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Theory of Structural Design†

by

W.S. Hemp, M.A., F.R.Ae.S.

(Professor of Aircraft Structures and Aero-Elasticity
and Head of the Department of Aircraft Design.)

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Abstract

The Theory of Structures is for the most part concerned with the calculation of stresses in a given structure under given external conditions of loading and temperature. The real problem of structural design, however, in aeronautics at any rate, is to find that structure, which will equilibrate the external loads, without failure or undue deformation, under such conditions of temperature, as may be appropriate, and which at the same time will have the least possible weight. The solution of this general design problem is obviously very difficult and cannot be resolved at the present time. However, on the basis of certain classical theorems due to Maxwell and Michell and using methods and suggestions derived from these theorems by H.L. Cox, one can make certain progress, and in addition point the way to profitable lines of research. The present paper reviews the classical results and their current application, develops the mathematical theory for the two-dimensional case and derives a number of special solutions. It is hoped that its publication will encourage research in this very important field.

1. Introduction

§1.1 Statement of the General Problem

The real problem of structural design is the disposal of material in such a way, that it will safely equilibrate given systems of applied force, under the appropriate physical conditions of, for example, temperature, without exceeding permissible amounts of deflection and at the same time using the minimum of material. This last requirement is crucial in aircraft construction. Practical considerations relating to manufacture, maintenance or function will of course force a departure from this ideal solution, but a knowledge of the optimum is clearly desirable as a control.

The Theory of Structures is for the most part concerned with stress or deflection analysis of given structures. This means that in practice, it can only be used in design by a process of trial and error, in which the structural layout and sizes are first guessed or very roughly calculated, and are then subjected to as complete analysis as the theory will permit. The results of these calculations are then used to modify the design to perhaps achieve a more uniform distribution of stress, and the thoroughgoing analysis repeated as a check. The theory ought to be in a position to tackle the design problem directly, that is, to begin with the given forces and to produce by calculation the best structure that will safely carry them. The present report is concerned with reviewing the present position in this little developed branch of the theory and with suggesting lines of research, which may lead to developments of knowledge, such as to make direct structural design a practical part of the normal techniques of engineering.

§1.2 History

The first important contribution to the Theory of Structural Design was made by Maxwell[†]. He proved a theorem about the equilibrium of a series of attracting and repelling centres of force and applied it to a frame structure in which the bars replaced the "actions at a distance" except in the case of the external forces, and by this means effectively obtained the result of equation (2.1). He commented upon the engineering significance of his theorem in these words:

"The importance of the theorem to the engineer arises from the circumstance that the strength of a piece is in general proportional to its section, so that if the strength of each piece is proportional to the stress which it has to bear, its weight will be proportional to the product of the stress multiplied by the length of the piece. Hence these sums of products give an estimate of the total quantity of material which must be used in sustaining tension and pressure respectively".
Noticing that Maxwell uses the word "stress" for what we should term "load" we see that in effect he has obtained equation (2.3) and has drawn the practical conclusion about the required weight of the structure.

Michell[††] made the second important contribution to our subject. He recognised the importance of Maxwell's result and enunciated equation (2.3) in its present form, applying it to the calculation of optimum structural weights. However he went much further than that by generalising one of Maxwell's proofs of his theorem by the method of virtual work, using, instead of Maxwell's uniform dilatation, the more general deformation of (2.6). This led him to sufficient conditions for a structure to be an optimum. He proved the geometric restriction of equation (3.3), which determines the classes of orthogonal sets of

† Ref.1 pp.175-7

†† Ref.2

curves along which the members of an optimum structure must lie and gave figures illustrating all the results of section §3.1 with the exception of the general integral of equations (3.8) and (3.11). He also gave an example of a three-dimensional structure for transmission of torque, the members of which lie on the surface of a sphere†.

These important contributions to our subject passed unnoticed for some forty years until Foulkes of the Department of Engineering at Cambridge University read Michell's paper and realised its theoretical importance. He drew the attention of H.L. Cox to the paper and by so doing created a champion for the cause of direct structural design. Cox has done much by lectures and papers†† to draw the attention of engineers to Maxwell's and Michell's results and to convince them of the important gains that may well be made by further development of this subject. His own important contributions to the practical application of, in particular, Maxwell's Theorem are outlined in section §2.3

† Reproduced in Ref.3, Fig.12

†† Refs. 3,4 and much unpublished material

2. Classical Theory for Frame Structures

§2.1 Maxwell's Theorem †

Consider any frame structure which equilibrates a set of forces $\bar{F}_i$ acting at points with position vectors $\bar{r}_i (i=1,2...n)$. Let T_t be the load carried in a typical tension member with length L_t and section area A_t. Let $(-T_c)$ be the load carried in a typical compression member with length L_c and section area A_c.

Impose a virtual displacement on the structure which consists of a uniform dilatation of space of magnitude $3e$, chosen so that the origin for the vectors $\bar{r}_i$ is at rest. Every linear element of space is extended by a strain e and so the virtual displacements at the points of application of the forces are $e\bar{r}_i$.

We can thus write,

$$\text{Virtual work of the external forces} \quad = \quad e\sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i$$

The change in energy of a tension member is

$$\left\{ \frac{(T_t/A_t + Ee)^2}{2E} - \frac{(T_t/A_t)^2}{2E} \right\} A_t L_t \quad = \quad T_t L_t e$$

correct to the first order in e, where E is Young's Modulus. The change in energy of a compression member is $-T_c L_c e$ and so applying the Principle of Virtual Work, we find, cancelling e, that

$$\sum_t T_t L_t - \sum_c T_c L_c = \sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i \qquad \ldots (2.1)$$

† Ref.1 pp.175-7

where $\sum_t$, $\sum_c$ are sums over the tension and compression members respectively.

If f_t and f_c are the permissible stresses in tension and compression and if we assume that all members are stressed to the limit, we can write,

$$T_t = A_t f_t \quad , \quad T_c = A_c f_c \qquad \ldots (2.2)$$

and substituting in (2.1) obtain,

$$f_t V_t - f_c V_c = \sum_i \bar{F}_i \cdot \bar{r}_i \qquad \ldots (2.3)$$

where V_t is the volume of all the tension members and V_c that of the compression members.

The total volume of the framework V is given by,

$$V = V_t + V_c \qquad \ldots (2.4)$$

and so using (2.3) we can write,

$$V = V_c\left(1 + \frac{f_c}{f_t}\right) + \frac{1}{f_t}\sum_i \bar{F}_i \cdot \bar{r}_i = V_t\left(1 + \frac{f_t}{f_c}\right) - \frac{1}{f_c}\sum_i \bar{F}_i \cdot \bar{r}_i \qquad \ldots (2.5)$$

We see then that of all the possible frameworks that equilibrate the forces $\bar{F}_i$ and satisfy the strength requirement (2.2), the lightest structure is the one which has the least volume of compression members or alternatively, the least volume of tension members†. In particular the framework, if it exists, all of whose members carry tension or alternatively compression loads only, is the lightest framework possible among all those which carry the given loads. The volume of this optimum structure is given by (2.5) with $V_t = 0$ or $V_c = 0$.

† H.L. Cox, Ref.3

§2.2 Michell's Theorem†

Consider once more as in §2.1 a series of external forces $\bar{F}_i$ acting at $\bar{r}_i$. Let D be a domain of space containing the points $\bar{r}_i$; in particular D can be the whole of space. Consider then all possible frameworks S, contained in D which equilibrate the forces $\bar{F}_i$ and which satisfy the limiting conditions of stress (2.2). Let us assume that there is a framework S^* which satisfies the following condition:

"There exists a virtual deformation of the domain D such that the strain along all the members of S^* is equal to $\pm$ e, where e is a small positive number, and where the sign agrees with the sign of the end load carried by the particular member, and further that no linear element of D has a strain numerically greater than e". ... (2.6)

Michell's Theorem states that the volume V^* of S^* is less than or equal to the volume V of any of the frameworks S.

First of all we notice from (2.2),(2.3) and (2.4) that,

$$V = \frac{(f_t + f_c)}{2f_t f_c}\left(\sum_t L_t T_t + \sum_c L_c T_c\right) - \frac{(f_t - f_c)}{2f_t f_c}\sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i \qquad \text{... (2.7)}$$

and so the frame with the least volume is that which has the least value of $\sum_t L_t T_t + \sum_c L_c T_c$. Secondly we apply the virtual deformation of (2.6)

† Ref.2

to any of the frameworks S. The virtual work of the external forces will be the same for all the frameworks and so the change in strain energy will be the same too. If e_t, e_c are the mean values of virtual direct strain taken along the lengths of typical tension and compression members, the change in strain energy calculated as in the proof of (2.1) is given by

$$\begin{matrix}\text{Change in Strain Energy} \\ \text{for any } S\end{matrix} = \sum_t T_t L_t e_t - \sum_c T_c L_c e_c$$

For the special case of S^* we find from (2.6) that,

$$\begin{matrix}\text{Change in Strain Energy} \\ \text{for } S^*\end{matrix} = \left(\sum_t T_t^* L_t^* + \sum_c T_c^* L_c^* \right) e$$

Equating these results we find,

$$\left(\sum_t T_t^* L_t^* + \sum_c T_c^* L_c^* \right) e = \sum_t T_t L_t e_t - \sum_c T_c L_c e_c \leqslant \left(\sum_t T_t L_t + \sum_c T_c L_c \right) e$$

since by (2.6) we have $|e_t| \leqslant e$, $|e_c| \leqslant e$. Dividing by the positive number e we see by (2.7) that,

$$V^* \leqslant V \qquad \dots (2.8)$$

The actual value of V^* follows from the Principle of Virtual Work. If the virtual displacements corresponding to (2.6) at the points of application of the forces, i.e. at $\bar{r}_i$, are $e\bar{v}_i$, we have, dividing out the e,

$$\sum_t T_t^* L_t^* + \sum_c T_c^* L_c^* = \sum_{i=1}^{n} \bar{F}_i \cdot \bar{v}_i$$

and so by (2.7),

$$V^* = \frac{(f_t + f_c)}{2f_t f_c} \sum_{i=1}^{n} \bar{\bar{F}}_i \cdot \bar{v}_i - \frac{(f_t - f_c)}{2f_t f_c} \sum_{i=1}^{n} \bar{\bar{F}}_i \cdot \bar{r}_i \qquad \ldots (2.9)$$

The character of the deformation of (2.6) imposes certain restrictions upon the layout of members in S^*. At a node of this framework the directions of the strains $\pm e$, which are along the lines of members of S^*, are principal directions of strain and must thus satisfy certain conditions of orthogonality. In a three dimensional framework, at a node with three members, there will be no restriction, if the loads in the members have the same sign, since in that case the virtual deformation is a pure dilatation and therefore isotropic; however, if one load is of opposite sign to the others, it must be at right angles to them. At a node with four members, there is again no restriction if all loads have the same sign; if one member has an opposite load to the other three, then it must be orthogonal to them all and so forces them to lie in a plane; finally if the members fall into pairs with opposite signed loads then one of these pairs must be in line and normal to the other two. The general nature of the restrictions is clear from these examples. Similar requirements follow for the layout of optimum two-dimensional frames. When the loads at a node have the same sign, there is no restriction. A node with two members carrying loads of opposite sign must have these members at right angles. A node with three members, one of whose loads is opposite to the two others, must have the two in line and the member with the opposite load at right angles, while one with four members, with two pairs having opposite signed loads, must have the pairs in line and orthogonal to one another.

The optimum structure S^* has another very important property. In a general sense, it has greater stiffness than any other structure of S which satisfies (2.2) without meeting the requirement of $(2.6)^{\dagger}$. Let us think of the structures S loaded by forces $\lambda \bar{F}_i$ acting at $\bar{r}_i$, where λ is a parameter which varies from 0 to 1. The stresses developed are λf_T in the tension members and λf_c in the compression members. The strain energy stored W is thus,

$$W = \frac{\lambda^2}{2E}\left(V_T f_T^2 + V_c f_c^2\right)$$

The displacement "corresponding to the force system $\bar{F}_i$" is by Castigliano's First Theorem given by

$$\left(\frac{\partial W}{\partial \lambda}\right)_{\lambda=1} = \frac{1}{E}\left(V_T f_T^2 + V_c f_c^2\right)$$

Substituting from (2.5) for V_t and V_c we find

$$\text{Displacement corresponding to } \bar{F}_i = \frac{1}{E}\left\{f_t f_c V + (f_t - f_c)\sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i\right\} \quad \ldots (2.10)$$

The fact that S^* has the greatest possible stiffness then follows by (2.8).

§2.3 Cox's Design Applications††

Applications of the theorems of Maxwell and Michell to simple design problems have been made by H.L. Cox. He has con. dered first of all the problem of three coplanar forces. In the case where their point of intersection lies within the triangle formed by their points of application, the optimum framework can consist of tension (or compression)

† H.L. Cox. Ref.3

†† Refs. 3,4.

members only. Some of his layouts are given in Fig. 2.1. Others can be obtained by superposition of these and analogous structures in suitable proportions yielding a series of redundant frameworks. Equation (2.5), with $V_c = 0$, shows that all these structures have equal weight. One can remark, as pertinent to the general philosophy, that we have here an infinity of solutions, ranging from mechanisms to simply stiff structures to structures of any degree of redundancy!

The case where the point of intersection of the three forces lies outside the triangle of points of application is more difficult. Cox gives solutions for a number of symmetrical cases including the case of parallel forces illustrated in Fig. 2.2. Here the structure consists of a circular rod, conceived as the limit of infinitesimal chords pinned end to end, two straight members and a continuum of spokes all lying along radii of the circle. The radial members are all in tension and the curved member in compression. Michell's criterion (2.6) is satisfied using a constant deformation with direct strain e radially and $-e$ circumferentially. The fact that this is a consistent strain system will be shown later (in § 3.1); it also follows readily from B.3 in polar coordinates. The structure of Fig. 2.2 is thus an optimum.

Cox uses this last construction to build up a structure for the transmission of a bending moment (see Fig. 2.3). He shows that for $l/d > 4$ this structure is considerably lighter than a "simple tie and strut" and that for larger values of l/d multiple constructions, on the lines of Fig. 2.3, can be even lighter. He produces a competitive 14-bar framework and a variation on Fig. 2.3, in which the circles are replaced by spirals, which for $l/d > 4$ is ligher than any other construction considered. These structures for the transmission of bending moments are not Michell optimum structures, since they fail to satisfy the orthogonality conditions for members with opposite signed loads (§2.2). They are however by Maxwell's Theorem the best of their "class".

