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A fast-convergence algorithm for reliability analysis based on the AK-MCS

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Highlights

- The algorithm performance of existing AK-MCS structures is sensitive to the Kriging trend and initial candidate size.
- A new DoE selection method is developed based on clustering theory.
- Multiple trends method is integrated into the new AK-MCS structure.
- The algorithm efficiency of the proposed structure is validated against existing methods using four engineering examples.

In the field of reliability engineering, assessing the probability of failure of an event is usually a computationally demanding task. One way of tackling this issue is by metamodelling, in which the original computational-expensive model is approximated by a simpler metamodel. A method called AK-MCS for Active learning reliability method combining Kriging and Monte Carlo Simulation, was developed for the metamodel construction, and proved effective in reliability analysis. However, the performance of the AK-MCS algorithm is sensitive to the candidate size and the Kriging trend. Moreover, it cannot take advantage of parallel computing, a highly efficient way to speed up the simulation process. Focusing on the identified issues, this study proposes three methods to improve algorithm performance: the candidate size control method, multiple trends method, and weighted clustering method. These three methods are integrated into the AK-MCS structure, with their individual and combined performances being tested using four examples. Results suggest that all three methods contribute to the improvement of algorithm efficiency. When the three methods working together, the computational time is reduced significantly, and in the meantime, higher accuracy can be achieved.

Key words: Reliability; Kriging; Adaptive refinement; Monte Carlo simulation

1. Introduction: Kriging in reliability analysis

Reliability analysis quantifies the influence of uncertainties on the safety of a product or a system. In the framework of probabilistic-based analysis, reliability is measured by the probability of failure P_f, which is defined as:

$$Reliability = 1 - P_f \tag{1.1}$$

Given a model $\mathcal{M}$, failure is considered when the model response $\mathcal{Y}$ of the input vector $x \in \mathbb{X}$ is negative [1]. Thus, the probability of failure can be expressed as:

$$P_f = \mathbb{P}(\mathcal{Y} \leq 0) = \mathbb{P}(\mathcal{M}(x) \leq 0) \tag{1.2}$$

Strategies to calculate the probability of failure P_f fall into two categories [2]. First is the gradient-based methods, such as the first-order (FORM) and second-order (SORM) reliability methods, and the most probable point (MPP)-based method [3]. The biggest problem of these methods is their unstable performance when dealing with high dimensional and nonlinear systems. Another category is the simulation-based methods, which include the Monte Carlo Simulation (MCS), importance sampling (IS) [4], line sampling [5] and adaptive sampling [6]. Simulation-based approaches conduct a repeating analysis of the model with different input variable values and could therefore deal with high-dimensional and nonlinear problems. For computational expensive models, simulation-based methods are usually criticized for requiring an enormous number of runs to achieve the target accuracy. Metamodels or surrogate models [7] solve this dilemma by building a simpler approximation ($\widehat{\mathcal{M}}$) of the original model ($\mathcal{M}$). When a metamodel is accurate enough, MCS can be easily carried out with little effort.

Building a metamodel consists of four steps [7]: (a) Choose a functional form for the original expensive model $\mathcal{M}$; (b) Select *Design of Experiments* (DoE) $\mathcal{X}$, where DoE is a set

of points in the design space that the actual response $\mathcal{Y}(\mathcal{X})$ needs to be calculated from $\mathcal{M}$; (c) Fit $\widehat{\mathcal{M}}$ using the actual response $\mathcal{Y}(\mathcal{X})$; (d) Assess the adequacy of the metamodel.

Different metamodel functional forms have been adopted in literature, such as polynomial chaos expansion [8], support vector machine [9], Bayesian inference [10] and Kriging method [11]. Among all the metamodels, Kriging is an exact interpolation method and provides both the predict response and the uncertainty associated with the prediction.

The essence of Kriging is a spatial interpolation method proposed by South African mining-engineer Krige who used it to evaluate mineral resources [11]. His work was then appreciated by Matheron in the sixties. During his research, he formalized Krige's work and developed a complete mathematical model called the theory of regionalized variables, which becomes the theoretical basis of Kriging method. A comprehensive review of Kriging method and applications can be found in [3]. Kriging combined with the active learning method was first introduced to solve the Efficient Global Optimization (EGO) [13]. It was then cooperated with a learning function called the Expected feasibility function (EFF) and used for reliability analysis by Ranjan [14].

One of the most basic and frequently used adaptive Kriging structure for reliability analysis is proposed by Echard et al. [15], called AK-MCS, for Active learning reliability method combining Kriging and Monte Carlo Simulation. AK-MCS treats the reliability calculation as a binary classification problem. Consequently, it tends to explore points near the limit state surface ($\mathcal{Y}(x) = 0$), rather than the points that are quite certain to fall into the failure or safe domain. This process is briefly summarized as:

(a) Generate a candidate set from the design space using MCS.

(b) Build or update Kriging (Ordinary) model from DoE.

(c) Calculate Kriging estimation and learning function (LF) value of each candidates.

(d) DoE enrichment and go to (b) until stopping criteria is satisfied.

(e) Algorithm stops and outputs the probability of failure.

Compared with other traditional simulation-based methods like MCS, fewer calls of the original expensive physical model are required in this structure. This is achieved by selecting and evaluating only the most "dangerous" points from the candidates in an adaptive manner. Therefore, the intuition behind the high efficiency of this method can be summarized as: the failure probability is the classification result of the AK-MCS model, while in MCS, it focuses not only on the accuracy of sign, but more demandingly, the actual response of each candidate.

Because of the high efficiency and generality of this method, it has been quickly applied in various fields in reliability engineering. Gaspar, B. et al [16] compared the adaptive Kriging method with the polynomial regression method using a finite element structural model in marine applications, and proved its efficiency and accuracy. Bdour, T. et al [17] used the AK-MCS method to evaluate the reliability of radiated susceptibility in coaxial shielded cables. In his work, a 5-dimensional non-linear application only required 211 runs of original models to achieve a similar failure probability as MCS. Varentsov, M. et al [18] utilized 10 Kriging methods in the large-eddy simulations for high-resolution local temperature mapping. To minimise experiment duration and material usage, Noack, M. et al [19] applied Ordinary Kriging to autonomous X-Ray scattering experiments control, and the obtained variogram is insensitive to local data variation.

Although AK-MCS reduces the computation burden significantly, multiple problems have been reported during the usage. For examples, the single candidate enrichment strategy cannot take advantage of the parallel computing; the choice of Kriging trend is quite arbitrary; a slow convergence is noticed when candidate size is inappropriate. A brief summary of the problems of AK-MCS with the corresponding methods is presented in Section 3. In this paper, focusing on the three issues mentioned above, three methods are proposed to speed up the convergence without compromising the algorithm accuracy. Firstly, a new clustering method

has been developed for DoE enrichment. It combines the K-means clustering and K-medoids clustering methods, selecting DoE based on their spatial distribution and estimated uncertainties. Secondly, based on the multiple surrogates theory [20][21], multiple Kriging metamodels are built in each iteration using four different trends, and only the best metamodel is selected. When building a metamodel is much cheaper than evaluating the physical model, multiple metamodels provide the extra benefit of decreasing the risk of poor fit [20]. Finally, a candidate size control method has been developed to choose an appropriate size of candidate set in an adaptive manner. The paper is organized as follows: Section 2 gives a brief reminder of Kriging metamodel theory. Section 3 reviews the problems in AK-MCS, and the existing strategies toward them. Section 4 describes the three proposed features and their implementation in AK-MCS. In Section 5, the proposed methods and structure are validated using four numerical examples.

2. Kriging theory

For a computational-expensive model $\mathcal{M}$, it maps the n-dimensional input vector $x \in \mathbb{X}$ to the one-dimensional output y. Kriging metamodel assumes $\hat{y}$ is a realization of a Gaussian process, and has the following expression to approximate $\mathcal{M}$ and its response $\hat{y}$ [22]:

$$\hat{y} = \widehat{\mathcal{M}}(x) = \boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x}) + \sigma^2 Z(x, \omega) \tag{2.1}$$

where the first term on the right-hand side of the equation $\boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x})$ represents the trend (or the mean of the Gaussian process), and $\boldsymbol{\beta}^T = [\beta_1, \dots, \beta_p]^T$ is the regression coefficient of the basis function $\boldsymbol{f}(\boldsymbol{x}) = [f_1, \dots, f_p]$. The second term consists of the variance of the Gaussian process σ^2, and a stationary Gaussian process with zero-mean, unit variance. The elementary event in the probability space is represented by ω, and it is characterized by a correlation function R which describes the correlation between two sample points, and its hyper-parameter $\boldsymbol{\theta}$.

