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A Review of Pressure Fluctuations in Turbulent Boundary Layers at Subsonic and Supersonic Speeds.*

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**G.M. Lilley, M.Sc., D.I.C., A.M.I.Mech.E., F.R.Ae.S.

SUMMARY

The equation for the pressure fluctuations in a turbulent boundary layer are derived with special reference to their values at a rigid wall. It is shown that in incompressible flow the pressure field is defined completely once the velocity field is known. The results obtained from the theory are compared with experiment. The work is extended to include the effects of compressibility and it is found that a treatment similar to that given by Phillips is appropriate. It is shown that under conditions of zero heat transfer the theory obtained in incompressible flow is only slightly modified by the effects of compressibility. However, at higher Mach numbers eddy Mach waves are formed and these modify to some extent the pressure distribution throughout the layer, and in particular at the wall. In addition strong radiation of sound occurs under conditions of supersonic flow external to the boundary layer.

Comparisons with experiment show moderate agreement. The theories as mentioned previously are for the case of zero external pressure gradient, but it is shown that even when these conditions are relaxed similar results occur. Some experimental results in support of this conclusion are presented.

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** Professor of Experimental Fluid Mechanics at the College of Aeronautics, Cranfield, Bletchley, Bucks., England.

CONTENTS

NOTATION

$\bar{a}$ =	a/a_w
a	speed of sound
$A(\underset{\sim}{x},t)$	source function
C_f	skin friction coefficients
G_o, G_i	Green functions for unbounded and image flows respectively
$H(y)$	source function
$\underline{k}$	two dimensional wave number
p	pressure
q	see equation (34)
$\underline{r}$	spatial separation
R	Reynolds number
R_{pu_2}, $R_{p\dot{u}_2}$	pressure-velocity and pressure-velocity gradient covariances
t	time
$\underline{U}$	mean flow velocity
$\underline{u}$	turbulent velocity
$\underline{v}$	velocity
U_c	convection velocity
dz_2	Fourier coefficient of u_2
τ	delay time; mean shear dU_1/dx_2
τ_w	wall shear stress
ω	frequency
$d\tilde{\omega}$	Fourier coefficient of pressure
$d\gamma$	Fourier coefficient of source function
$\alpha,\beta,\gamma,\delta$; $\alpha^*,\beta^*,\gamma^*,\delta^*$	coefficients
δ	Boundary layer thickness
δ_1	displacement thickness
ζ =	$a/a_w \; d\tilde{\omega}$
λ =	a_w/u_τ

u_τ	=	$\sqrt{\tau_w/\rho_w}$ shear velocity
μ		viscosity
$\Pi(\underline{k},\omega)$		spectrum function (pressure)
Φ_{22}		spectrum function (velocity component u_2)
ϕ		source function

Subscripts

w	wall value
∞	external to boundary layer
0	incompressible value
i,j	tensor notation
x_1	streamwise direction
x_2	normal to the wall
x_3	transverse direction

Other symbols are defined where they occur in the text.

1. Introduction

The problem of pressure fluctuations in a turbulent boundary layer has been studied theoretically by Kraichnan(1) and others for the case of incompressible flow. In particular Kraichnan found on making certain assumptions about the structure of the mean and turbulent flows that $\sqrt{\overline{p_w^2}}$ was proportional to the local wall mean shear stress τ_w. This result is based on the assumption that the dominant source term involves an interaction between the mean shear and the turbulence, whereas in the case of isotropic turbulence, Batchelor(2) and others have shown that the mean square fluctuating pressure is determined by quadruple velocity correlations. The experiments of Hodgson(3) and Wooldridge and Willmarth(4) provide confirmation that the dominant interaction involves the product of the mean shear and the turbulence, and these results also confirm to some extent the theoretical predictions of Kraichnan as modified by Lilley and Hodgson(5).

These results, both theoretical and experimental, apply to the case where the external pressure gradient is zero. One of the difficulties associated with experiments on pressure fluctuations in turbulent boundary layers is the difficulty of making measurements in the boundary layer itself, and secondly the readings of wall pressure fluctuations made with a finite size of pressure transducer require extrapolation to zero microphone diameter. The corrections associated with the finite size of the pressure transducer are very significant, and have been determined to some extent in the work by Hodgson and Wooldridge and Willmarth, while Corcos(6) has performed both experiments and presented a theory for these corrections. At high Mach numbers we find that apart from the work by Phillips(7) no theory is available and it is in this area that the present paper is primarily concerned.

The work by Phillips is related to the problem of sound radiation from supersonic turbulent shear layers, where he shows that the radiated noise arises from eddy Mach waves which are generated by some wave numbers of the turbulence in those layers of the shear flow for which the difference between the mean velocity of the fluid outside and the local eddy convection velocity of the turbulence is greater than the speed of sound outside the shear layers. The problem of the pressure fluctuations inside a supersonic turbulent boundary layer present an analogous problem. However, Phillips theory needs to be modified somewhat in our problem and it is necessary to find the pressure fluctuations over a range of Mach numbers including both subsonic and supersonic, whereas Phillips asymptotic theory is strictly only applicable for very high supersonic Mach numbers. The present theory is compared with the experimental results of Kistler and Chen.[8]

2. Incompressible Flow Theory

The equations of motion and continuity are

$$\rho_0\left(\frac{\partial v_i}{\partial t} + \frac{\partial v_i v_j}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu_0 \nabla^2 v_i \qquad \ldots\ldots\ldots (1)$$

$$\partial v_i/\partial x_i = 0 \qquad \ldots\ldots\ldots (2)$$

where $\underset{\sim}{v}$ is the velocity, p is the pressure, ρ_0 is the density and μ_0 is the viscosity. The equation for the fluctuating pressure is found from the divergence of equation 1, and gives

$$\nabla^2 p = -\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}\left(v_i v_j - \overline{v_i v_j}\right) \qquad \ldots\ldots\ldots (3)$$

If $\underset{\sim}{v} = \underset{\sim}{U} + \underset{\sim}{u}$ where $\underset{\sim}{U}$ is the mean flow velocity and $\underset{\sim}{u}$ is the turbulent velocity then it can easily be shown for a boundary layer flow that

$$\nabla^2 p = -2\rho_0 \frac{\partial U_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} - \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}\left(u_i u_j - \overline{u_i u_j}\right) \qquad \ldots (4)$$
$$= A(\underset{\sim}{x}, t)$$

since the remaining terms such as $2\,\frac{\partial U_2}{\partial x_1}\,\frac{\partial u_1}{\partial x_2}$ are small by comparison.

It is seen that the source terms in Poisson's equation for the pressure involve an interaction between the mean shear $\left(\partial U_1/\partial x_2\right)$ and the turbulent gradient $\left(\partial u_2/\partial x_1\right)$ which we call the (M-T) interaction while the remaining source term represents the turbulent-turbulent (T-T) interaction. In this equation x_1 is measured in the direction of the mean flow outside the boundary layer, while x_2 is measured normal to the wall. We see that in the case of a uniform flow the (M-T) source term is absent and the equation for the pressure reduces to that used by Batchelor (1951) in his investigation on pressure fluctuations in isotropic turbulence.

In the case of boundary layer flows large values of the mean shear $(\partial U_1/\partial x_2)$ exist throughout the boundary layer, and in this case it might be expected that the two source terms have at lease equal significance. In fact the analysis which follows shows that in the case of zero external pressure gradient the dominant source term involves the (M-T) interaction.

The general solution of equation (4) for the fluctuating pressure can be written in the form

$$p(\underset{\sim}{x},t) = -\frac{1}{4\pi}\int_0^\infty dx_2'\iint_{-\infty}^{\infty} dx_1'dx_3'\,(G_o + G_i)\,A(\underset{\sim}{x}',t) - \frac{1}{2\pi}\iint_{-\infty}^{\infty} dx_1'dx_3'\;G_o\;\partial p/\partial x_2' \qquad \text{.........} (5)$$

where the volume integral is taken over the entire boundary layer, and the surface integral is taken over the wall at $x_2' = 0$. The Green functions G_o, G_i are given respectively by

$$G_o = |\underset{\sim}{x} - \underset{\sim}{x}'|^{-1};\quad G_i = |\underset{\sim}{x} - \underset{\sim}{x}'^*|^{-1} \qquad \text{.........} (6)$$

where $\underset{\sim}{x}'^*$ is the image point with respect to the wall.

However, from the equation of motion, equation (1), we see that since the velocity vanishes at the wall our equation for the pressure becomes

$$p(\underset{\sim}{x},t) = -\frac{1}{4\pi}\int_0^\infty dx_2'\iint_{-\infty}^{\infty} dx_1'dx_3'\,(G_o + G_i)\,A(\underset{\sim}{x}',t) - \frac{\mu_0}{2\pi}\iint_{-\infty}^{\infty} dx_1'dx_3'\,(G_o\,\partial^2 u_2'/\partial x_2'^2)_{x_2'} = 0 \qquad \text{.........} (7)$$

and shows that the pressure fluctuations can be determined from a knowledge of the velocity field.

The covariance between the pressures at any point $(\underset{\sim}{x},t)$ and $(\underset{\sim}{x}'',t')$ is given by

$$R_{pp} \equiv \overline{p(\underset{\sim}{x},t)p''(\underset{\sim}{x}'',t')} = -\frac{1}{4\pi}\int_0^\infty dx_2' \iint_{-\infty}^{\infty} dx_1' dx_3' (G_o + G_i)\ \overline{A(\underset{\sim}{x}',t)p''(\underset{\sim}{x}'',t')}$$

$$-\frac{\mu_o}{2\pi}\iint_{-\infty}^{\infty} dx_1' dx_3'\ G_o\ \frac{\partial^2}{\partial x_2'^2}\ \overline{u_2'(\underset{\sim}{x}',t)p''(\underset{\sim}{x}'',t')}$$

$$\ldots\ldots\ldots \quad (8)$$

Thus in order to obtain the pressure covariance anywhere in the shear layer the following covariances are required:

$$\overline{p''(\underset{\sim}{x}'',t')u_2'(\underset{\sim}{x}',t)}; \quad \overline{p''(\underset{\sim}{x}'',t')u_i'u_j'(\underset{\sim}{x}',t)}$$

together with their spatial derivatives. For convenience we will write these covariances as R_{pu_2} and $R_{pu_iu_j}$ respectively.

Now the covariance $\overline{pu_2''}$ can be obtained from equation (7) giving

$$R_{pu_2}(\underset{\sim}{x}'',\underset{\sim}{x}';t',t) = \frac{\rho_o}{4\pi}\int_0^\infty dx_2' \iint_{-\infty}^{\infty} dx_1' dx_3' (G_o + G_i) \left\{ 2\,\frac{\partial U_1}{\partial x_2'}\ \frac{\partial}{\partial x_1'}\ \overline{u_2'u_2''} + \frac{\partial^2}{\partial x_i' \partial x_j'}\ \overline{u_i'u_j'u_2''} \right\}$$

$$-\frac{\mu_o}{2\pi}\iint_{-\infty}^{\infty} dx_1' dx_3' \left(G_o\ \frac{\partial^2}{\partial x_2'}\ \overline{u_2'u_2''} \right)_{x_2' = 0} \quad \ldots\ldots\ldots(9)$$

and a similar expression exists for $\overline{pu_i'u_j''}$. Our knowledge of the double velocity correlations is fairly complete for shear flow turbulence, but little is known about the related triple and quadruple velocity correlations. If we assume that the joint-probability distribution of $\underset{\sim}{u}$ and $\underset{\sim}{u}'$ is normal then as shown by Batchelor the triple velocity correlations are zero, while the quadruple correlations reduce to a sum of products of the double velocity correlations. Although the assumption of a normal joint probability distribution would simplify the analysis, in our case its use has less justification than in the problem of pressure fluctuations in an isotropic turbulence investigated by Batchelor (loc cit). Our analysis will therefore be devoted to the contributions to the pressure fluctuations from the double velocity correlations, and the effect of the triple and quadruple velocity correlations will be found by difference when our results are compared with experiment. The data on which the double velocity correlations will be based is obtained from the work of Grant[9] (1958) and Klebanoff[10] (1954).