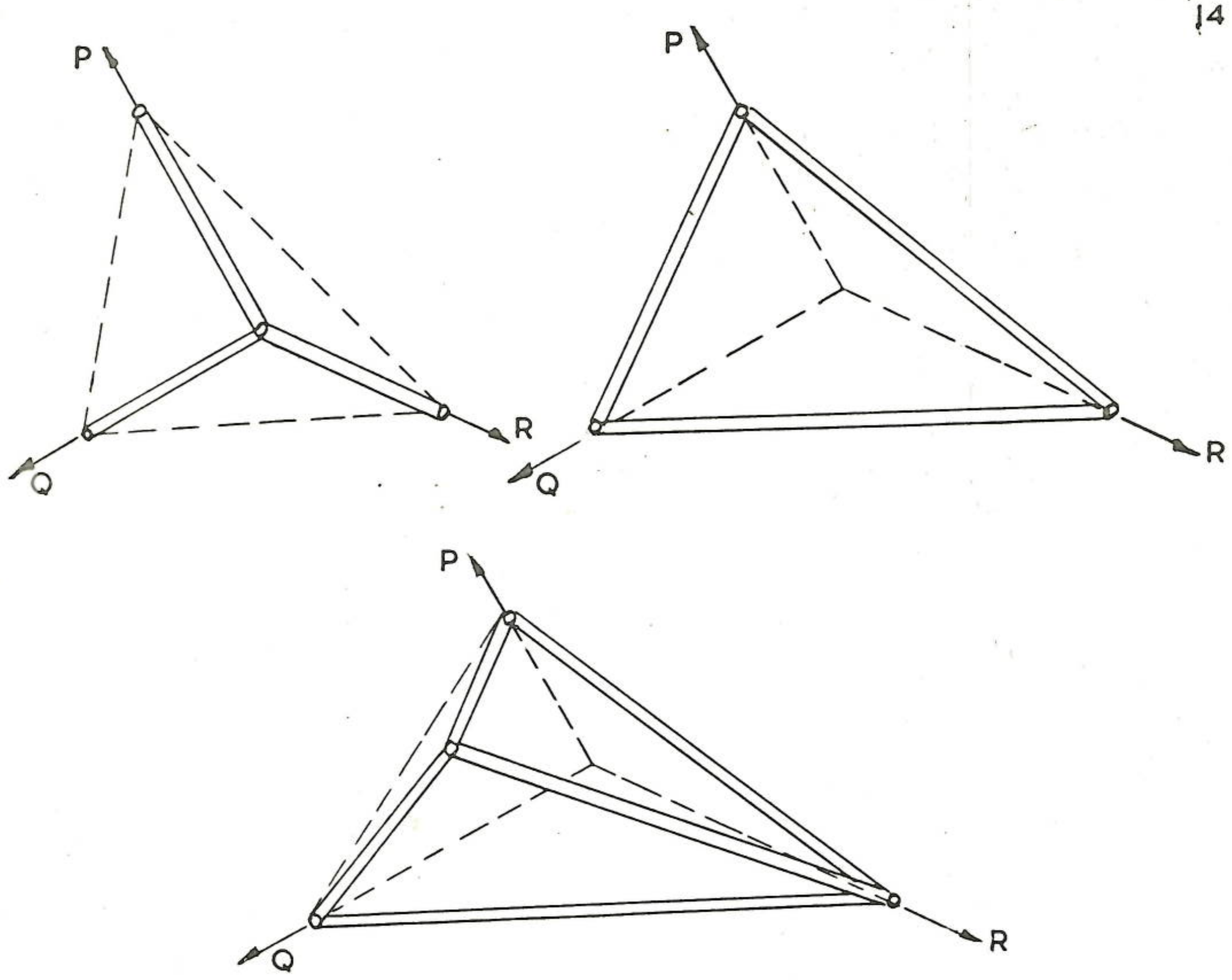

FIG 2·1

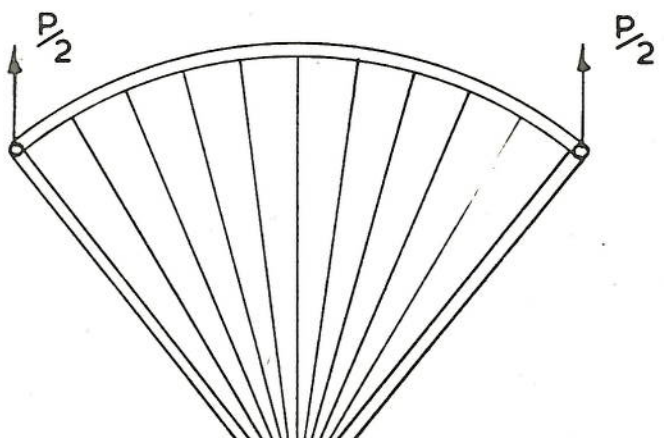

FIG 2·2

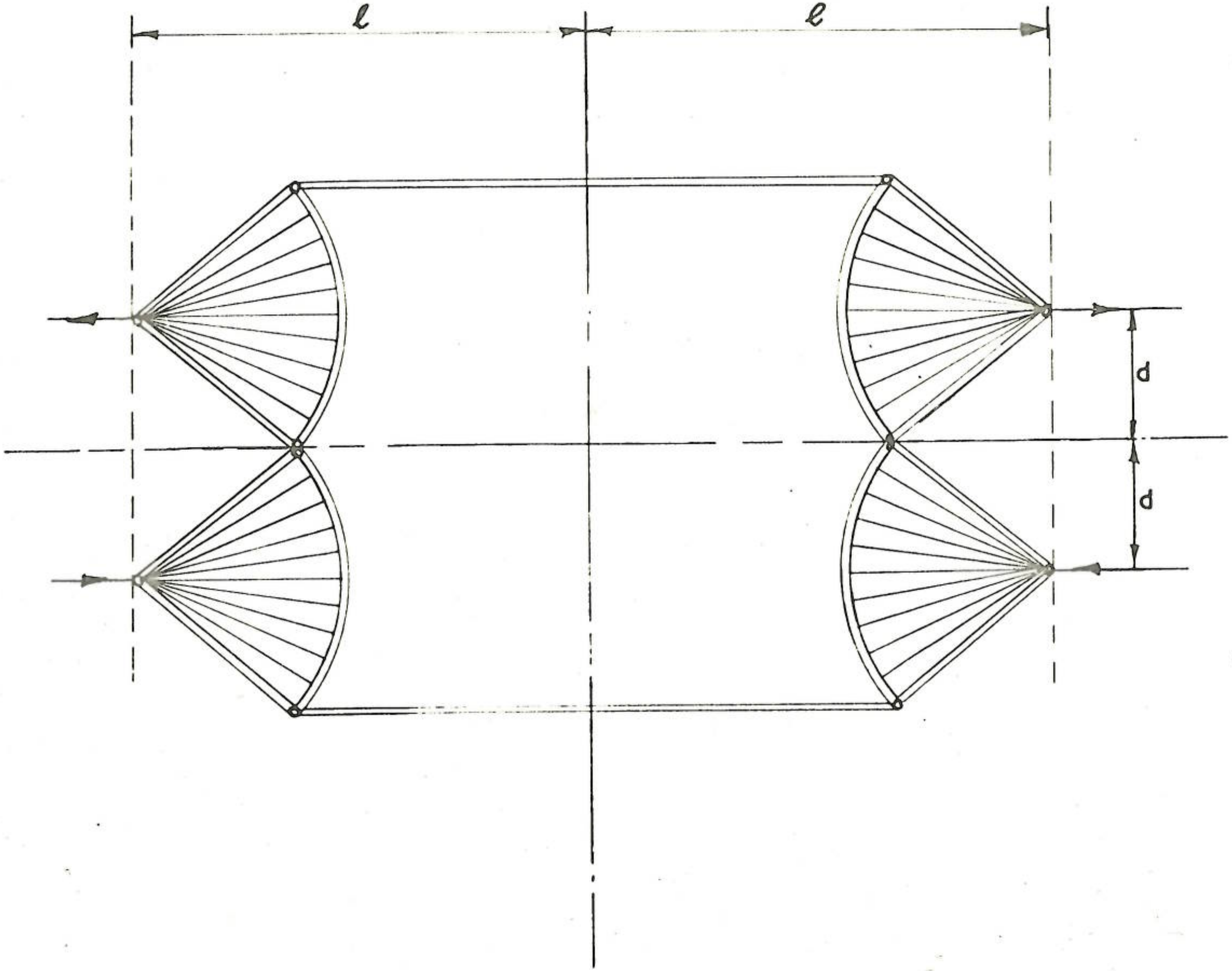

FIG 2.3

3. Developments in Two-Dimensions

§3.1 Lines of Principal Strain

The deformations, associated with two-dimensional Michell optimum structures, are by the criterion (2.6) of two kinds. In the first kind the principal strains at a point are equal in magnitude and sign and so correspond to a state of uniform dilatation. The lines of principal strain are thus completely indeterminate and as remarked before there is no restriction on the layout of the corresponding structure. The situation is quite different for the second kind of deformation in which the principal strains are equal and opposite, say $+e$ and $-e$; here the lines of principal strain are restricted to certain classes of orthogonal curves. This can be seen as follows.

Let us take the lines of principal strain as coordinate curves for a system of curvilinear coordinates (α, β). The formulae of Appendix A will then apply with $\varpi = \frac{\pi}{2}$ and those of Appendix B as they stand. The state of strain under consideration is defined by

$$e_{\alpha\alpha} = e \;, \quad e_{\beta\beta} = -e \;, \quad e_{\alpha\beta} = 0 \qquad \ldots (3.1)$$

Substitution from (3.1) in the compatibility equation (B.3) yields, cancelling $(-2e)$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) - \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0$$

Equation (A.17) which must be satisfied in any coordinate system yields with $\varpi = \frac{\pi}{2}$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0$$

We see then that our lines of principal strain must be such as to imply,

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = 0 , \qquad \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0 \qquad \ldots (3.2)$$

Reference to (A.16) shows that these results can be written

$$\frac{\partial^2\psi_1}{\partial\alpha\partial\beta} = 0 \qquad \text{or} \qquad \frac{\partial^2\psi_2}{\partial\alpha\partial\beta} = 0 \qquad \ldots (3.3)$$

where these last are not independent since $\psi_2 = \psi_1 + \pi/2$. This equation can be integrated as,

$$\psi_1 = F_1(\alpha) + F_2(\beta) = \psi_2 - \frac{\pi}{2} \qquad \ldots (3.4)$$

where F_1 and F_2 are arbitrary functions. Equation (A.16) then gives the following results, which also follow directly from (3.2)

$$\frac{1}{B}\frac{\partial A}{\partial\beta} = -F_1'(\alpha) , \qquad \frac{1}{A}\frac{\partial B}{\partial\alpha} = F_2'(\beta) \qquad \ldots (3.5)$$

The form of equation (3.3) shows that our lines of principal strain have the same form as the slip lines for two-dimensional perfect plastic flow†. This means that we can make use of much of the known developments in this field. Some of the integrations which follow, parallel corresponding processes in Plasticity Theory, but as the methods used are standard mathematics, it cannot be said that we are really using the analogy. However, it may well be that this analogy could yield fruitful suggestions.

† See Ref.5, Equation (23)

The integration of (3.5) is by no means a straightforward process. However progress can be made at the expense of a slight restriction of generality. Let us assume that the derivatives $F_1'(\alpha)$ and $F_2'(\beta)$ of (3.5) maintain a constant sign over the region of the plane in which our structures will lie. Reference to (A.16) shows that, since $\frac{\partial\psi_1}{\partial\alpha}$ and $\frac{\partial\psi_2}{\partial\beta}$ cannot vanish, there can be no points of inflexion on our lines of principal strain. This is the meaning of our restriction in geometrical terms.

Let us now apply the transformation of (A.18) to (3.5). We find

$$\frac{1}{\bar{B}}\frac{\partial\bar{A}}{\partial\bar{\beta}} = \mp\frac{dF_1}{d\bar{\alpha}} \;, \qquad \frac{1}{\bar{A}}\frac{\partial\bar{B}}{\partial\bar{\alpha}} = \pm\frac{dF_2}{d\bar{\beta}}$$

where the upper or lower signs must be taken according as $\frac{d\varphi_1}{d\bar{\alpha}} \cdot \frac{d\varphi_2}{d\bar{\beta}}$ is positive or negative. Choosing φ_1 and φ_2 so that,

$$F_1\{\varphi_1(\bar{\alpha})\} = \pm\bar{\alpha} \quad , \quad F_2\{\varphi_2(\bar{\beta})\} = \pm\bar{\beta}$$

or since F_1' and F_2' do not vanish, writing,

$$\varphi_1(\bar{\alpha}) = F_1^{-1}(\pm\bar{\alpha}) \quad , \qquad \varphi_2(\bar{\beta}) = F_2^{-1}(\pm\bar{\beta})$$

where F_1^{-1}, F_2^{-1} are the functions inverse to F_1, F_2 and the upper and lower signs are now to be taken accordingly as $F_1' F_2'$ is positive or negative†, we can write the transform of (3.5), omitting the bars, in the form,

$$\frac{1}{B}\frac{\partial A}{\partial\beta} = -1 \quad , \qquad \frac{1}{A}\frac{\partial B}{\partial\alpha} = 1 \qquad \ldots (3.6)$$

Equations (A.16) in conjunction with (3.6) show that with an appropriate choice of reference direction for ψ_1 we can write in our new coordinates,

†This is equivalent to the previous convention since $F_1'\frac{d\varphi_1}{d\bar{\alpha}} = \pm 1$, $F_2'\frac{d\varphi_2}{d\bar{\beta}} = \pm 1$ and so $(F_1' \cdot F_2') \cdot (d\varphi_1/d\bar{\alpha} \cdot d\varphi_2/d\bar{\beta}) = 1$

$$\psi_1 = \alpha + \beta = \psi_2 - \frac{\pi}{2} \qquad \ldots (3.7)$$

When A,B have been found equation (3.7) together with (A.5) and (A.8) will enable the determination of intrinsic equations for the lines of principal strain.

Equations (3.6) yield,

$$\frac{\partial^2 A}{\partial\alpha\partial\beta} + A = 0 , \qquad \frac{\partial^2 B}{\partial\alpha\partial\beta} + B = 0$$

the first of which can be integrated in the form,†

$$A = H_1(\alpha) + H_2(\beta) - \int_0^\alpha d\xi \int_0^\beta \{H_1(\xi) + H_2(\eta)\} J'\{(\xi-\alpha)(\beta-\eta)\} d\eta \qquad \ldots (3.8)$$

where H_1, H_2 are arbitrary functions and $J(\omega)$ is the Bessel Function††

$$J(\omega) = 1 + \omega + \frac{\omega^2}{(2!)^2} + \ldots + \frac{\omega^n}{(n!)^2} + \ldots \qquad \ldots (3.9)$$

$$\left.\begin{aligned} \text{which satisfies } \omega J''(\omega) + J'(\omega) - J(\omega) &= 0 \\ J(0) &= 1 \end{aligned}\right\} \qquad \ldots (3.10)$$

The first of (3.6) and (3.8) then give

$$B = -H_2'(\beta) + \alpha H_2(\beta) + \int_0^\alpha H_1(\xi) d\xi + \int_0^\alpha d\xi \int_0^\beta \{H_1(\xi) + H_2(\eta)\}(\xi-\alpha) J'\{(\xi-\alpha)(\beta-\eta)\} d\eta \qquad \ldots (3.11)$$

† Ref.6 Tome III §499

†† $J(\omega) = I_0(2\sqrt{\omega})$ for $\omega > 0$ and $J(\omega) = J_0(2\sqrt{-\omega})$ for $\omega < 0$.

