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The Calculation of the Profile Drag of Aerofoils and Bodies of Revolution at Supersonic Speeds

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SUMMARY

The effects of viscosity on the aerodynamic characteristics of wings and bodies at supersonic speeds can be assessed if we can calculate (a) the development of the boundary layers in the laminar and turbulent states, (b) the interaction of the boundary layers and main stream away from the neighbourhood of shock waves, (c) the effects of shock wave-boundary layer interaction. Comprehensive methods are developed and discussed for dealing with problems (a) and (b), problem (c) is discussed in the light of existing experimental data but more systematic data are required before quantitative prediction of shock wave-boundary layer interaction effects in any particular case can be confidently made. Fortunately, for many practical cases of interest these latter effects are small.

The detailed results of calculations made on the lines described in this paper for a wide range of aerofoil thickness, body fineness ratio, Reynolds number, Mach number and transition position will be given in a subsequent report.

MEP

NOTATION

x	distance measured along the surface (or meridian profile)
y	distance measured normal to the surface
r	radial distance from axis in axi-symmetric flow
r_o	radius of cross-section of axi-symmetric body
ρ	density
μ	coefficient of viscosity
T	temperature
ω	exponent in viscosity - temperature relation (i.e. $\mu \propto T^{\omega}$)
u	velocity component in x-direction
v	velocity component in y-direction
k	coefficient of heat conduction
c_p	coefficient of specific heat at constant pressure
c_v	coefficient of specific heat at constant volume
γ	c_p/c_v
σ	$\mu c_p/k$ (Prandtl number)
M	Mach number
δ	boundary layer thickness
θ	momentum thickness $= \int_o^{\delta} \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right).dy$, in two dimensions
	$= \int_o^{\delta}\left(1 + \frac{y}{r_o}.\cos\vartheta\right)\frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right)dy$, in axi-symmetric flow
δ^*	displacement thickness
	$= \int_o^{\delta}\left(1 - \frac{\rho u}{\rho_1 u_1}\right).dy$, in two dimensions
	$= \int_o^{\delta}\left(1 + \frac{y}{r_o}.\cos\vartheta\right)\left(1 - \frac{\rho u}{\rho_1 u_1}\right).dy$, in axi-symmetric flow
H	δ^*/θ
ϑ	angle between tangent to meridian profile and axis

/c_f ...

NOTATION

(Contd.)

c_f	local skin friction coefficient
C_F	overall skin friction coefficient
c	wing chord
τ	frictional stress
ℓ	standard length
R	Reynolds number based on ℓ
R_x	Reynolds number based on x.
A_δ	boundary layer area $= \int_0^\delta 2\pi\,(r_o + y\cos\vartheta)\,dy$ (axi-symmetric flow)
A_{δ^*}	displacement area $= 2\pi\, r_o . \delta^*$ (axi-symmetric flow)
f, g	functions defined by equation 2
Λ	function defined by equation 3
$h(M_1, R_o)$	function defined by equation 7
$C_1, C_2, \&\ n$	constants (see equations 5, 6 and 7)
$F(x), G(x)$	functions defined by equation 8
d	a reference length.

Suffix 1 refers to quantities measured at the outer edge of the boundary layer, suffix w to quantities at the surface, suffix o to quantities measured at some reference station e.g. the undisturbed stream or just aft of the leading edge shock, suffix i to incompressible flow, suffixes p and a to two dimensional and axi-symmetric flow respectively, and a dash denotes differentiation with respect to x.

1. Introduction

It is now well established that the concept of the boundary layer is qualitatively as valid for high speed flow as for low speed flow. However, to reach the point at which we can begin to estimate the drag, lift and pitching moments of an aerofoil or body at supersonic speeds allowing for the effects of viscosity, we must clarify as far as possible our ideas on:-

(a) The development of the boundary layer in both the laminar and turbulent states.

(b) The interaction of boundary layer and main stream away from the neighbourhood of shock waves.