3. The Double Velocity Correlation Functions R_{22}

Let us now consider functions defining R_{22} which fit the measured correlations of Grant (loc cit) with fair accuracy. It can be shown that in the 'inner region' of the boundary layer, and including the constant stress layer, that R_{22} is composed of contributions from small and big eddies. The scale of the smaller energy containing eddies is proportional to their distance from the wall, while in this region the scale of the big eddies is roughly independent of the distance from the wall. If we write a_s and a_b as the amplitude of the small and big eddy contributions to R_{22} we find that a reasonable fit with the results of Grant is obtained when

$$R_{22}(x_2;r_1,0,0) = a_s\left(1-\frac{0.42|r_1|}{x_2}\right)\exp(-0.83|r_1|/x_2) + a_b\left(1-\frac{(r_1/\delta_1)^2}{4}\right)\exp\left(-(r_1/\delta_1)^2\right)8 \qquad \ldots\ldots (10)$$

with $a_s = 0.9$ and $a_b = 0.1$ as shown in Figure 1. The one dimensional spectrum function corresponding to this correlation is:

$$\phi_{22}(x_2;k_1) = \frac{a_s\delta_1}{2\pi\alpha_1\delta_1}\,\frac{(1+3k_1^2/\alpha_1^2)}{(1+k_1^2/\alpha_1^2)^2} + \frac{a_b\delta_1}{4\sqrt{\pi}}(\ell_1/\delta_1)^3(k_1\delta_1)^2\exp(-k_1^2\ell_1^2/4) \qquad \ldots\ldots\ldots (11)$$

where $\alpha_1 = 0.83/x_2$ and $\ell_1 = \sqrt{8}\,\delta_1$. It is seen from Figure 2 that ϕ_{22} is flat at low wave numbers, has a shallow peak near $k_1\delta_1 = 0.7$, corresponding to the contribution from the big eddies, and falls off at high wave numbers like k_1^{-2}.

Due to the dependence of the scale of the small eddies on the distance from the wall, the amplitude of ϕ_{22} falls at low wave numbers as x_2 is decreased, while at high wave numbers ϕ_{22} extends to higher and higher wave numbers as x_2 is decreased, although necessarily some high wave number cut off will eventually be reached. It is interesting to note that apart from the shallow peak at $k_1\delta_1 = 0.7$ the shape of the spectrum function is almost entirely the result of contributions from the smaller eddies.

The results of inserting a high pass filter, such as used by Wooldridge and Willmarth(11)(1962), so that all wave numbers below $|k_1| = |K_1|$ are eliminated can easily be demonstrated. Since the assumed form for R_{22} is an even function of r_1

$$\phi_{22}(x_2;k_1) = \frac{1}{\pi}\int_0^\infty \cos k_1 r_1 R_{22}(x_2;r_1,0,0)dr_1 \quad \text{.........} \ (12)$$

and since ϕ_{22} is an even function of k_1

$$R_{22}(x_2;|r_1|,0,0) = 2\int_0^\infty \phi_{22}(x_2;k_1)\cos(k_1|r_1|)dk_1 \quad \text{.........} \ (13)$$

The truncated form for ϕ_{22}, associated with the insertion of the high pass filter, leads to

$$R_{22}^T(x_2;r_1,0,0) = 2\int_{K_1}^\infty \phi_{22}(x_2;k_1)\cos k_1 r_1 dk_1$$

$$= R_{22}(x_2;r_1,0,0) - 2\int_0^{K_1} \phi_{22}(x_2;k_1)\cos(k_1 r_1)dk_1 \quad \text{.....}(14)$$

and when $K_1\delta_1 \ll 1$ we find very nearly that $R_{22}^T - R_{22} = \frac{-a_s}{\pi}\ \frac{\sin K_1 r_1}{r_1\alpha_1}$ .... (15)

since the big eddy contribution can be shown to be negligible.

Thus for $\frac{r_1}{\delta_1} < 2$ when $\frac{x_2}{\delta_1} = 1$, and $K_1\delta_1 = 0.2$

$$\Delta R_{22} = 0.07 \qquad \text{.........} (16)$$

which is sufficient to modify greatly the shape of the R_{22} correlation. In fact for this value of the cut-off wave number the zero point for the truncated R_{22} (R_{22}^T) occurs at smaller values of r_1 and in addition R_{22}^T has a pronounced negative loop. For these reasons correlation functions obtained from filtered signals need careful analysis as to their true interpretation.

Our assumed form for R_{22} is not entirely satisfactory since it does not satisfy the continuity relation:

$$\iint_{-\infty}^{\infty} R_{22}(x_2;\underset{\sim}{r})\, dr_1 dr_3 = 0 \qquad \text{.........} (17)$$

unless a negative loop exists in the r_3 direction. But this is absent from Grants measurements near the wall and so strictly

$$\int_{-\infty}^{\infty} R_{22}(x_2;r_1,0,0)dr_1$$

should equal zero. However, errors arising from this cause can be shown to be small unless the value of R_{22} at large separations is under consideration. It is interesting to note however that our truncated form of R_{22} does satisfy the above continuity relation, whereas in general it will not.

4. Pressure-velocity Covariance $R_{pu_2} \equiv \overline{p(\underset{\sim}{x},t)u_2''(\underset{\sim}{x}'',t')}$

If we assume that the triple velocity correlation is zero we find that R_{pu_2} is obtained from R_{22} together with the variation of the mean shear across the boundary layer. If the form for R_{22} given in section 3 above is substituted into equation (9) R_{pu_2} can be obtained.

Provided the pressure is not measured at the wall it can be shown that $\overline{pu_2}$, that is R_{pu_2} for zero separation and time delay, is very nearly equal to, or at least proportional to, $-\rho_0\overline{q^2u_2}$, showing that it depends on triple velocity correlations or in other words on the turbulent-turbulent interaction. It can also be shown that the (M-T) contribution to $\overline{pu_2}$ is identically zero if R_{22} is symmetric with respect to r_1. It is important in this case to note the dominant role of the turbulent-turbulent interaction, but it must be emphasised that this only applies to contributions to $\overline{pu_2}$ and it does not follow that the (M-T) term is zero or negligible for R_{pu_2}. The importance of the triple velocity correlation can be found to some extent by re-writing the equation for the pressure in the following form

$$\nabla^2(p + \rho_0\, q^2/3) = -\rho_0\frac{\partial^2}{\partial x_i\partial x_j}\,(v_iv_j - \overline{v_iv_j} - q^2/3\ \delta_{ij}) \qquad \ldots(18)$$

This equation is in the form proposed by Patterson[12], but clearly similar equations can be proposed with different coefficients for the velocity term on the left-hand side of the equation. If structural similarity exists in the inner region of the boundary layer it follows directly that

$$-\overline{pu_2} \sim \rho_0\overline{q^2u_2} \qquad \ldots\ldots\ldots (19)$$

If we now turn to the case of R_{pu_2} where the pressure is measured at the wall we find from equation (9) that the correlation is zero when $r_1 = 0$ and is antisymmetric with respect to r_1. It is however symmetric with respect to r_3. This covariance has been subjected to fairly full experimental treatment by Willmarth and Wooldridge (1962), and a comparison between the calculated and measured values is shown in Figure 3.

5. The Turbulent Energy Equation

Our results for the pressure-velocity product can be checked by reference to the role they play in the turbulent energy balance.

Now from the equations of motion and continuity the turbulent energy equation can be derived in the form (see Townsend 1956)[14].

$$-\rho_0 \overline{u_1 u_2} \frac{dU_1}{dx_2} - \frac{d}{dx_2} \overline{(p + \rho_0 q^2/2) u_2}$$

$$= \rho_0 \epsilon - \mu_0 \frac{d^2}{dx_2^2} (\overline{q^2}/2 + \overline{u_2^2}) \qquad \ldots\ldots\ldots (20)$$

where the terms represent in turn the production, convective diffusion, dissipation, and the work done by viscous stresses.

On integration we find

$$\overline{pu_2} = - \int_0^{x_2} \rho_0 \epsilon dx_2 + \mu_0 \frac{d}{dx_2} \left(\overline{q^2}/2 + \overline{u_2^2}\right)$$

$$- \rho_0 \frac{\overline{q^2 u_2}}{2} + \int_0^{x_2} \left(-\rho_0 \overline{u_1 u_2}\right) \frac{dU_1}{dx_2} dx_2 \qquad \ldots\ldots\ldots (21)$$

since $\overline{pu_2} = \overline{q^2 u_2} = \frac{d}{dx_2} \left(\overline{q^2}/2 + \overline{u_2^2}\right) = 0$ at $x_2 = 0$.

In the region close to the wall the last two terms in (21) are negligible, while in the constant stress region

$$- \overline{u_1 u_2} \frac{dU_1}{dx_2} \approx \epsilon \qquad \ldots\ldots\ldots (22)$$

and

$$\overline{pu_2} \approx -\rho_0 \overline{q^2 u_2}/2 \qquad \ldots\ldots\ldots (23)$$

which can be deduced from the results of Laufer[13] (1953) as plotted by Townsend (1956).

Thus in the constant stress layer the dominant contribution to $\overline{pu_2}$ is from the (T-T) interaction whereas closer to the wall it is dominated by the viscous terms. If we now compare these results with equation (9) we see that the (M-T) and (T-T) terms must be small when $\underset{\sim}{x}' = \underset{\sim}{x}''$ is close to the wall. $\overline{pu_2}$ is then given by the surface integral which is the only viscous term in equation (9). For larger values of $\underset{\sim}{x}' = \underset{\sim}{x}''$ this viscous term is itself negligible. However the (M-T) term is also zero if $\overline{u_2'u_2''}$ is symmetric about $s_1 = 0$ (where $\underset{\sim}{s} = \underset{\sim}{x}'' - \underset{\sim}{x}'$), as noted by Corcos[17] (1962). Thus the only remaining term, the (T-T) interaction, can contribute to $\overline{pu_2}$ in this region. This conclusion needs some qualification since Grants' measurements of R_{22} show some slight asymmetry about $s_1 = 0$ and hence the (M-T) contribution is not entirely negligible, but our main result dealing with the dominant role of the (T-T) term is barely modified.

If it can be verified that in this constant stress region

$$\overline{(p + \rho q^2/2)u_2} \sim 0 \qquad \text{.........} \; (24)$$

it would add support to the conclusion that turbulent motions exist in which the total energy is conserved during convection with the velocity u_2. The random mixing jet hypothesis of Grant (loc cit) supports such a conclusion and in such a model u_2 would be the random outward jet velocity.