A given DoE set has a length of n, $\boldsymbol{x} = \{x^1, \dots, x^n\}$, the corresponding response set from model $\mathcal{M}$ is $\boldsymbol{y} = \mathcal{M}(x) = \{y^1, \dots, y^n\}$, the regression coefficient β and variance σ^2 of the process can be computed:

$$\beta = (\boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{F})^{-1} \boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{y} \tag{2.2}$$

$$\sigma^2 = \frac{1}{n} (\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta})^T \boldsymbol{R}^{-1} (\boldsymbol{y} - \mathbf{F}\boldsymbol{\beta}) \tag{2.3}$$

where $\boldsymbol{R}$ is the correlation matrix $\boldsymbol{R}_{i,j} = R(x_i, x_j; \boldsymbol{\theta})$, and $\mathbf{F}$ is the information matrix $\mathrm{F}_{ij} = f_j(x_i)$. Hyper-parameter $\boldsymbol{\theta}$ and variance σ^2 are normally unknown, and have to be estimated using methods like maximum likelihood principle and cross-validation [23].

Finally, the predicted mean $\mu_{\widehat{y_0}}$ and variance $\sigma^2_{\widehat{y_0}}$ of the realization at a sample point x_0 is giving by Kriging as [1]:

$$\mu_{\widehat{y_0}} = \boldsymbol{f}(x_0)^T \boldsymbol{\beta} + r_0^T(x_0) \boldsymbol{R}^{-1} (\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta}) \tag{2.4}$$

$$\sigma^2_{\widehat{y_0}} = \sigma^2 \left(1 - r_0^T(x_0) \boldsymbol{R}^{-1} r_0(x_0) + u^T(x_0) (\boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{F})^{-1} u(x_0)\right) \tag{2.5}$$

where $r_0(x_0) = R(x_0 - x^i; \boldsymbol{\theta})$ represents the correlation between x_0 and DoE $x^i \in x, i = 1, \dots, n$, and

$$u(x_0) = \boldsymbol{F}^T \boldsymbol{R}^{-1} r_0(x_0) - \boldsymbol{f}(x_0) \tag{2.6}$$

Recall that Kriging assumes each prediction point is subjected to a Gaussian distribution, and hence, the probability density function (PDF) at x_0 can be written as:

$$\widehat{y_0} = \widehat{\mathcal{M}}(x_0) \sim N(\mu_{\widehat{y_0}}, \sigma^2_{\widehat{y_0}}) \tag{2.7}$$

The probability of the sign of $\widehat{y_0}$ is easily read:

$$\mathcal{P}(\widehat{\mathcal{G}_0} \leq 0) = \Phi\left(\frac{0 - \mu_{\widehat{\mathcal{G}_0}}}{\sigma_{\widehat{\mathcal{G}_0}}}\right) \quad 2.8$$

$$\mathcal{P}(\widehat{\mathcal{G}_0} \geq 0) = 1 - \Phi\left(\frac{0 - \mu_{\widehat{\mathcal{G}_0}}}{\sigma_{\widehat{\mathcal{G}_0}}}\right) \quad 2.9$$

$\Phi(\cdot)$ denotes the cumulative probability function of the Gaussian distribution. Note that in this paper, $\mathcal{P}(\cdot)$ represents the probability associated with the epistemic uncertainty in Kriging prediction at a single sampling point, and $\mathbb{P}(\cdot)$ is the probability associates with the actual overall failure rate P_f. The overall failure rate equals to the number of sample points with a negative response divided by the total number of sample points. Therefore, to estimate the failure probability P_f using Kriging, the expression in Eq 1.2 can be rewritten as:

$$\widehat{P_f} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\hat{\mathcal{G}} \leq 0}(x_i) \quad 2.10$$

Where $\mathbb{I}_{\hat{\mathcal{G}} \leq 0}$ is the indicator function such that $\mathbb{I}_{\hat{\mathcal{G}} \leq 0}(x) = 1$ for sample x in the failure domain and $\mathbb{I}_{\hat{\mathcal{G}} \leq 0}(x) = 0$ for sample x in the safe domain.

Coefficient of variation (CoV) is used to check the convergence of MCS, and it is also a measure of the estimation accuracy:

$$CoV\left(\widehat{P_f}\right) = \sqrt{\frac{1 - \widehat{P_f}}{n\widehat{P_f}}} \quad 2.11$$

It is used to as a convergence criterion to adaptively determine the MCS size until a certain value CoV_{thr} is reached:

$$CoV\left(\widehat{P_f}\right) \leq CoV_{thr} \quad 2.12$$

Eq 2.12 indicates that for a reliability problem with a low failure probability, the MCS sample size required to achieve high accuracy needs to be increased.

3. Problems encountered in AK-MCS: a short review

In this section, the main components of AK-MCS, along with the associated problems, are reviewed and discussed. First, the issues addressed in the literature and their solutions are summarized as follow:

(a) Sampling strategy of candidates and initial DoE

In AK-MCS, MCS is used to generate candidates. Instead of directly applying the joint PDF to generate the candidates, the Latin Hypercube Sampling (LHS) has been a popular alternative to generate more scattered samples within the design domain [1] [20]. MCS normally requires a large sample size to achieve an acceptable variation coefficient. The solution to this problem is using variance reduced sampling methods. Importance sampling (IS) is cooperated in the AK-MCS framework [24][25] for sampling the candidate points from an importance density centred about the MPP identified by FORM. When facing low failure probability problems, sampling methods such as subset simulation [26], MCMC-IS [27] and MCMC-LS [5], have been proved effective in reducing computational costs.

(b) Trend of Kriging model

The AK-MCS uses ordinary Kriging, which adopts a constant but unknown value, as the trend. It is a simple but effective assumption since there is no prior knowledge of the original model. But in some applications, the ordinary Kriging shows a slow convergence [28]. This problem can be solved by using other forms of function as Kriging trends. Inspired by the high interpolating performance of PCE, Schobi [1] applies PCE as the trend and develops the PC-Kriging structure. Vahedi, J [20] builds four Kriging models at each iteration with constant,

linear and quadratic regression trends, respectively. And by computing the Kullback–Leibler (KL) divergence [29] and learning function values, the best Kriging is chosen for the final metamodel.

(c) Learning function

Leaning function is the core of the AK-MCS algorithm. It helps determine the critical region in the domain, where the next batch of DoE is sampled. Two learning functions are used in the original work: EFF and U function. EFF is first proposed in [30]. It indicates how well the true response of a candidate point is expected to satisfy a constraint over a defined region. However, it tends to approximate the whole limit surface rather than the boundary. Hence, a new learning function that reflects misclassification closer to the predicted limit state is proposed, and called the U function[15]:

$$U(x) = \frac{|\mu_{\hat{g}}(x)|}{\sigma_{\hat{g}}(x)} \tag{3.1}$$

Where $\mu_{\hat{g}}(x)$ and $\sigma_{\hat{g}}(x)$ are mean and standard deviation of a Gaussian distribution at candidate x given by Kriging prediction.

A weighted U function proposed by Sundar [28] suggests that the adaptive performance can be further improved by multiplying the original value with its PDF. Sun et al. [31] develop a Least Improvement Function (LIF) that accounts for the joint PDF of the inputs in the DoE selection. A learning function called H function based on information entropy theory is developed in [5]. Zhang et al. [32] derived a reliability-based expected improvement function (REIF) from the folded-normal distribution for rate event estimations.

