REPORT NO. 141

November, 1960.

THE COLLEGE OF AERONAUTICS

CRANFIELD

The response time of wind tunnel pressure measuring systems

- by -

G. M. Lilley, M.Sc., D.I.C., A.F.R.Ae.S., A.M.I.Mech.E.

and

D. Morton, D.C.Ae. *

of the Department of Aerodynamics

SUMMARY

The time response of a wind tunnel pressure measuring system, comprising a pressure transducer of fixed volume and a length of capillary tubing, is analysed and the results compared with experiments. It is shown that the approximate analysis of Kendall (1958), in which the friction losses at any given time are assumed equal to the steady state losses, has a wide range of validity, provided the L/R ratio for the capillary tube is large and the inlet and exit losses are included as equivalent lengths of the capillary tube.

---

* Now a Research Engineer at Aircraft Research Association Ltd., Bedford.

CONTENTS

		Page
	Summary	
	Notation	
1.	Introduction	1
2.	The simplified pressure measurement system	1
3.	Assumptions	1
4.	The approximate solution of Kendall (1958)	2
5.	Unsteady flow in a capillary tube	4
6.	Steady flow in a long capillary tube	8
7.	The solution of the pressure equation for the unsteady flow in a reservoir connected to a capillary tube	11
8.	Apparatus used in the experiments	16
9.	Description of tests	16
10.	Discussion	16
11.	Conclusions	17
12.	Acknowledgement	18
13.	References	18
	Appendix	19
	Table I	20
	Figures	

NOTATION

a, a_o	speed of sound
c_1, c_2, c_3	constants
C, $\overline{C}$	parameter and its average value respectively
$f(^r/R)$	see equation 5.8
$k = \dfrac{\pi R^4 p_1}{8 \mu VL}$	response parameter
L	length of capillary tube
L_e	effective length
m	rate of mass flow
p	pressure
p^*	pressure at $z = L$
p_a	atmospheric pressure
p_o, p_1	initial and final pressures respectively
P =	$\dfrac{p^2 - p_o^2}{p^2}$
P_o =	$\dfrac{p_1^2 - p_o^2}{p_1^2}$
Q	rate of volume flow
(r, θ, z)	cylindrical polar co-ordinates
R	radius of capillary tube
s	Laplace Transform operator
t	time, also non-dimensional time $\left(\dfrac{t\, a_o}{L}\right)$
t_s	stabilisation time
t_c	time constant

Notation (Continued)

V	reservoir volume
w	axial velocity component
w_1	axial velocity component at $r = 0$
$\bar{w}$	average velocity at any section
z	distance along tube, also non-dimensional axial length (z/L)
α	see equation (7.12)
γ	$= \dfrac{2\pi R^2 L}{V \bar{C}}$
$\underset{\sim}{\lambda}$	complex variable
μ	viscosity
ρ	density
τ_w	wall shear stress
τ	see equation (7.4)
τ_1	value of τ corresponding to $t = 1$

1. Introduction

In pressure measuring systems for intermittent supersonic wind tunnels it is important that the response time of the system for stabilization of the pressures is less than the overall available running time of the tunnel. Also in intermittent supersonic wind tunnel testing, it is often necessary to measure several pressures in a short time. Due to the limited space available, in cases where the pressure transducer has to be installed in the model itself, it is necessary to use a selector switch (e.g. 'Scanivalve') to select the pressure tappings in turn. To determine the speed at which such a selector switch may be operated and in order to gain useful information from such a system, a knowledge of the response times of 'transducer - capillary tube' combinations is required. The approximate calculation of the response time of a simplified system, comprising a capillary tube connected to a constant volume reservoir, has been given by Kendall (1958) and others. The purpose of the present note is to give a more exact treatment of the response time for the same simplified system used by Kendall, and to give the results of some experiments to check the theory. It should be noted that whereas Kendall considered the case of a sudden decrease in the applied pressure to the transducer, for convenience, we have considered the case of a sudden increase in the applied pressure to the transducer, although the theory is applicable to the case of a sudden decrease in pressure also.

2. The simplified pressure measurement system

A capillary tube of diameter 2R and length L is connected to a reservoir of volume V. The initial pressure in the tube and reservoir is p_o. At time $t = 0$ the pressure at the tube inlet is raised to p_1. The problem is then to find the variation of pressure with time in the reservoir, and to find the stabilization time, which is the time taken for the pressure in the reservoir to rise within 1% of the pressure difference $(p_1 - p_o)$.

3. Assumptions

1. The flow is laminar* and the distribution of velocity across any section of the tube is independent of the distance along the tube.

2. The velocity in the reservoir is zero.

---

* This assumption is reasonable since in most practical systems the Reynolds number of the pipe flow will be less than 1000 (based on radius).

3. Inlet and outlet losses in the capillary tube are neglected.

4. The flow is isothermal. (See footnote after equation 5.4).

5. The flow does not involve slip.

4. The approximate solution of Kendall (1958)

In addition to the assumptions listed under paragraph 3 Kendall makes the assumption that the pressure squared distribution along the tubing is linear with distance for all time. Thus the flow along the tube at any time is determined from the end pressure difference. If p is the variable pressure at z = L then the rate of mass flow m(t) in the tube is given by (see paragraph 6 below):

$$m(t) = \frac{\pi R^4}{16 \mu a_o^2} \quad \frac{p_1^2 - p^2}{L} \tag{4.1}$$

where R = tube radius

L = tube length

μ = viscosity

a_o = speed of sound (for isothermal flow)

The rate of change of fluid mass in the reservoir of volume V is

$$\frac{d\,V\rho}{dt} = \frac{V\,d\rho}{dt} = \frac{V}{a_o^2} \quad \frac{dp}{dt} \tag{4.2}$$

since $p = \rho\, a_o^2$ in isothermal flow.

Now the rate of mass flow in the tube must equal the rate of change of fluid mass in the reservoir, so that

$$\frac{dp}{dt} = \frac{\pi R^4}{16 \mu V} \quad \frac{p_1^2 - p^2}{L} \tag{4.3}$$

with $p = p_o$ when $t = 0$.

The solution of 4.3 is

$$\frac{p_1 - p}{p_1 + p} = \frac{p_1 - p_o}{p_1 + p_o} \exp\left(- \frac{\pi R^4 p_1}{8 \mu V L} t \right) \tag{4.4}$$

If $\frac{p_1}{p_o} \gg 1$ then for small t

$$\frac{p}{p_1} = \frac{1}{2} \left(1 - \exp(-kt) \right) \tag{4.5}$$

where $k = \dfrac{\pi R^4 p_1}{8 \mu V L}$.

Similarly for large time with $\frac{p_1}{p_o} \gg 1$

$$\frac{p}{p_1} = 1 - 2 \exp(-kt) \tag{4.6}$$

If we define the stabilization time t_s as the time taken for the pressure in the reservoir to reach within 1% of the overall pressure difference $(p_1 - p_o)$

$$t_s = \frac{\ln\left(\dfrac{200\, p_1}{p_1 - p_o}\right)}{k} \tag{4.7}$$

and when $p_1/p_o \gg 1$

$$t_s = \frac{5.3}{k} \tag{4.8}$$

We find for air at 15°C that

$$k = \frac{\pi R^2 L}{8V} \cdot \frac{R^2}{L^2} \cdot \frac{p_1}{p_a} \cdot 5.68 \times 10^9 \text{ sec}^{-1} \tag{4.9}$$

where p_a = atmospheric pressure.