We also note that in the region close to the wall the total dissipation exceeds the production of turbulent energy and is therefore a region of energy defect. Thus the inner region takes on a double role of transferring energy towards and away from the wall. The details of such a transfer mechanism have been given by Grant and Townsend.

6. The Correlation Function $R_{p\dot{u}_2} = \overline{p(\underset{\sim}{x};t') \frac{\partial u_2'}{\partial t'} (\underset{\sim}{x}';t')}$

Following Wooldridge and Willmarth (loc cit) we will assume for small eddies (or small separations) that

$$\frac{\partial}{\partial t} \sim U_1 \frac{\partial}{\partial x_1}$$

and so

$$R_{p\dot{u}_2} = - U_1(x_2') \, \overline{p(\underset{\sim}{x};t) \frac{\partial u_2'}{\partial x_1'} (\underset{\sim}{x}';t')} \qquad \text{.........} \; (25)$$

The correlation coefficient is written

$$\bar{R}_{p\dot{u}_2} = \frac{\overline{p(\underset{\sim}{x};t)\,\frac{\partial u'_2}{\partial x'_1}(\underset{\sim}{x}';t')}}{\sqrt{\overline{p(\underset{\sim}{x})^2}}\ \sqrt{\overline{\left(\frac{\partial u'_2}{\partial x'_1}\right)^2}}} \qquad \ldots\ldots\ldots (26)$$

and in what follows we will only consider the case when the pressure is measured at the wall. The wall pressure covariance can then be obtained approximately from the distribution over all space of $\bar{R}_{p\dot{u}_2}$ since

$$\bar{R}_{pp}(0,t;\underset{\sim}{x},t+\tau) \equiv \frac{\overline{p(0,t)p(\underset{\sim}{x},t+\tau)}}{\overline{p(0)^2}}$$

$$= -\frac{\rho_0}{\pi}\int_0^\infty dy_2\ \frac{\partial U_1}{\partial y_2}\iint_{-\infty}^{\infty}\frac{dy_1\,dy_3}{|\underset{\sim}{x}-\underset{\sim}{y}|}\,\bar{R}_{p\dot{u}_2}(\underset{\sim}{x},t+\tau;\underset{\sim}{y},t)\sqrt{\overline{\left(\frac{\partial u_2}{\partial y_1}\right)^2}}\Bigg/\sqrt{\overline{p(0)^2}}$$

if the (T-T) interaction is omitted as well as the surface integral. This form is in any case only valid if the integrals are dominated by the values of $R_{p\dot{u}_2}$ at small separations, since it is only for small separations that Taylor's hypothesis, used in equation 25, is valid.

If we write

$$\Lambda(\bar{y}_2;\ \tau) \equiv \bar{R}_{p\dot{u}_2}(0,y_2,0;\tau)$$

where $\bar{y}_2 = y_2/\delta_1$ and assume throughout the inner region that

$$\frac{\partial \bar{U}_1}{\partial \bar{y}_2} = \frac{1}{K\bar{y}_2} \qquad \ldots\ldots\ldots (28)$$

where $K = 0.4$ and $\bar{U}_1 = U_1/u_\tau$

with $\qquad u_\tau = \sqrt{\tau_W/\rho_0}$

we have from (8) that

$$\bar{R}_{pp}(\underset{\sim}{x};\tau)\,\frac{u_\tau}{U_\infty}\sqrt{\overline{p_w^2}}/\tau_w \;=\; \frac{1}{\pi}\int_0^\infty F(\bar{y}_2)\,d\bar{y}_2 \iint_{-\infty}^{\infty} \frac{dy_1 dy_3}{|\underset{\sim}{x}-\underset{\sim}{y}|}\cdot\frac{\bar{R}^{\bullet}_{pu_2}(\underset{\sim}{x},\underset{\sim}{y};\tau)}{\bar{R}^{\bullet}_{pu_2}(y_2;\tau)} \qquad \ldots (29)$$

$$\text{where } F(\bar{y}_2) = \frac{\Lambda(\bar{y}_2)}{k\bar{y}_2}\sqrt{\overline{\left(\frac{\partial u_2}{\partial y_1}\right)^2}}\,\frac{\delta_1}{U_\infty}$$

since $\Lambda(\bar{y}_2)$ is roughly independent of τ for small values of τ $(0 \leqslant \frac{\tau U_\infty}{\delta_1} < 1)$ (see Wooldridge and Willmarth loc cit).

For $\tau = 0$ we find a good fit (see Figure 4) with the data of Willmarth and Wooldridge, if

$$\frac{\bar{R}_{p\dot{u}_2}(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{y}};0)}{\bar{R}_{p\dot{u}_2}(\bar{y}_2;0)} = C(\underset{\sim}{\xi}, \bar{y}_2;0) \qquad \ldots\ldots\ldots (30)$$

$$= \left[e^{-\alpha_1(\xi_1^2+\xi_3^2)} + \frac{1}{9}\left(1-\frac{8}{3}\alpha_1\xi_1^2\right)e^{-\alpha_1(\xi_1^2+\xi_3^2)/3}\right]$$

where $\underset{\sim}{\xi} = (\underset{\sim}{\bar{x}} - \underset{\sim}{\bar{y}})$ and α_1 is a function of $\bar{y}_2$. This form for $C(\underset{\sim}{\xi},\bar{y}_2;0)$ satisfies the continuity relation

$$\iint_{-\infty}^{\infty} C \;\; d\xi_1 d\xi_3 = 0.$$

The second term in C which clearly arises from the big eddies is important, only for large separation.

It is found that the spectrum function is dominated by the term arising from the smaller eddies and the only effect of the big eddy term is to cause a small hump near $k_1\delta_1 = 1$ as shown in Figure 5.

However in view of the presence of extraneous disturbances at low frequencies Wooldridge and Willmarth inserted a high pass filter which has the effect of modifying the value of $R_{p\dot{u}_2}$ at large separations. In addition, as already stated, $\overline{p \frac{\partial u'_2}{\partial x'_1}}$ can only be derived from $R_{p\dot{u}_2}$ when the separations are small. For these several reasons we will only evaluate $\sqrt{\overline{p_w^2}}$ although we can show qualitatively that the estimated values of $\bar{R}_{pp}$, the wall pressure covariance, is in reasonable agreement with the measurements of Wooldridge and Willmarth.

On inserting the measured values of $F(\bar{y}_2)$ and performing the integration numerically we find that the small eddy contribution to $\sqrt{\overline{p_w^2}}/\tau_w$ is 2.5.

The effect of including the second term in C is to reduce this value slightly. It is found that the dominant contribution to the wall pressure fluctuation comes from a region between the wall and $y_2/\delta_1 = 1$.

The accuracy of the integration is poor since, as already indicated, the accuracy in fitting curves to the experimental data is poor when the correlation coefficient is very small. A rough estimate of the accuracy involved indicates that the coefficient of τ_w in the above expression for $\sqrt{\overline{p_w^2}}$ lies between 1.7 and 3 compared with a measured value of about $2.2\tau_w$.

The conclusion from this and the previous sections indicate that the pressure fluctuations at the wall is largely controlled by the turbulence-mean shear interaction and the region where this source term is dominant extends up to about $2\delta_1$ from the surface. As already stated this does not necessarily apply to the case when the pressure is measured away from the surface, and in fact we have shown that for $\overline{pu_2}$ the (M-T) interaction is exactly zero. Although $\overline{p^2}$ has not been calculated with any accuracy at distances away from the surface, approximate calculations indicate that the mean shear turbulence interaction contributes more than 50% to its overall value and the remainder is made up from the T-T interaction.

7. The Wall Pressure Covariance and Comparison with Experiment

We have shown above that the wall pressure covariance can be evaluated in terms of the mean flow and turbulent velocity fields. We have also shown, at least for the case of zero external pressure gradient, that the wall pressure fluctuations are dominated by contributions from the mean shear turbulence interaction. We have also discussed the difficulties in finding the magnitude of the remaining turbulence-turbulence interaction. On the assumption that the latter contribution is negligible, Hodgson (1962) has evaluated the necessary integrals numerically to obtain the longitudinal and transverse wall pressure spatial fluctuations and the autocorrelation. The comparison between his calculations and measurements made both in a wind tunnel and on the wing of a glider, are shown in Figures 6 and 7. The comparison between the calculated and measured autocorrelation as obtained from the glider experiments is particularly encouraging and adds support that the (M-T) interaction is dominant in this case.

Figure 7 shows that whereas the calculated transverse spatial correlation falls to zero beyond separation distances of the order of the boundary layer thickness, measurements made in wind tunnels invariably tend to a constant value for large separations. It can easily be shown that this asymptotic value decreases as the magnitude of these extraneous disturbances tends to zero.

Figure 8 shows the comparison between the calculated and measured power spectral density obtained from the glider measurements, and although the agreement is fair it should be noted that the ratio of the microphone diameter to the boundary layer displacement thickness is equal to 2.93, and 4.1 and hence it is necessary to correct the measured data for the finite size of the microphone. This has the effect of broadening out the spectrum curve and increases its level towards the higher frequencies.

From the measurements of Willmarth and Wooldridge, Hodgson and others, of the space-time correlations of the wall pressure fluctuations, it is found that the pressure field is convected past the wall transducer at a speed of approximately $0.8U_\infty$. This convection speed is a function of frequency and reduces to values of nearer $0.6U_\infty$ as the frequency is increased. An important result from the work of Hodgson is that the integral of the autocorrelation is zero in confirmation with the theory, and leads to a falling power spectral density at low frequencies. This result has not been obtained from experiments in wind tunnels (for example the experiments of Wooldridge and Willmarth) due to the presence of extraneous disturbances as discussed above.

It has already been mentioned that corrections are required to allow for the finite size of the pressure tranducer, and the magnitude of such corrections can be determined to some extent from the experimental work of Hodgson, and Wooldridge and Willmarth, while Corcos has performed both experiments and has presented a theory for these corrections. It would appear, however, that our state of knowledge about these corrections is only fair and more work is required in order to determine this effect with certainty.

The pressure measurements made on the glider recently by Eaton and Goddard[(18)] (1963) have shown that in the region of adverse pressure gradient the values of $\sqrt{\overline{p_w^2}}$ do not change greatly from their values in zero pressure gradient. The effect of variation of Reynolds number cannot be deduced from these results. It is however noted, since all velocities are now conveniently expressed in terms of the local external velocity, U_∞, that

$$\sqrt{\overline{p_w^2}} \Big/ \rho_0 U_\infty^2$$

should increase slightly as separation is approached, corresponding to the increase in

$$\overline{q^2} \Big/ U_\infty^2$$

The experimental results confirm this.

8. Compressible Flow Theory

From the theory in incompressible flow as presented above, we have shown that for the case of zero external pressure gradient with a linear relation between $\overline{p_w^2}$ and τ_w exists. We have also presented some evidence to support the view that the larger contribution to the wall pressure fluctuations arises from the (M-T) interaction. In the case of a compressible flow we might expect similar relations to apply and indeed this is confirmed by the measurements of Kistler and Chen (1962) who show from wind tunnel experiments of the wall pressure fluctuations under conditions of zero heat transfer up to Mach numbers of 5, that

$$\sqrt{\overline{p_W^2}} = a(R, M_\infty)\ \tau_W$$

where $a(R, M_\infty)$ is a slowly varying function of both the Reynolds number (R), and the freestream Mach number (M_∞).

