrix of a composite element
  - ↓ eqn 18
  - P-multiplied Laplace Transform of the compliance matrix of the composite for Lower bound stiffness

↓

Linear regression to find the best fit of equation (31) to each of the elements of the p-multiplied Laplace Transform of the compliance matrix of the composite

↓

OUTPUT S[0:N,1:6,1:6] describes the time-dependence of the compliance matrix for the upper and lower bounds of the stiffness of the composite