(c) The nature and effects of boundary layer - shock wave interaction.

Problem (a) for the laminar boundary layer can now be regarded as solved for practical purposes, for the turbulent boundary layer the position is less certain but it is possible to suggest acceptable lines of attack. Problem (b) can on plausible if not mathematically rigid grounds be reduced to the familiar problem of determining the effective displacement of the surface which can be related to the displacement thickness of the boundary layer. Our knowledge on problem (c) is far from complete, but some data are available from which we can draw useful if interim deductions.

Thus, it will be seen that though we have not yet available a complete theory on which to base the estimation of viscous effects at high speeds, we have arrived at an interesting stage at which a semi-empirical attack can be developed on some of the problems of major practical interest.

This note is concerned with describing methods that have been adopted for the calculation of the profile drag of aerofoil sections and bodies of revolution. The results of such calculations covering comprehensive ranges of Mach number, Reynolds number, transition position and thickness or fineness ratio will be given in a subsequent report.

It should be noted that the term profile drag is here used to denote the drag arising from the viscosity of the medium and is a combination of the skin friction drag and the form drag. The latter at supersonic speeds is here defined as the change in wave drag due to the effective modification of the wing or body shape caused by the boundary layer. The inviscid wave drag, contrary to the practice adopted by some authors, is here excluded

/from the ...

from the definition of profile drag. It may be remarked that the profile drag at supersonic speeds is not determinable, as it is at low speeds, from the momentum loss in the viscous wake, since this loss includes in part the wave drag which is also manifest in a loss of momentum downstream of the wing or body. The profile drag must therefore be determined directly as the sum of the separately determined skin friction and form drags. The following paragraph will discuss in this context the problems (a), (b) and (c) above and our methods of dealing with them.

2. The development of the boundary layer

2.1. Two dimensions

It will be appreciated that for the purpose of estimating profile drag we require a method of following the development of the boundary layer which must be reliable as far as providing assessments of overall characteristics are concerned, e.g. momentum thickness and skin friction, but which need not provide characteristics of the boundary layer in detail. For the laminar layer various methods are available for this purpose, the one that has been adopted here is that described in Ref. 1, because of its relative simplicity and of the fact that it is not restricted to any particular values of the Prandtl number (σ) or the exponent of the viscosity temperature relation (ω). It requires a simple graphical or numerical integration to determine the momentum thickness at any point, the formula being

$$\left[\rho_1^2\ \theta^2\right]_{x_1} = \frac{4\ \mu_o}{f\ u_1 g} \int_o^{x_1} \rho_1\ u_1^{\ g-1} dx \qquad \ldots\ldots\ldots\ldots (1)$$

where x is the distance measured along the surface, ρ_1 and u_1 are the local values of the density and velocity, respectively, just outside the boundary layer, θ is the momentum thickness of the boundary layer, f and g are functions of a reference Mach number M_o, e.g. the undisturbed stream Mach number or the Mach number just aft of the leading edge shock, and are given by

$$\left.\begin{aligned} f &= 9.072\left[1 + 0.365(\gamma-1)\sigma^{\frac{1}{2}}\ M_o^2\right]^{1-\omega} \\ g &= 9.18 + 1.436\ M_o^2 - \frac{f}{3}\left[1 + \frac{(\gamma-1)}{2}\ \sigma^{\frac{1}{2}}\ M_o^2\right]^{\omega} \end{aligned}\right\} \qquad \ldots . (2)$$

and μ_o is the viscosity corresponding to the reference conditions.