If we assume therefore that the interactions existing in incompressible flow still apply apart from modifications arising from compressibility effects, and if we neglect all diffusive effects then our equations in dimensional form can be written in the form proposed by Phillips (1960) as follows

$$\frac{u_\tau^2}{a_\infty^2}\frac{D^2p}{Dt^2} + \frac{a^2}{a_\infty^2}\nabla^2 p + \frac{d}{dx_2}\frac{a^2}{a_\infty^2}\frac{\partial p}{\partial x_2} = \frac{\rho_a}{\rho_W}A(\underset{\sim}{x}, t) \qquad \text{.... (31)}$$

where $p \equiv p/\rho_W u_\tau^2$; $\underset{\sim}{x} \equiv \rho_W u_\tau x/\mu_W$; $t \equiv t u_\tau^2 \rho_W/\mu_W$; $u_\tau \equiv \sqrt{\tau_W/\rho_W}$

$$A(\underset{\sim}{x}, t) \equiv -\left(2\frac{\partial U_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x_1} \quad \text{and} \quad U_1 \equiv U_1(x_2) \text{ only.}$$

In this equation we have assumed that the mean values of the density, viscosity and shear stress at the wall are independent of the streamwise distance. We have also assumed that the speed of sound, a, is only a function of the distance normal to the wall. These simplifying assumptions imply the neglect of convection and scattering of sound waves by the turbulence and of fluctuations in the speed of sound. Our equation does however include the effect of fluctuations in density (sound waves) and hence pressure fluctuations result from fluctuations in both the vorticity and sound modes, where in view of our remarks regarding the analogy between the incompressible and compressible flow problems, the vorticity mode is expected to provide the larger contribution. If this were not so the mean flow in a compressible boundary layer could not be derived from a simple transformation of the corresponding results in an incompressible flow which is in contradiction to the work of Morkovin[15] (1962) and Coles[16] (1961).

It is of course not necessary for us to neglect diffusive effects at this stage since these can be incorporated into the right hand side of the equation above.

A solution to equation (31) can be obtained by finding its Fourier transform and defining the three dimensional Fourier-Stieltjes transforms of $p(\underset{\sim}{x},t)$ and $A(\underset{\sim}{x},t)$ as follows:

$$p(\underset{\sim}{x},t) = \int e^{i(k_1x_1+k_3x_3+\omega t)} \quad d\tilde{\omega}(x_2;\underset{\sim}{k},\omega)$$

$$A(\underset{\sim}{x},t) = \int e^{i(k_1x_1+k_3x_3+\omega t)} \quad d\gamma(x_2;\underset{\sim}{k},\omega)$$

where $\underset{\sim}{k}$ is the wave number vector in the (x_1,x_3) plane and ω is the frequency. The equation for the Fourier coefficient $d\,\tilde{\omega}$ is

$$d\tilde{\omega}'' + \frac{d}{dx_2}\left(\ln \frac{a^2}{a_\infty^2}\right) d\tilde{\omega}' - \left(k^2 - \frac{u_\tau^2}{a^2}(\omega + U_1k_1)^2\right) d\tilde{\omega} = (\rho/\rho_w)d\gamma \quad \ldots\ldots (32)$$

and on eliminating the first derivative by the use of the dependent variable ζ with

$$\zeta = (a/a_w)\, d\tilde{\omega}$$

we find that

$$\zeta'' - \left(k^2 - \frac{u_\tau^2}{a^2}(\omega + U_1k_1)^2 + a''/a\right)\zeta = (\rho a/\rho_w a_w)d\,\gamma \qquad \ldots\ldots\ldots\ldots (33)$$

It can be seen that this equation reduces to the corresponding equation in incompressible flow when $a = a_w \rightarrow \infty$.

In his analysis Phillips chose non-dimensional co-ordinates such that the width of the shear layer was unity and the Mach number of the external flow was very high, and finally obtained the sound radiation from the shear layer by a solution which neglected terms of order $1/M_\infty$. In our problem we have chosen boundary layer coordinates such that $U_1 \rightarrow U_\infty$ as $x_2 \rightarrow \infty$.

It is not possible for us to estimate the radiation of sound from the boundary layer, since our solution does not include the far field outside the boundary layer, but nevertheless one boundary condition which we must apply involves the magnitude of the disturbance pressure at the edge of the boundary layer.

9. The Solution of the Pressure Equation

The solution is required of

$$\zeta'' - b(x_2)\zeta = h(x_2)$$

where primes denote differentiation with respect to x_2, and

$$b(x_2) = k^2 - \frac{u_\tau^2}{a^2}(\omega + U_1 k_1)^2 + \frac{a''}{a}$$

$$h(x_2) = \frac{\rho a}{\rho_w a_w} \, d\gamma(x_2; \underset{\sim}{k}, \omega)$$

We note that

$$\zeta(0; \underset{\sim}{k}, \omega) \equiv d\tilde{\omega}(0; \underset{\sim}{k}, \omega)$$

Now

$$\frac{u_\tau^2}{a_w^2} = (C_f/2)\, M_\infty^2 \quad \ll 1$$

for all M_∞ and so we can choose $\lambda = \frac{a_w}{u_\tau} \gg 1$ for all M_∞, as shown in Figure 9. Hence we require a solution of (33) for large values of λ. If we introduce the new dependent variable

$$y = x_2 u_\tau^2 / a_w^2$$

and put

$$q(y) = k^2 \frac{a_w^2}{u_\tau^2} - \left(\frac{\omega + U_1 k_1}{\bar{a}}\right)^2$$

where $\bar{a} = a/a_w$, then

$$\zeta_{yy} - \lambda^2 q(y)\zeta = H(y) \qquad \qquad \text{.........} \; (34)$$

with

$$H(y) = \frac{\rho a}{\rho_w a_w}\left(\frac{a_w}{u_\tau}\right)^4 d\gamma + \frac{a_{yy}}{\bar{a}}\zeta$$

The term in $\bar{a}_{yy}$ has been included as a source term for convenience since its numerical value is small compared with the dominant source term as can be inferred form Figure 10. A typical variation of $q(y)$ across the boundary layer is shown in Figure 11.

The boundary conditions are as follows:

$$d\tilde{\omega}'(0) = \zeta'(0) = 0$$

and either $$d\tilde{\omega}'(\infty) = \zeta(\infty) = 0$$

or, ζ must be bounded as $x_2 \rightarrow \infty$, for the cases respectively of zero disturbance at the edge of the boundary layer and outward propagating waves. For $y \rightarrow \infty$, $H(y) \sim 0$, $U_1 = U_\infty$, $\bar{a} = a_\infty/a_W$, so that (34) becomes:

$$\zeta_{yy} - \lambda^2 \left(k^2\lambda^2 - \frac{a_W^2}{a_\infty^2} (\omega + U_\infty k_1)^2 \right) \zeta = 0$$

and the solution must be exponentially decreasing or oscillatory so that ζ is bounded as $y \rightarrow \infty$. Thus when

$$(k\lambda)^2 > \left[\frac{a_W}{a_\infty} (\omega + U_\infty k_1) \right]^2$$

we obtain the exponentially decreasing solution while for

$$\left[\frac{a_W}{a_\infty} (\omega + U_\infty k_1) \right]^2 > (k\lambda)^2$$

we obtain the oscillatory solution which physically corresponds to a radiated pattern of Mach waves as discussed by Phillips (loc cit). In fact with $k_1 = k \cos \theta$ and for a convection speed of the turbulence U_c such that $\omega = -k_1 U_c$ eddy Mach waves will be propagated outwards for those wave numbers defined by

$$\cos^2\theta > \left[M_\infty (1 - U_c/U_\infty) \right]^{-2}$$

Thus the eddy Mach waves are generated by those wave numbers in the outer region of the boundary layer for which

$$\pi - \theta_m < \theta < \pi + \theta_m \text{ and } - \theta_m < \theta < \theta_m$$

where $$\theta_m = \cos^{-1}\left[1/(1-U_c/U_\infty)M_\infty\right].$$

For the remaining wave numbers $\zeta \longrightarrow 0$ as $y \longrightarrow \infty$.

At the wall if $\omega = -U_c k_1$, $q(0)$ vanishes when

$$\cos^2\theta_c = a_w^2/U_c^2 = \frac{1+\frac{\gamma-1}{2}M_\infty^2}{M_\infty^2} \quad \frac{U_\infty}{U_c}$$

for the case of zero heat transfer.

Thus for those wave numbers in the region of the wall for which

$$\pi - \theta_c < \theta < \pi + \theta_c \text{ and } - \theta_c < \theta < \theta_c$$

eddy Mach waves exist since $q(y_1) < 1$ and the solution will be oscillatory.

Hence there are certain wave numbers of the turbulence in the region $0 \leq y < y_o$ where $q(y)$ is negative and similarly in the region $y_1 < y < \infty$. In the region $y_o < y < y_1$, $q(y)$ is positive, although for some wave numbers $q(y)$ is positive for all values of y.

Thus for certain wave numbers equation (34) has two transition points at $y = y_o$ and $y = y_1$ where $q(y) = 0$.

<u>Case I $q(y) > 0 \quad 0 \leq y \leq \infty$</u>

For the case where $q(y)$ is everywhere positive the solution of equation (34) can be found if terms in $1/\lambda$ are neglected. On insertion of the boundary conditions at the wall and at $y = \infty$ we find that

$$d\tilde{\omega}(0) = - \frac{1}{\lambda q(0)^{\frac{1}{4}}} \int_0^\infty \frac{H(y')}{q(y')^{\frac{1}{4}}} \exp\left(-\lambda \int_0^{y'} q^{\frac{1}{2}} dy''\right) dy' \qquad \ldots. (35)$$

This equation reduces to that found in incompressible flow when $\lambda q^{\frac{1}{2}} = a_w^2 k/u_\tau^2$.

Case II $q(y) < 0 \quad y_1 < y < \infty$ where $q(y_1) = 0$

A solution is required in the region around $y = y_1$ where $q(y_1) = 0$. But this solution must be the analytic continuation of the solution found around $y = y_1$ for $q(y) > 0$ and this is presented under case IV below.