In Figure 1 the variation of pressure with time is shown together with the small and large time approximations. In this example

$$R = 0.1 \text{ cm}$$

$$L = 200 \text{ cm}$$

$$V = 3000 \text{ c.c.}$$

$$p_1/p_a = 1 \qquad p_1/p_o = 1000$$

$$t_s = 14 \text{ secs.}$$

5. Unsteady flow in a capillary tube

Let us use cylindrical polar co-ordinates r, θ, z with z measured along the tube axis. If the radial and circumferential components of the velocity are zero the equations of continuity and motion become if $w \equiv w(t, r, z)$ and $p \equiv p(t, r, z)$

Equation of Continuity
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho w}{\partial z} = 0 \tag{5.1}$$

Equations of Motion
$$0 = -\frac{\partial p}{\partial r} \tag{5.2}$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho w^2}{\partial z} = -\frac{\partial p}{\partial z} + \mu \nabla^2 w \tag{5.3}$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and μ is constant since the flow is isothermal.

If the z differential of 5.3 . is subtracted from the t differential of 5.1 we obtain

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 \rho w^2}{\partial z^2} - \mu \nabla^2 \frac{\partial w}{\partial z} \tag{5.4}$$

We can eliminate the density ρ by substituting the isothermal relation.*

---

* The reason for assuming the flow is isothermal is that since all velocities are assumed small (in fact for large time they must necessarily be so - for very small time they will in general not be so) the dissipation energy and pressure work terms will be very small since they are of $O(w^2)$. Hence the temperature changes must necessarily be small and we are justified in neglecting them. It is for this reason that we do not need to introduce the energy equation into our formulation of this problem.

$$p = \rho a_o^2 \tag{5.5}$$

Thus

$$\frac{\partial^2 p}{\partial t^2} = a_o^2 \frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p w^2}{\partial z^2} - \mu a_o^2 \nabla^2 \frac{\partial w}{\partial z} \tag{5.6}$$

Some simplification results if we integrate the equations of motion and continuity with respect to r^2 and make the assumption that the radial velocity distribution is independent of z. Thus on writing $\overline{w}(z,t)$ as the average velocity at any section we have

$$\overline{w}(z, t) = \int_0^1 w \, d(r^2/_{R^2}) \tag{5.7}$$

and if $$w(r, z, t) = \overline{w}(z, t) \, f(r/_R) \tag{5.8}$$

with*
$$c_1 = \int_0^1 f(\eta) \, d\eta^2 = 1$$
$$c_2 = \int_0^1 f^2(\eta) \, d\eta^2$$
$$c_3 = -\left(\frac{\partial f}{\partial \eta}\right)_{\eta=1}$$

the integral form of the equations of continuity and motion become

$$\frac{\partial p}{\partial t} + \frac{\partial p \overline{w}}{\partial z} = 0 \tag{5.9}$$

and
$$\frac{1}{a_o^2}\left(\frac{\partial p \overline{w}}{\partial t} + c_2 \frac{\partial}{\partial z} p \overline{w}^2\right) = -\frac{\partial p}{\partial z} - \frac{2 c_3 \mu \overline{w}}{R^2} + \mu \frac{\partial^2 \overline{w}}{\partial z^2}$$
..... (5.10)

respectively, on making use of (5.5) and (5.2).

---

* For a parabolic velocity distribution

$c_2 = 4/3$ and $c_3 = 4$.

If we subtract the z - derivative of (5.10) from the time derivative of (5.9) there results

$$\frac{\partial^2 p}{\partial t^2} - a_o^2 \frac{\partial^2 p}{\partial z^2} = \frac{2 c_3 \mu a_o^2}{R^2} \frac{\partial \overline{w}}{\partial z} - \mu a_o^2 \frac{\partial^3 \overline{w}}{\partial z^3} + c_2 \frac{\partial^2 p \overline{w}^2}{\partial z^2} \quad \ldots\ldots \quad (5.11)$$

which is the 'integral form' of (5.6), and is the wave equation for pressure fluctuations in a tube allowing for viscous dissipation. Of course equation (5.11) can only be solved when $\overline{w}$ is replaced by a function of p.

Now from (5.10) we find that

$$\mu \overline{w} = - \frac{R^2}{2 c_3} \frac{\partial p}{\partial z} - \frac{R^2}{2 c_3 a_o^2} \left(\frac{\partial p \overline{w}}{\partial t} + c_2 \frac{\partial p \overline{w}^2}{\partial z} \right) + \frac{\mu R^2}{2 c_3} \frac{\partial^2 \overline{w}}{\partial z^2} \quad \ldots\ldots \quad (5.12)$$

and if R/L is sufficiently small so that $\frac{p_1 R^2}{a_o \mu L} < 1$ we may retain only terms in R^4 giving

$$\mu \overline{w} = - \frac{R^2}{2 c_3} \frac{\partial p}{\partial z} - \frac{R^4}{4 c_3^2} \frac{\partial^3 p}{\partial z^3} + \frac{R^4}{8 c_3^2 a_o^2 \mu} \frac{\partial^2 p^2}{\partial z \partial t} + 0(R^6) \quad (5.13)$$

If (5.13) is substituted into either (5.11) or (5.9) and only terms in R^2 are retained we find that

$$\frac{\partial p}{\partial t} \approx \frac{R^2}{4 c_3 \mu} \frac{\partial^2 p^2}{\partial z^2} \quad (5.14)$$

showing that p^2 is only proportional to z in the steady case.

If we multiply (5.14) by 2p then

$$\frac{\partial p^2}{\partial t} = \frac{R^2 p}{4 \mu c_3} \frac{\partial^2 p^2}{\partial z^2} \quad (5.15)$$

which is a one-dimensional diffusion equation, where the diffusivity $\frac{R^2 p}{4 \mu c_3}$, is a function of pressure.

The solution of (5.15) is made more complicated by the form of the boundary conditions. Thus on writing $P = \dfrac{p^2 - p_o^2}{p_1^2}$ with

$$P_o = \frac{p^2 - p_o^2}{p_1^2} \quad , \quad \bar{t} = \frac{t\, a_o}{L} \quad ; \quad \bar{z} = z/L$$ (1.15) becomes

(on dropping the bars on t and z)

$$\frac{\partial P}{\partial t} = \frac{R^2\, p_1}{4\, c_3 \mu a_o\, L} \; \frac{p}{p_1} \; \frac{\partial^2 P}{\partial z^2} \tag{5.16}$$

with the boundary condition (associated with a step function change in pressure at the tube inlet)

$$\begin{array}{lll} P = 0 & t < 0 & \text{all } z \\ P = P_o & z = 0 & t > 0 \\ P = P_o & \text{all } z & t = \infty \end{array} \tag{5.17}$$

and

$$\frac{\partial P}{\partial t} = - \frac{\pi R^4\, p_1}{2\, c_3\, \mu\, a_o V} \; \frac{p}{p_1} \; \frac{\partial P}{\partial z} \quad \begin{array}{l} \text{at } z = 1, \\ \text{for } \bar{t} > 1 \end{array}$$

$$= 0 \quad \text{at } z = 1 \text{ for } \bar{t} < 1$$

This last boundary condition arises from the necessary condition that the rate of mass flow out of the tube equals the rate of change of mass in the reservoir of volume V.