The skin friction coefficient, $c_f = \dfrac{2\tau_w}{\rho_o u_o^2}$, where τ_w

/is the ...

is the local intensity of skin friction, is given by (see Ref.1)

$$c_f = \frac{(\Lambda + 12)}{3R\, f} \frac{u_1}{u_o} \cdot \frac{\ell}{\theta}, \qquad \text{.............(3)}$$

where

$$\Lambda = R \left[\frac{du_1}{dx} \frac{\theta^2 f^2}{u_o \cdot \ell} \frac{\rho_1}{\rho_o} \frac{\mu_w}{\mu_o} \right]$$

ℓ is a standard reference length, e.g. the wing chord,

$R = \frac{u_o \ell \rho_o}{\mu_o}$, i.e. the reference Reynolds number, and

μ_w is the value of μ at the surface and is given by

$$\frac{\mu_w}{\mu_o} = \left\{ 1 + \frac{(\gamma-1)}{2} M_o^2 \left[1 + \frac{u_1^2}{u_o^2} (\sigma^{\frac{1}{2}} - 1) \right] \right\}^{\omega}.$$

If we express all quantities non-dimensionally in terms of their reference condition values (suffix o) then equation (1) becomes

$$\left[\rho_1^2 \, \theta^2 \right]_{x_1} = \frac{4}{Rf \, u_1{}^g} \int_o^{x_1} \rho_1 u_1{}^{g-1} dx \qquad \text{............(1A)}$$

and equation 3 becomes

$$c_f = \frac{(\Lambda + 12) \cdot u_1}{3\, R\, f\, \theta}, \qquad \text{............(3A)}$$

where

$$\Lambda = R \left[u_1' \, \theta^2 \, f^2 \, \rho_1 \mu_w \right]$$

and

$$\mu_w = \left\{ 1 + \frac{(\gamma-1)}{2} M_o^2 \left[1 + u_1^2 \, (\sigma^{\frac{1}{2}} - 1) \right] \right\}^{\omega}.$$

Knowing the distribution of c_f along a surface to the transition point we can then readily determine by a simple integration the contribution of the laminar layer to the skin friction drag.

As in incompressible flow, θ is assumed to be continuous at the transition point, since otherwise a discontinuity in θ there would imply an infinite local viscous stress.

For the turbulent boundary layer the following approach has been adopted. The momentum equation of the boundary layer[2] can be written

$$\theta' + \theta \left\{ \frac{u_1'}{u_1} (H + 2) + \frac{\rho_1'}{\rho_1} \right\} = \frac{\tau_w}{\rho_1 u_1^2}, \qquad \text{............(4)}$$

/where the ...

where the dash denotes differentiation with respect to x, and H is the ratio of the displacement thickness of the boundary layer (δ^*) to the momentum thickness (θ). To solve this equation we require to know how H is varying with x and we require a further relation between τ_w and θ. Now for incompressible flow past a flat plate at zero incidence the velocity power law leads to the relations

$$\left.\begin{aligned} \frac{\tau_w}{\rho_1 u_1^2} &= C_1 R_x^{-1/n}, \\ \text{and} \quad \frac{u_1\theta}{\nu_1} &= \frac{n}{n-1} \cdot C_1 (R_x)^{\frac{n-1}{n}} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(5)$$

where C_1 and n are related constants, and $R_x = \dfrac{u_1 x}{\nu_1}$.
From these relations it follows that

$$\frac{\tau_w}{\rho_1 u_1^2} = C_2 \left(\frac{u_1\theta}{\nu_1}\right)^{-\frac{1}{n-1}}, \qquad \ldots\ldots\ldots\ldots(6)$$

where $C_2 = C_1^{\frac{n}{n-1}} \left(\dfrac{n-1}{n}\right)^{-\frac{1}{n-1}}$.

We now generalise equation (6) for compressible flow by assuming that

$$\frac{\tau_w}{\rho_1 u_1^2} = C_2 \left(\frac{u_1\theta}{\nu_1}\right)^{-\frac{1}{n-1}} \cdot h(M_1) \qquad \ldots\ldots\ldots\ldots(7)$$

where $h(M_1)$ is an as yet unspecified function of M_1. Further, we assume equation (7) holds locally on the surface of the aerofoil, so that (4) becomes

$$\theta' + \theta\left\{\frac{u_1'}{u_1}(H+2) + \frac{\rho_1'}{\rho_1}\right\} = C_2\left(\frac{u_1\theta}{\nu_1}\right)^{-\frac{1}{n-1}} h(M_1).$$