(d) DoE enrichment method

In Echard's work, the criterion of DoE enrichment is governed by the learning function values. The candidate with the minimum U function value or the maximum EFF function value is selected as the best next point and added into the DoE set. When it comes to adaptive Kriging modelling, higher efficiency can be achieved by selecting not a single, but multiple candidates for the next iteration. This issue was first addressed in Ginsbourger's work [33]. It proposes two strategies: Kriging Believer (KB) and Constant Liar (CL), for multi-candidate selection. Other strategies towards the parallelization issue mainly rely on clustering techniques. In many research [1][34][35], K-means clustering has been used widely to make a compromise between the uncertainty of candidates, and the distances between cluster centres. In [20], KL divergence is used as an indicator of uncertainty between multiple Kriging models.

(e) Stopping criteria

In the active learning process, the most frequently used stopping criteria are based on the values of learning function EFF and U. To be specific, the learning process stops when the maximum value of EFF is smaller than a threshold ($\max\big(EFF(x)\big) < 0.001$), or when the minimum value of U function is larger than a threshold ($\min\big(U(x)\big) > 2$) [15]. However, it is argued that the two criteria focus more on the accuracy near the limit state rather than the probability of failure itself [1]. Therefore, an alternative criterion proposed in [1] suggests that the algorithm stops learning when:

$$\Delta\hat{P}_f = \frac{\hat{P}_f^+ - \hat{P}_f^-}{\hat{P}_f^0} \leq \epsilon_{\hat{P}_f} \tag{3.2}$$

for two consecutive iteration steps, where $\epsilon_{\hat{P}_f} = 5\%$ [9]. $\hat{P}_f^0$ and the upper bound $\hat{P}_f^+$ and lower bound $\hat{P}_f^-$ of failure probability are defined as:

$$\hat{P}_f^0 = \mathbb{P}\big(\mu_{\hat{g}}(x) \leq 0\big) \tag{3.3}$$

$$\hat{P}_f^{\pm} = \mathbb{P}\big(\mu_{\hat{g}}(x) \mp k\sigma_{\hat{g}}(x) \leq 0\big) \tag{3.4}$$

For a given confidence level $(1-\gamma)$, $k = \Phi^{-1}\left(1-\frac{\gamma}{2}\right)$. Normally, $\gamma = 5\%$ is sufficient, and hence has $k = \Phi^{-1}(97.5\%) = 1.96$.

Another technique to evaluate the convergence of failure probability is using cross-validation [34][36]. Xiao [36] constructs multiple sub-Kriging models with part of the DoE at each iteration. Based on the central limit theorem, Wang, Z [37] proposes a convergence criterion using the maximum error estimation combined with EFF.

Apart from the above basic components in the AK-MCS framework, an adaptive sampling region (ASR) is suggested to be calculated at the beginning of each iteration ([35][37]). ASR excludes candidates with small PDF value and selects DoE from the region with high PDF values. Based on the similar idea, Yang et al. [38] proposed a concept called the truncated candidate region (TCR), in which candidates have 80% probability to be negative in a failure mode.

The present work aims to improve the efficiency of AK-MCS, focusing on three identified issues of the original model: (a) Stopping criteria in Eq 3.2 shows a higher efficiency than the learning function-based criteria; however, it is very sensitive to the size of MCS candidates. A over large or over small size of the candidate set shows an extremely slow convergence, and hence increasing the computational loads; (b) The performance of Kriging model is highly sensitive to the trend. For most cases, the number of calls required from the physical model can be reduced significantly when a proper trend is provided for the metamodel construction. (c) The K-means clustering method is effective in dividing sampling regions but cannot guarantee a proper distance between DoE. For instance, Figure 1a shows the candidate selection result obtained from the clustering method in [35]. Candidates with the highest uncertainty of each cluster are selected as DoE (black dots). However, because K-mean clustering does not consider uncertainties of candidates during the clustering process, consequently, some DoE are spatially close to each other (see DoE in the purple square).

Calculating all those DoE in the physical model is considered inefficient, because without prior knowledge, it is very likely that information provided by at least one of those DoE is redundant.

To address the above issues, the present work proposes an adaptive Kriging structure using a new weighted clustering method, multiple Kriging trends and candidate size control method. In the end, four case studies targeting different engineering problems are performed to showcase the high efficiency of the proposed algorithm.

4. Methodology

4.1 A new weighted clustering DoE enrichment method

Clustering candidates and using the cluster centres as the next best DoE set is a good practice to ensure the uniformity of DoE among candidates. However, it ignores the uncertainty estimated by the learning function. If the selected cluster centres have low uncertainty, a further calculation of their true responses contributes little to the convergence of the algorithm. A solution proposed in [35] suggests to choose the candidates with the highest uncertainty value in each cluster as the DoE for the next iteration. However, during the practice, very close distances are observed between two DoE in some iterations. As mentioned in the previous section, if two DoE in an iteration are spatially close to each other, they may provide similar information for the Kriging refinement. These problems are addressed by a proposed method inspired by K-means and weighted K-medoids algorithms.

Both K-means and K-medoids clustering methods group the data by minimizing the sum of distances between data and the cluster centroid. The difference between the two methods is the selection of cluster centres. The cluster centres in K-means are the arithmetic mean of all points in each cluster, and therefore do not have to be one of the points and are easy to compute. In contrast, the cluster centres (medoids) of K-medoids are chosen from actual input points. K-medoid clustering falls into the NP-hard problem category and can take a large

amount of time to converge [39]. Therefore, the proposed method first uses K-means to divide the candidate space into m clusters, and in each cluster, its centre is selected using the weighted K-medoids clustering when the number of clusters equals 1. In the k^{th} cluster, the weighted distance of the i^{th} sample (the potential cluster center) is defined as:

$$\mathrm{d}_{w,k,i} = \sum_{i=1}^{m_k} \frac{d_{euclidean,i}}{{U_{cc,i}}^{\alpha}} \tag{4.1}$$

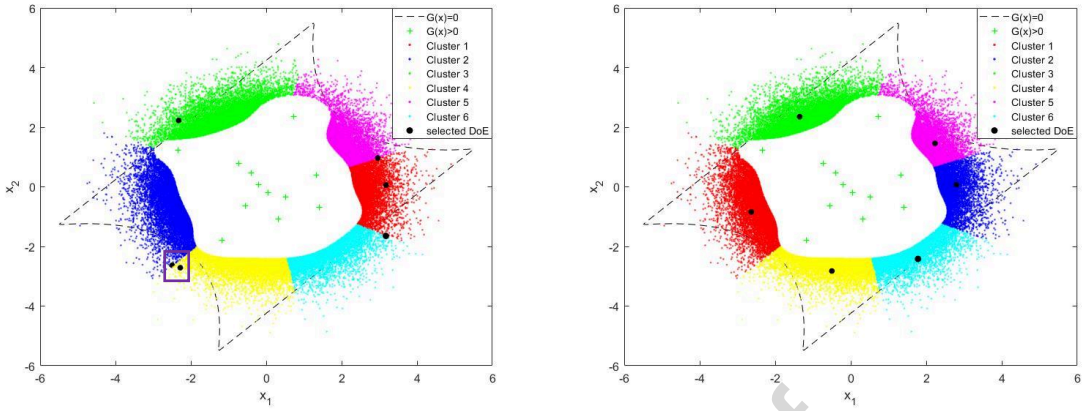
where m_k represents the number of sample points in the k^{th} cluster, $U_{cc,i}$ is the normalized U function value of the i^{th} sample ($U_{cc} = \frac{U-U_{min}}{U_{max}-U_{min}} + 1$, where U_{max} and U_{min} are maximum and minimum value of the original U function, respectively), and $d_{euclidean,i}$ is the Euclidean distance between the i^{th} sample and the rest samples in the cluster. α is the relaxation factor to adjust the weight of uncertainty in the expression. Normally, $\alpha = 1$ is used, but when a certain convergence is achieved, it punishes the uncertainty term, avoiding the algorithm keeping search a small region during iterations.

Like K-medoids, in this algorithm, the cluster centre is the sample point that has the minimum weighted distance between it and the rest points in the cluster. All cluster centres are selected as DoE for the next iteration and can be written as:

$$\mathrm{DoE}_k^{j+1} = \min d_{w,k,i} \tag{4.2}$$

The proposed cluster centres are a compromise between spatial representativeness and classification uncertainty. The comparison of the DoE selection methods (K-means method and the proposed method) at one of the iterations in the four-branch example is shown in Figure 1. Candidates in all the clusters exhibit high estimation uncertainties ($U(\mathrm{x}) > 2$). Visually, DoE selected by the proposed weighted clustering method has an evener distribution among the candidates than the K-means. Moreover, the estimated values of accuracy indicator $\Delta\hat{P}_f$ of the two methods in the current iteration are 2.90 and 3.21, respectively. Since a larger $\Delta\hat{P}_f$

indicates a less accuracy estimation, the proposed method provides a better way of DoE selection in this example, and therefore improves the performance of Kriging estimation.