(The solution can of course be written in terms of μ^* and t^* where

$$t^* = \left(\frac{3}{2} \int_{y_1}^{y} \sqrt{q^*} \; dy \right)^{2/3}$$

$$\mu^* = \frac{2}{3} \lambda t^{*\,3/2} \quad \text{and}$$

$$q^* = -q > 0$$

Hence $t = t^* e^{i\pi}$; $q = q^* e^{i\pi}$; $\mu = \mu^* e^{i3\pi/2}$)

Case III $q(y) > 0 \quad y_o < y < y_1$ where $q(y_o) = 0$

The solution around $y = y_o$ where $q(y_o) = 0$ is

$$\zeta(y) = (s/q)^{\frac{1}{4}} \left\{ \eta^{\frac{1}{3}} I_{\frac{1}{3}}(\eta)\, \alpha(y) + \eta^{\frac{1}{3}} K_{\frac{1}{3}}(\eta)\, \beta(y) \right\}$$

where $$s = \left(\frac{3}{2} \int_{y_o}^{y} \sqrt{q} \; dy \right)^{\frac{2}{3}} \qquad q > 0$$

$$\eta = \frac{2}{3} \lambda s^{3/2}$$

$$\alpha(y) = A + \frac{(\frac{2}{3})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \int_0^s \phi(s)\, \eta^{\frac{1}{3}} K_{\frac{1}{3}}(\eta)\, ds \qquad \ldots\ldots\ldots (36)$$

$$\beta(y) = B - \frac{(\frac{2}{3})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \int_0^s \phi(s)\, \eta^{\frac{1}{3}} I_{\frac{1}{3}}(\eta)\, ds \qquad \ldots\ldots\ldots (37)$$

with $\phi(s) = \dfrac{H(y)}{(q/s)^{\frac{3}{4}}}$ + small terms.

A and B are constants to be determined from the boundary conditions.

Case IV $\quad q(y) > 0 \quad y_o < y < y_1$ where $q(y_1) = 0$

The solution found above is not valid near $y = y_1$ where $q(y_1) = 0$ even though s is finite at $y = y_1$. A solution around $y = y_1$ is

$$\zeta(y_1) = (t/q)^{\frac{1}{4}} \left\{ \mu^{\frac{1}{3}} I_{\frac{1}{3}}(\mu)\, \gamma(y) + \mu^{\frac{1}{3}} K_{\frac{1}{3}}(\mu) \quad \delta(y) \right\}$$

where $t = \left(\frac{3}{2} \int_y^{y_1} \sqrt{q} \quad dy \right)^{\frac{2}{3}} \quad q > 0$

$$\mu = \tfrac{2}{3}\lambda \; t^{3/2}$$

$$\gamma(y) = C + \frac{(\frac{2}{3})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \int_0^t \phi(t) \; \mu^{\frac{1}{3}} K_{\frac{1}{3}}(\mu) \; dt \qquad \text{............ (38)}$$

$$\delta(y) = D - \frac{(\frac{2}{3})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \int_0^t \phi(t) \; \mu^{\frac{1}{3}} I_{\frac{1}{3}}(\mu) \; dt \qquad \text{............ (39)}$$

with $\phi(t) = \dfrac{H(y)}{(q/t)^{\frac{3}{4}}}$ + small terms

C and D are constants to be determined from the boundary conditions.

Case V $\quad q < 0 \quad 0 < y < y_o \quad q(y_o) = 0$

In this region although q is negative the solution given for Case III will still be valid. However s and η will now be complex numbers.

(The solution can of course be written in terms of η^* and s^*

where $s^* = \left(\frac{3}{2} \int_y^{y_o} q^* \quad dy \right)^{\frac{2}{3}}$

$$\eta^* = \tfrac{2}{3}\, \lambda \; s^{*\,3/2}$$

and $\quad q^* = -q > 0$

Hence $s = s^* e^{i\pi}$; $q = q^* e^{i\pi}$; $\eta = \eta^* e^{i3\pi/2}$ and it follows that the modified Bessel functions with imaginary arguments can be written in terms of Bessel functions of the first kind.

10. Evaluation of the constants A,B,C and D.

The boundary conditions are:

$$\frac{\partial \zeta}{\partial y} = 0 \qquad y = 0$$

and ζ is bounded as $y \to \infty$. We will assume, following Phillips, that only outgoing Mach waves will exist or

$$\zeta(y) \sim \exp(-i\mu^*) \text{ as } y \to \infty.$$

The remaining two conditions are found by matching the solutions found for ζ and $\frac{d\zeta}{dy}$ around $y = y_o$ and $y = y_1$ respectively at some convenient station in $y_o < y < y_1$.

In the appendix this station is taken at $y = Y$ where

$$\omega + U_1(Y)k_1 = 0$$

It is shown in the appendix that A,B,C,D can be evaluated in terms of weighted integrals of the source function $H(y)$ taken over certain regions of the boundary layer.

11. The Pressure Spectrum and the Mean Square Pressure

The spectrum function can be obtained from the Fourier coefficient $d\tilde{\omega}(0)$ of the wall pressure fluctuations. Thus for the case $q(y) > 0$ everywhere, that is for eddies moving subsonically with respect to the speed of sound at the wall, it is found that

$$\Pi(0;\underset{\sim}{k},\omega) \equiv \frac{\overline{d\tilde{\omega}(0;\underset{\sim}{k},\omega)\, d\tilde{\omega}^*(0;\underset{\sim}{k},\omega)}}{dk_1 dk_3 d\omega}$$

$$= \frac{4k_1^2\lambda^2}{q_w^{\frac{1}{2}}} \int_0^\infty \Phi_{22}(y,z;\underset{\sim}{k},\omega)\, \frac{dU_1}{dy}\, \frac{\overline{\rho a e}}{q^{\frac{1}{4}}}^{-\eta}\, dy.$$

$$\int_{-y}^{\infty} \frac{dU_1}{dz}\, \frac{\overline{\rho a e}}{q^{\frac{1}{4}}}^{-\eta}\, dz$$

on writing $\eta = \int_0^y \sqrt{q}\ dy$, and Φ_{22} is the spectrum function of the normal velocity. For the other wave numbers similar expressions for $\Pi(0;\underset{\sim}{k},\omega)$ can be obtained.

Now in the range of supersonic freestream Mach numbers up to about 5 it is found that the contribution to the pressure from the wave numbers generating eddy Mach waves at the wall is roughly comparable with the contribution from the wave numbers moving subsonically with respect to the speed of sound at the wall.

The remainder of the calculation involves the replacing of the compressible flow quantities by the equivalent incompressible values using Coles' transformation(16). It is found that the spectrum function is not greatly changed from its value in incompressible flow.

An integration over all wave numbers and frequencies enables us to find the mean square pressure and the results of the numerical integration show that $\sqrt{\overline{p_w^2}}/\tau_w$ increases slowly with Mach number. In fact $\sqrt{\overline{p_w^2}}$ ecreases due to the reduction in skin friction coefficient with Mach number and increases as a result of the dominant source region moving nearer the wall with increase in Mach number.

Although as stated previously the present analysis cannot be used to determine the sound radiation from the boundary layers, it is shown that the outward radiation in the outer part of the layer is given by equation A.23 in the Appendix.

The comparison of these results with the only available experimental data of Kistler and Chen(8) is fair as shown in Figure 12.

12. Conclusions

It is shown that in incompressible flow the available theoretical and experimental data support the conclusion that the pressure fluctuations at the wall in a turbulent boundary layer are dominated by the interaction between the mean shear and the turbulence.

In incompressible flow few experimental data exist but this data shows that up to Mach numbers of the order of 5 similar source terms must exist since the changes in level of the pressure fluctuations with M_∞ together with the changes in the power spectral density are reasonably small. A theory is presented which supports this conclusion. It is shown that even though eddy Mach waves exist both near the wall and in the outer region when the turbulence is moving supersonically with respect to the speeds of sound in the external flow, and at the wall, the overall increase in pressure level is small at Mach numbers up to 5.

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19. Laufer, J.

APPENDIX.

The Solution of the Pressure Equation for the Wall Pressure

In Section 9 the solution of the pressure equation for a compressible boundary layer has been obtained apart from the evaluation of four constants which are to be determined from the boundary conditions. For those wave numbers which generate eddy Mach waves both near the wall and in the outer region of the boundary layer the solution is as follows:

$$\zeta(y) = -\left(\frac{s^*}{q^*}\right)^{\frac{1}{4}}\left|\eta^{*\frac{1}{3}}J_{\frac{1}{3}}(\eta^*)\alpha^*(y) + \frac{i\pi}{2}e^{i\pi/3}\eta^{*\frac{1}{3}}\mathcal{J}_{\frac{1}{3}}(\eta^*)\beta^*(y)\right| 0 \leq y < y_o \quad \ldots\ldots \text{A.1}$$

$$\zeta(y) = \left(\frac{s}{q}\right)^{\frac{1}{4}}\left|\eta^{\frac{1}{3}}I_{\frac{1}{3}}(\eta)\,\alpha(y) + \eta^{\frac{1}{3}}K_{\frac{1}{3}}(\eta)\,\beta(y)\right| y_o \leq y < y_1 \quad \ldots\ldots \text{A.2}$$

$$\zeta(y) = \left(\frac{t}{q}\right)^{\frac{1}{4}}\left|\mu^{\frac{1}{3}}I_{\frac{1}{3}}(\mu)\gamma(y) + \mu^{\frac{1}{3}}K_{\frac{1}{3}}(\mu)\,\delta(y)\right| y_o < y \leq y_1 \quad \ldots\ldots \text{A.3}$$

$$\zeta(y) = -\left(\frac{t^*}{q^*}\right)^{\frac{1}{4}}\left|\mu^{*\frac{1}{3}}J_{\frac{1}{3}}(\mu^*)\gamma^*(y) + \frac{i\pi}{2}e^{i\pi/3}\mu^{*\frac{1}{3}}\mathcal{J}_{\frac{1}{3}}(\mu^*)\delta^*(y)\right| y_1 < y < \infty \quad \ldots\ldots \text{A.4}$$

with

$$\alpha(y) = A + \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}}\int_o^s \phi(s)\eta^{\frac{1}{3}}K_{\frac{1}{3}}(\eta)\,ds \quad \ldots\ldots \text{A.5}$$

$$\beta(y) = B - \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}}\int_o^s \phi(s)\,\eta^{\frac{1}{3}}I_{\frac{1}{3}}(\eta)\,ds \quad \ldots\ldots \text{A.6}$$

$$\gamma(y) = C + \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}}\int_o^t \phi(t)\,\mu^{\frac{1}{3}}K_{\frac{1}{3}}(\mu)\,dt \quad \ldots\ldots \text{A.7}$$

$$\delta(y) = D - \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}}\int_o^t \phi(t)\,\mu^{\frac{1}{3}}I_{\frac{1}{3}}(\mu)\,dt \quad \ldots\ldots \text{A.8}$$

$$\alpha^*(y) = A - \frac{\pi\left(\frac{2}{3}\right)^{\frac{1}{3}}}{2\sin\frac{\pi}{3}\,\lambda^{\frac{2}{3}}}\int_o^{s^*}\phi(s^*)\left(J_{\frac{1}{3}}(\eta^*) + J_{-\frac{1}{3}}(\eta^*)\right)\eta^{*\frac{1}{3}}\,ds^* \quad \ldots\ldots \text{A.9}$$

$$\beta^*(y) = B - \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}}\int_o^{s^*}\phi(s^*)J_{\frac{1}{3}}(\eta^*)\eta^{*\frac{1}{3}}ds^* \quad \ldots\ldots \text{A.10}$$

$$\gamma^*(y) = C - \frac{\pi}{2}\,\frac{(\frac{2}{3})^{\frac{1}{3}}}{\sin\frac{\pi}{3}\lambda^{\frac{2}{3}}}\int_0^{t^*} \phi(t^*)\left(J_{\frac{1}{3}}(\mu^*) + J_{-\frac{1}{3}}(\mu^*)\right)\mu^{*\frac{1}{3}}dt^* \quad \ldots\ldots \text{A.11}$$

$$\delta^*(y) = D - \frac{(\frac{2}{3})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}}\int_0^{t^*} \phi(t^*)J_{\frac{1}{3}}(\mu^*)\,\mu^{*\frac{1}{3}}dt^* \quad \ldots\ldots\ldots\ldots \text{A.12}$$

The application of the boundary condition

$$\frac{\partial\zeta}{\partial y} = 0 \text{ at } y = 0 \text{ leads to}$$

$$d\tilde{\omega}(0) = \zeta(0) = \frac{(3/2)^{1/6}\,\beta^*(0)}{\lambda^{1/6}q_w^{*\frac{1}{4}}\,\eta_w^{*\frac{1}{2}}\,J_{-\frac{2}{3}}(\eta^*_w)} \quad \ldots\ldots\ldots\ldots \text{A.13}$$

for λ large, where the subscript, w, refers to conditions at y = 0, and

$$\alpha^*(0) = \frac{\pi}{\sqrt{3}}\left(1 - J_{\frac{2}{3}}(\eta^*_w)\Big/J_{-\frac{2}{3}}(\eta^*_w)\right)\beta^*(0) \quad \ldots\ldots\ldots\ldots \text{A.14}$$

provided the term in brackets does not vanish.