We have obtained in (3.8), (3.11) general integrals of our equations for A,B depending upon two arbitrary functions H_1 and H_2. These results however are not very simple in form.

An important special case occurs when $A = kB$, where k is a positive constant. The integrals for this are most easily obtained from (3.6) directly. We find,

$$A = kB = Ke^{k\alpha-\beta/k} \qquad \ldots (3.12)$$

where K is a positive constant. The intrinsic equations for the lines of principal strain are by (A.5), (A.8) and (3.7) given by

$$\left.\begin{aligned} s_1 &= \frac{K}{k}\,e^{-\beta/k}\left\{e^{k(\psi_1-\beta)} - 1\right\} \\ s_2 &= -Ke^{k\alpha}\left\{e^{-(\psi_2-\alpha-\pi/2)/k} - 1\right\} \end{aligned}\right\} \qquad \ldots (3.13)$$

where we have taken s_1 as measured from $\alpha = 0$ and s_2 from $\beta = 0$. The Cartesian forms follow from (A.7) and (A.10) and with a particular choice of origin can be written,

$$x + iy = \frac{K}{(k+i)}\,e^{(k+i)\psi_1-\beta\,(k+1/k)} + K_1$$

and

$$x + iy = \frac{K}{(ki-1)}\,e^{(i-1/k)\psi_2+\alpha\,(k+1/k)+\pi/2k}$$

Substituting from (3.7) for ψ_1 and ψ_2 and identifying the resulting expressions, which are special cases of (A.3), we find that $K_1 = 0$ and

$$x + iy = \frac{K}{\sqrt{1+k^2}}\,e^{k\alpha-\beta/k\,+\,i\{\alpha+\beta-\tan^{-1}(1/k)\}} = \rho e^{i\omega} \qquad \ldots (3.14)$$

introducing polar coordinates ρ, ω. The equations for the coordinate curves in polar coordinates follow from (3.14), which gives,

$$\rho = \frac{K}{\sqrt{1+k^2}} e^{k\alpha-\beta/k}, \qquad \omega = \alpha+\beta - \tan^{-1}(1/k)$$

and so we find,

$$\left.\begin{array}{ll} \underline{\alpha\text{-curves}} & \rho = \dfrac{K}{\sqrt{1+k^2}} e^{k\omega-\beta(k+1/k)+k\tan^{-1}(1/k)} \\ \\ \underline{\beta\text{-curves}} & \rho = \dfrac{K}{\sqrt{1+k^2}} e^{-\omega/k+\alpha(k+1/k)-(1/k)\tan^{-1}(1/k)} \end{array}\right\} \quad \ldots (3.15)$$

We see that both these sets of curves are equi-angular spirals with angles† $\tan^{-1}(1/k)$ and $\tan^{-2}(-k)$ respectively. The two sets are orthogonal and circulate the origin in opposite directions.

The solutions obtained so far, besides ruling out inflexions, rule out the case where a set of coordinate curves are straight lines. If one of the α-curves is a straight line $\frac{\partial\psi_1}{\partial\alpha} = 0$ or by (3.4) $F_1'(\alpha) = 0$ on this line, which means of course $\frac{\partial\psi_1}{\partial\alpha} = 0$ everywhere and so all the α-curves are straight. This means by (3.5) that,

$$A = F_3'(\alpha), \quad B = F_2'(\beta)\, F_3(\alpha) + G(\beta)$$

where F_3 and G are arbitrary functions.

Choosing α as the length along our straight lines, i.e. taking $F_3(\alpha) = \alpha$, and since $\frac{\partial\psi_2}{\partial\beta} = F_2'(\beta)$, which does not vanish if the β curves have no inflexions, choosing β as the angle ψ_2 , i.e. taking $F_2'(\beta) = 1$, we can write:

$$A = 1, \quad B = \alpha + G(\beta) \qquad \ldots (3.16)$$

†$\tan\varphi = r d\theta/dr$, where φ is the angle between the radius vector and the tangent.

The α-curves are straight lines depending upon a single parameter β and so envelope the "evolute" of the β-curves for which they are the normals. The β-curves are of course the "involutes" and both sets of coordinate curves are in this case completely defined by the "evolute" which by (3.16) has the equation

$$\alpha + G(\beta) = 0 \qquad \ldots (3.17)$$

The evolute must of course be outside the region surveyed by our coordinate system.

An interesting special case is obtained when the evolute degenerates to a point. Our coordinate curves then become the set of rays through the point and the set of concentric circles. This is the layout used in Fig. 2.2 and our present result shows that this is a Michell optimum design.

The case where both sets of coordinate curves are straight lines is almost trivial. Here we can take α, β as Cartesian coordinates and write,

$$A = B = 1 \qquad \ldots (3.18)$$

In summary we can say that the layouts of Michell optimum structures, for the case where the associated principal strains are equal and opposite and where inflexions are ruled out, take the forms defined by, firstly, (3.8) and (3.11), which depend upon two arbitrary functions, and includes the special case (3.12), secondly (3.16), which depends upon one arbitrary function and finally (3.18) which has nothing arbitrary about it at all.

§3.2 Conditions of Equilibrium

The considerations of §3.1 give guidance for the choice of layout for an optimum structure. The determination of the required sizes of members results from a consideration of equilibrium conditions. The investigation of this matter necessitates a choice of structural form. In this section we will continue to deal with frameworks, but will specialise our studies to the case of continuous distributions with perhaps concentrated members along isolated lines, for example along edges. We shall thus be treating plane structures consisting of double arrays of closely spaced fibres, which for the optimum case must lie along the lines of principal strain for Michell's virtual deformation of (2.6), i.e. along the α and β coordinate curves considered in §3.1.

The case where Michell's principal strains are equal must be considered first. Here there is no restriction upon layout at all as long as the structure transmits the applied loads by members entirely in tension or alternatively compression. The example of §2.3 shows that there may well be an infinite number of alternative structures, which by Maxwell's Theorem are of equal weight. It is quite clear that this multiplicity is a general property and so our problem is really to pick out simple yet adequate designs from the infinite possibilities. We shall therefore restrict ourselves to orthogonal layouts of fibres.

Since our structural elements are continuously distributed, their magnitude is properly described by their equivalent thicknesses t_1 and t_2 in the α and β directions respectively. This means for example that across a width $Bd\beta$ normal to the α-direction, there pass members whose total cross section area is $t_1 Bd\beta$. Now in the present case the stress in all the members could be f_t † and so, in the notation of (B.4),

$$T_1 = t_1 f_t \quad , \quad T_2 = t_2 f_t \quad , \quad S = 0 \qquad \dots (3.19)$$

Substituting in (B.4) we then obtain,

$$\frac{\partial}{\partial\alpha}(Bt_1) - \frac{\partial B}{\partial\alpha} t_2 = 0 \quad , \quad \frac{\partial}{\partial\beta}(At_2) - \frac{\partial A}{\partial\beta} t_1 = 0$$

which may be written,

$$\frac{\partial t_1}{\partial\alpha} + \frac{1}{B}\frac{\partial B}{\partial\alpha}(t_1 - t_2) = 0 \ , \quad \frac{\partial t_2}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}(t_1 - t_2) = 0 \qquad \dots (3.20)$$

The boundary conditions (B.5) for the case where there is no edge member becomes,

$$\left.\begin{aligned} f_t t_1 \sin\theta &= F_n \sin\theta + F_t \cos\theta \\ f_t t_2 \cos\theta &= F_n \cos\theta - F_t \sin\theta \end{aligned}\right\} \qquad \dots (3.21)$$

Our problem for any given layout is to find solutions $t_1 \geqslant 0$ and $t_2 \geqslant 0$ of (3.20) which satisfy (3.21) on the boundary. We shall discuss the possibility of resolving this problem in §3.3.

The case where Michell's principal strains are equal and opposite and where our intersecting fibres carry opposite signed stresses f_t and $-f_c$, will have stress resultants given by,

$$T_1 = t_1 f_t \quad , \quad T_2 = -t_2 f_c \quad , \quad S = 0 \qquad \dots (3.22)$$

† Or, of course $(-f_c)$. However (3.20) is the same in both cases.

Equations (B.4) and (B.5) give for this case,

$$\frac{\partial}{\partial\alpha}(Bf_t t_1) + \frac{\partial B}{\partial\alpha} f_c t_2 = 0 \quad , \quad \frac{\partial}{\partial\beta}(Af_c t_2) + \frac{\partial A}{\partial\beta} f_t t_1 = 0 \qquad \ldots (3.23)$$

and,

$$\left.\begin{aligned} f_t t_1 \sin\theta &= F_n \sin\theta + F_t \cos\theta \\ f_c t_2 \cos\theta &= -F_n \cos\theta + F_t \sin\theta \end{aligned}\right\} \qquad \ldots (3.24)$$

These equations are similar to those for the previous case, but it must be remembered of course that here the coordinate curves are limited to the special forms studied in §3.1. In the general case covered by (3.6) we can write,

$$\frac{1}{(-Bf_t t_1)} \frac{\partial}{\partial\beta} (Af_c t_2) = -1 \quad , \quad \frac{1}{(Af_c t_2)} \frac{\partial}{\partial\alpha} (-Bf_t t_1) = 1$$

These are of the same form as (3.6) itself and so possess integrals analogous to (3.8) and (3.11). We then find,

$$\left.\begin{aligned} Af_c t_2 &= K_1(\alpha) + K_2(\beta) - \int_0^\alpha d\xi \int_0^\beta \{K_1(\xi) + K_2(\eta)\} J\{(\xi-\alpha)(\beta-\eta)\} d\eta \\ Bf_t t_1 &= K_2'(\beta) - \alpha K_2(\beta) - \int_0^\alpha K_1(\xi) d\xi - \int_0^\alpha d\xi \int_0^\beta \{K_1(\xi) + K_2(\eta)\}(\xi-\alpha) \\ &\qquad \times J'\{(\xi-\alpha)(\beta-\eta)\} d\eta \end{aligned}\right\} \ldots (3.25)$$

where K_1 , K_2 are arbitrary and A,B,J are given by (3.8),(3.9),(3.11).

For the special case of (3.12) we find from (3.23) that,

$$f_t \frac{\partial t_1}{\partial\alpha} + k(f_t t_1 + f_c t_2) = 0 \quad , \quad f_c \frac{\partial t_2}{\partial\beta} - \frac{1}{k} (f_t t_1 + f_c t_2) = 0$$

It is convenient to transform this as follows

$$\left.\begin{aligned} f_t t_1 &= \tau_1\, e^{-k\alpha+\beta/k} , \quad f_c t_2 = \tau_2\, e^{-k\alpha+\beta/k} \\ \text{where,} \quad \frac{1}{\tau_2}\frac{\partial}{\partial\alpha}(-\tau_1/k) &= 1 , \quad \frac{1}{(-\tau_1/k)}\frac{\partial\tau_2}{\partial\beta} = -1 \end{aligned}\right\} \quad \ldots (3.26)$$

Comparison with (3.6) gives once more a general solution of the form,

$$\left.\begin{aligned} \tau_2 &= L_1(\alpha)+L_2(\beta)-\int_0^\alpha d\xi\int_0^\beta \{L_1(\xi)+L_2(\eta)\}J\{(\xi-\alpha)(\beta-\eta)\}d\eta \\ \tau_1/k &= L_2'(\beta)-\alpha L_2(\beta)-\int_0^\alpha L_1(\xi)d\xi-\int_0^\alpha d\xi\int_0^\beta\{L_1(\xi)+L_2(\eta)\}(\xi-\alpha)J'\{(\xi-\alpha)(\beta-\eta)\}d\eta \end{aligned}\right\} \ldots (3.27)$$

where L_1 and L_2 are arbitrary functions. A special solution analogous to (3.12) can be written,

$$\tau_2 = \frac{k_1}{k}\tau_1 = K_1\, e^{-k_1\alpha+\beta/k_1} \quad \ldots (3.28)$$

where k_1 and K_1 are positive constants.

The special case of (3.16) gives for (3.23):

$$\frac{\partial}{\partial\alpha}[f_t t_1\{\alpha + G(\beta)\}] + f_c t_2 = 0 , \quad \frac{\partial}{\partial\beta}(f_c t_2) = 0$$

which yields,

$$f_t t_1 = \frac{G_1(\beta) - F(\alpha)}{G(\beta) + \alpha} , \quad f_c t_2 = F'(\alpha) \quad \ldots (3.29)$$

where, $F(\alpha)$ and $G_1(\beta)$ are arbitrary functions. Finally, the particular case of (3.18), gives the obvious result that t_1 and t_2 must be constants.

§3.3 Formulation of the General Problem

The problem of determining the optimum arrangement of fibres, which will equilibrate a given system of forces applied to the boundary of a region, can now be formulated. The two systems of fibres must carry constant stresses f_1 and f_2 which must have the values,

$$f_1 = f_t \text{ or } -f_c , \quad f_2 = f_t \text{ or } -f_c \qquad \ldots (3.30)$$

The layout of the fibres must determine an orthogonal† curvilinear coordinate system (α,β) for which the functions A,B satisfy, by (A.17) with $\varpi = 0$ and (3.2)

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) &= 0 \quad (f_1 f_2 > 0) \\ \frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) &= 0 \quad (f_1 f_2 < 0) \end{aligned}\right\} \qquad \ldots (3.31)$$

The equivalent thicknesses t_1 and t_2 of the fibres must satisfy the differential equations of equilibrium, which by (B.4) take the form,

$$\frac{\partial}{\partial\alpha}(Bf_1 t_1) - \frac{\partial B}{\partial\alpha} f_2 t_2 = 0 \quad , \quad \frac{\partial}{\partial\beta}(Af_2 t_2) - \frac{\partial A}{\partial\beta} f_1 t_1 = 0 \qquad \ldots (3.32)$$

The boundary, which we assume known in the intrinsic form $\varphi = \varphi(\sigma)$, where φ is the angle between the reference direction used for ψ_1 in Appendix A and the positive direction along the tangent to the boundary, must be expressed as in (A.20) in terms of $b_1(\sigma)$ and $b_2(\sigma)$. The relevant equations are (A.22), (A.16) and (A.11), with $\varpi = 0$; we can thus write,

† This is of course not essential for the case $f_1 f_2 > 0$, but is assumed in the interest of simplicity!

$$\left.\begin{aligned}
\varphi(\sigma) &= \int_{(\alpha_o\ \beta_o)}^{\{b_1(\sigma),b_2(\sigma)\}} \left(-\frac{1}{B}\frac{\partial A}{\partial \beta}\,d\alpha + \frac{1}{A}\frac{\partial B}{\partial \alpha}\,d\beta\right) + \theta \\
\cos\theta &= A\{b_1(\sigma),\ b_2(\sigma)\}\ .\ b_1'(\sigma) \\
\sin\theta &= B\{b_1(\sigma),\ b_2(\sigma)\}\ .\ b_2'(\sigma)
\end{aligned}\right\} \qquad \ldots (3.33)$$

where (α_o, β_o) is the point at which ψ_1 is assumed to be zero. The equilibrium conditions at the boundary must be satisfied. The appropriate equations follow from (B.5) with $T = 0$ i.e.

$$\left.\begin{aligned}
f_1 t_1 \sin\theta &= F_n \sin\theta + F_t \cos\theta \\
f_2 t_2 \cos\theta &= F_n \cos\theta - F_t \sin\theta
\end{aligned}\right\} \qquad \ldots (3.34)$$

Finally the solution for t_1 and t_2 must obviously satisfy the conditions:

$$t_1 \geqslant 0 \quad , \qquad t_2 \geqslant 0 \qquad \ldots (3.35)$$

The mathematical problem presented by the equations (3.30) to (3.35) is a formidable one. Furthermore we have no guarantee that, for any boundary and any distribution of force, a solution exists. Special solutions given in §3.4 below show that the problem can be solved in certain cases, but a study of the boundary conditions (3.34) shows that for some loading conditions, solutions cannot exist unless the boundary has special forms. A pertinent problem would thus appear to be the determination of the restriction that must be imposed upon the forces and the boundaries to ensure the existence of a Michell optimum design.