According to our assumptions we have not considered in detail the wave motion set up in the tube as a result of the step-function change in the inlet pressure. We see from equation (5.6) that the shock wave propagated down the tube would be attenuated and decelerated, and that the pressure would not begin to rise in the reservoir until a time $t = L/a_o$ after the initiation of the shock wave. Consequent reflections of the shock wave would result in further changes in pressure but these changes in pressure will be small, and in any case the times over which they occur will be small compared with the stabilisation time provided $L/R \gg 1$.

Hence provided we accept the fact that for our purpose the details of the wave motion in the tube are not required in detail, we see that our problem reduces from a pressure wave problem to a pressure diffusion problem.

Before the solution of (5.16) together with (5.17) is attempted we will return to the corresponding steady flow problem in order to justify the approximate forms used for $\overline{w}$ above.

6. Steady flow in a long capillary tube

For steady axi-symmetric flow in a tube of constant radius R the equations of continuity and motion are

Equation of Continuity
$$\frac{\partial \rho w}{\partial z} = 0 \tag{6.1}$$

Equations of Motion
$$\frac{\partial \rho w^2}{\partial z} = - \frac{\partial p}{\partial z} + \mu \nabla^2 w \tag{6.2}$$

$$0 = - \frac{\partial p}{\partial r} \tag{6.3}$$

Equation (6.1) shows that ρw is a function of r only while equation (6.3) shows that p is a function of z only. If the flow is isothermal $p = \rho a_o^2$, say, and then

$$\frac{\partial \rho w^2}{\partial z} = - \frac{w^2}{a_o^2} \frac{dp}{dz} \tag{6.4}$$

with (6.2) becoming

$$0 = - \left(1 - w^2/a_o^2\right) \frac{dp}{dz} + \frac{\mu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right) + \mu \frac{\partial^2 w}{\partial z^2} \quad \ldots\ldots \tag{6.5}$$

If ρ_o and w_o are taken as the reference density and velocity respectively then

$$0 = -\left(1 - w^2/a_o^2\right)\frac{d\ {}^{p}/\rho_o\, w_o^2}{d\ {}^{z}/L} + \frac{L}{R\, R_e}\left[\frac{1}{\overline{r}}\ \frac{\partial}{\partial \overline{r}}\left(\overline{r}\ \frac{\partial w/w_o}{\partial \overline{r}}\right) + \frac{R^2}{L^2}\ \frac{\partial^2 w/w_o}{\partial\,({}^{z}/L)^2}\right] \tag{6.6}$$

where $\overline{r} = r/R$, and $R_e = \dfrac{\rho_o w_o R}{\mu}$. It is assumed that $\dfrac{1}{\overline{r}}\ \dfrac{\partial}{\partial \overline{r}}\left(\overline{r}\ \dfrac{\partial w/w_o}{\partial \overline{r}}\right)$ and $\dfrac{\partial^2 w/w_o}{\partial\,({}^{z}/L)^2}$ are the same order of magnitude. It follows that when $w/a_o \ll 1$ and $\dfrac{R}{L} \ll 1$ equation (6.5) approximates to

$$0 = -\frac{dp}{dz} + \frac{\mu}{r}\ \frac{\partial}{\partial r}\left(r\ \frac{\partial w}{\partial r}\right) \tag{6.7}$$

The first integral of (6.7) with respect to r is

$$0 = -\frac{r}{2}\ \frac{dp}{dz} + \mu\frac{\partial w}{\partial r} \tag{6.8}$$

and the second integral is

$$w_1 - w = -\frac{r^2}{4\mu}\ \frac{dp}{dz} \tag{6.9}$$

where w_1 is the axial velocity at $r = 0$.

Hence
$$w_1 = -\frac{R^2}{4\mu}\ \frac{dp}{dz} \tag{6.10}$$

$$\tau_w = -\mu\left(\frac{\partial w}{\partial r}\right)_{r=R} = -\frac{R}{2}\ \frac{dp}{dz} \tag{6.11}$$

$$\frac{w}{w_1} = 1 - r^2/_{R^2} \tag{6.12}$$

$$\overline{w} = w_1/2 \tag{6.13}$$

as in standard Poiseuille flow except that dp/dz is not constant. (See equation 6.15 below).

The rate of volume flow

$$Q = \pi R^2\ \overline{w} = -\frac{\pi R^4}{8\mu}\ \frac{dp}{dz} \tag{6.14}$$

If the flow is isothermal $p = a_o^2$, say, and then the rate of mass flow is

$$m = \rho\pi R^2 \overline{w} = -\frac{\pi R^4}{16\mu a_o^2} \frac{dp^2}{dz} \tag{6.15}$$

Under steady flow conditions the rate of mass flow will be constant so that if the pressures at z = 0 and z = L are p_1 and p^* respectively

$$m = \frac{\pi R^4}{16\mu a_o^2} \frac{p_1^2 - p^{*2}}{L} \tag{6.16}$$

and

$$\overline{w}(z) = \frac{R^2}{16\mu z} \frac{p_1^2 - p^2}{p}$$

$$= \frac{R^2}{16\mu z} (p_1 - p) \left(\frac{p_1 + p}{p}\right) \tag{6.17}$$

whereas under conditions of constant density

$$\overline{w} = \frac{R^2}{8\mu z} (p_1 - p) = \text{const.} \tag{6.18}$$

Hence as $p \to p_1$ in equation (6.17)

$$\overline{w} \to \frac{R^2}{8\mu z} (p_1 - p) \text{ and } \overline{w} \text{ is then independent of } z.$$

In the latter case equation (6.7) is exact and not approximate since then $w \equiv w(r)$ only with

$$\rho w \frac{\partial w}{\partial z} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial z^2} = 0.$$

We see from (6.14) that under steady flow conditions

$$\overline{w} = -\frac{R^2}{8\mu} \frac{dp}{dz} \tag{6.19}$$

and hence the approximation to $\overline{w}$ made in paragraph 5 for the unsteady case is essentially that of replacing $\overline{w}$ by its value in the steady case, allowing for the fact that the velocity distribution in the unsteady case will not be exactly parabolic.

Equation (6.16) gives the expression used by Kendall (see 4.1 above) for the rate of mass flow through the tube.

7. The solution of the pressure equation for the unsteady flow in a reservoir connected to a capillary tube.

It was shown above that the approximate pressure equation is

$$\frac{4\, c_3\, \mu\, a_o L}{R^2\, p_1} \; \frac{\partial P}{\partial t} = \frac{p}{p_1} \; \frac{\partial^2 P}{\partial z^2} \tag{7.1}$$

with the boundary conditions

$$\begin{array}{llll} P = 0 & t < 0 & \text{all } z & \\ P = P_o & z = 0 & t > 0 & (7.2) \\ P = P_o & \text{all } z & t = \infty & \end{array}$$

$$\left.\begin{array}{ll} \dfrac{\partial P}{\partial t} = 0 & 0 < t < 1 \\ \quad = - \dfrac{\pi R^4\, p_1}{2\, c_3 \mu a_o V} \; \dfrac{p}{p_1} \; \dfrac{\partial P}{\partial z} & t \geqslant 1 \end{array}\right\} \quad z = 1$$

In our notation $P = \dfrac{p^2 - p_o^2}{p_1^2}$, with $P_o = \dfrac{p_1^2 - p_o^2}{p_1^2}$,

and t and z are non-dimensional quantities. $z = 0$ corresponds to the open mouth of the capillary tube while at $z = 1$ the tube, of length L, is connected to the reservoir of volume V, having the initial and final pressures of p_o and p_1 respectively. The significance of $t = 1$ is that according to our assumptions the shock wave of initial strength $p_1 - p_o$ at time $t = 0$ takes a time $t = 1$ (non-dimensional time) to travel the length L of the capillary tube.