(a) (b)

Figure 1 Comparison of DoE enrichment (m=6) results using: (a) K-means method, (b) proposed weighted clustering method

4.2 The multiple trends Kriging

The trend is a crucial ingredient in building a Kriging metamodel, and the selection of it varies with the types of data. Traditionally, depending on the functional basis of the trend in Kriging, three types of Kriging model are commonly used: simple Kriging, ordinary Kriging and universal Kriging [40]. The trend in simple Kriging is a known function. In ordinary Kriging, the trend has a constant yet unknown value:

$$\boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x}) = \beta_0 f_0(x) = \beta_0 \tag{4.3}$$

Ordinary Kriging is the most frequently used form, and it is also applied in the original AK-MCS structure. The trend in universal Kriging is a general polynomial such as the linear combination of the previous two trends. Because no prior knowledge of the data is available, therefore, ordinary Kriging and three universal Kriging trends: linear, quadratic and PCE, are selected to construct the Kriging metamodel in this study. For a l-dimensional system, the linear trend has:

$$\boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x}) = \beta_0 + \sum_{i=1}^{l} \beta_i x_i \quad 4.4$$

For the quadratic trend:

$$\boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x}) = \beta_0 + \sum_{i=1}^{l} \beta_i x_i + \sum_{i=1}^{l} \sum_{j=1}^{l} \beta_{ij} x_i \, x_j \quad 4.5$$

PCE is another metamodel method which approximates a model by the sum of orthonormal polynomials [41]:

$$\hat{y} = \widehat{\mathcal{M}}^{PCE}(x) = \sum_{\varphi \in \mathbb{N}^n} a_\varphi \phi_\varphi(x) \quad 4.6$$

where $\phi_\varphi(x)$ are multivariate polynomials and a_φ are the coefficients to be determined. Substitute $\boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x})$ in Eq 2.1 with Eq 4.6, PC-Kriging metamodel can be written as[42]:

$$\hat{y} = \widehat{\mathcal{M}}^{PCK}(x) = \sum_{\varphi \in \mathbb{N}^n} a_\varphi \phi_\varphi(x) + \sigma^2 Z(x, \omega) \quad 4.7$$

Details about PCE and PC-Kriging can refer to [42] [43].

In the proposed multiple trends Kriging, four Kriging metamodels are built with different trends at each iteration. Then, each metamodel gives its own estimation for the same candidate set, where the corresponding $\Delta\hat{P}_f$ of each metamodel is calculated. As $\Delta\hat{P}_f$ describes the level of misclassification, the Kriging metamodel shows the lowest $\Delta\hat{P}_f$ value represents the most confident estimation and is therefore selected as the best Kriging in the current iteration. The following steps of DoE enrichment and failure probability calculation are carried out based on the best Kriging. Note that at different iterations, the best Kriging can be different.

4.3 The candidate size control method

In the original AK-MCS method, the sample size of MCS candidates is defined before the learning process. During the learning process, the candidates remain the same at each iteration, where the failure probability is calculated. Candidates are only enriched when Eq 2.12 is not satisfied. Normally, for a reliability problem with an expected failure probability of 10^{-N}, the required candidate size is $10^{N+2} \sim 10^{N+4}$[44]. However, in most engineering cases, there is no prior knowledge of the failure probability, and therefore the size of MCS candidates is hard to determine. In the construction of Kriging metamodel, failure patterns could be lost if the candidate size is too small. And if the size is unnecessarily large, the computational burden increases because it is likely to require more DoE for the algorithm to reach the CoV convergence defined by Eq 3.2 [35]. Therefore, the algorithm has low efficiency when the size of MCS candidates is inappropriate. The effect of candidate size on the total number of DoE required to build the Kriging metamodel is demonstrated using the four branches function and presented in Table 1. It shows that when the candidate size is around 10^5, the AK-MCS algorithm has the highest efficiency and requires the least amount of DoE to converge. Therefore, a method optimizing the MCS candidate size adaptively is proposed to reduce the number of iterations without compromising the estimation accuracy.

In the proposed algorithm, the initial size of candidate set can be a random value. Although a good guess at the beginning could speed up the convergence process, the candidate size is adjusted gradually by the algorithm during iterations. The initial Kriging metamodel is constructed in the same manner as the original AK-MCS method, except with a larger threshold value $\epsilon_{\hat{P}_{f,1}}$. Because $\epsilon_{\hat{P}_{f,1}}$ is much larger than the target value of stopping criterion (normally, $\epsilon_{\hat{P}_f} \leq 5\%$), the initial metamodel requires a smaller amount of DoE to converge. Then, the metamodel calculates the predicted failure probability $\hat{P}_{f,1}$ based on the initial candidates set.

For a $\hat{P}_{f,1}$, the corresponding size of MCS candidate set $n_{target,1}$ can be calculated from Eq 2.11. And in the next iteration, $\delta \cdot n_{target,1}$ is used as the candidate size. Here, δ is a relaxation factor that controls the candidate size slightly larger than the required value. It ensures that the criterion in Eq 2.12 can be satisfied when there is a small fluctuation of $\hat{P}_{f,i}$ during the update of candidate size. During the usage, $\delta = 1.2$ is a good practice. Note that $\hat{P}_{f,i}$ (i represents the number of iteration) at the first few iterations is a rough estimation of the true P_f, and its accuracy depends on the value of $\epsilon_{\hat{P}_{f,i}}$. As $\epsilon_{\hat{P}_{f,i}}$ decreases, the estimation of $\hat{P}_{f,i}$ is closer to the true value; meanwhile, more DoE are required. In practice, an initial $\epsilon_{\hat{P}_{f,1}}$ value between 0.4 to 0.8 is recommended. In the following iterations, $\epsilon_{\hat{P}_{f,i}}$ is reduced by half every time convergence is achieved, until $\epsilon_{\hat{P}_{f,i}}$ equals to or smaller than the target value. When $n_{target,i}$ is calculated and compared with the candidates size of the current iteration n_i, three possible situations are considered: 1) $\delta \cdot n_{target,i} > n_i$, 2) $\delta \cdot n_{target,i} < n_i/2$, 3) $n_i/2 < \delta \cdot n_{target,i} < n_i$. The first situation represents the current candidate size is not large enough to satisfy Eq 2.12, and therefore needs to be enriched. In this situation, more candidates are drawn from the joint PDF and added to the candidate set. In the second situation, the current candidate size is too large for the required accuracy. Hence, a new candidate set with a size of $\delta \cdot n_{target,i}$ is generated for the next iteration. The candidate size in the last situation is considered a proper size, i.e., it is large enough to satisfy Eq 2.12, but not over large to jeopardize the algorithm efficiency. Therefore, it remains the same for the next iteration. The algorithm stops until both criteria in Eq 2.12 and Eq 3.2 are satisfied.

The framework of AK-MCS method implemented with the three proposed features is summarized as follow:

(a) Generate the initial MCS candidate set with a size of n_1 using LHS.

(b) Initialize $\mathrm{i} = 1, \mathrm{j} = 1$.

(c) Define the initial DoE set $\boldsymbol{x} = \{x^1, \dots, x^m\}$

(d) Evaluate the responses $\mathcal{y}(\boldsymbol{x}) = \{\mathcal{y}^1, \dots, \mathcal{y}^m\}$.

(e) Construct four Kinging metamodels using four trends: constant, linear, quadratic and PCE respectively.

(f) Predict the responses of MCS candidate set and calculate $\Delta\hat{P}_f$ of the four Kriging models. The model with minimum $\Delta\hat{P}_f$ is selected as the best Kriging model for the current iteration.

(g) Check convergence criterion $\Delta\hat{P}_f \leq \epsilon_{\hat{P}_{f,i}}$. If it is satisfied, go to step (h). If not, select next best candidates using weighted clustering method and add to DoE set, then, go to step (d).