If we express all quantities non-dimensionally in terms of their reference condition values (suffix o), then this equation becomes

$$\theta' + \theta\left\{\frac{u_1'}{u_1}(H+2) + \frac{\rho_1'}{\rho_1}\right\} = C_2 R^{-\frac{1}{n-1}} \left(\frac{u_1\theta}{\nu_1}\right)^{-\frac{1}{n-1}} h(M_1).$$

Assuming isentropic conditions outside the boundary layer we can readily show that

$$\frac{\rho_1'}{\rho_1} = -M_1^2 \frac{u_1'}{u_1},$$

and hence this equation can be written in the form

$$\theta' + \theta . F(x) = C_2 R^{-\frac{1}{n-1}} \theta^{-\frac{1}{n-1}} G(x)$$

where $$F(x) = \frac{u_1'}{u_1}\left\{(H+2) - M_1^2\right\} \qquad \text{............(8)}$$

and $$G(x) = \left(\frac{u_1}{\nu_1}\right)^{-\frac{1}{n-1}} h(M_1) .$$

Equation (8) can be integrated to give

$$\left(\theta^{\frac{n}{n-1}}\right)_{x_1}\left[\exp. \int_T^{x_1} F(x). \frac{n}{n-1} . dx\right] - \theta_T^{\frac{n}{n-1}}$$

$$= \frac{n}{n-1} C_2 R^{-\frac{1}{n-1}} \int_T^{x_1} G(x). \exp.\left[\int_T^{x} F(x). \frac{n}{n-1} . dx\right]. dx \ (9)$$

where suffix T refers to the transition point. Our problem is then effectively solved when we have determined the function $h(M_1)$, the relation between H and x and have decided on the values of the constant C_2 and n.

To determine $h(M_1)$, we note that for fully turbulent flow along a flat plate at zero incidence $F(x) = 0$, and $G(x) = h(M_1)$, and hence equation (9) yields

$$\theta^{\frac{n}{n-1}} = \frac{n}{n-1} . C_2 R^{-\frac{1}{n-1}} h(M_1). x ,$$

from which it follows that

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{d\theta}{dx} = \left(\frac{n-1}{n}\right)\left[\frac{n}{n-1} C_2\right]^{\frac{n-1}{n}} . R_x^{-1/n}\left[h(M_1)\right]^{\frac{n-1}{n}}$$

$$= C_1 . R_x^{-1/n}\left[h(M_1)\right]^{\frac{n-1}{n}}$$

$$= \left(\frac{\tau_w}{\rho_1 u_1^2}\right)_i \left[h(M_1)\right]^{\frac{n-1}{n}} , \qquad \text{............(10)}$$

where $\left(\frac{\tau_w}{\rho_1 u_1^2}\right)_i$ is the value of $\frac{\tau_w}{\rho_1 u_1^2}$ in incompressible flow (see equation 5) at the same value of R_x. Thus

$$\left.\begin{aligned} h(M_1) &= \left[\left(\frac{\tau_w}{\rho_1 u_1^2}\right) \Big/ \left(\frac{\tau_w}{\rho_1 u_1^2}\right)_i\right]^{\frac{n}{n-1}} , \\ &= \left[c_f / c_{fi}\right]^{\frac{n}{n-1}} . \end{aligned}\right\} \qquad \text{............(11)}$$

/We require ...