An approximate solution of the non-linear equation (7.1) with the non-linear boundary conditions (7.2) can be obtained if we assume that the term p/p_1 on the right hand side of (7.1) can be replaced by a function of t only. Thus if $p^*(t)$ is the value of p at $z = 1$ then let

$$\frac{p}{p_1} = \frac{C(t)\, p^*(t)}{p_1} \tag{7.3}$$

where C is an adjustable parameter, whose value is of the order of unity.

If

$$\tau = \int_0^t \frac{R^2 p_1 C}{4\, c_3 \mu a_o L} \frac{p^*}{p_1}\, dt' \tag{7.4}$$

then (7.1) and (7.2) become respectively

$$\frac{\partial P}{\partial \tau} = \frac{\partial^2 P}{\partial z^2} \tag{7.5}$$

with the boundary conditions

$$\begin{aligned} P &= 0 & \tau &< 0 & &\text{all } z \\ P &= 0 & z &= 0 & \tau &> 0 \\ P &= P_o & &\text{all } z & \tau &= \infty \\ \frac{\partial P^*}{\partial \tau} &= 0 & z &= 1 & 0 < \tau &< \tau_1 \\ &= -\gamma \frac{\partial P^*}{\partial z} & z &= 1 & \tau &\geqslant \tau_1 \end{aligned} \tag{7.6}$$

where $P^* = \dfrac{p^{*2} - p_o^2}{p_1^2}$, $\gamma = \dfrac{2\pi R^2 L}{V \overline{C}}$ is assumed to have a constant value, by replacing C by its average value $\overline{C}$, and

$$\tau_1 = \int_0^1 \frac{R^2 p_1 C}{4\, c_3 \mu a_o L} \frac{p^*}{p_1}\, dt\, .$$

If s is the Laplace Transform operator the 'subsidiary equation' is

$$\overline{P} = \frac{1}{s} \frac{d^2 \overline{P}}{dz^2} \tag{7.7}$$

where $\overline{P} = \int_0^\infty P\, e^{-s\tau}\, d\tau$, and the boundary conditions are replaced by

$$\overline{P} = P_o/s \qquad z = 0$$
$$\overline{P}^* = -\frac{\gamma e^{s\tau_1}}{s}\frac{d\overline{P}}{dz} \qquad z = 1 \tag{7.8}$$

The solution of (7.7) satisfying (7.8) is

$$\overline{P}/P_o = \frac{\sinh(1-z)\sqrt{s} + \frac{\gamma e^{s\tau_1}}{\sqrt{s}}\cosh(1-z)\sqrt{s}}{s\left[\sinh\sqrt{s} + \frac{\gamma e^{s\tau_1}}{\sqrt{s}}\cosh\sqrt{s}\right]} \tag{7.9}$$

and when z = 1

$$\overline{P}^*/P_o = \frac{\gamma e^{s\tau_1}}{s\left[\sqrt{s}\sinh\sqrt{s} + \gamma e^{s\tau_1}\cosh\sqrt{s}\right]} \tag{7.10}$$

From the inversion theorem we obtain

$$P^*/P_o = \frac{\gamma}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{e^{\lambda(\tau+\tau_1)}\, d\lambda}{\lambda\left[\sqrt{\lambda}\sinh\sqrt{\lambda} + \gamma e^{\lambda\tau_1}\cosh\sqrt{\lambda}\right]}$$
$$= 0 \qquad 0 < \tau < \tau_1$$
$$= 1 - 2\gamma\sum_{\nu=1}^{\infty}\frac{\exp(-\alpha_\nu^2\tau)}{\alpha_\nu\left[(1+\gamma e^{-\alpha_\nu^2\tau_1})\sin\alpha_\nu + \alpha_\nu(1+2\gamma\tau_1 e^{-\alpha_\nu^2\tau_1})\cos\alpha_\nu\right]} \quad \text{for } \tau \geq \tau_1 \tag{7.11}$$

where $\pm\alpha_1$, $\pm\alpha_2$ ...... are the roots of the equation

$$\alpha\tan\alpha = \gamma e^{-\alpha^2\tau_1} \tag{7.12}$$

Now in our problem $\gamma = \dfrac{2\pi R^2 L}{V\overline{C}}$, which is roughly the ratio of the tube volume to the reservoir volume, and in general $\gamma e^{-\alpha^2\tau_1} << 1$.
In this case $\alpha_1^2 = \gamma e^{-\alpha_1^2\tau_1}$ with $\sin\alpha_1 \approx \alpha_1$ and $\cos\alpha_1 \approx 1$, and on retaining only the first term in the series in (7.11) and replacing t by <u>real time,</u>

$$\frac{P^*}{P_o} = 0 \qquad 0 < t < L/a_o$$

$$= 1 - \exp\left(- \frac{k}{\overline{C}} \int_{L/a_o}^{t} \frac{C\,p^*}{p_1}\,dt'\right) \qquad (7.13)$$

$$\text{for } t \geq {}^{L}/a_o$$

where $\quad k = \dfrac{\pi R^4 p_1}{2\, c_3 \mu V L}$.

The solution to (7.13) easily follows when C is a constant. On differentiating both sides of (7.13) with respect to t and rearranging we find that

$$\frac{dp^*}{dt} = 0 \qquad 0 < t < {}^{L}/a_o$$

$$= \frac{k}{2}\left(\frac{p_1^2 - p^{*2}}{p_1}\right) \qquad \text{for } t \geq {}^{L}/a_o \qquad (7.14)$$

which we found above was the approximate equation derived by Kendall. The solution of (7.14) is therefore

$$\frac{p_1 - p^*}{p_1 + p^*} = \frac{p_1 - p_o}{p_1 + p_o} \exp\left[-k\,(t - L/a_o)\right] \qquad (7.15)$$

and the stabilisation time, for $p_1/p_o >> 1$, is

$$t_s = \frac{5.3}{k} + \frac{L}{a_o} \qquad (7.16)$$

It would appear at first sight a little surprising that our various assumptions, all of which seem justifiable, merely add up to Kendall's local steady flow approximation. The only modification to Kendall's result is the addition of the term L/a_o.