Some guidance can be obtained in relation to the ambiguities of (3.30) and to other more general matters by a consideration of (3.34).

If $F_t = 0$ everywhere, we have:

$$\left.\begin{array}{ll} & \theta = 0 \text{ and } f_2 t_2 = F_n \\ \text{or} & \theta = \pi/2 \text{ and } f_1 t_1 = F_n \\ \text{or} & f_1 t_1 = f_2 t_2 = F_n \end{array}\right\} \quad \ldots (3.36)$$

and so if F_n has the same sign, say positive, everywhere, we must take $f_1 = f_2 = f_t$. If on the other hand F_n is sometimes positive and sometimes negative, the boundary must be built up of pieces of coordinate curves, α-curves $(\theta = 0)$ say where $F_n > 0$ and β-curves $(\theta = \pi/2)$ for $F_n < 0$. We shall then have $f_2 = f_T$ and $f_1 = -f_c$ and furthermore the boundary must have right angled corners at all the zeros of F_n! If $F_n = 0$ everywhere we first notice that $\theta = 0$ and $t_2 = 0$ or $\theta = \pi/2$ and $t_1 = 0$ at the zeros of F_t i.e. one of the coordinate curves must touch the boundary at these points. At these zeros too $\tan\theta$ and $\cot\theta$ change sign and so $f_1 t_1$ has the same sign everywhere and likewise $f_2 t_2$, but opposite to $f_1 t_1$. It follows that one of f_1, f_2 is f_t and the other $(-f_c)$, everywhere on the boundary. In the intermediate cases where the resultant of F_n, F_t is neither normal nor tangential to the boundary one can by an appropriate choice of θ arrange that the signs of f_1, f_2 are the same or opposed, but variation of the general direction of the external force from outwards to inwards will give rise to similar problems like those induced by the zeros of F_n.

The equations (3.31) which determine the coordinate curves have, at least in their form appropriate to $f_1 f_2 < 0$†, been studied thoroughly in §3.1.

†These can of course be used for the case $f_1 f_2 > 0$ as well.

Reasonably general integrals have been obtained, although their form is not too convenient. However, if one adopts the point of view that the business of optimum structures is still partly an art, then the material obtained can form the basis for the construction of an enormous variety of layouts, which can be used as judgement, intuition or even hunches may direct, as trial arrangements for the solution of any given problem. Alternatively, they can be used as in §3.4, below, to construct artificial problems.

Once appropriate values of f_1 and f_2 are chosen and a layout decided with determinate A and B, then our remaining problems can be resolved. The problem presented by (3.33), that of determining values of α, β and θ on the boundary can be resolved, if not analytically, at least graphically, by drawing out the boundary and superimposing a grid of α and β curves. The problem of the determination of the sizes of members t_1 and t_2, equations (3.32), (3.34), can then be resolved, perhaps by the analysis of §3.2, but certainly by the usual numerical methods of integrating step by step the hyperbolic differential equations along their characteristic lines. This last step however faces us with new difficulties. It is not usual to have to integrate hyperbolic differential equations subject to boundary conditions on closed curves and rightly so, since, as is easily seen, restrictions have to be imposed upon the possible boundary values on different parts of a closed curve. Consider the problem of integrating (3.32) in the region bounded by the curve ABCD of Fig.3.1. This curve is the transform of our real bounding curve in a plane where α, β are rectangular coordinates. We assume by (3.34) that the values of t_1 and t_2 are known on ABCD. Since t_1, t_2 are known on AB they can be found at all points within the closed region ABE bounded by the curve AB and the characteristics AE, BE through its end points. Similarly values of t_1, t_2 can be found in the closed region BCF using the given values on BC. We shall then have two distinct determinations of t_1 and t_2 on the line BF. These must agree with one another and so must imply restrictions upon the boundary conditions on BC i.e. on F_n and F_t. In fact a knowledge of t_1, t_2 on BF and t_2 alone on BC determines t_1 and t_2 in BCF and in particular on BC! It may be that a specially chosen layout will avoid these difficulties, but this is a question which cannot be answered with our existing knowledge.

§3.4 Special Solutions

The general problem is clearly too difficult to solve and so, as in other fields, we must turn to "inverse methods", which assume a solution, or at any rate a layout, and examine what particular problems are solved by this assumption.

Let us begin with the special case of (3.16) which arises when G = 0. Here the evolute becomes a point and the coordinates (α, β) become polar coordinates, with radii and concentric circles for coordinate curves. With the values $A = 1$, $B = \alpha$ the equations (3.32) give,

$$f_1 t_1 = \frac{F_1(\alpha) + F_2(\beta)}{\alpha} , \quad f_2 t_2 = F_1'(\alpha) \qquad \ldots (3.37)$$

where F_1, F_2 are arbitrary functions. On circular boundaries $(\theta = \pi/2)$ and radial boundaries $(\theta = 0)$ we have by (3.34),

$$\left.\begin{array}{lll} \text{circular boundaries} & F_n = f_1 t_1, & F_t = 0 \\ \text{radial boundaries} & F_n = f_2 t_2, & F_t = 0 \end{array}\right\} \qquad \ldots (3.38)$$

The part of the solution which depends on F_1 can be used to illustrate the points made in discussing (3.36). If we take $F_1 > 0$, but with $F_1' < 0$, we must write $f_1 = f_t$ and $f_2 = -f_c$. Taking the region bounded by two radii and two circles, we have the case where the radial boundaries are under normal pressure and the circular boundaries under normal tension, and, as predicted, we have four right-angled corners.

The case where $F_1 = -P\alpha (P > 0)$, $F_2 = 0$ requires $f_1 = f_2 = -f_c$ and solves the problem of a circular disc under radial pressure P by filling in the circle with fibres of constant equivalent thicknesses

$$t_1 = t_2 = P/f_c \qquad \ldots (3.39)$$

The case $F_1 = 0$, $F_2 > 0$ consists of radial spokes transmitting tensions of varying amounts. The point $\alpha = 0$ is now a singular point and is in general a "centre of pressure" and the point of application of a "concentrated force". Applied to a wedge this solution gives a concentrated tension whose line of action lies within the angle of the wedge. The case where F_2 is constant gives a symmetrical load, whose reaction can be collected by a circular member in compression. Adding radial edge members gives us Fig. 2.2 once again.

A second example is furnished by (3.12) which gives coordinate curves in the form of equi-angular spirals (3.15). A special solution for t_1 and t_2 for the case $f_1 = f_t$, $f_2 = -f_c$ is given in (3.26) and (3.28). Let us adopt this with $k_1 = k$. We then obtain by (3.14),

$$f_t t_1 = f_c t_2 = \frac{K_1 K^2}{(1 + k^2)\rho^2} \qquad \ldots (3.40)$$

The origin $\rho = 0$ must be excluded. Let us consider a region of the plane bounded internally by a circle ρ = constant. The loading at this boundary will be of constant magnitude by (3.40) and of constant inclination to the radius by the defining property of equi-angular spirals and so will consist of a uniform normal tension and a uniform tangential traction. For a suitable value of k ($k = 1$, when $f_t = f_c$) the normal tension will be zero and we shall be left with the tractions whose resultant will be a couple. (Fig.3.2).

The origin $\rho = 0$ is thus a "centre of pressure" and a centre of "concentrated torque". This solution can be applied to the transmission of torque from the inner boundary to the outer boundary of a circular ring (Fig.3.2).

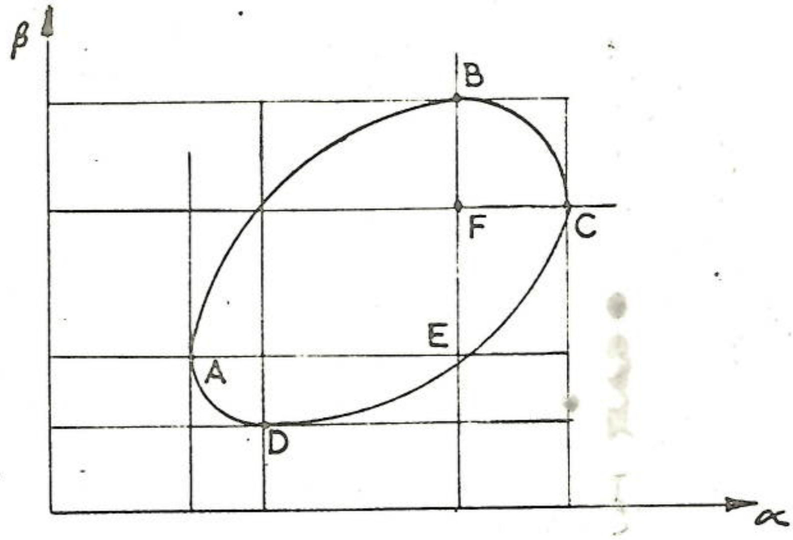

FIG 3·1

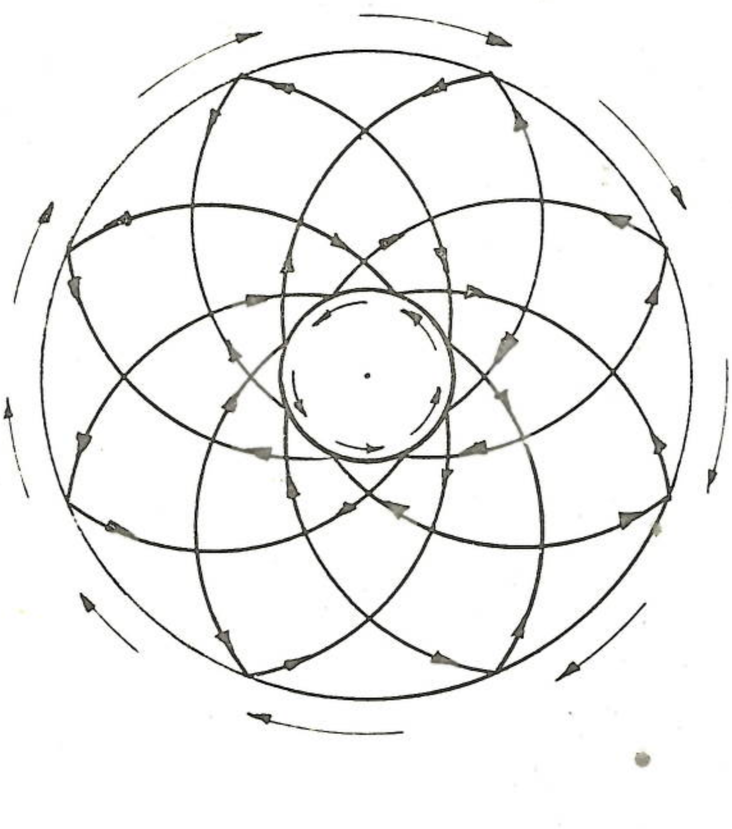

FIG 3·2

4. Plates

§4.1 Michell Type Theorem for a Plate

Consider the set of plates P of varying thickness t, which equilibrate given coplanar external forces with a stress distribution, having principal stresses f_1 and f_2,† which satisfy the maximum shear stress criterion of yielding, namely,

$$\text{(Greatest minus least of } f_1,\ f_2,\ 0) \leq Y \qquad \ldots (4.1)$$

where Y is the tension-compression yield stress. We assume that there is a plate P* of the set P, with principal stresses f_1^*, f_2^*, which satisfies the conditions:

(1) (Greatest minus least of f_1^*, f_2^*, 0) = Y

(2) There exists a virtual strain system, with principal strains e_1 and e_2 and principal directions coinciding with those of the principal stresses f_1^*, f_2^*, whose magnitude depends upon a constant number $e > 0$ and is such that, if

$$\epsilon_1 = \frac{e_1(1-2\nu^2)-\nu e_2}{(1-\nu^2)}, \qquad \epsilon_2 = \frac{e_2(1-2\nu^2)-\nu e_1}{(1-\nu^2)},$$

then, in regions (a) where $f_1^* \geq f_2^* \geq 0$ and $f_1^* = Y$, one has $\epsilon_1 = e$, $\epsilon_2 = 0$
" " (b) " $f_1^* \geq 0 \geq f_2^*$ " $f_1^* - f_2^* = Y$, " " $\epsilon_1 = e$, $\epsilon_2 = -e$
" " (c) " $f_2^* \geq f_1^* \geq 0$ " $f_2^* = Y$, " " $\epsilon_1 = 0$, $\epsilon_2 = e$
" " (d) " $f_2^* \geq 0 \geq f_1^*$ " $f_2^* - f_1^* = Y$ " " $\epsilon_1 = -e$, $\epsilon_2 = e$
" " (e) " $0 \geq f_1^* \geq f_2^*$ $-f_2^* = Y$ " " $\epsilon_1 = 0$, $\epsilon_2 = -e$
" " (f) " $0 \geq f_2^* \geq f_1^*$ $-f_1^* = Y$ " " $\epsilon_1 = -e$, $\epsilon_2 = 0$

... (4.2)

† We assume conventional plate theory, in spite of the varying t.

Let us now apply the virtual strain system of condition (2) in (4.2) to all the plates P. Since the virtual work of the external forces is the same in all cases, the increments of strain energy induced in the plate will also have the same value. If the stress components in a general plate P, referred to the principal directions of our virtual strain, are f_{11}, f_{22} and f_{12}, then the increment of strain energy W for this plate is given by,

$$W = \iint \frac{t}{2E}\Big[\{f_{11}+\frac{E}{(1-\nu^2)}(e_1+\nu e_2)\}^2+\{f_{22}+\frac{E}{(1-\nu^2)}(e_2+\nu e_1)\}^2$$
$$-2\nu\{f_{11}+\frac{E}{(1-\nu^2)}(e_1+\nu e_2)\}\{f_{22}+\frac{E}{(1-\nu^2)}(e_2+\nu e_1)\}+2(1+\nu)f_{12}^2$$
$$-\{f_{11}^2+f_{22}^2-2\nu f_{11}f_{22}+2(1+\nu)f_{12}^2\}\Big]\,dA$$

where E is Young's modulus, ν is Poisson's ratio and the integration with respect to area is taken over the region of the plane, which is occupied by the plate P. Developing the terms in the integrand and neglecting second order terms in the strains, we find, using the definitions of ϵ_1 and ϵ_2 in (4.2) that,

$$W = \iint t(f_{11}\epsilon_1 + f_{22}\epsilon_2)dA \qquad \ldots (4.3)$$

For the plate P* we have $f_{11} = f_1^*$, $f_{22} = f_2^*$ and so taking account of the conditions of (4.2) we find,

$$W^* = \iint t^*\, Y\, e\, dA = Y\, e\, V^* \qquad \ldots (4.4)$$

where W* and t* apply to P* and V* is its volume.