(h) Check whether $\epsilon_{\hat{P}_{f,i}} \leq \epsilon_{\hat{P}_f}$ is satisfied. If yes, go to step (i). If no, update $\epsilon_{\hat{P}_{f,i+1}} = \max(\epsilon_{\hat{P}_{f,i}}/2, \epsilon_{\hat{P}_f})$, select next best candidates using weighted clustering method and add to DoE set, then, go to step (d).

(i) Compute $CoV(\hat{P}_f)$ and check convergence criterion $CoV(\hat{P}_f) \leq CoV_{thr}$. If it is satisfied, algorithm ends and output Kriging metamodel. If not, go to step (j).

(j) Calculate the required candidate size $n_{target,i}$ from Eq 2.11, and:

if $\delta \cdot n_{target,j} > n_j$: enrich candidate set with its size $n_{j+1} = \delta \cdot n_{target,j}$, go to step (f).

if $\delta \cdot n_{target,j} < n_j/2$: no action is taken and candidate set remains the same $n_{j+1} = n_j$, go to step (f).

if $n_j/2 < \delta \cdot n_{target,j} < n_j$: draw a new candidate set with a size of $n_{j+1} = \delta \cdot n_{target,j}$, go to step (f).

The flowchart of the proposed structure is presented in Figure 2. Note that the three added features are independent of each other and can be implemented individually in the AK-MCS algorithm. In the next section, four test functions are used to showcase the increasing efficiency of using the proposed algorithm. And the individual contribution of each feature will be discussed.

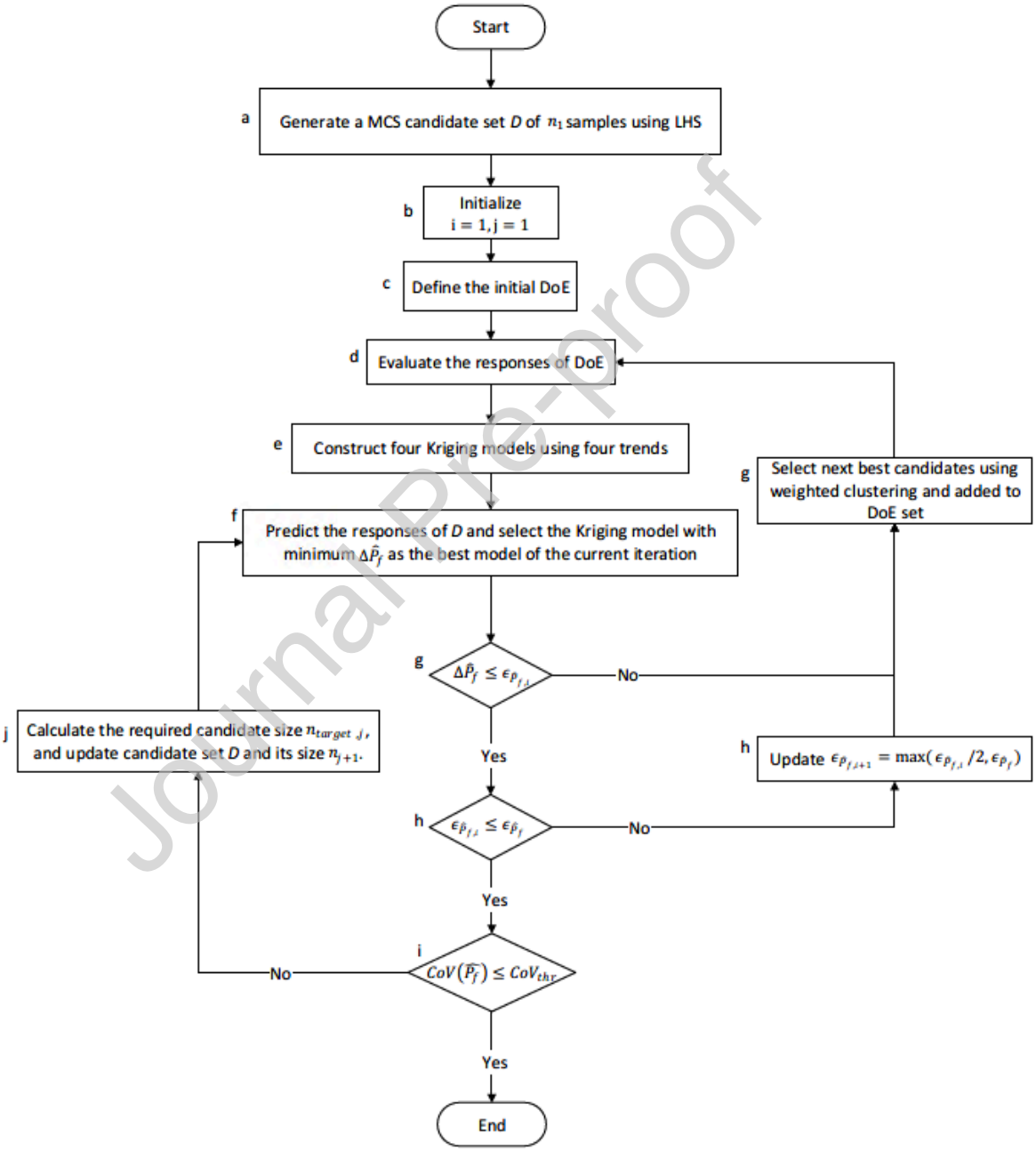


Figure 2 Flowchart of the proposed method

5. Numerical validations and discussions

The performance of the new structure - AK-MCS with the implementation of three proposed methods, is illustrated and investigated using four examples, including a two-dimensional function to observe the behaviour of the algorithm; an engineering problem with small failure probability ($<10^{-3}$); a mathematic function with high dimensions (>10); and a practical fluid dynamic problem. The performance of the proposed structure is compared against the MCS and the original AK-MCS method to demonstrate its efficiency. In all four examples, LHS is used for initial DoE sampling, and U function is chosen as the learning function. Note that the first three examples are computationally cheap, so that direct MCS solutions are available and can be used for validation. Whereas for the last example, the result obtained from the original AK-MCS is used as the benchmark.

5.1 Example 1: series system with four branches

The first example is a classic test function of reliability problem and has been explored in many studies [15][37][14]. It is a series system consists with four branches, and each branch represents a failure pattern. The limit state function is defined as:

$$G(x_1, x_2) = min\begin{Bmatrix} 3 + 0.1(x_1 - x_2)^2 - \frac{(x_1+x_2)}{\sqrt{3}} \\ 3 + 0.1(x_1 - x_2)^2 + \frac{(x_1+x_2)}{\sqrt{3}} \\ (x_1 - x_2) + \frac{6}{\sqrt{2}} \\ -(x_1 - x_2) + \frac{6}{\sqrt{2}} \end{Bmatrix} \quad 5.1$$

where two independent variables x_1 and x_2 are subjected to the standard normal distribution.

The proposed method is compared with AK-MCS methods and the results are presented in Table 1 Reliability results of Example 1. The performances of different methods are demonstrated in terms of the number of calls of physical model, N_{call}, the estimated probability of failure, $\hat{P}_f$, coefficient of variance, $CoV(\hat{P}_f)$, and the relative error, $RE_{\hat{P}_f}$ ($RE_{\hat{P}_f} =$

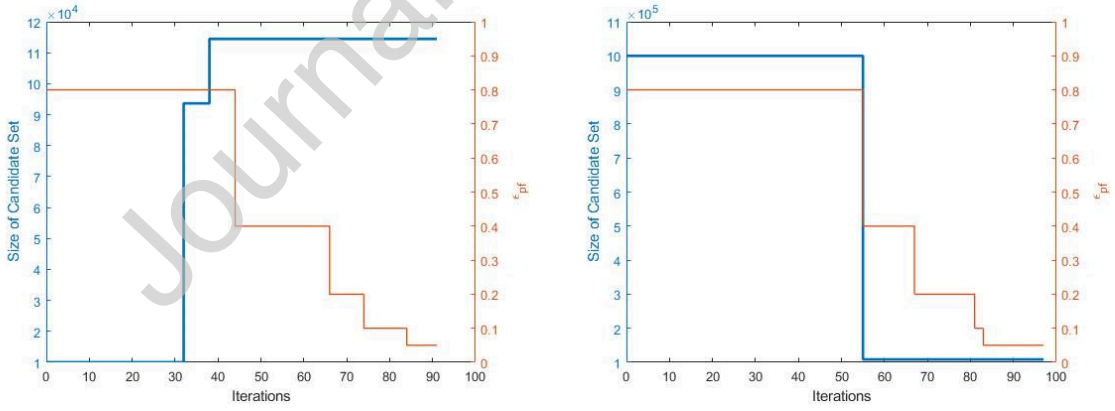
$(\hat{P}_{f,MCS} - \hat{P}_f)/\hat{P}_{f,MCS} \times 100\%)$. All the methods were evaluated 20 times, and results provided in Table 1 are the average of the 20 repeating runs. In this example, $\epsilon_{\hat{P}_f} = 5\%$ and $CoV_{thr} = 5\%$ are used as convergence criteria. Initial DoE size for all adaptive methods is 12.