We require therefore data on the ratio c_f/c_{fi} on a flat plate and its variation with Mach number. The available experimental evidence is scanty and is not as consistent as one would wish. Perhaps the most reliable data are those provided by Coles[3], who has determined local skin friction coefficients on a flat plate by a direct method of force measurement and his results for the ratio c_f/c_{fi} as a function of Mach number M_o are reproduced in Fig. 1. These results were obtained at a Reynolds number of 8×10^6. Other experimental results showing a similar fall of c_f/c_{fi} with Mach number, although not fully agreeing quantitatively with Coles' results, have been reported by Wilson[4] and Eckert[5]. There are various theories which have been developed from which values for c_f/c_{fi} may be derived, but all such theories are extrapolations from incompressible flow theory and the results depend critically on the assumptions underlying the mode of extrapolation. However, the theory that gives closest agreement with the results of Coles is that developed by Cope[6] based on an extrapolation of the familiar 'log' law of incompressible flow. Monaghan[7], following what are essentially the same lines of argument as Cope, obtained the following interesting result. Write C_{FW} for the overall skin friction coefficient based on the density at the surface (ρ_w), i.e.

$$C_{FW} = \frac{\int_o^c \tau_w \, dx}{\frac{1}{2}\rho_w u_1^2} \text{ , where } c \text{ is the chord length ,}$$

and define a Reynolds number $R_w = \dfrac{u_1 c \rho_w}{\mu_w} \cdot \dfrac{T_1}{T_w} = R\left(\dfrac{\mu_1}{\mu_w}\right)\left(\dfrac{\rho_w}{\rho_1}\right)\dfrac{T_1}{T_w}$

where T_1 is the temperature just outside the boundary layer and T_w is the temperature at the wall. Then Monaghan's assumptions led him to deduce that the relation between C_{Fw} and R_w is the same as that between C_F and R in incompressible flow. Similarly, the relation between the correspondingly defined local values of the skin friction coefficient c_{fw} and the local Reynolds number R_{xw} is also the same as that between c_f and R_x in incompressible flow. Corresponding to the accepted empirical incompressible flow relation between C_F and R, due to Prandtl and Schlichting[8], viz.

$$C_F = 0.455 / (\log R)^{2.58} \qquad \text{............(12)}$$

Schlichting[9] has also deduced the relation

$$c_f = (2 \log R_x - 0.65)^{2.3} \qquad \text{............(13)}$$

/If we ...

If we accept Monaghan's result, then we have that for compressible flow

$$c_{fw} = (2 \log R_{xw} - 0.65)^{-2.3}$$

and hence

$$c_f \cdot \frac{\rho_1}{\rho_w} = \left\{ 2 \log \left[R_x \cdot \frac{\mu_1}{\mu_w} \left(\frac{\rho_w}{\rho_1}\right) \frac{T_1}{T_w} \right] - 0.65 \right\}^{-2.3}$$

Using the gas law and the relation $\mu \propto T^{\omega}$ we therefore have

$$c_f = \frac{T_1}{T_w} \left[2 \log R_x - 0.65 + 2(2+\omega) \log \left(\frac{T_1}{T_w}\right) \right]^{-2.3}.$$

Hence

$$\frac{c_f}{c_{fi}} = \frac{T_1}{T_w} \left\{ \frac{2 \log R_x - 0.65}{2 \log R_x - 0.65 + 2(2+\omega) \log \frac{T_1}{T_w}} \right\}^{2.3} \quad \text{...(14)}$$

Similarly, from equation (10) we deduce that

$$\frac{C_F}{C_{Fi}} = \frac{T_1}{T_w} \left\{ \frac{\log R}{\log R + (2+\omega) \log \frac{T_1}{T_w}} \right\}^{2.58} \quad \text{............(15)}$$

We are confining ourselves to the case of zero heat transfer, and for our purposes there can be little error in using Squire's[10] relation for the ratio T_1/T_w, viz.

$$\frac{T_1}{T_w} = \left[1 + \frac{(\gamma - 1)}{2} M_1^2 \sigma^{1/3} \right]^{-1} \quad \text{............(16)}$$

In Fig. 1 the relation given by equation (14) for $R_x = 8 \times 10^6$, $\sigma = 0.72$ and $\omega = 8/9$ is compared with the experimental results of Coles[3] and it will be seen that they agree remarkably well. This leads us then to look with some confidence to the relations given by equations (12) and (13) to enable us to deduce the function $h(M_1)$.