However the solution to (7.13) is based on the fact that C is a constant. But when C is taken as a function of time and noting that $C/\overline{C}$ is less than unity, and takes on its smallest values for small values of $t - L/a_o$, we see that the effective value of k in (7.15) will be less* than Kendall's k

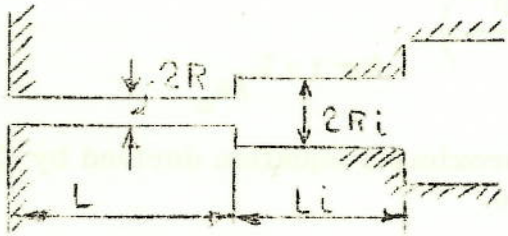
---

* It is assumed that the velocity distribution is parabolic so that $c_3 = 4$.

for small values of $t - L/a_o$, and will approach its value for large time. This part of the difference between the theoretical and experimental curves shown in Fig. 8 may be due to this overestimate of k at small times.

The above solution can be extended to the case of a series of tubes of different lengths and diameters by reducing them to an 'equivalent length' of constant diameter tubing. In addition allowance can be made for entry and exit losses, as well as the loss at a rapid expansion, by adding 'equivalent lengths' of constant diameter tubing. For example, if in addition to a tube of radius R and length L there are a number of tubes of radius R_i each of length L_i, then the 'equivalent length' of tube of radius R is

$$L_e = L + \sum L_i \left(\frac{R}{R_i}\right)^4 \tag{7.17}$$



The equivalent length of tubing of radius R for the above arrangement making allowances for entry, exit and rapid expansion losses is

$$L_e = L + L_i \left({}^R/R_i\right)^4 + \frac{R_e}{16} \cdot \frac{R}{L} \cdot L \left(1.48 + \frac{R^4}{R_i^4}\right) \tag{7.18}$$

where $R_e = \dfrac{\rho_1 \overline{w}_1 R}{\mu}$

and suffix '$_1$' denotes conditions at the entry to the tube of radius R*.

---

* It might be noted that if the entry hole (say a diameter of $2R_o$) is of different diameter than the tube of diameter 2R, the term in brackets in (7.18) is replaced by $\left(1.48 \dfrac{R^4}{R_o^4} + \dfrac{R^4}{R_i^4} + 1\right)$,

when $\dfrac{R_o}{R} < 1$. The equivalent length of tube is thereby increased many times.

In most pressure systems for supersonic wind tunnels, where $R_e < 1000$, the last term can be neglected. Further details of the application of the 'equivalent tube' method to the response of pressure measuring systems should be made to Kendall (1958) and Heyser (1958).

8. Apparatus used in the experiments

The basic apparatus consisted of a Langham Thompson type UP.4/150/325 No. 806 pressure transducer having a range of ±15 p.s.i. and a chamber volume of 0.0941 cu.in. (1540 mm^3). This was connected in turn to various lengths of steel capillary tubing having a bore of 1 mm. diameter. The other end of the capillary tube was connected to the vacuum chamber of a 'Speedivac' vacuum pump and the chamber was sealed off with a cellophane diaphragm (Fig. 2). The output of the Langham Thompson pressure transducer was connected to a 'Tektronix' type 545 oscilloscope fitted with a 'Polaroid' camera for permanent recording.

The pressure in the capillary tube and transducer chamber was first reduced to approximately 10 mm of mercury. The subsequent bursting of the diaphragm produced a good approximation to a step input of pressure to the system. Also fitted to the vacuum chamber was an S.L.M. PZ.14 piezoelectric pressure transducer. This was used initially to measure the pressure in the vacuum chamber but later, due to its very rapid response, it was used to trigger the oscilloscope. In the latter case it was used with a direct coupled amplifier.

9. Description of tests

The tests were made with capillary tube lengths from zero (i.e. the transducer mounted flush with the wall of the vacuum chamber) to a maximum of 68 in. The longer tubes comprised 2 or 3 shorter lengths butted together and sealed with short pieces of rubber tubing. The results are shown in Table 1 and Figures 3 to 6 inclusive.

10. Discussion

The response of the S.L.M. transducer to the bursting of the diaphragm is shown in Fig. 3. It can be seen that the pressure in the vacuum chamber reaches its final value (atmospheric) after 300 micro seconds and then, after about 5 overshoots, finally settles down to a steady value after 2.5 milliseconds. This is the 'step input' to the capillary tubing.

Since the stabilisation time, that is the time required for the pressure to reach 99% of its final value, is difficult to measure in practice, a more accurate measure of the response is found from the determination of the 'time constant', t_c, of the system. We will define the 'time constant' as the time required for the pressure to reach its final value at the maximum response rate. It will be seen from Figs. 4, 5 and 6 that the measurement of the maximum slope can be made with reasonable accuracy.

Fig. 7 shows that the experimental results agree with our predictions and those of Kendall, provided the tube length is not too short and the inlet and exit losses are included. When the tube length is zero the above theory is not applicable but a simple calculation for this case (see Appendix) shows that the 'time constant' will be about 12 milliseconds, in agreement with the experimental results. On the other hand, Fig. 8 shows that the experimental time variation of the pressure in the reservoir at small times differs greatly from the theoretical. A partial explanation for this difference lies in the inexactness of our approximations for small times especially the neglect of the initial wave motion. It has already been explained above that a more exact solution of our equations should give a smaller slope at small times than that shown in Fig. 8, and this would have the effect of bringing the theoretical curve nearer the experimental curve. The difference between the theory and experiment is further demonstrated in Fig. 9 and in this case it would appear that we have overestimated the losses due to inlet and exit.

The high frequency signals superimposed on both the Langham Thompson and S.L.M. transducer traces are due to the gauges 'ringing'. The ringing frequencies are seen to be 1500 c/s for the Langham Thompson and 5000 c/s for the S.L.M. transducers.

When the diaphragms were burst they always shattered leaving small pieces of cellophane free to block up the end of the capillary tube. Although the possibility of this occurring was reduced by placing the end of the capillary tube under the centre of the diaphragm, it did occur in cases where the transducer was mounted flush with the wall of the vacuum chamber. Fortunately when this happened it could easily be detected from the oscilloscope record (see Fig. 5d (49); Fig. 6d (44); Figs. 4b and 4c (61)).

11. Conclusions

The response of a pressure measuring system to a step-function input is analysed and it is shown that the approximate solution of Kendall (1958) has a wide range of validity. This solution gives results in reasonable agreement with experiment provided the length/diameter ratio of the capillary tubing is sufficiently large and allowance is made for the inlet and exit losses.

12. Acknowledgement

The authors wish to express their thanks to Mr. J. R. Busing for assistance in formulating the experiment and in the analysis of the experimental data.

13. References

1. Kendall, J.M. Optimized design of systems for measuring low pressures in supersonic wind tunnels. (Presented at AGARD meeting, London, March 1958).

2. Heyser, A. Development of pressure measuring devices for a blow-down wind tunnel at the DVL. (Presented at AGARD meeting, London, March 1958).

APPENDIX

The response time of a zero length capillary tube pressure system

When a pressure transducer is subjected to a step function input the response time will be a function of the orifice size and chamber volume. If we assume the orifice discharge coefficient is 0.64, then on equating the rate of mass flow through the orifice to the rate of increase of mass in the chamber we find that

$$0.64 \; \rho_t \, V_t \quad \frac{\pi R^2}{V} = \frac{d\rho}{dt} \qquad (A.1)$$

where R and V are the orifice radius and chamber volume respectively and $\rho_t \; V_t$ is the rate of mass flow per unit area of orifice. In the case where the applied pressure ratio exceeds the critical pressure for sonic conditions at the orifice we have, for small times that

$$\frac{dp}{dt} = \frac{0.64. \; \gamma \; p_1}{1.728 \; a_1} \quad \frac{\pi R^2}{V} \quad a^2 \qquad (A.2)$$

On the assumption that a^2 is constant, the time constant t_c, when $p_1 >> p_o$, is given by

$$t_c = \frac{1.728}{0.64. \; \gamma \, a_1} \quad \frac{V}{\pi R^2} \qquad (A.3)$$

In our experiment $V = 1540 \text{ mm}^3$ and $R = 0.5$ mm so that

$$t_c \approx 12 \text{ millisecs}$$

which is equal to the value found by experiment.