Table 1 Reliability results of Example 1

Method	N_{call}	$\widehat{P_f}$	$CoV(\widehat{P_f})$	$RE_{\hat{P}_f}$
MCS	2.4×10^6	4.47×10^{-3}	0.97%	-
Original AK-MCS (10^4)	164	4.38×10^{-3}	4.88%	2.00%
Original AK-MCS (10^5)	116	4.40×10^{-3}	4.69%	1.56%
Original AK-MCS (10^6)	127	4.44×10^{-3}	2.17%	0.70%
AK-MCS+ Size Control (10^4)	120	4.38×10^{-3}	4.21%	2.00%
AK-MCS+ Size Control (10^6)	109	4.56×10^{-3}	4.19%	1.96%
AK-MCS+ Weighted Clustering	120	4.42×10^{-3}	2.18%	1.23%
AK-MCS+ K-means Clustering	132	4.42×10^{-3}	1.48%	1.67%
AK-MCS+ Multiple Trends	78	4.42×10^{-3}	1.50%	1.29%
AK-MCS+ All 3 Features	84	4.47×10^{-3}	4.45%	0.67%

The benchmark probability of failure is obtained using MCS with 2.4×10^6 samples. First, to illustrate the influence of candidate size on the AK-MCS efficiency, three different sizes (10^4, 10^5 and 10^6) are tested with the original AK-MCS method. All three cases manage to approximate the probability of failure well and show similar error rates. However, in terms of the algorithm efficiency, the candidate size of 10^5 requires the least number of calls of the physical model, and therefore is considered as the idealist candidate size among the three cases. The underlying reason is that, to satisfy $CoV_{thr} = 5\%$, the required size is around 9×10^4 for a problem with $\hat{P}_f = 4.47 \times 10^{-3}$, which is close to the second candidate size. While the size

of 10^6 is over conservative for the CoV criterion ($CoV\left(\widehat{P_f}\right) = 2.17\%$), an over small size ($10^4$) results in continuous candidate enrichment, hence low algorithm efficiency. When applying the proposed candidate size control algorithm to the original AK-MCS structure (AK-MCS+SC), two initial candidate sizes: 10^4 and 10^6 are evaluated with $\epsilon_{\hat{P}_{f,1}} = 0.8$. As shown in Table 1, for the same candidate size, the number of calls required by the proposed size control method is significantly reduced compared with the original AK-MCS. In the meantime, the accuracy of the failure probability estimation is not compromised. The update histories of candidate size of the two cases are shown in Figure 3. In both cases, AK-MCS+SC could quickly identify the appropriate sizes based on the preliminary estimation of failure probability, and in the later iterations, the candidate sets remain unchanged. The benefit of AK-MCS+SC is that it eliminates the candidate size impact on the algorithm. Since the failure probability is unknown before the analysis, the proposed method ensures the high efficiency of algorithm regardless of the choices of the initial candidate size. In other simulations, the candidate size is fixed as 10^6 to showcase the improved efficiency provided by other proposed features.



(a) (b)

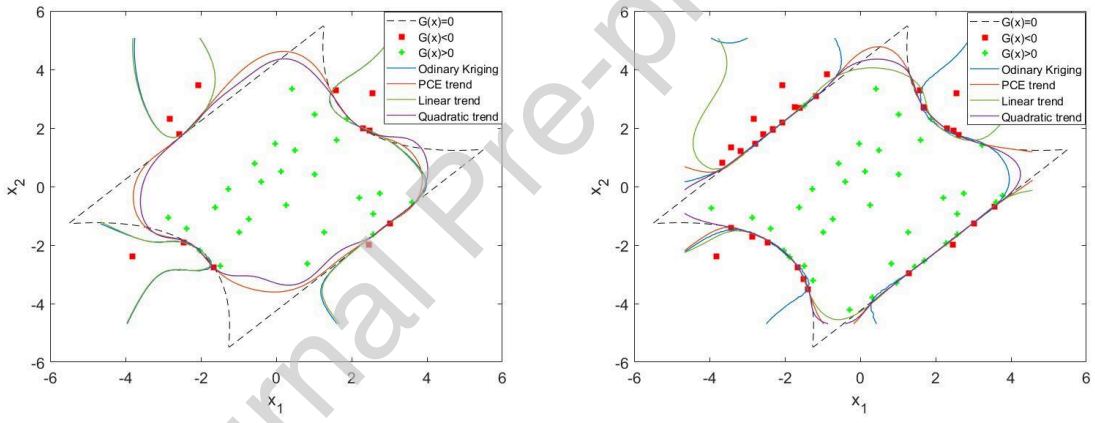
Figure 3 Update history of candidate size: (a) $n_{initial} = 10^4$, (b) $n_{initial} = 10^6$

For the weighted clustering algorithm, $k = 6$ is considered. It means that six candidates are added into the DoE set and evaluated by the physical model at each iteration. The result

shows that when AK-MCS is combined with the weighted clustering method (AK-MCS+WC), 120 calls are required to converge. Since the algorithm evaluates 6 DoE simultaneously, the computational time is reduced by a factor of 6. However, it is a surprising result that this combination requires even a smaller number of calls than the original method, considering the essence of AK-MCS+WC is to reduce computational time by enabling parallel computation, not by reducing the total amount of calls. This can be explained by the characteristic of the function itself. Because there are four failure patterns (four branches) in the physical model, AK-MCS+WC could explore all four failure patterns at a single iteration rather than being limited with just one or two patterns (Figure 1a). Consequently, algorithm performance can be further improved. For other applications (Example 2 to 4), although the computational time is reduced, more calls may be required. To further demonstrate the effectiveness of AK-MCS+WC, its performance is compared with AK-MCS using k-means clustering for DoE selection (AK-MCS+Kmeans). In Table 1, it shows that AK-MCS+WC requires 12 calls less than AK-MCS+Kmeans, therefore proves that weighted clustering provides a better DoE selection strategy than the traditional K-means. The comparison of DoE selections at one iteration is given in Figure 1.

AK-MCS with the proposed multiple trends algorithm (AK-MCS+MT) provides a significant improvement in performance by requiring the least number of calls (78). This is because, in this example, PCK trend and quadratic trend are selected in higher frequency among the four trends during iterations. Whereas in other AK-MCS based algorithms described above, ordinary Kriging is used. To visualize the effect of Kriging trend on the metamodel estimation, comparisons of the estimated limit state boundaries established using four Kriging trends at iteration 26 and 59 (last iteration) are shown in Figure 4. At these two iterations, the quadratic trend and the PCK trend are selected as the best trend, respectively. Figure 4 suggests that in the final iteration, Kriging metamodels built by all four trends can identify the four

failure patterns of the test function and approximate the limit state well. But it is also obvious that PCE and quadratic trends provide more accurate estimations in this example, especially at early iterations, where estimations given by these two trends are much closer to the real limit state. This implies that if only one Kriging trend can be selected and used through all iterations, PCK or quadratic trend is a better choice than the other two trends for this test function. But for other functions, the best trend can be different. Nevertheless, since there is no prior knowledge of the possible best trend, the proposed multiple trends method helps discover the best trend for each iteration, thus increasing the efficiency. However, it is noticed that the error rate of AK-MCS+MT is higher than other combinations, and therefore a relatively tighter convergence criterion is recommended for this combination.