At first sight it would seem that equations (11) and (14) provide the answer, but it will be noted that they lead to the result that $h(M_1)$ is a slowly varying function of R_x and therefore of x, and the possibility of any variation of $h(M_1)$ with x on a flat plate was ignored in the derivation of equation (11). If we allow for this possibility and write $h(M_1, x)$ instead of $h(M_1)$, then revising the argument that led to equation (10) we see that

$$\theta^{\frac{n}{n-1}} = \frac{n}{n-1} C_2 R^{-\frac{1}{n-1}} \int_0^x h(M_1, x) \,.\, dx$$

$$= \frac{n}{n-1} C_2 R^{-\frac{1}{n-1}} \,.\, x \,.\, \overline{h(M_1, x)}, \qquad \text{/where ...}$$

where $\overline{h(M_1,x)}$ is a mean value of $h(M_1,x)$ from the leading edge to the position x. Hence, putting $x = 1$, the non-dimensional chord length, we have at the trailing edge

$$\theta^{\frac{n}{n-1}} = \left[\frac{n}{n-1} \cdot C_2 R^{-\frac{1}{n-1}}\right] \overline{h(M_1,1)}$$

But $\theta = 2 C_F$, and $\left[\frac{n}{n-1} \cdot C_2 R^{-\frac{1}{n-1}}\right]^{\frac{n-1}{n}} = 2 C_{Fi}$,

and therefore

$$\overline{h(M_1,1)} = (C_F / C_{Fi})^{\frac{n}{n-1}} \quad \ldots\ldots\ldots\ldots (17)$$

If we are to ignore the slow variation of $h(M_1)$ with R_x in our general method of profile drag calculation, then it seems logical to take a mean value averaged over the whole chord. Consequently, we propose to use equation (17) to determine $h(M_1)$ and not equation (11), i.e. we shall take

$$h(M_1) = (C_F / C_{Fi})^{\frac{n}{n-1}} \quad \ldots\ldots\ldots\ldots (18)$$

the ratio (C_F / C_{Fi}) being assumed to be given for each Reynolds number considered by equation (15). This ratio is shown plotted in Fig. 2 for $R = 10^6$, 10^7 and 10^8, and the validity of equation (18) can to some extent be justified a posteriori by the relatively small differences between the three curves shown.

For the variation of H with x we assumed that H is a function of M_1 only, the relation being the same as that for flow past a flat plate at zero incidence and Mach number M_1. Making the plausible assumption that the total energy is constant across the boundary layer Cope[6] has evaluated the relations between H and M_1 for boundary layer velocity distributions following various power laws. These relations are reproduced in Fig. 3 and it will be seen that H is not particularly sensitive to the power law assumed, and for the purposes of this investigation it is probably sufficiently accurate to take the relation appropriate to the 1/9th power law, as this is consistent with the value of n equal to 6, which for reasons described in the next paragraph was the value chosen.

In order to choose suitable values of n and C_2, values were sought that gave the best fit with the accepted empirical incompressible overall skin friction coefficient formula for a flat plate at zero incidence with fully turbulent boundary layer viz.

$$C_F = 0.455 / \left[\log_{10} R\right]^{-2.58}, \quad \ldots\ldots\ldots\ldots (12)$$

/over the ...

over the range of Reynolds number considered which was 10^6 to 10^8. It is clearly possible to choose values of n and C_2 to give a close fit with this formula over small specified ranges of Reynolds number, allowing the values of n and C_2 to change from one range to the next. However, it was found that by taking $n = 6$, and $C_2 = 0.00878$, which lead in the incompressible flow case to the relation

$$C_F = 0.0450\, R^{-1/6} \qquad \ldots\ldots\ldots\ldots(19)$$

agreement to within 0.0001 was obtained with the relation given in equation (12) over the Reynolds number range considered. This is illustrated in Fig. 4 where the two relations (equations 12 and 19) are compared. It is doubtful whether these values of n and C_2 could be applied much outside the Reynolds number range considered without exceeding the above order of error, and were it decided to extend the range of Reynolds number then other and more suitable values of n and C_2 would be required for the extensions.