When $\quad J_{\frac{2}{3}}(\eta^*_w) = J_{-\frac{2}{3}}(\eta^*_w)$

$$d\tilde{\omega}(0) = \zeta(0) = -\frac{i\pi}{2}\,e^{i\pi/3}\,\eta_w^{*\frac{1}{2}}\mathcal{J}_{\frac{1}{3}}(\eta^*_w)\beta^*(0)\Big/(\tfrac{2}{3})^{1/6}\,\lambda^{1/6}\,q^{*\,\frac{1}{4}}_w \quad \ldots\ldots \text{A.15}$$

for λ large and

$$\alpha^*(0) = \frac{-\frac{i\pi}{8}\left(\left(\frac{q^*_w}{s^*_w}\right)^{3/2} + q^{*\prime}_w\right)e^{i\pi/3}\mathcal{J}_{\frac{1}{3}}(\eta^*_w)\beta^*(0)}{\lambda J_{-\frac{2}{3}}(\eta^*_w)q^{*\;3/2}_w} \quad \ldots\ldots \text{A.16}$$

Thus A is given in terms of B and $d\tilde{\omega}(0)$ is found when the value of B has been obtained.

If only outward propagating eddy Mach waves are allowed in the region $y \to \infty$ we find that

$$\zeta(y) \sim -\frac{1}{\sqrt{2}\pi(\frac{2}{3})^{1/6}\,\lambda^{1/6}\,q^*(\infty)^{\frac{1}{4}}}\;\frac{e^{i5\pi/12}\left(e^{i\pi/3} - e^{-i\pi/3}\right)\gamma^*(\infty)}{(1 + e^{i\pi/3})} \quad \ldots\text{A.17}$$

with

$$\gamma^*(\infty) = \pi e^{i\pi/6} \delta^*(\infty) \qquad \text{......... A.18}$$

and so C is given in terms of D.

The next condition is found by matching the value of ζ and $\frac{d\zeta}{dy}$ found from the solutions about $y = y_o$ and $y = y_1$ respectively. We thus find that for λ large

$$\left(\alpha(Y) + \gamma(Y) \right) I_{-\frac{2}{3}}(\eta) = \left(\beta(Y) + \delta(Y) \right) K_{-\frac{2}{3}}(\eta) \qquad \text{........ A.19}$$

$$\left(\alpha(Y) - \gamma(Y) \right) I_{\frac{1}{3}}(\eta) = \left(\delta(Y) - \beta(Y) \right) K_{\frac{1}{3}}(\eta) \qquad \text{........ A.20}$$

and for large η

$$\alpha(Y) = \gamma(Y) = 0$$

The matching station is taken at $y = Y$ with $y_o < Y < y_1$ where

$$\omega + U_1(Y)k_1 = 0$$

Hence both A and C can be evaluated. Thus

$$A = - \frac{(\frac{3}{2})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \int_{y_o}^{Y} \phi(s)\, \eta^{\frac{1}{3}} K_{\frac{1}{3}}(\eta) \frac{ds}{dy}\, dy \qquad \text{......... A.21}$$

and

$$C = - \frac{(\frac{3}{2})^{\frac{1}{3}}}{\lambda^{\frac{2}{3}}} \int_{y_1}^{Y} \phi(t) \mu^{\frac{1}{3}} K_{\frac{1}{3}}(\mu) \frac{dt}{dy}\, dy \qquad \text{......... A.22}$$

and we can find B and D from (A.16) and (A.18) respectively.

The outward radiation given by (A.17) becomes using these expressions for C and D

$$d\tilde{\alpha}(\infty, \underset{\sim}{k}, \omega) \sim ie^{i\pi/4} e^{-i\mu^*} \frac{\sqrt{\pi}}{\lambda^{3/2}} \frac{H(y_1)}{\sqrt{q^{*\prime}(y_1)}} \frac{a_w/a_\infty}{q^*(\infty)^{\frac{1}{4}}} \qquad \text{......... A.23}$$

If we insert the values for $H(y_1)$ and $q^{*\prime}(y_1)$ respectively the spectrum function can be shown to have the form

$$\Pi(\infty;\, \underset{\sim}{k},\omega) \sim 2\pi \frac{a_w}{a} \frac{dU_1}{dx_2} \frac{a_w^2}{a_\infty^2} \cdot \frac{\lambda^2 |k_1| \Phi_{22}}{q^*(\infty)^{\frac{1}{2}} k} \qquad \text{.......... A.24}$$

where a, $\frac{dU_1}{dx_2}$ and Φ_{22} are all evaluated at $y = y_1$.

It is now necessary to identify the frequencies ω at a given wave number $\underset{\sim}{k}$ in the source function in the plane $y = y_1$, which arise from disturbances below this plane, say at $y = Y$, where the speed of convection is $U_c(Y)$ and $\omega = -U_c(Y)k_1$. Since the convection speed is positive ω will have the opposite sign to $\cos\theta$, where $k_1 = k\cos\theta$, and this imposes a cut off wave number angle at each $y = y_1$ as shown in figs. 13a and 13b. Thus only a finite range of frequencies contributes to the pressure wave number spectrum for any given wave number. A diagram illustrating the generation of eddy Mach waves is shown in Figures 14a and 14b.

On inserting a suitable form for Φ_{22}, which is a reasonable approximation to the measured spectrum function except at very high numbers, and using Coles' relations between the incompressible and compressible properties of the boundary layer it is found that

$$\frac{\overline{p^2}(\infty)}{p_\infty^2} \; \frac{U_c/U_\infty \quad \delta_2/\delta}{M_\infty^3} \simeq 1 \times 10^{-8} \qquad \text{......... A.25}$$

A comparison with the experimental results of Laufer[18] is shown in Figure 15.

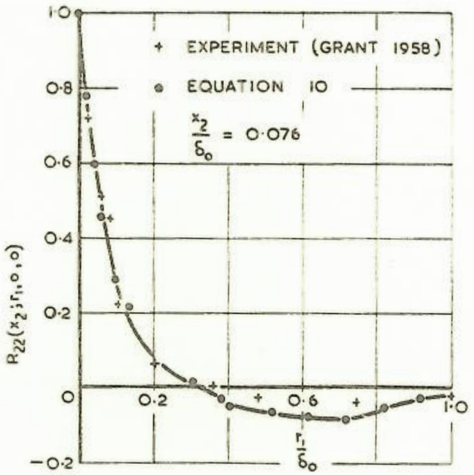


FIG. 1. VELOCITY CORRELATION COEFFICIENT R_{22} (SEPARATION IN STREAMWISE DIRECTION.)

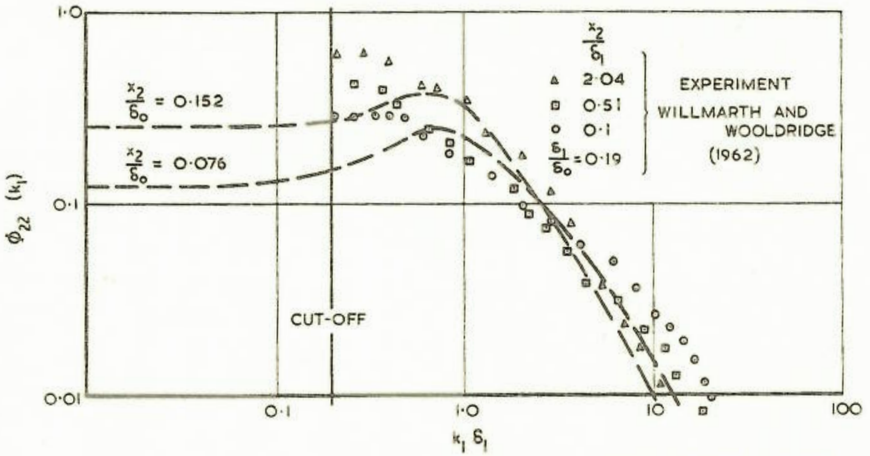


FIG. 2. COMPARISON BETWEEN EXPERIMENT AND ASSUMED ONE DIMENSIONAL SPECTRUM FUNCTION ϕ_{22}

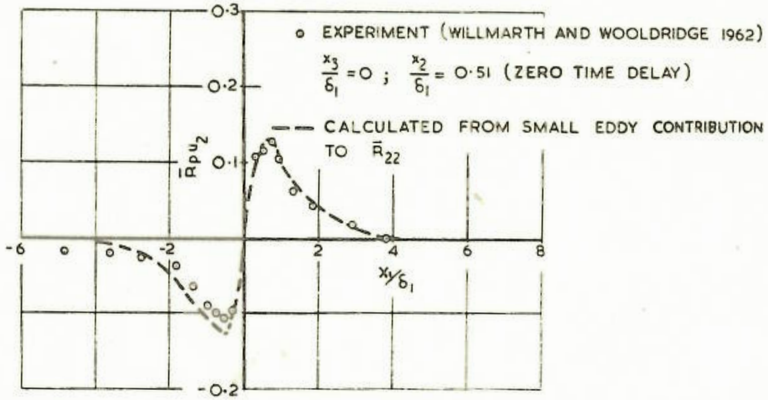


FIG. 3. COMPARISON BETWEEN CALCULATED AND MEASURED CORRELATION COEFFICIENT $\bar{R}pu_2$.