TABLE I

Fig. No.	Record No.	Length of 1 mm bore capillary tube (in.)	Vertical scale mV/cm	Horizontal scale ms/cm
3a	64	0	10	10
3b	63	0	10	2
3c	62	0	10	0.2
4a	61	0.01	5	2
4b	-	0.01	5	2
4c	-	0.01	5	2
5a	60) 59) 58)	9.9	5	2
5b	43) 42) 41)	5.8	5	5
5c	48	9.8	5	5
	47	13.8	5	5
	46	18.8	5	5
5d	51	42.9	5	10
	50	26.8	5	10
	49	22.8	5	10
6a	40) 39) 38)	31.0	5	10
6b	54	55.4	5	10
	53	50.9	5	10
	52	46.9	5	10
6c	57	67.9	5	10
	56	63.9	5	10
	55	59.9	5	10
6d	45) 44)	67.88	5	10

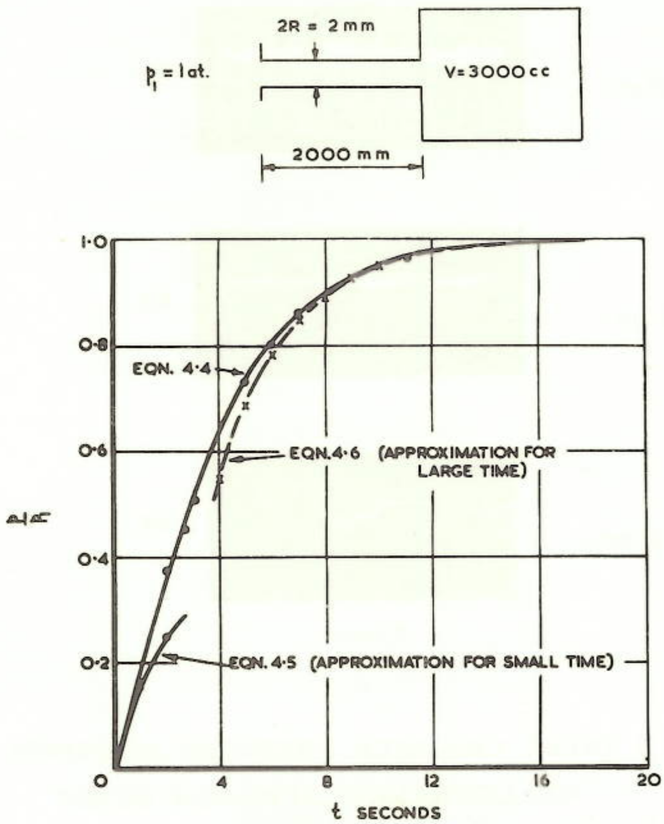


FIG. 1. RESERVOIR PRESSURE AS A FUNCTION OF TIME (KENDALL)

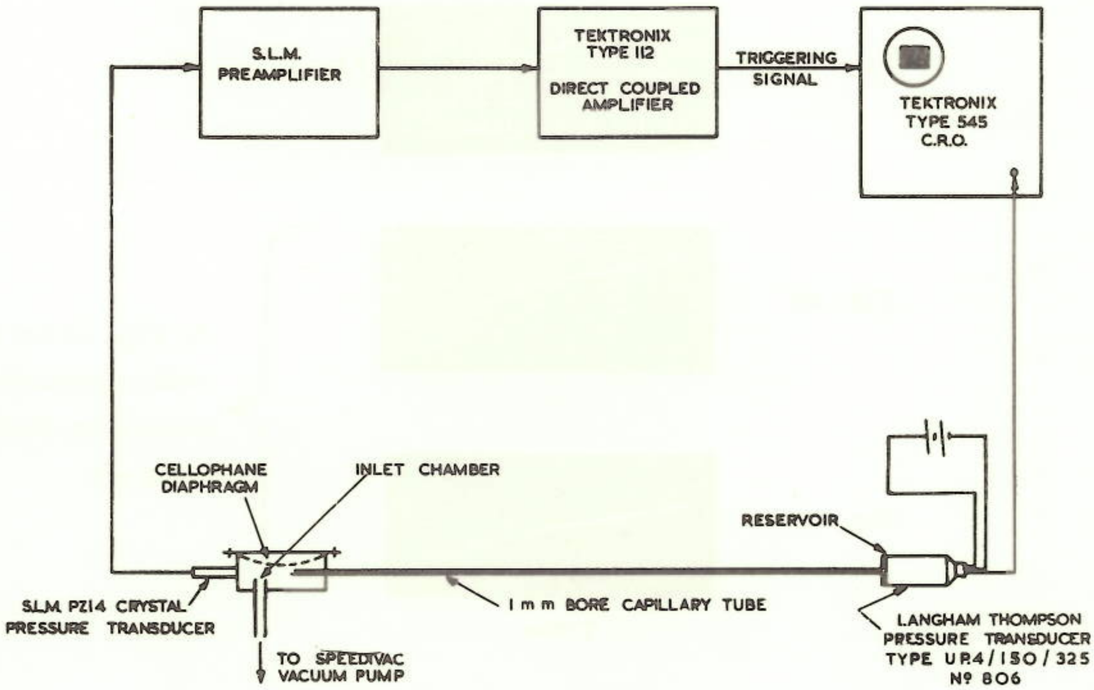


FIG. 2. LAYOUT OF APPARATUS

Fig. 3a
Time base 10 ms/cm.

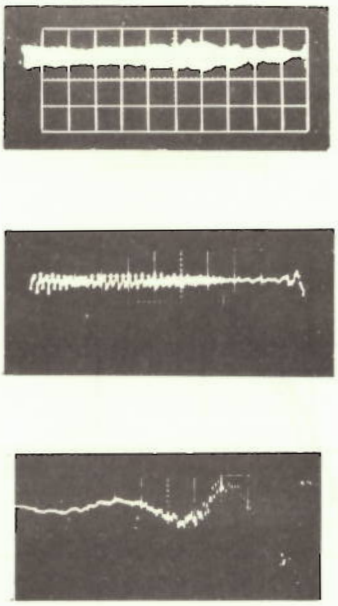
64

Fig. 3b
Time base 2 ms/cm.

63

Fig. 3c
Time base 200 μs/cm.

62



Time

FIG. 3 INLET CHAMBER PRESSURE RESPONSE TO CELLOPHANE DIAPHRAGM BURST

(S.L.M. PZ 14)

(Vertical Scale 10 mV/cm.)