Equation W* and W we find,

$$YeV^* = \iint t(f_{11}\epsilon_1 + f_{22}\epsilon_2)dA$$

$$= \underset{(a)}{\iint tf_{11}edA} + \underset{(b)}{\iint t(f_{11} - f_{22})edA} + \underset{(c)}{\iint tf_{22}edA}$$

$$+ \underset{(d)}{\iint t(f_{22} - f_{11})edA} + \underset{(e)}{\iint t(-f_{22})edA} + \underset{(f)}{\iint t(-f_{11})edA} \quad \ldots (4.5)$$

where the region of integration has been split into the regions designated (a) to (f) in (4.2) and use has been made of the values of ϵ_1 , ϵ_2 given in this same equation. Now by the properties of principal stresses and (4.1) we see that,

$$|f_{11}| \leqslant \max.\{|f_1|,|f_2|\} \leqslant Y \text{ and similarly } |f_{22}| \leqslant Y.$$

Again by (4.1) the max. shear stress in the plane of the plate cannot exceed Y/2 and so by a known formula,

$$\left(\frac{f_{11} - f_{22}}{2}\right)^2 + f_{12}^2 \leqslant \left(\frac{Y}{2}\right)^2$$

This implies $|f_{11} - f_{22}| \leqslant Y$. Hence,

$$\pm f_{11} \leqslant Y \ , \ \pm f_{22} \leqslant Y \ , \ \pm(f_{11} - f_{22}) \leqslant Y \quad \ldots (4.6)$$

Applying (4.6) to (4.5) we deduce that,

$$YeV^* \leqslant \iint tYedA = YeV$$

where V is the volume of a general plate P. Since Ye is positive we have,

$$V^* \leqslant V \quad \ldots (4.7)$$

The plate P* which satisfies (4.2) has as small a volume as any other plate of the set P.

The actual volume V^* can be calculated from the virtual work of the external forces. If the forces are specified as in §2.2 by $\bar{F}_i$ and if the virtual displacements, at the points of application of these forces, corresponding to the strain system of (4.2) are $e\bar{v}_i$ then, the virtual work is $\sum_i \bar{F}_i \, e\bar{v}_i$ and so by (4.4) we find,

$$V^* = \frac{1}{Y} \sum_i \bar{F}_i \cdot \bar{v}_i \qquad \ldots (4.8)$$

§4.2 Development of the Theory for a Plate

The virtual strain system of (4.2) has principal strains e_1, e_2 given by,

$$e_1 = \frac{(1-\nu^2)\{(1-2\nu^2)\epsilon_1 + \nu\epsilon_2\}}{(1-5\nu^2+4\nu^4)}, \qquad e_2 = \frac{(1-\nu^2)\{\nu\epsilon_1 + (1-2\nu^2)\epsilon_2\}}{(1-5\nu^2+4\nu^4)} \qquad \ldots (4.9)$$

where ϵ_1, ϵ_2 are defined in (4.2). We see that the strains are constant and further that $(e_1 - e_2)$ does not vanish. Substitution in (B.3) then gives an equation, which combined with (A.17) with $\varpi = \pi/2$, gives, just as in (3.2) the equations

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) = \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = 0 \qquad \ldots (4.10)$$

The theory of §3.1 is thus valid for this case too and the principal strain lines determined there can be used as principal stress lines for optimum plate designs.

The equations of equilibrium can be written by writing

$$T_1 = tf_1 \;, \quad T_2 = tf_2 \quad , \quad S = 0^{\dagger} \qquad \ldots (4.11)$$

in (B.4), which yields,

$$\frac{\partial}{\partial \alpha}(Btf_1) - \frac{\partial B}{\partial \alpha}\, tf_2 = 0 \;, \quad \frac{\partial}{\partial \beta}(Atf_2) - \frac{\partial A}{\partial \beta}\, tf_1 = 0 \qquad \ldots (4.12)$$

In addition by (4.2) the stresses must satisfy one of the following,

$$f_1 = \pm Y \;, \quad f_2 = \pm Y \;, \quad f_1 - f_2 = \pm Y \qquad \ldots (4.13)$$

This gives us three equations for t, f_1, f_2, which would appear to be sufficient. However the strains correspondong to f_1 and f_2 must be compatible. This means by (B.3) and (4.10) that,

$$\frac{\partial}{\partial \alpha}\left\{\frac{B}{A}\frac{\partial}{\partial \alpha}(f_2 - \nu f_1)\right\} + \frac{\partial}{\partial \beta}\left\{\frac{A}{B}\frac{\partial}{\partial \beta}(f_1 - \nu f_2)\right\}$$

$$- \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial}{\partial \alpha}\{(1+\nu)(f_1 - f_2)\} + \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial}{\partial \beta}\{(1+\nu)(f_1 - f_2)\} = 0 \qquad \ldots (4.14)$$

Finally by (4.11) and (B.5) we have at an unreinforced boundary,

$$\left.\begin{aligned} tf_1 \sin\theta &= F_n \sin\theta + F_t \cos\theta \\ tf_2 \cos\theta &= F_n \cos\theta - F_t \sin\theta \end{aligned}\right\} \qquad \ldots (4.15)$$

Our present problem has yielded a superfluity of equations. Equations (4.10), (4.12), (4.13) and (4.14) are six equations between the five unknowns A, B, t, f_1 and f_2. This is very restrictive on kinds of solutions and may well mean that only very special distributions of F_n, F_t can be accommodated in (4.15). One way of obtaining consistency is, as we shall see in §4.3, to assume that f_1 and f_2 are constants. This makes (4.14) an identity, but removes f_1 and f_2 from the list of variables.

† We now omit the * on f_1 and f_2 for simplicity.

§4.3 Constant Stress Solutions

Let us now assume that the stresses f_1, f_2 are constant. Equation (4.13) gives one relation between them, but does not fix them completely. However (4.1) restricts their values to certain ranges, but among all the possibilities, every ratio f_1/f_2 between $-\infty$ and $+\infty$ can occur. We can thus leave the values of f_1 and f_2 open for the moment, until indeed we have to consider the boundary conditions.

The equilibrium equations (4.12) integrate in the forms:

$$\frac{\partial}{\partial\alpha}\{f_1 \log t + (f_1 - f_2) \log B\} = 0 \ , \quad \frac{\partial}{\partial\beta}\{f_2 \log t - (f_1 - f_2) \log A\} = 0$$

We see immediately that if $f_1 = f_2$,

$$t = \text{constant} \quad (f_1 = f_2) \qquad \ldots (4.16)$$

Substituting in the boundary conditions (4.15) and writing by (4.13) $f_1 = f_2 = \pm Y$ we then find

$$F_n = \pm tY \ , \quad F_t = 0 \qquad \ldots (4.17)$$

The case $f_1 = f_2$ thus solves any problem of uniform normal stress applied to a boundary by the not unexpected solution of a uniform thickness plate!

Assuming from now on that $f_1 \neq f_2$ and writing,

$$p_1 = \frac{f_2}{(f_1 - f_2)} \ , \qquad p_2 = -\frac{f_1}{(f_1 - f_2)} \qquad \ldots (4.18)$$

we see that the equilibrium conditions imply,

$$A = F_1'(\alpha) t^{p_1} \ , \qquad B = F_2'(\beta) t^{p_2}$$

where F_1 and F_2 are arbitrary functions. However using a transformation like (4.18) namely,

$$\bar{\alpha} = F_1(\alpha) , \quad \bar{\beta} = F_2(\beta)$$

which is reversible, since from the nature of A,B and t, F_1' and F_2' are one signed namely positive, we find

$$A = \bar{A}\, F_1'(\alpha) , \quad B = \bar{B}\, F_2'(\beta)$$

and so, omitting the bars, we can write finally,

$$A = t^{p_1} , \qquad B = t^{p_2} \qquad \ldots (4.19)$$

The equations (4.10) integrate in the form

$$\frac{\partial A}{\partial \beta} = B\, F_3(\alpha) , \qquad \frac{\partial B}{\partial \alpha} = A\, F_4(\beta) .$$

when F_3, F_4 are arbitrary functions. Substitution from (4.19) gives on integration, since $p_1 + p_2 + 1 = 0$,

$$\frac{p_1}{(2p_1+1)}\, t^{2p_1+1} = \beta F_3(\alpha) + F_5(\alpha), \quad \frac{p_2}{(2p_2+1)}\, t^{2p_2+1} = \alpha F_4(\beta) + F_6(\beta) \qquad \ldots (4.20)$$

unless $p_1 = p_2 = -\frac{1}{2}$ and then

$$-\tfrac{1}{2}\log t = \beta F_3(\alpha) + F_5(\alpha), \quad -\tfrac{1}{2}\log t = \alpha F_4(\beta) + F_6(\beta) \qquad \ldots (4.21)$$

where F_5 and F_6 are arbitrary functions. The consistency of (4.20) and (4.21) give respectively,

$$\frac{p_1 p_2}{(2p_1+1)(2p_2+1)} = \{\beta F_3(\alpha) + F_5(\alpha)\}\{\alpha F_4(\beta) + F_6(\beta)\} \qquad \ldots (4.22)$$

$$\beta F_3(\alpha) + F_5(\alpha) = \alpha F_4(\beta) + F_6(\beta) \qquad \ldots (4.23)$$

The special case $p_1 = p_2 = -\frac{1}{2}$ gives, by (4.18), $f_1 = -f_2$. Equation (4.13) then shows that we must have

$$f_1 = \pm Y/2 \quad , \quad f_2 = \mp Y/2 \qquad \dots (4.24)$$

which may be compared to the Michell solution of Section 3 with $f_1 = f_T$, $f_2 = -f_c$. Equation (4.23) has the solution,

$$F_3(\alpha) = c_1\alpha + c_3 \; , \quad F_4(\beta) = c_1\beta + c_2 \; , \quad F_5(\alpha) = c_2\alpha + c_4$$
$$F_6(\beta) = c_3\beta + c_4$$

where c_1, c_2, c_3, c_4 are constants, and so by (4.19), (4.21) we can write,

$$A = B = t^{-1/2} = e^{c_1\alpha\beta + c_2\alpha + c_3\beta + c_4} \qquad \dots (4.25)$$

A special case can be studied using (3.12) with k = 1 if we write,

$$c_1 = 0 \; , \quad c_2 = 1 \; , \quad c_3 = -1 \; , \quad e^{c_4} = K$$

The lines of principal stress are equi-angular spirals with angle $\pi/4$ [see(3.15)]. Fig.3.2 applies to this case and shows that the external loading for a circular hole is a uniform tangential traction, whose resultant is a torque. Comparison between (4.25) and (3.14) shows that t varies as,

$$t = \frac{1}{2\rho^2} \qquad \dots (4.26)$$

where ρ is the distance from the centre of the circle. We thus see that the optimum design for the transmission of a uniformly applied torque from a circular hole in a plate consists of a plate whose thickness varies inversely as the square of the distance from the centre of the hole. Inspection of (3.40) for the case of a fibre mesh shows direct comparison with this case.

Another interesting example can be obtained by writing,

$$c_1 = c_3 = 0$$

Equation (4.16) gives $\frac{\partial\psi_1}{\partial\alpha} = 0$ and so the α-curves are straight lines. Also $\frac{\partial\psi_2}{\partial\beta} = c_2$ and so $\psi_2 = c_2\beta$ for an appropriate reference direction.

Equation (4.6) gives,

$$s_2 = \beta e^{c_2\alpha + c_4} = \frac{\psi_2}{c_2} e^{c_2\alpha + c_4}$$

measuring s_2 from $\beta = 0$. This is a circle of radius ρ given by,

$$\rho = \frac{1}{c_2} e^{c_2\alpha + c_4}$$

Our principal directions are thus radii and concentric circles and the load on a circular hole is a uniform normal tension or pressure. Equation (4.25) shows that the optimum variation of t is given by,

$$t = \frac{1}{c_2^2 \rho^2} \qquad \ldots (4.27)$$

Another special case is given by $p_1 = 0$, $p_2 = -1$ or $f_2 = 0$, $f_1 = \pm Y$. For this we have,

$$A = 1, \quad B = t^{-1} = \alpha F_4(\beta) + F_6(\beta) \qquad \ldots (4.28)$$

The α-curves are straight lines which envelope an evolute given by

$$\alpha F_4(\beta) + F_6(\beta) = 0 \qquad \ldots (4.29)$$

and the β-curves are the involutes. The degenerate case of concentric circles and radii can be applied to a wedge under tension due to normal traction on the circular boundaries, but <u>not</u> to a hole in a plate since in that case the displacements given by (B.2) are not consistent; we have in fact a dislocation.

The general case of arbitrary p_1, p_2 yields by (4.22),

$$F_3(\alpha) = \frac{p_1 c_1}{(2p_1+1)(c_2\alpha+c_3)} \qquad F_5(\alpha) = \frac{p_1 c_4}{(2p_1+1)(c_2\alpha+c_3)}$$

$$F_4(\beta) = \frac{p_2 c_2}{(2p_2+1)(c_1\beta+c_4)} \qquad F_6(\beta) = \frac{p_2 c_3}{(2p_2+1)(c_1\beta+c_4)}$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Equation (4.20) then gives,

$$t = \left(\frac{c_1\beta+c_4}{c_2\alpha+c_3}\right)^{(f_1-f_2)/(f_1+f_2)} \qquad \ldots (4.30)$$

and (4.19)

$$A = \left(\frac{c_1\beta+c_4}{c_2\alpha+c_3}\right)^{f_2/(f_1+f_2)}, \qquad B = \left(\frac{c_1\beta+c_4}{c_2\alpha+c_3}\right)^{-f_1/(f_1+f_2)}. \quad (4.31)$$

The resulting coordinate curves are of spiral form and as in the case of (4.25) with $c_1 \neq 0$ reveal the novel feature of containing "points of inflexion". These were ruled out in our previous discussion in §3.1.