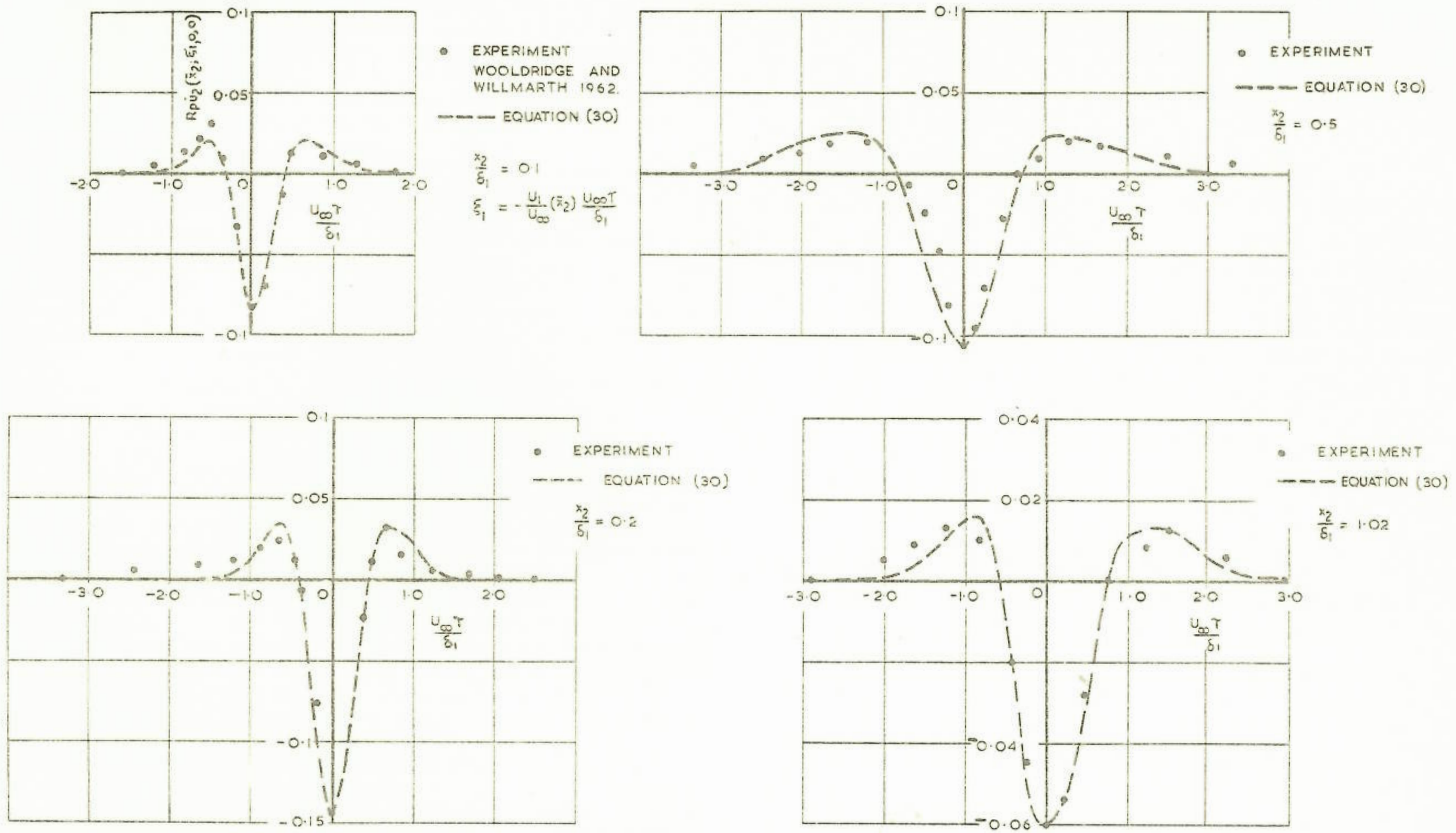


FIG. 4 COMPARISON BETWEEN MEASURED (WILLMARTH AND WOOLDRIDGE 1962) AND ASSUMED VALUES FOR THE CORRELATION $R_{p\dot{u}_2}$ WITH ZERO TIME DELAY.

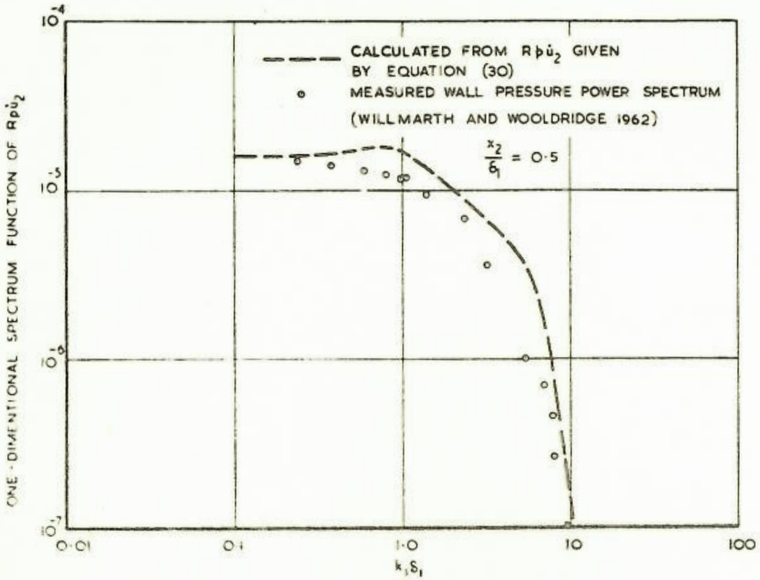


FIG. 5. COMPARISON BETWEEN ASSUMED $\phi_{p\dot{u}_2}(k_1)$ AND WALL PRESSURE POWER SPECTRUM.

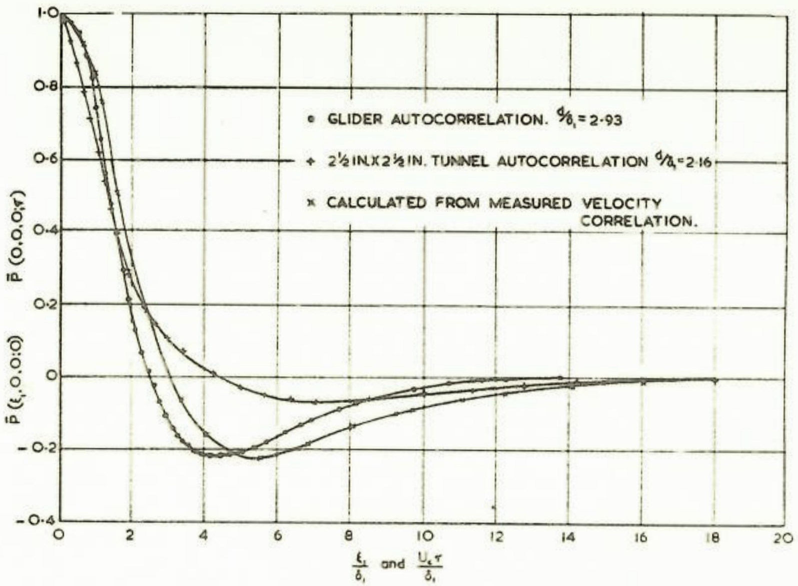


FIG. 6. COMPARISON OF SPATIAL CORRELATION AND AUTOCORRELATION OF WALL PRESSURE FLUCTUATIONS IN A TURBULENT BOUNDARY LAYER. (TAKEN FROM REF. 3.)

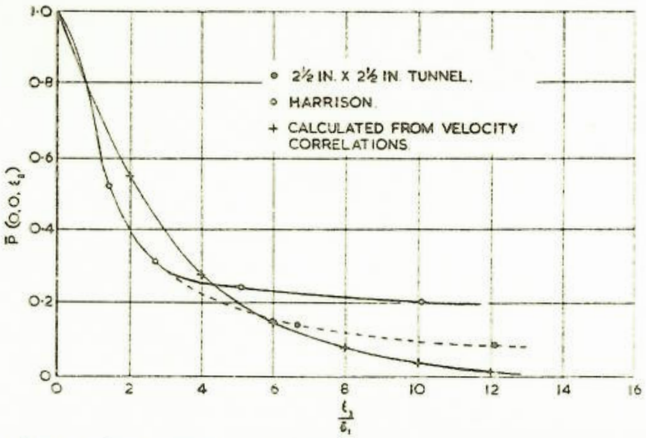


FIG. 7. $2\frac{1}{2}$ IN. X $2\frac{1}{2}$ IN. TUNNEL. TRANSVERSE SPATIAL CORRELATION OF WALL PRESSURE FLUCTUATIONS IN A TURBULENT BOUNDARY LAYER. (TAKEN FROM REF. 3.)

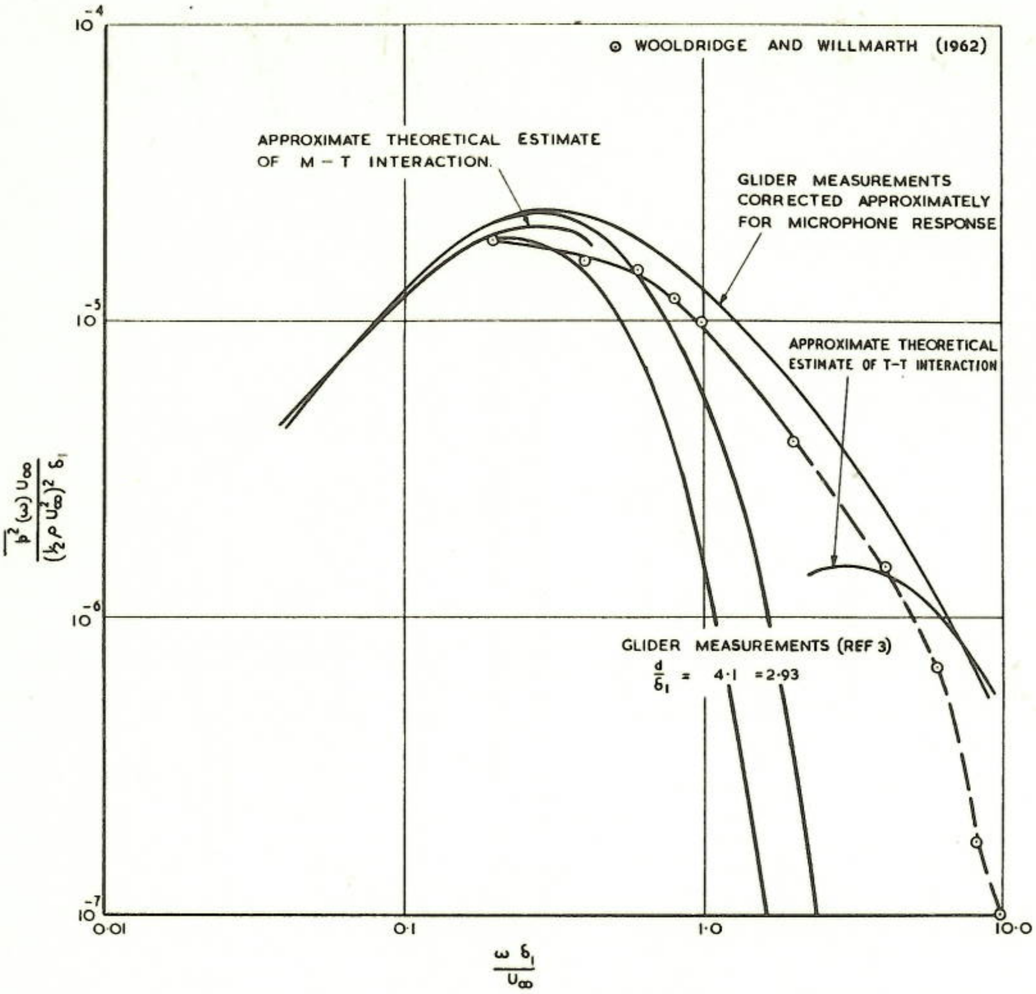


FIG.8. CALCULATED AND MEASURED POWER SPECTRAL DENSITY OF SURFACE PRESSURE FLUCTUATIONS IN A TURBULENT BOUNDARY LAYER

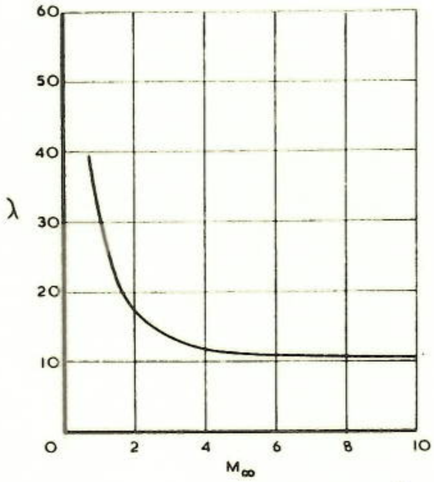


FIG. 9. VARIATION OF λ WITH MACH NUMBER. (ZERO HEAT TRANSFER)