Fig. 4a

61

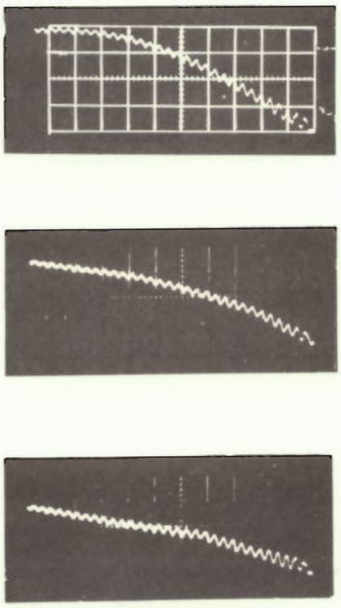
Fig. 4b

-

Fig. 4c

-

As Fig. 4a but with orifice partially blocked by diaphragm fragment.



Time

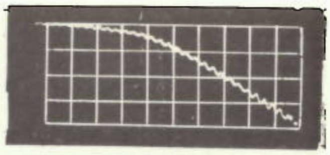
FIG. 4 TIME VARIATION OF RESERVOIR PRESSURE

L = 0.01 in.

Time base 2 ms/cm.

(Vertical Scale 5 mV/cm.)

FIG. 5a
L = 0.9 in.
Time base 2 ms/cm.

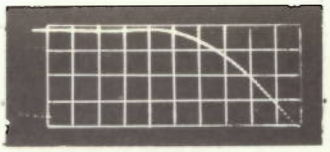


60

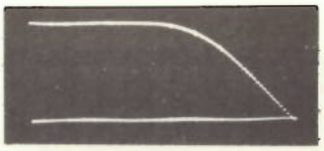
59

58

FIG. 5b
L = 5.8 in.
Time base 5 ms/cm.



43



42

41

FIG. 5 TIME VARIATION OF RESERVOIR PRESSURE

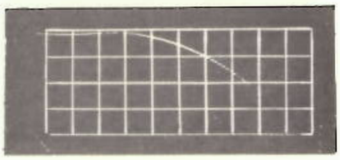
(Vertical Scale 5 mV/cm.)

FIG. 5c

Time base 5 ms/cm.

L

9.8 in.

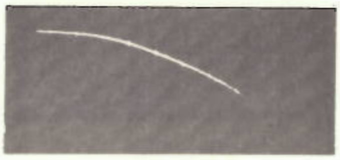


48

13.8 in.

47

18.8 in.

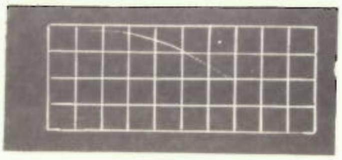


46

FIG. 5d

Time base 10 ms/cm.

42.9 in.



51

26.8 in.

50

22.8 in.

49

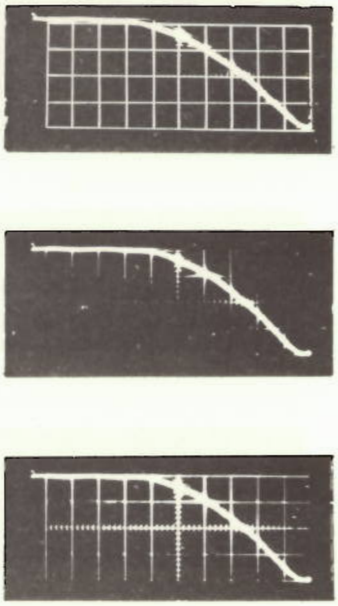
FIG. 5 TIME VARIATION OF RESERVOIR PRESSURE

(Vertical Scale 5 mV/cm.)

FIG. 6a

L = 31 in.

Time base 10 ms/cm.



40

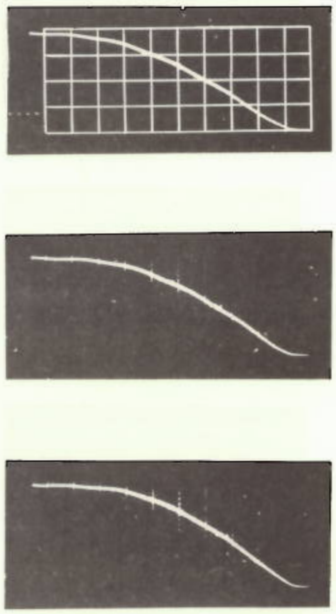
39

38

FIG. 6b

Time base 10 ms/cm.

L



55.4 in. 54

50.9 in. 53

46.9 in. 52

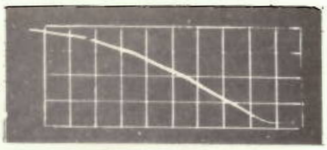
FIG. 6 TIME VARIATION OF RESERVOIR PRESSURE

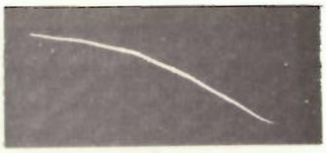
(Vertical Scale 5 mV/cm.)

FIG. 6c

Time base 10 ms/cm.

L

67.9 in.  57

63.9 in.  56

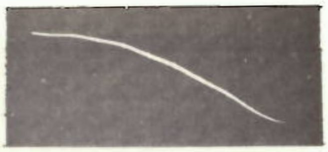
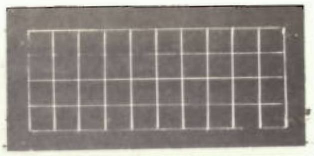
59.9 in.  55

FIG. 6d

L = 67.88 in.

Time base 10 ms/cm.

 -

 45

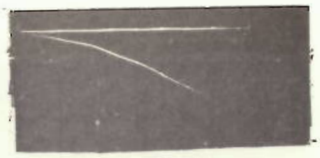
 44

FIG. 6 TIME VARIATION OF RESERVOIR PRESSURE

(Vertical Scale 5 mV/cm.)