§4.4 Alternative Approach

A direct attack upon the problem of the optimum design of plate structures has been suggested by the present writer[†]. Let us refer our plate to rectangular Cartesian axes $O(x,y)$. The components of the

[†]Ref. 7

stress tensor f_{xx}, f_{yy} and f_{xy}, when multiplied by the variable thickness t, may be derived from a stress function U by,

$$tf_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad tf_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad tf_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \qquad \ldots (4.32)$$

This ensures that equilibrium is satisfied throughout the plate. Compatibility of strain requires,

$$\frac{\partial^2}{\partial x^2}(f_{yy} - \nu f_{xx}) + \frac{\partial^2}{\partial y^2}(f_{xx} - \nu f_{yy}) = 2(1+\nu)\frac{\partial^2 f_{xy}}{\partial x \partial y} \qquad \ldots (4.33)$$

A yield condition, such as the Mises-Hencky criterion, must be imposed, giving,

$$f_{xx}^2 + f_{yy}^2 - f_{xx} \cdot f_{yy} + 3f_{xy}^2 \leqslant 3q^2 \qquad \ldots (4.34)$$

where q is the yield stress for pure shear. Finally equilibrium conditions at the boundary give,

$$l(tf_{xx}) + m(tf_{xy}) = F_x, \quad l(tf_{xy}) + m(tf_{yy}) = F_y \qquad \ldots (4.35)$$

where (l, m) are direction cosines for the outward normal and (F_x, F_y) are Cartesian components of the external traction per unit length of boundary.

Elimination of the stress components gives:

$$\left(\frac{1}{t}\right)\nabla^4 U + 2\frac{\partial}{\partial x}\left(\frac{1}{t}\right)\frac{\partial}{\partial x}(\nabla^2 U) + 2\frac{\partial}{\partial y}\left(\frac{1}{t}\right)\frac{\partial}{\partial y}(\nabla^2 U)$$

$$+ \frac{\partial^2}{\partial x^2}\left(\frac{1}{t}\right)\left(\frac{\partial^2 U}{\partial x^2} - \nu\frac{\partial^2 U}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{t}\right)\left(\frac{\partial^2 U}{\partial y^2} - \nu\frac{\partial^2 U}{\partial x^2}\right)$$

$$+ 2(1+\nu)\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{t}\right) \cdot \frac{\partial^2 U}{\partial x \partial y} = 0 \qquad \ldots (4.36)$$

$$(\nabla^2 U)^2 - 3\left\{ \frac{\partial^2 U}{\partial x^2} \cdot \frac{\partial^2 U}{\partial y^2} - \left(\frac{\partial^2 U}{\partial x \partial y}\right)^2 \right\} \leqslant 3q^2 t^2 \qquad \ldots (4.37)$$

$$\frac{d}{ds}\left(\frac{\partial U}{\partial y}\right) = F_x , \qquad \frac{d}{ds}\left(\frac{\partial U}{\partial x}\right) = -F_y \qquad \ldots (4.38)$$

where s is the arc length of the boundary. Assuming that the optimum design is given by taking the equality in (4.37), we can eliminate t from (4.36),(4.37) and obtain a fourth order equation for U to be solved subject to (4.38). It may be that a numerical process like Relaxation could be used to resolve this formidable problem, but we cannot say, with our present knowledge, that a physically acceptable solution exists. Again, since the equations are non-linear, it may be that several solutions are possible, but in this case we can presumably pick out the lightest one of the alternatives. Finally since we have not used a condition of least weight, we cannot really be sure that our procedure gives us the lightest structure†. It is conceivable that the use of the inequality in (4.37) might yield a structure of less weight, although a structure which is just about to yield everywhere at its working load is clearly a very good engineering design.

Our equations can be put into a variational form. Let us vary the stress components and the thickness by amounts δf_{xx}, δf_{yy}, δf_{xy} and δt subject to the conditions of equilibrium and a yielding condition like (4.34), with only an equality sign, being satisfied. The maintenance of the equilibrium conditions in the varied state yields

†Equation (4.41) below only gives stationary weight with the Haigh Yielding Criterion W = constant!

$$\iint \left\{ e_{xx}\delta(tf_{xx}) + e_{yy}\delta(tf_{yy}) + 2e_{xy}\delta(tf_{xy}) \right\} dxdy = 0 \qquad \ldots (4.39)$$

where e_{xx}, e_{yy}, e_{xy} are the components of the strain tensor. Introducing the Strain Energy density W, which is given by,

$$W = \frac{1}{2E}\left\{ (f_{xx} + f_{yy})^2 + 2(1+\nu)(f_{xy}^2 - f_{xx}f_{yy}) \right\}$$

$$= \frac{(1-2\nu)}{6E}(f_{xx}+f_{yy})^2 + \frac{(1+\nu)}{3E}(f_{xx}^2+f_{yy}^2-f_{xx}f_{yy} + 3f_{xy}^2) \quad .. (4.40)$$

we can write (4.39) in the form,

$$\iint (t\delta W + 2W\delta t)dxdy = 0 \qquad \ldots (4.41)$$

Introducing the Mises-Hencky yield condition and using the second expression for W in (4.40), we can write (4.41) as,

$$\iint \left[t(f_{xx}+f_{yy})(\delta f_{xx}+\delta f_{yy}) + \left\{ (f_{xx}+f_{yy})^2 + \frac{6(1+\nu)}{(1-2\nu)} q^2 \right\} \delta t \right] dxdy = 0 \qquad \ldots (4.42)$$

The variational equation (4.42) can be used for approximate solution of optimum design problems. One might begin by finding a stress function U, which satisfies (4.38), and which depends upon a number of arbitrary parameters or functions. The thickness t follows from (4.37), with the equal sign, and the stress components from (4.32). Equation (4.42) then gives, by the usual processes of the calculus of variations, equations for the parameters or functions, which though complicated will undoubtedly yield to the process of numerical analysis.

5. Suggestions for Lines of Research

The theories discussed in previous sections exhibit many gaps both in their scope and in their foundations. Application of the theories is held up in many cases by mathematical difficulties and in others by the need to carry out possible, but lengthy, special investigations. Future research can therefore take two forms. In the first place, one can fill in the details in those parts of the subject, where the next steps are reasonably clear and where the necessary mathematical techniques are known. Secondly one can tackle the fundamental problems, attempting to clear up some of the mysteries and to broaden the coverage of the theories. The pedestrian first form may well be suggestive for solution of the more profound problems of the second.

Reasonably straightforward problems include:

(1) Systematic study of the coordinate curves corresponding to the general integrals for A,B obtained in equations (3.8), (3.11).

(2) Systematic study of the possible forms that can be assumed by involute curves and their normals. Inspection of known solutions with a view to application to design problems. (See equations (3.16), (3.29)).

(3) Use of the analogy with slip lines in plastic flow to make use of known results in this field. In particular one might study equation (3.12) with k a complex number.

(4) Detail study of the constant stress solutions for plates, [see equations (4.25), (4.28) and (4.30),(4.31)]. This might throw some light on the use of coordinate curves with inflexions, which are ruled out in the general study based on (3.8),(3.11).

(5) Development of practical methods either analytical or graphical-numerical for the determination of t_1 and t_2 using equations of the type (3.32) and (3.34) when the coordinate curves are known. Study of the restrictions imposed on the external forces.

(6) Use of the experience gained from research projects like (1),(2), (3),(4) above to develop the art of drawing in a set of coordinate curves to meet a given loaded boundary at qualitatively appropriate angles so that consistent signs can be given to f_1 and f_2.

(7) Solution of a number of simple problems using the variational equation (4.42).

Fundamental investigations into the existing theory include:

(8) Proof of an existence theorem for a Michell optimum framework to equilibrate a finite number of given forces.

(9) Investigation of the equations of section §3.3. Development of techniques for their solution. Proof of an existence theorem for a Michell optimum layout of fibres.

(10) Study of the equations of section §4.2 with a view to developing solutions for plates with variable stresses. Existence theorem for the optimum plate.

(11) Investigation of the general problem of integrating hyperbolic partial differential equations with boundary conditions on closed curves.

Investigations directed towards broadening the scope of the existing theory include:

(12) Development of Michell type theorems for other types of construction, e.g. reinforced plates and shells.

(13) Development of less restrictive conditions than the Michell type for plates. The restrictions imposed by the theorem of §4.1 seem to rule out most practical problems.

(14) Development of theories to deal with several alternative loading conditions and stiffness requirements.

(15) Investigation of the stability of Michell optimum structures.

6. Conclusions

(1) The classical theorems, such as that of Maxwell, can be used with considerable success to produce economical layouts for frame structures.

(2) The theorem of Michell can be used to create a complete theory for the economical design of a plate-like structure consisting of a double array (weave) of closely spaced fibres. It may well be that the optimum layouts can only be achieved by restrictions on the distribution of the external forces or in some cases on the shape of the plate, but the development of techniques for the solution of the mathematical problems involved, should lead to many solutions of practical interest.

(3) The extension of Michell's methods to continuous plates leads to interesting, but rather restricted results. It would seem that here, a less specialised approach is called for and it may well be that the variational theorem of section §4.4 and its associated approximate methods of analysis could be of greater use in the problem of plate design.

(4) The subject of this report offers great scope for research, both in the detail development of the classical methods and their extension to theories of greater breadth.

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Appendix A - Curvilinear Coordinates

Consider a plane referred to rectangular Cartesian axes $O(x,y)$ (Fig.A.1). Let us define functions $\alpha(x,y)$, $\beta(x,y)$ in a region R of the plane, which are continuous and possess continuous first derivatives in R and for which the Jacobian is positive at all points in R, i.e.

$$J = \frac{\partial\alpha}{\partial x} \cdot \frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial y} \cdot \frac{\partial\beta}{\partial x} > 0 \qquad \text{... (A.1)}$$

We write,

$$\alpha(x,y) = \alpha , \qquad \beta(x,y) = \beta \qquad \text{... (A.2)}$$

where α, β are parameters. Equation (A.2) then defines two sets of curvilinear arcs, which continuously fill the region R as the parameters α, β are varied. We observe that one curve of each set passes through each point of R. If (α,β) are taken as represented by points referred to a Cartesian coordinate system (α,β) in a plane, then (α,β) will vary in a region R' of this plane for (x,y) in R.

The condition (A.1) is known† to ensure that (A.2) can be solved for x,y in the neighbourhood of any point (α,β) of R', i.e. we can write,

$$x = x(\alpha,\beta) , \qquad y = y(\alpha,\beta) \qquad \text{... (A.3)}$$

where the functions $x(\alpha,\beta)$, $y(\alpha,\beta)$ are continuous, as well as all the first derivatives with respect to α, β. The validity of (A.3) for the whole region R' does not follow in general from our hypotheses††. We shall therefore make the additional assumption, that any pair of values (α,β) in R' determines one and only one point (x,y) of our region R. The points of R and R' are thus in one to one correspondence with one another and so the parameter pair (α,β) can be used as "curvilinear coordinates" for points in R.

†Ref.6 Tome I §40.

††Idem. §117

Differentiation of (A.2) gives

$$d\alpha = \frac{\partial\alpha}{\partial x}\,dx + \frac{\partial\alpha}{\partial y}\,dy \quad , \qquad d\beta = \frac{\partial\beta}{\partial x}\,dx + \frac{\partial\beta}{\partial y}\,dy$$

Solution for dx, dy yields,

$$dx = \frac{1}{J}\left(\frac{\partial\beta}{\partial y}\,d\alpha - \frac{\partial\alpha}{\partial y}\,d\beta\right) \quad , \qquad dy = \frac{1}{J}\left(-\frac{\partial\beta}{\partial x}\,d\alpha + \frac{\partial\alpha}{\partial x}\,d\beta\right)$$

which is always valid by (A.1). The following important relationships then follow,

$$\frac{\partial x}{\partial\alpha} = \frac{1}{J}\frac{\partial\beta}{\partial y}, \quad \frac{\partial x}{\partial\beta} = -\frac{1}{J}\frac{\partial\alpha}{\partial y}, \quad \frac{\partial y}{\partial\alpha} = -\frac{1}{J}\frac{\partial\beta}{\partial x}, \quad \frac{\partial y}{\partial\beta} = \frac{1}{J}\frac{\partial\alpha}{\partial x} \quad .. \text{(A.4)}$$

The curves given by β = constant, along which α varies, are termed "α-coordinate curves"; those given by α = constant, along which β varies, are termed "β-coordinate curves". "Positive directions" on these curves are defined as those directions for which the corresponding coordinate α or β is increasing. On an α-curve (β = constant) we have,

$$dx = \frac{\partial x}{\partial\alpha}\,d\alpha \quad , \qquad dy = \frac{\partial y}{\partial\alpha}\,d\alpha$$

and so the corresponding element of arc ds_1 is given by

$$ds_1 = A\,d\alpha \qquad ... \text{(A.5)}$$

where, using (A.4) and (A.1),

$$A = +\left\{\left(\frac{\partial x}{\partial\alpha}\right)^2 + \left(\frac{\partial y}{\partial\alpha}\right)^2\right\}^{1/2} = +\frac{1}{J}\left\{\left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2\right\}^{1/2} > 0 \qquad ... \text{(A.6)}$$

The arc length s_1 is measured in the positive direction on the α-curve. Now if ψ_1 is the angle between Ox and the positive tangent† to the α-curve measured with the convention that a rotation Ox to Oy is positive, we have, (see Fig.A.1),

†The one whose direction coincides with the positive direction on the curve.

$$\left.\begin{aligned} \cos\psi_1 &= \frac{dx}{ds_1} = \frac{1}{A}\frac{\partial x}{\partial \alpha} = \frac{1}{AJ}\frac{\partial \beta}{\partial y} \\ \sin\psi_1 &= \frac{dy}{ds_1} = \frac{1}{A}\frac{\partial y}{\partial \alpha} = -\frac{1}{AJ}\frac{\partial \beta}{\partial x} \end{aligned}\right\} \qquad \ldots (A.7)$$

where (A.4) has been used once more.