$$\lambda = \frac{1}{M_\infty \sqrt{C_f/2}}$$

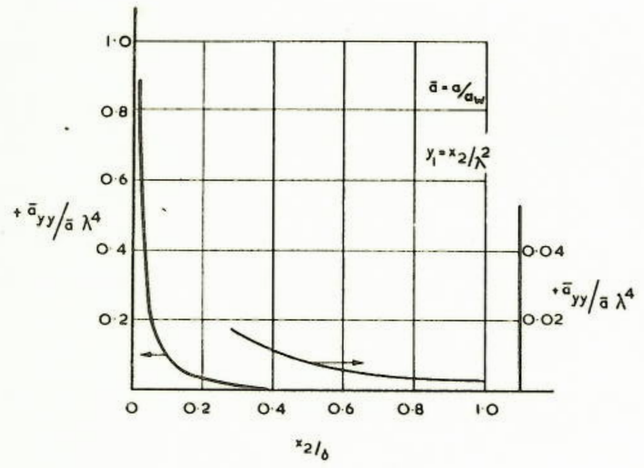


FIG. 10. VARIATION OF $+\bar{a}_{yy}/\bar{a}\,\lambda^4$ ACROSS BOUNDARY LAYER.

$M_\infty = 5$

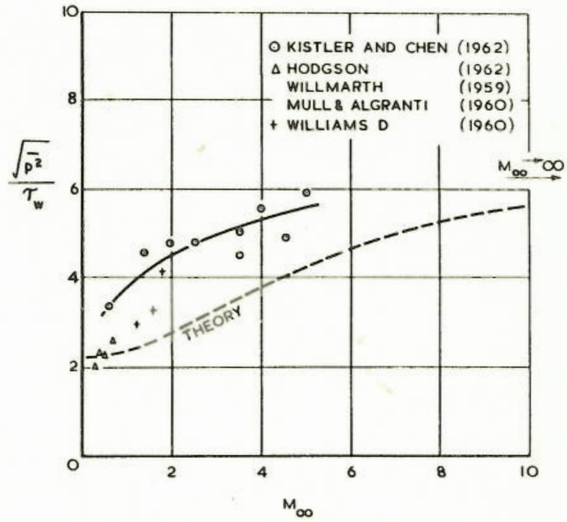


FIG.12. WALL PRESSURE FLUCTUATIONS IN A TURBULENT BOUNDARY LAYER.

M_∞ = FREESTREAM MACH NUMBER.

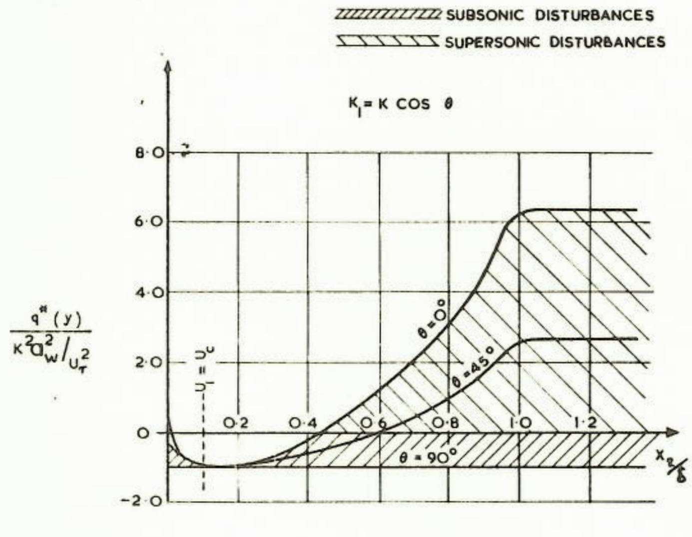


FIG. II. DISTRIBUTION OF $q^*(y)$ WHEN $\omega = U_c\,k_1$.

$M_\infty = 6{\cdot}8$ ZERO HEAT TRANSFER

— LINES y_1 = CONSTANT

- - - $\omega = -U_c K_1$

—⊥— CUT-OFF WAVE NUMBER (FUNCTION OF y_1)

FORMULA A

$$\left| \frac{\omega u_r}{a_\infty K} + \bar{U}_1 M_\infty \cos\theta \right| = \frac{a}{a_\infty}$$

$$q^*(\infty) \equiv K^2 \lambda^2 \left[\left(\frac{\omega u_r}{a_\infty K} + M_\infty \cos\theta \right)^2 - 1 \right] > 0$$

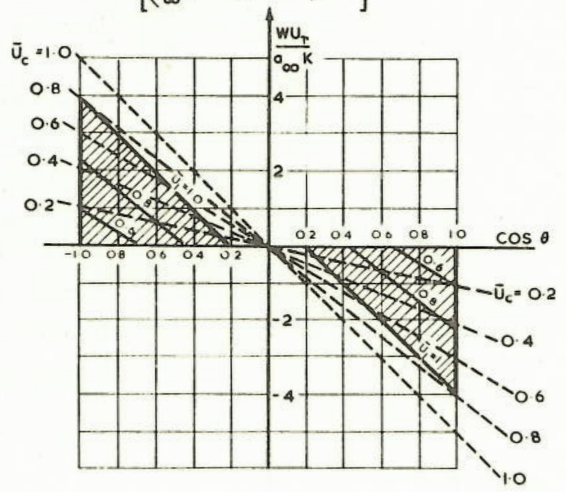
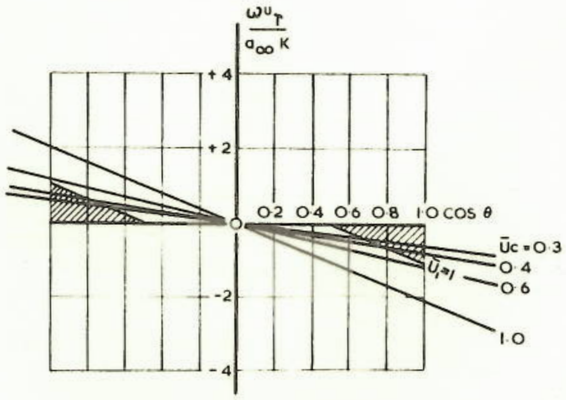


FIG. 13a. RANGE OF FREQUENCIES GENERATING 'EDDY' MACH WAVES OUTSIDE BOUNDARY LAYER.

M_∞ = 5 (ZERO HEAT TRANSFER)



M_∞ = 2 (ZERO HEAT TRANSFER)

FIG 13b. RANGE OF FREQUENCIES GENERATING 'EDDY' MACH WAVES OUTSIDE BOUNDARY LAYER.

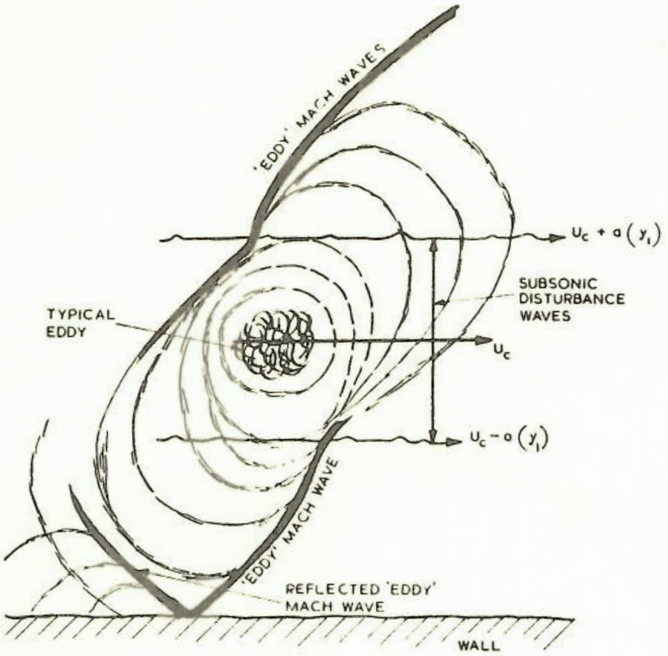


FIG. 14a. DIAGRAM SHOWING SCHEMATICALLY THE GENERATION OF 'EDDY' MACH WAVES.

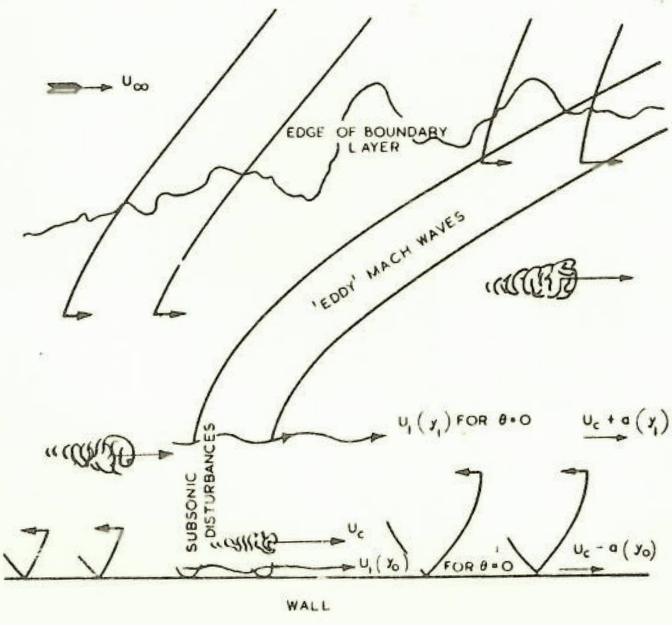


FIG. 14b. DIAGRAM SHOWING SCHEMATICALLY THE GENERATION OF 'EDDY' MACH WAVES.

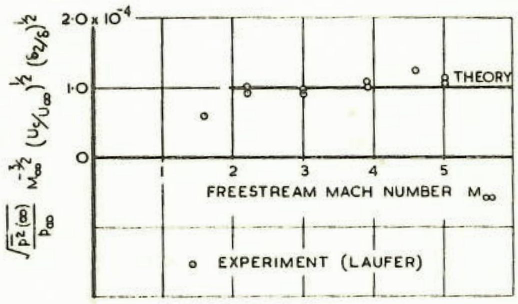


FIG. 15. PRESSURE FLUCTUATION OUTSIDE TURBULENT BOUNDARY LAYER. (CORRECTED FOR ONE WALL) ZERO HEAT TRANSFER.