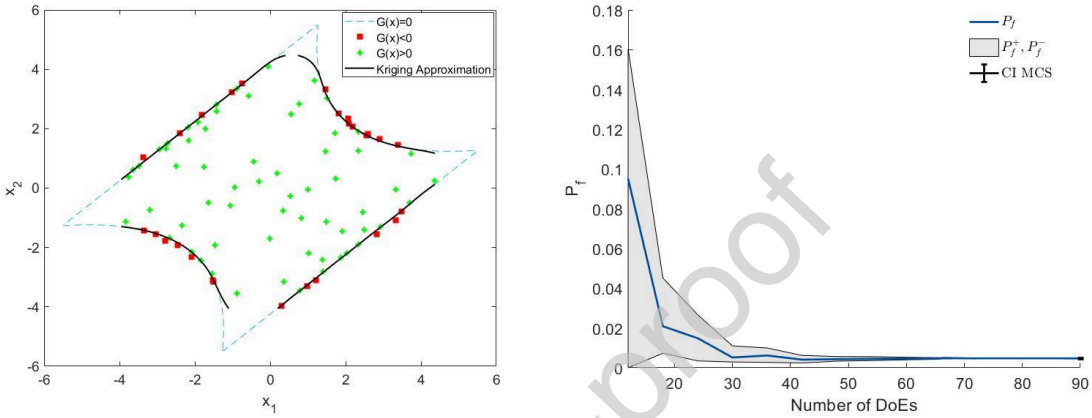


(a) (b)

Figure 4 Estimated limit states by four trends at: (a) iteration no.26, (b) iteration no.59

When all three features are integrated into the AK-MCS structure (AK-MCS+3F), it requires the second least amount of calls and has the lowest error rate among all the methods. For the computational time, thanks to the weighted clustering method, AK-MCS+3F is the most time-saving method for this example. It indicates a high efficiency of the proposed algorithm, as well as the robustness when three features work together, i.e., during iterations, the relatively high error rate because of the multiple trends method is gradually reduced every

time the value of $\epsilon_{\hat{P}_{f,i}}$ is halved. A typical DoE distribution, the corresponding limit state boundary estimation and convergence history using AK-MCS+3F are presented in Figure 5. Results show that the proposed method could quickly discover the four failure patterns (in the 4th iteration). And after that, each batch of DoE added are uniformly distributed in high uncertainty regions, until a quick and smooth convergence is achieved.



(a) (b)

Figure 5 Results of AK-MCS+3F: (a) DoE selection, (b) Convergence history

5.2 Example 2: Modified cantilever tube

This example aims to examine the performance of proposed methods in dealing with problems with relatively small probability of failure. The original engineering problem is presented in [45] and its schematic description is shown in Figure 6. A cantilever tube is subjected to three external forces, F_1, F_2, P and a torsion T, the limit state function is defined as:

$$g(x) = S - \sigma_{max} \tag{5.2}$$

where S is the yield stress, and σ_{max} is the maximum von Mises stress which is given as:

$$\sigma_{max} = \sqrt{{\sigma_x}^2 + 3{\tau_{zx}}^2} \tag{5.3}$$

Normal stress σ_x is calculated as:

$$\sigma_x = \frac{P+F_1 sin\theta_1+F_2 sin\theta_2}{A} + \frac{Mc}{I} \tag{5.4}$$

The first term represents the normal stress due to the axial force, where A is the tube area and $\theta_1 = 5^{\circ}$, $\theta_2 = 10^{\circ}$. The second term represents the normal stress due the bending moment M, which is given by:

$$M = F_1 L_1 cos\theta_1 + F_2 L_2 cos\theta_2 \quad 5.5$$

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] \quad 5.6$$

$$c = d/2 \quad 5.7$$

$$I = \frac{\pi}{64}[d^4 - (d - 2t)^4] \quad 5.8$$

The torsional stress τ_{zx} due to torsion T is defined as:

$$\tau_{zx} = Td/2J \quad 5.9$$

where $J = 2I$.

The probability information of nine random variables in this example is given in Table 2.

The benchmark failure probability obtained from MCS method is $\hat{P}_f = 2.05 \times 10^{-5}$. For AK-MCS based methods, the initial sizes of DoE and candidate set are 12 and 10^6, respectively. $\epsilon_{\hat{P}_f} = 5\%$, $\epsilon_{\hat{P}_{f,1}} = 0.8$ and $CoV_{thr} = 5\%$ are used for convergence check. Results obtained from different methods are summarised in Table 3.

In this example, due to the small failure probability, AK-MCS+SC method enriches the candidate set when an initial convergence is achieved. However, because the first convergence cannot be satisfied at very early iterations, therefore the improved performance brought by the AK-MCS+SC method is limited. Note that by increasing $\epsilon_{\hat{P}_{f,1}} = 0.8$, initial convergence can be satisfied quickly, but the algorithm performance becomes unstable. This is because the estimation given by the initial convergence is less accurate as $\epsilon_{\hat{P}_{f,1}}$ increases. Again, AK-MCS+MT and AK-MCS+WC methods show high efficiency in reducing the number of calls without compromising accuracy. Note that when AK-MCS+MT is used in this example, PCE

trend is selected in the highest frequency while the ordinary Kriging is rarely used. This indicates the significant effect of Kriging trend selection on the algorithm performance, since the number of calls is reduced by more than half by applying the AK-MCS+MT. The computational time is reduced by more than 10 times when all three proposed features are used, which is a remarkable improvement in the real time usage.

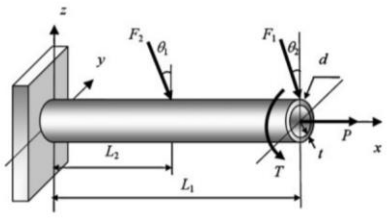


Figure 6 Schematic description of Example 2 [45]

Table 2 Random Variables in Example 2

Variable	**Distribution**	**Parameter 1**	**Parameter 2**
t_1	Normal	5 mm (mean)	0.1 mm (std)
d	Normal	42 mm (mean)	0.5 mm (std)
L_1	Uniform	119.75 mm (lb)	120.25 mm(ub)
L_2	Uniform	59.75 mm (lb)	60.25 mm (ub)
F_1	Normal	3.0 kN (mean)	0.3 kN (std)
F_2	Normal	3.0 kN (mean)	0.3 kN (std)
P	Gumbel	12.0 kN (mean)	1.2 kN (std)
T	Normal	90.0 kN (mean)	9.0 Nm (std)
S_y	Normal	240.0 kN (mean)	24.0 MPa (std)

Table 3 Reliability results of Example 2

Method	N_{call}	$\widehat{P_f}$	$CoV(\widehat{P_f})$	$RE_{\widehat{P}_f}$

MCS	4×10^{7}	2.05×10^{-5}	3.45%	-
Original AK-MCS	106	2.10×10^{-5}	4.85%	2.21%
AK-MCS+ Size Control	99	2.01×10^{-5}	4.82%	1.95%
AK-MCS+ Multiple Trends	65	2.11×10^{-5}	4.75%	2.81%
AK-MCS+ Weighted Clustering	126	2.08×10^{-5}	4.70%	1.01%
AK-MCS+ All 3 Features	66	2.10×10^{-5}	4.78%	1.92%

5.3 Example 3: High-dimensional problem

The last example aims to test the capability of the proposed method in dealing with high-dimensional problem. In this example, the test function has a series of independent random variables which all subject to a lognormal distribution with a mean $\mu = 1$ and standard deviation $\sigma = 0.2$. The performance function is defined as[46]:

$$G(x_1, \cdots, x_n) = \left(n + 3\sigma\sqrt{n}\right) - \sum_{i=1}^{n} x_i \tag{5.10}$$

In the present study, $n = 20$ is considered.

The benchmark probability of failure $\hat{P}_f = 2.26 \times 10^{-3}$ is calculated by MCS using 10^7 samples. The initial sizes of DoE and candidate set for AK-MCS based method are 12 and 10^6 respectively, and convergence criterion are $\epsilon_{\hat{P}_f} = 5\%$, $\epsilon_{\hat{P}_{f,1}} = 0.8$ and $CoV_{thr} = 5\%$. Comparisons of performance given by different methods are given in Table 4.