Corresponding results follow for the β-curves. We find in a similar way that

$$ds_2 = B d\beta \qquad \ldots (A.8)$$

$$B = +\left\{\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right\}^{1/2} = +\frac{1}{J}\left\{\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2\right\}^{1/2} > 0 \qquad \ldots (A.9)$$

$$\left.\begin{aligned} \cos\psi_2 &= \frac{dx}{ds_2} = \frac{1}{B}\frac{\partial x}{\partial \beta} = -\frac{1}{BJ}\frac{\partial \alpha}{\partial y} \\ \sin\psi_2 &= \frac{dy}{ds_2} = \frac{1}{B}\frac{\partial y}{\partial \beta} = \frac{1}{BJ}\frac{\partial \alpha}{\partial x} \end{aligned}\right\} \qquad \ldots (A.10)$$

Let the angle between the positive tangents to the α, β curves at a point be denoted by ϖ. We can then write,

$$\varpi = \psi_2 - \psi_1 \qquad \ldots (A.11)$$

The value of $\sin\varpi$ then follows from (A.7),(A.10) and (A.1). We find,

$$\begin{aligned} \sin\varpi &= \sin(\psi_2-\psi_1) = \sin\psi_2\cos\psi_1 - \cos\psi_2\sin\psi_1 \\ &= \frac{1}{ABJ^2}\left(\frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial x}\right) \\ &= \frac{1}{ABJ} > 0 \text{ , by (A.1),(A.6),(A.9)} \end{aligned}$$

It follows that ϖ must lie between 0 and π or,

$$0 < \varpi < \pi \qquad \ldots (A.12)$$

The direction of rotation at any point from a positive tangent on an α-curve to a positive tangent on a β-curve is thus the same over the whole region R. The value of $\cos\varpi$ follows in a similar way, we find,

$$\cos\varpi = \frac{1}{AB}\left(\frac{\partial x}{\partial\alpha} \cdot \frac{\partial x}{\partial\beta} + \frac{\partial y}{\partial\alpha} \cdot \frac{\partial y}{\partial\beta}\right) = -\frac{1}{ABJ^2}\left(\frac{\partial\alpha}{\partial x} \cdot \frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial y} \cdot \frac{\partial\beta}{\partial y}\right) \quad .. \text{(A.13)}$$

The length of the linear element ds between (α,β) and $(\alpha+d\alpha, \beta+d\beta)$ is given by,

$$ds^2 = dx^2 + dy^2 = \left(\frac{\partial x}{\partial\alpha}\,d\alpha + \frac{\partial x}{\partial\beta}\,d\beta\right)^2 + \left(\frac{\partial y}{\partial\alpha}\,d\alpha + \frac{\partial y}{\partial\beta}\,d\beta\right)^2$$

Using (A.6),(A.9) and (A.13) we find,

$$ds^2 = (A d\alpha)^2 + (B d\beta)^2 + 2AB\,d\alpha\,d\beta\cos\varpi \qquad \text{... (A.14)}$$

If ψ is the angle between directions defined by $(d\alpha,d\beta)$, $(\delta\alpha,\delta\beta)$ at a point we have,

$$\cos\psi = \frac{dx}{ds} \cdot \frac{\delta x}{\delta s} + \frac{dy}{ds} \cdot \frac{\delta y}{\delta s}\ , \qquad \sin\psi = \frac{dx}{ds} \cdot \frac{\delta y}{\delta s} - \frac{dy}{ds} \cdot \frac{\delta x}{\delta s}$$

Substituting for $\frac{dx}{ds}$ etc. in terms of derivatives with respect to α, β and using (A.6),(A.9) and (A.13) we find,

$$\left.\begin{aligned}\cos\psi &= A^2\frac{d\alpha}{ds} \cdot \frac{\delta\alpha}{\delta s} + B^2\frac{d\beta}{ds} \cdot \frac{\delta\beta}{\delta s} + AB\left(\frac{d\alpha}{ds} \cdot \frac{\delta\beta}{\delta s} + \frac{d\beta}{ds} \cdot \frac{\delta\alpha}{\delta s}\right)\cos\varpi \\ \sin\psi &= AB\left(\frac{d\alpha}{ds} \cdot \frac{\delta\beta}{\delta s} - \frac{d\beta}{ds} \cdot \frac{\delta\alpha}{\delta s}\right)\sin\varpi\end{aligned}\right\} \qquad \text{... (A.15)}$$

Let us now assume that the functions $x(\alpha,\beta)$, $y(\alpha,\beta)$ possess all their second derivatives with respect to α, β. The following important formulae can then be demonstrated:

$$\left.\begin{aligned}\frac{\partial\psi_1}{\partial\alpha} &= -\frac{1}{B\sin\varpi}\left\{\frac{\partial A}{\partial\beta} - \frac{\partial}{\partial\alpha}(B\cos\varpi)\right\} \\ \frac{\partial\psi_2}{\partial\beta} &= \frac{1}{A\sin\varpi}\left\{\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(A\cos\varpi)\right\}\end{aligned}\right\} \qquad \text{... (A.16)}$$

The proofs are straightforward and will not be g...en in detail. They are obtained by differentiation of (A.6), (A.9) and (A.13) with respect to α, β and (A.7) with respect to α and (A.10) with respect to β. Simple transformation then yields the formulae of (A.16). Recalling (A.11) we can then obtain by elimination of ψ_1 and ψ_2 the following important relation between A, B and ϖ :

$$\frac{\partial}{\partial\alpha}\left[\frac{1}{A\sin\varpi}\left\{\frac{\partial B}{\partial\alpha} - \frac{\partial}{\partial\beta}(A\cos\varpi)\right\} - \frac{\partial\varpi}{\partial\beta}\right] + \frac{\partial}{\partial\beta}\left[\frac{1}{B\sin\varpi}\left\{\frac{\partial A}{\partial\beta} - \frac{\partial}{\partial\alpha}(B\cos\varpi)\right\}\right] = 0 \qquad \dots \text{(A.17)}$$

The parameters α, β corresponding to definite coordinate curves are not unique and this fact can be used on occasion to simplify calculations. Let us write,

$$\alpha = \varphi_1(\bar{\alpha}) , \qquad \beta = \varphi_2(\bar{\beta}) \qquad \dots \text{(A.18)}$$

where φ_1, φ_2 are continuous functions, with continuous first derivatives, which do not vanish in the ranges of $\bar{\alpha}$, $\bar{\beta}$ which correspond to the region R$'$. The equations (A.18) can be solved uniquely for $\bar{\alpha}$, $\bar{\beta}$ and so it is possible to take ($\bar{\alpha}$, $\bar{\beta}$) as new curvilinear coordinates in our region R. The coordinate curves are of course unchanged. If $\bar{A}$, $\bar{B}$ and $\bar{\varpi}$ are the new functions which define ds as in (A.14) we find,

$$\left.\begin{array}{l} \bar{A} = A\left|\dfrac{d\varphi_1}{d\bar{\alpha}}\right| , \qquad \bar{B} = B\left|\dfrac{d\varphi_2}{d\bar{\beta}}\right| \\ \text{and } \bar{\varpi} = \varpi \text{ or } \pi-\varpi \text{ according as } \dfrac{d\varphi_1}{d\bar{\alpha}} \cdot \dfrac{d\varphi_2}{d\bar{\beta}} \text{ is positive or negative} \end{array}\right\} \qquad \dots \text{(A.19)}$$

Finally, let us consider the analytical representation of an arbitrary curve, for example the boundary of region R (Fig. A.1). Any curve can be written

$$\alpha = b_1(\sigma) , \qquad \beta = b_2(\sigma) \qquad \dots \text{(A.20)}$$

where σ is a parameter and b_1 , b_2 are continuous functions, with continuous first derivatives, which do not vanish simultaneously. The parameter σ can be taken as the arc length of the curve. This requires, as substitution from (A.20) into (A.14) shows, that

$$\{A(\sigma)b_1'(\sigma)\}^2 + \{B(\sigma)b_2'(\sigma)\}^2 + 2\{A(\sigma)b_1'(\sigma)\}\{B(\sigma)b_2'(\sigma)\}\cos\{\varpi(\sigma)\} = 1 \qquad \ldots \text{(A.21)}$$

where $A(\sigma) = A\{b_1(\sigma), b_2(\sigma)\}$ etc. If θ is the angle between the positive tangent to an α-curve and the positive tangent to our arbitrary curve (A.20) at any point σ , we find writing $d\alpha = ds/A$, $d\beta = 0$ and $\delta s = \delta\sigma$ in (A.15) that,

$$\left.\begin{aligned} \cos\theta &= A(\sigma)b_1'(\sigma) + B(\sigma)b_2'(\sigma)\cos\{\varpi(\sigma)\} \\ \sin\theta &= B(\sigma)b_2'(\sigma)\sin\{\varpi(\sigma)\} \end{aligned}\right\} \qquad \ldots \text{(A.22)}$$

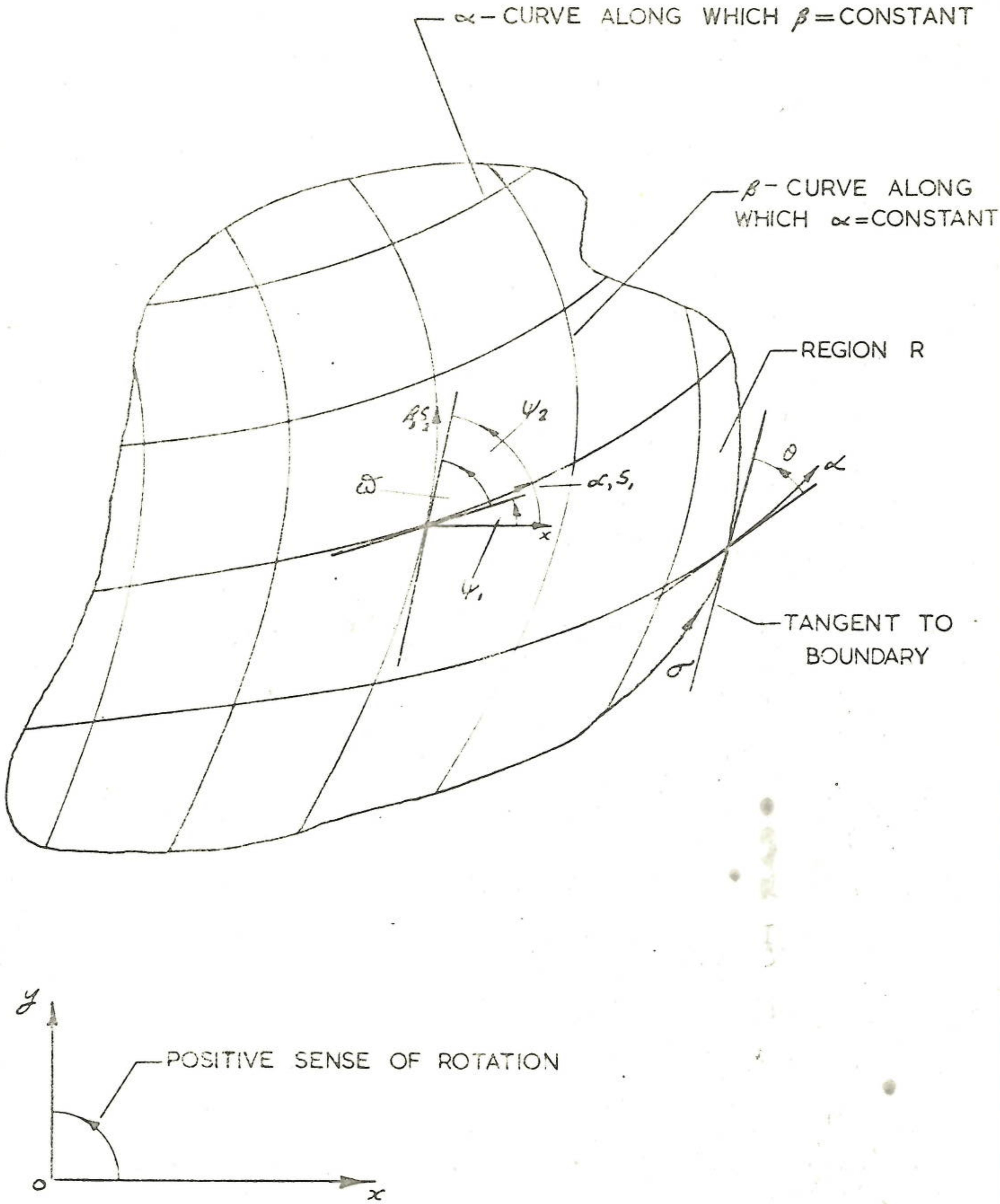

∝ – CURVE ALONG WHICH β = CONSTANT
β – CURVE ALONG WHICH ∝ = CONSTANT
REGION R
ψ₂
θ
∝
ϖ
∝, s₁
x
ψ₁
TANGENT TO BOUNDARY
σ
y
POSITIVE SENSE OF ROTATION
O
x

FIG A.1

Appendix B - Deformation and Equilibrium Conditions in Orthogonal Curvilinear Coordinates

Consider a two-dimensional plane structure referred to orthogonal curvilinear coordinates (α,β). The condition of orthogonality means that we must write

$$\varpi = \pi/2 \qquad \ldots (B.1)$$

in all the formulae of Appendix A.

Let the vector displacement at any point (α,β) be resolved into orthogonal components (u,v) parallel to the positive directions of the tangents to the α and β coordinate curves. The components of direct strain $e_{\alpha\alpha}$, $e_{\beta\beta}$ in the directions of these tangents and the half shear strain $e_{\alpha\beta}$ associated with these directions can be calculated in the usual way using Fig.B.1. We find, using (A.16),

$$\left.\begin{aligned} e_{\alpha\alpha} &= \frac{1}{A}\frac{\partial u}{\partial\alpha} - \frac{v}{A}\frac{\partial\psi_1}{\partial\alpha} = \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{v}{AB}\frac{\partial A}{\partial\beta} \\ e_{\beta\beta} &= \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{u}{B}\frac{\partial\psi_2}{\partial\beta} = \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{u}{AB}\frac{\partial B}{\partial\alpha} \\ e_{\alpha\beta} &= \frac{1}{2}\left\{\frac{1}{A}\frac{\partial v}{\partial\alpha} + \frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{u}{A}\frac{\partial\psi_1}{\partial\alpha} - \frac{v}{B}\frac{\partial\psi_2}{\partial\beta}\right\} \\ &= \frac{1}{2}\left\{\frac{B}{A}\frac{\partial}{\partial\alpha}(v/B) + \frac{A}{B}\frac{\partial}{\partial\beta}(u/A)\right\} \end{aligned}\right\} \qquad \ldots (B.2)$$

If we continue to use the coordinates (α,β) to describe the deformed state, we must make certain changes to our metric. The quantities A,B must be replaced by $A(1 + e_{\alpha\alpha})$ and $B(1 + e_{\beta\beta})$ respectively, since ds_1 and ds_2 of (A.5), (A.8) are increased by the factors $(1 + e_{\alpha\alpha})$, $(1 + e_{\beta\beta})$, and $\varpi = \pi/2$ must be replaced by $\varpi = \pi/2 - 2e_{\alpha\beta}$, by the definition of shear strain.

If these new values be substituted in (A.17) and the resulting formula developed correctly to the first order in the strains, we find, cancelling the finite part using (A.17) with $\varpi = \pi/2$, that

$$\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial e_{\beta\beta}}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial e_{\alpha\alpha}}{\partial\beta}\right) - 2\,\frac{\partial^2 e_{\alpha\beta}}{\partial\alpha\partial\beta}$$

$$- \frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\frac{\partial B}{\partial\alpha}\left(e_{\alpha\alpha} - e_{\beta\beta}\right)\right\} + \frac{\partial}{\partial\beta}\left\{\frac{1}{B}\frac{\partial A}{\partial\beta}\left(e_{\alpha\alpha} - e_{\beta\beta}\right)\right\}$$

$$- 2\,\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial A}{\partial\beta}\,e_{\alpha\beta}\right) - 2\,\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial B}{\partial\alpha}\,e_{\alpha\beta}\right) = 0 \qquad \ldots \text{(B.3)}$$

This is the equation of compatibility of strain in our present coordinate system.