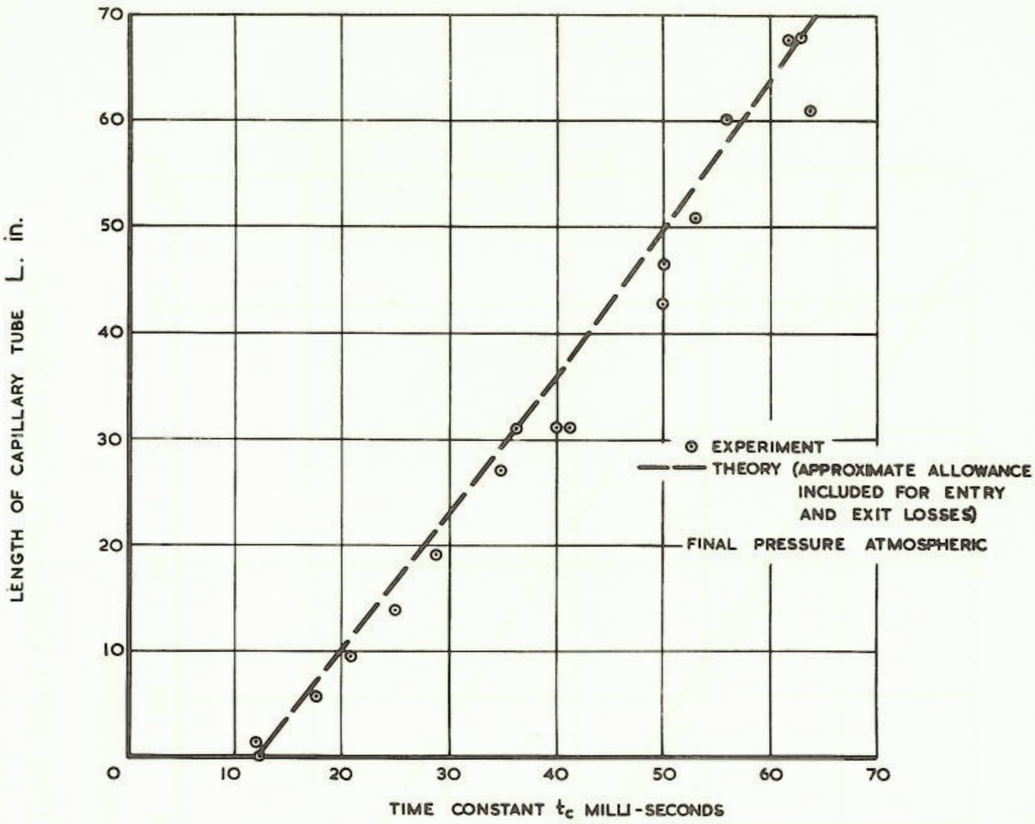


FIG. 7. VARIATION OF TIME CONSTANT WITH CAPILLARY TUBE LENGTH.
METAL TUBE 1mm INSIDE DIAMETER

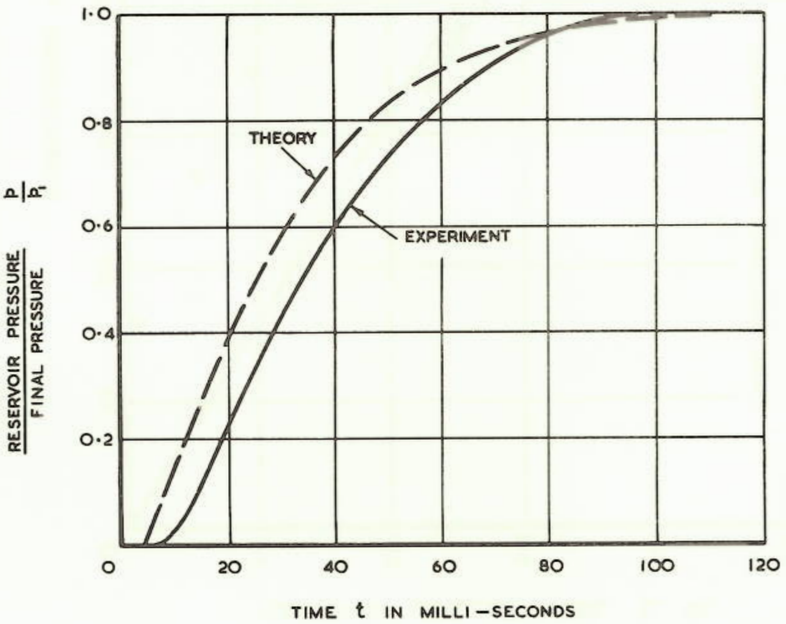


FIG.8. COMPARISON BETWEEN THEORY AND EXPERIMENT
TEST Nº 53 (FIG. 6b)
L = 50·9 EFFECTIVE LENGTH = 67 INS

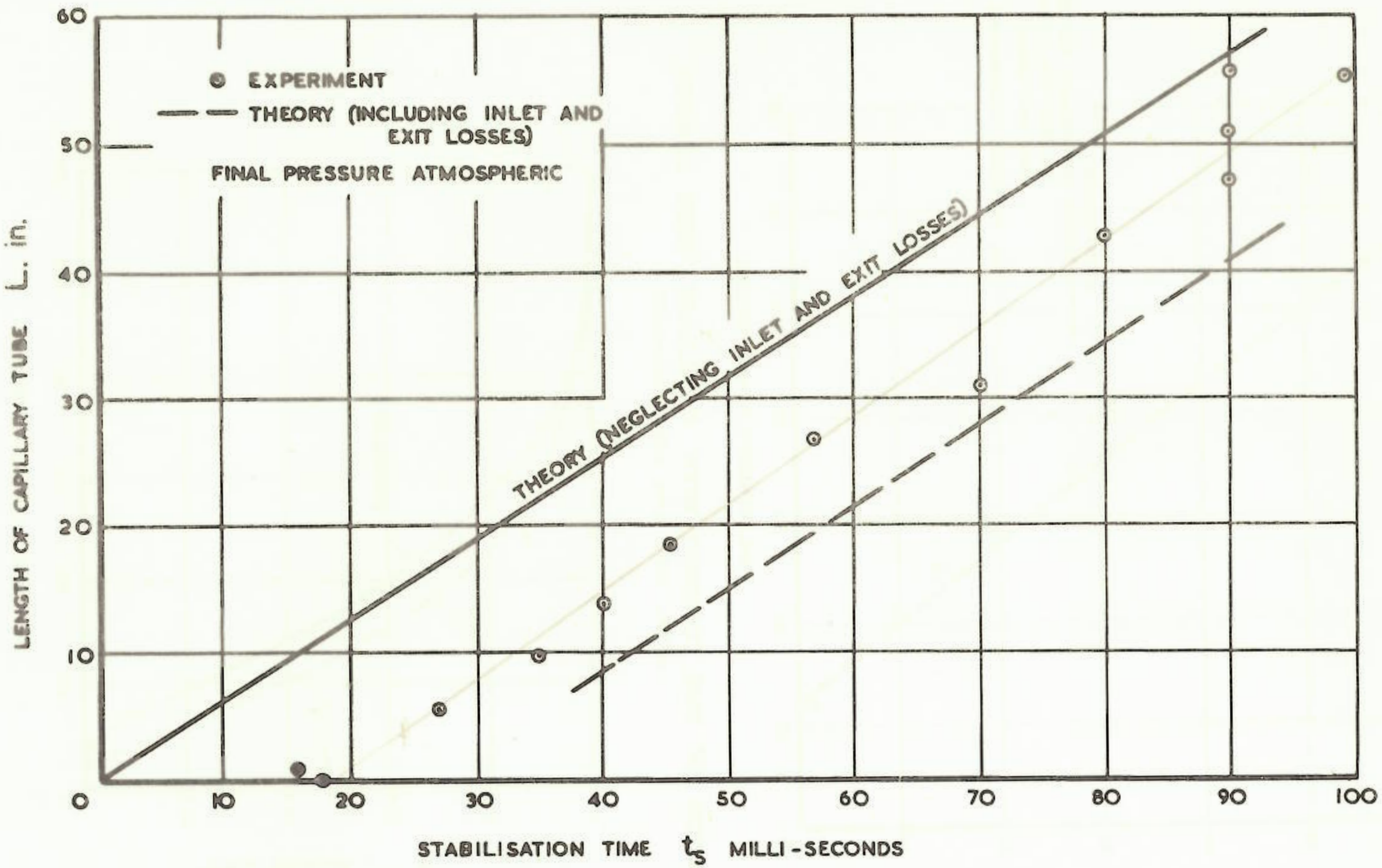


FIG. 9. VARIATION OF STABILISATION TIME WITH CAPILLARY TUBE LENGTH

METAL TUBE 1mm INSIDE DIAMETER