This example requires a relatively larger number of calls to achieve target accuracy than the previous three examples. A noticeable improvement in efficiency is observed using AK-MCS+SC because the initial size is much larger than the optimized size. Moreover, the reduction of N_{call} becomes less dramatic when AK-MCS+MT is used. This is because, in this example, ordinary Kriging is selected most frequently, as well as PCE trend, during iterations. The other two methods, once again, show good performance in terms of reducing computational time and achieving high accuracy.

Table 4 Reliability results of Example 3

Method	N_{call}	$\widehat{P_f}$	$CoV(\widehat{P_f})$	$RE_{\widehat{P}_f}$
MCS	10^7	2.26×10^{-3}	2.10%	-
Original AK-MCS	302	2.29×10^{-3}	2.95%	1.15%
AK-MCS+ Size Control	230	2.25×10^{-3}	4.47%	0.53%
AK-MCS+ Multiple Trends	225	2.27×10^{-3}	2.96%	0.31%
AK-MCS+ Weighted Clustering	426	2.28×10^{-3}	2.96%	0.66%
AK-MCS+ All 3 Features	234	2.28×10^{-3}	4.15%	0.53%

5.4 Example 4: Response time of a smoke detector

The last example is a computational fluid dynamic (CFD) model dealing with a fire safety engineering problem. The fast response of smoke detectors is critical for building evacuation during fire hazards. The activation of detectors is determined by the smoke obscuration of the chamber. In this example, a two-dimensional computation domain with the grid size of $0.01\ m \times 0.01\ m$ is constructed to evaluate the smoke obscuration in an open chamber (Figure 7). Based on Heskestad's model [47], the performance of a smoke detector can be described by the characteristic length l of the detector. Soot injecting from the inlet vent is detected by the smoke detector mounted on the ceiling. Once the detected smoke obscuration reached a certain level, the detector is activated. Failure is considered if the computed detection time exceeds the required response time. Other variables listed in Table 5 includes: the soot injection velocity u, the initial soot mass fraction of the chamber c_0, required response time t_0, the characteristic length l of the detector, and the detector activation obscuration a.

Due to the long computational hour of the CFD simulation, benchmark analysis using MCS is not carried out. In this case, stricter convergence criteria are applied to the original AM-MCS method to obtain a relatively accurate solution. The convergence criteria for the

benchmark AK-MCS are $\epsilon_{\hat{P}_f} = 1\%$ and $CoV_{thr} = 1\%$, whereas for other methods, 5% is used for both thresholds. The initial sizes of DoE and candidate set are 12 and 10^6 for all cases. The comparison of the original AK-MCS and proposed methods are given in Table 6.

Results in Table 6 Reliability results of Example show that each proposed feature improves the algorithm performance at a certain level. This example requires only a small number of DoE to achieve convergence, and therefore the benefit provided by AK-MCS+SC is minor. But for AK-MCS+ MT and AK-MCS+WC methods, the computational time is reduced significantly, and high accuracy is achieved. Again, when three features work together, it costs the least computational time, proving the high efficiency of the proposed method in dealing with practical engineering problems.

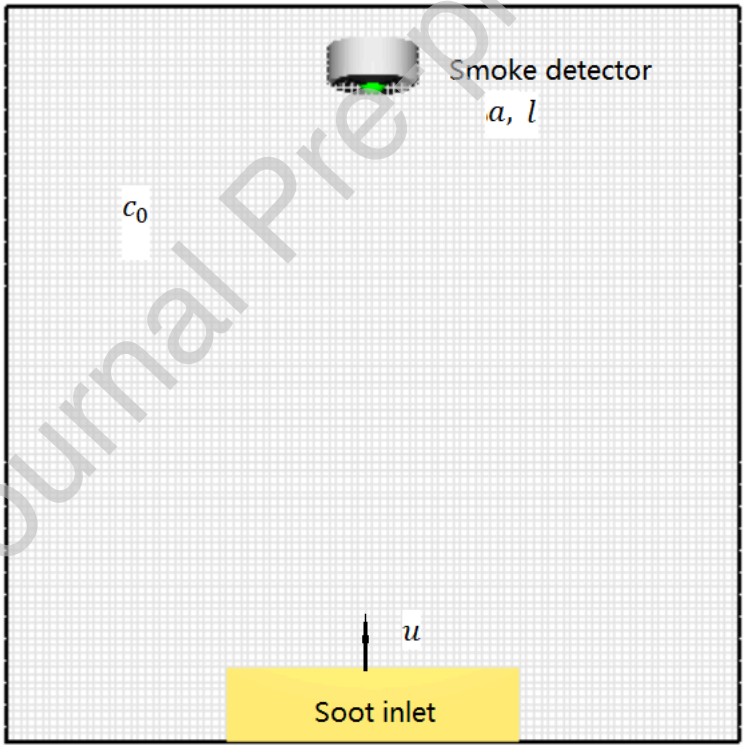


Figure 7 Schematic description of Example 4

Table 5 Random Variables in Example 4

Variable	***Distribution***	***Mean***	***Standard Deviation***

$\boldsymbol{l}$	Normal	2.0 (m)	0.1 (m)
$\boldsymbol{c_0}$	Normal	100 (mg/kg)	10 (mg/kg)
$\boldsymbol{t_0}$	Normal	8 (s)	0.2 (s)
$\boldsymbol{a}$	Normal	3.24 (%/m)	0.324 (%/m)
$\boldsymbol{u}$	Normal	0.1 (m/s)	0.01 (m/s)

Table 6 Reliability results of Example 4

Method	N_{call}	$\widehat{P_f}$	$CoV(\widehat{P_f})$	$RE_{\widehat{P}_f}$
Original AK-MCS (benchmark)	121	3.48×10^{-2}	2.47%	-
Original AK-MCS (10^6)	72	3.52×10^{-2}	2.48%	1.07%
AK-MCS+ Size Control (10^6)	70	3.40×10^{-2}	4.72%	2.04%
AK-MCS+ Multiple Trends	58	3.53×10^{-2}	2.87%	1.45%
AK-MCS+ Weighted Clustering	78	3.54×10^{-2}	3.05%	2.17%
AK-MCS+ All 3 Features	66	3.53×10^{-2}	4.81%	1.39%

6. Conclusion

This paper proposes a highly efficient reliability analysis algorithm based on the AK-MCS structure by integrating three individual features: weighted clustering method, multiple trends method and candidate size control method. The weighted clustering method provides a DoE selection strategy to enable parallel computing. It overcomes the drawback of applying traditional clustering methods, such as K-means, in the DoE selection by considering not only the spatial distribution of the candidates, but also the estimation uncertainty. The multiple trends method aims to eliminate the impact of the poor choice of Kriging trend on the algorithm performance. During iterations, four Kriging metamodels using different trends are built, and only the one with lowest estimation uncertainty is selected as the best metamodel for the current

iteration. Finally, because the candidate set size has a noticeable influence on the algorithm efficiency, the candidate size control method is developed to determine the appropriate candidate size in an adaptive manner. It calculates the candidate size based on convergence criteria and failure probability of each iteration, and then enriches or draws new samples accordingly.

The individual and combined effects of proposed features on the performance of AK-MCS algorithm are explored using four examples. In all examples, each feature shows the capability of improving algorithm efficiency, i.e., reducing the number of calls without compromising the error rate. Compared with K-means, the weighted clustering method provides better performance by choosing more representative DoE. In the meantime, because the computational time to calculate those DoE required by the weighted clustering method is divided by a factor of k (k is the number of clusters or the computer cores for parallel computing), the computational burden to obtain the probability of failure is reduced drastically. The multiple trend method requires a much fewer DoE when ordinary Kriging is rarely selected during iterations. And for the candidate size control method, its benefit is dominating when dealing with problems requiring a large amount of DoE. The last two features aim to increase the robustness of the algorithm, so that high efficiency can be achieved despite initial settings. When all three features integrated with AK-MCS, it requires the least amount of computational time, and high accuracy is observed. In general, the proposed method improves the original AK-MCS performance for a wide range of applications (e.g., problems with high non-linearity, high-dimensions, small probability of failure). In real-life usage, it is recommended to use all three features when no prior knowledge is available. However, since each feature focuses on different problems, users could also choose individual features according to the demand of problems.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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