Let our structure carry stress resultants T_1, T_2 and S, where T_1 is the direct load per unit length parallel to the α-curve, T_2 that parallel to the β-curve and S the shear per unit length. The differential equation of equilibrium can be set up by considering an element $d\alpha$ by $d\beta$ as in Fig.B.2. We find,

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha}(B\,T_1) + \frac{\partial}{\partial\beta}(A\,S) + \frac{\partial A}{\partial\beta}\,S - \frac{\partial B}{\partial\alpha}\,T_2 &= 0 \\ \frac{\partial}{\partial\alpha}(B\,S) + \frac{\partial}{\partial\beta}(A\,T_2) - \frac{\partial A}{\partial\beta}\,T_1 + \frac{\partial B}{\partial\alpha}\,S &= 0 \end{aligned}\right\} \qquad \ldots \text{(B.4)}$$

where (A.16) has been used to eliminate ψ_1 and ψ_2. Fig.B.2 also verifies the equality of complementary shears per unit length.

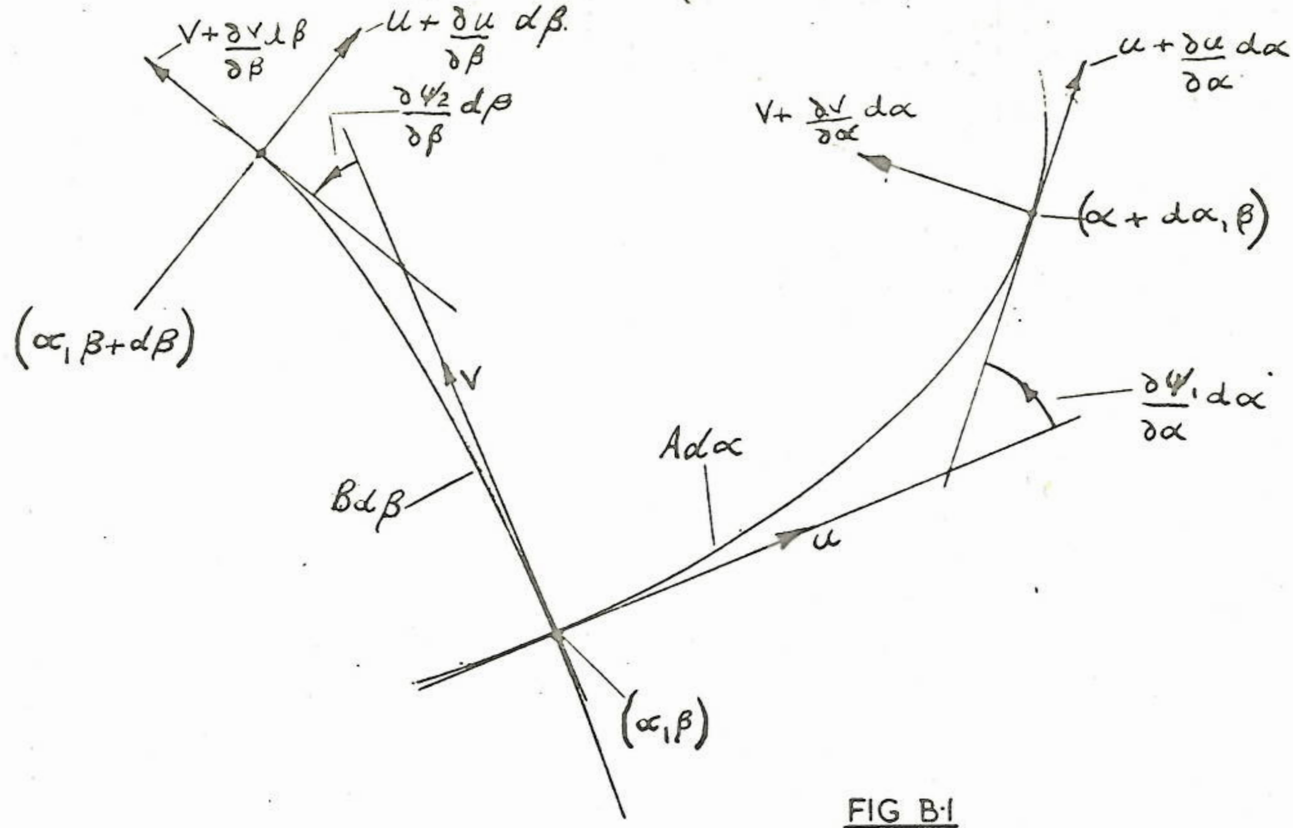
The equilibrium conditions at a boundary can be written using Fig.B.3. The angle between the α-curve and the tangent to the boundary is θ. The direction cosines of the normal to the boundary referred to the local directions of α and β at a point σ on the boundary are then $(\sin\theta, -\cos\theta)$. Hence if F_n, F_t are the components of external traction per unit length in the directions of the normal and tangent respectively and if T is the end load in an edge member at the boundary, which contributes components

$$- T/\rho \text{ and } \frac{\partial T}{\partial \sigma} \quad \text{to } F_n, F_t$$

respectively, we can write

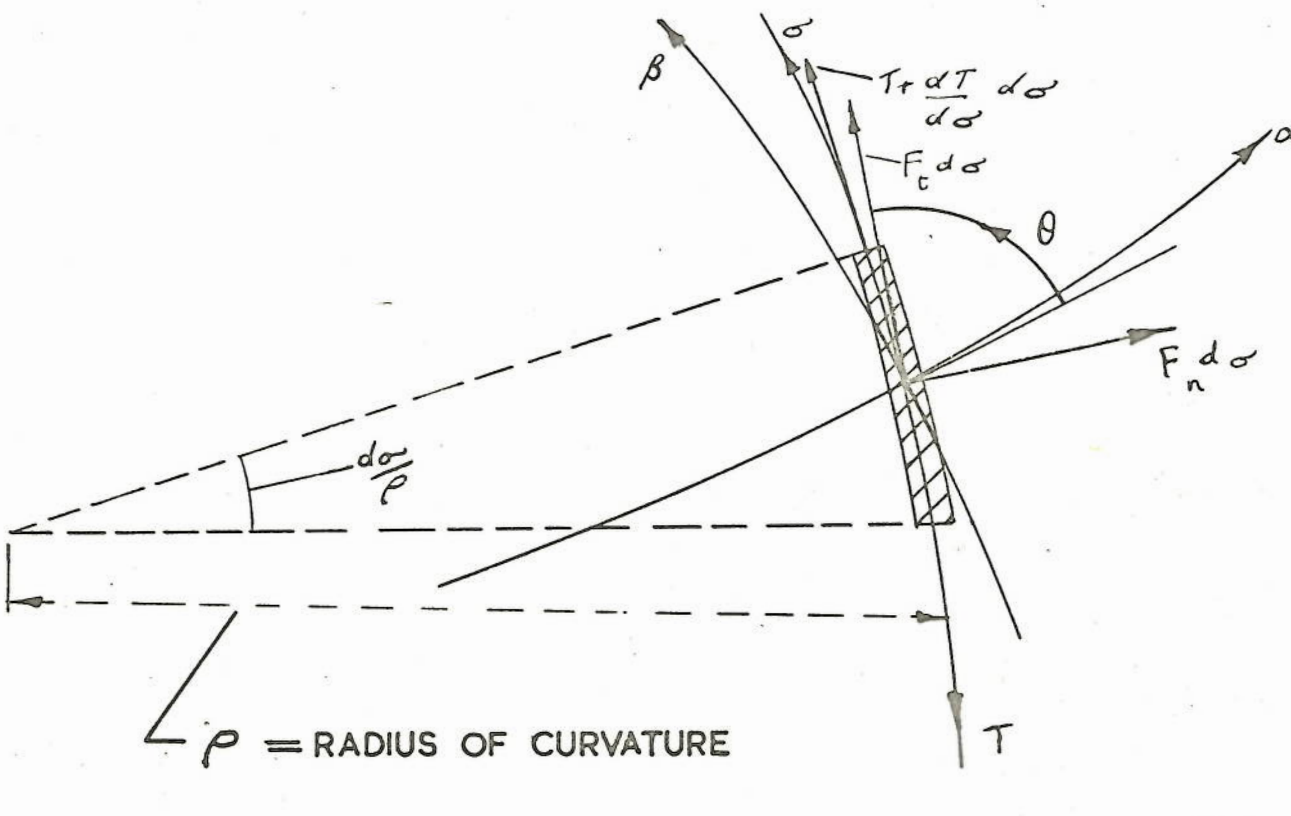
$$\left.\begin{aligned} T_1\sin\theta - S\cos\theta &= (F_n - T/\rho)\sin\theta + (F_t + \frac{dT}{d\sigma})\cos\theta \\ S\sin\theta - T_2\cos\theta &= -(F_n - T/\rho)\cos\theta + (F_t + \frac{dT}{d\sigma})\sin\theta \end{aligned}\right\} \quad \ldots \text{(B.5)}$$

The arc length σ and the angle θ were introduced in equations (A.20), (A.21) and (A.22). The radius of curvature ρ is defined in Fig. B.3.



$V + \frac{\partial V}{\partial \beta} d\beta$
$u + \frac{\partial u}{\partial \beta} d\beta$
$\frac{\partial \psi_2}{\partial \beta} d\beta$
$u + \frac{\partial u}{\partial \alpha} d\alpha$
$V + \frac{\partial V}{\partial \alpha} d\alpha$
$(\alpha + d\alpha, \beta)$
$(\alpha, \beta + d\beta)$
V
$\frac{\partial \psi_1}{\partial \alpha} d\alpha$
$A d\alpha$
$B d\beta$
u
(α, β)

FIG B·1



β
σ
$T + \frac{dT}{d\sigma} d\sigma$
$F_t d\sigma$
α
θ
$F_n d\sigma$
$\frac{d\sigma}{\rho}$
ρ = RADIUS OF CURVATURE
T

FIG B·3

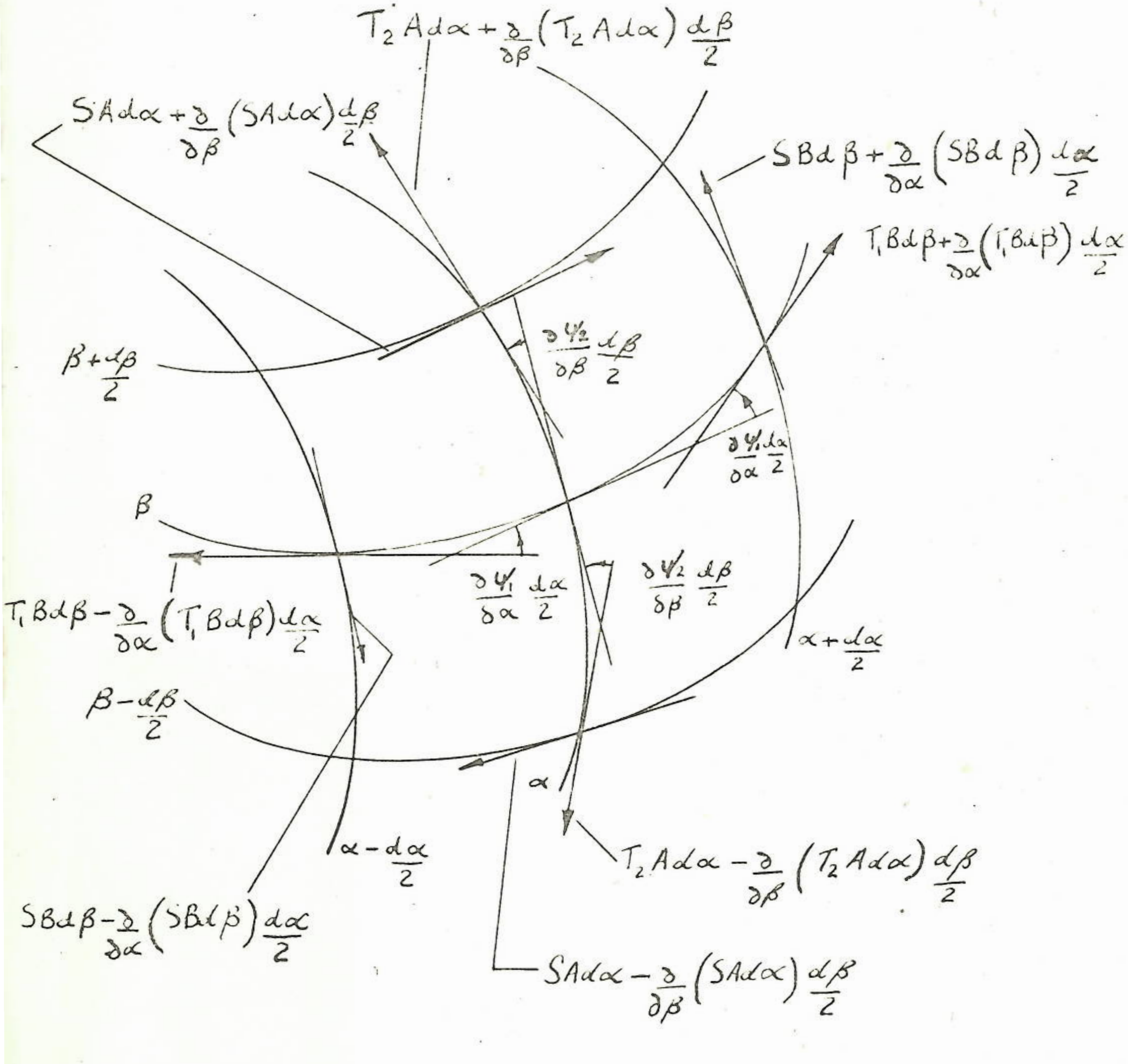

$T_2 A d\alpha + \frac{\partial}{\partial\beta}(T_2 A d\alpha)\frac{d\beta}{2}$
$SA d\alpha + \frac{\partial}{\partial\beta}(SA d\alpha)\frac{d\beta}{2}$
$SB d\beta + \frac{\partial}{\partial\alpha}(SB d\beta)\frac{d\alpha}{2}$
$T_1 B d\beta + \frac{\partial}{\partial\alpha}(T_1 B d\beta)\frac{d\alpha}{2}$
$\beta + \frac{d\beta}{2}$
$\frac{\partial\psi_2}{\partial\beta}\frac{d\beta}{2}$
$\frac{\partial\psi_1}{\partial\alpha}\frac{d\alpha}{2}$
β
$T_1 B d\beta - \frac{\partial}{\partial\alpha}(T_1 B d\beta)\frac{d\alpha}{2}$
$\frac{\partial\psi_1}{\partial\alpha}\frac{d\alpha}{2}$
$\frac{\partial\psi_2}{\partial\beta}\frac{d\beta}{2}$
$\alpha + \frac{d\alpha}{2}$
$\beta - \frac{d\beta}{2}$
α
$\alpha - \frac{d\alpha}{2}$
$T_2 A d\alpha - \frac{\partial}{\partial\beta}(T_2 A d\alpha)\frac{d\beta}{2}$
$SB d\beta - \frac{\partial}{\partial\alpha}(SB d\beta)\frac{d\alpha}{2}$
$SA d\alpha - \frac{\partial}{\partial\beta}(SA d\alpha)\frac{d\beta}{2}$

FIG B·2