Having determined θ as a function of x we can obtain the skin friction distribution making use of equation (7), thus:-

$$\left.\begin{aligned} c_f = \frac{2\tau_w}{\rho_1 u_1^2} &= 2\, C_2\, R^{-\frac{1}{n-1}} \cdot \rho_1 u_1^2 \left[\frac{u_1 \theta}{\nu_1}\right]^{-\frac{1}{n-1}} \cdot h(M_1) \\ &= .01756\ \rho_1 u_1^2 \left[\frac{u_1 \theta\, R}{\nu_1}\right]^{-\frac{1}{n-1}} h(M_1) \end{aligned}\right\} \ldots\ldots\ldots(20)$$

We note that the displacement thickness δ^* is given by

$$\delta^* = H.\theta \qquad \ldots\ldots\ldots\ldots(21)$$

2.2. Axi-symmetric flow

Mangler[11] has demonstrated that for flow in the laminar boundary layer on a body of revolution a transformation exists which will correlate the flow with that in a laminar boundary layer in two dimensions. The proviso is made that the body is sufficiently slender for the boundary layer thickness to be small compared with r_o, the radius of cross section of the body. The following discussion reproduces his results but involves a different approach which has some intrinsic interest.

The momentum equations of the boundary layer in two dimensional

/and in ...

and in axi-symmetric flow are.-

$$\theta_p' + \theta_p\left[(H_p+2)\ \frac{u_1'}{u_1} + \frac{\rho_1'}{\rho_1}\right] = \frac{\tau_{wp}}{\rho_1 u_1^2} \qquad \ldots\ldots\ldots\ldots(22)$$

and

$$\theta_a' + \theta_a\left[(H_a+2)\ \frac{u_1'}{u_1} + \frac{\rho_1'}{\rho_1} + \frac{r_o'}{r_o}\right] = \frac{\tau_{wa}}{\rho_1 u_1^2}\ , \qquad \ldots\ldots\ldots(23)$$

where suffix p refers to two dimensional flow and suffix a to axi-symmetric flow. It should be noted that in axi-symmetric flow the displacement and momentum thicknesses are defined by

$$\left.\begin{aligned} \delta_a^* &= \int_o^\delta \left(1 + \frac{y}{r_o}\cdot\cos\vartheta\right)\left(1 - \frac{\rho u}{\rho_1 u_1}\right)\cdot dy \\ \text{and} \qquad \theta_a &= \int_o^\delta \frac{\rho u}{\rho_1 u_1}\left(1 + \frac{y}{r_o}\cdot\cos\vartheta\right)\left(1 - \frac{u}{u_1}\right)\cdot dy \end{aligned}\right\}\ldots.(24)$$

where ϑ is the angle between the tangent to the meridian of the body and the axis; x and y are measured along and normal to the meridian.

It will now be assumed that to every axi-symmetric boundary layer there exists a two dimensional boundary layer such that at corresponding stations the boundary layer velocity profiles are the same except for a change in the scale of y, u_1 being the same. The conditions in the external flows will be assumed to be the same and isentropic, and the plausible assumption will also be made that the relation between temperature (T) and u at corresponding stations in the boundary layer are the same. Consequently, the relations between ρ and u as well as the temperatures and viscosities at the wall (T_w and μ_w) are the same. Accepting the proviso that the boundary layer thickness is small compared with r_o, it follows that the term $\frac{y}{r_o}\cos\vartheta$ can be neglected compared with unity in equation (24) and consequently

$$\frac{\theta_a}{\theta_p} = \frac{\delta_a}{\delta_p}\ ,\ \text{and}\ H_a = H_p\ .$$

But, it follows from the above assumptions that

$$\frac{\tau_{wa}}{\tau_{wp}} = \frac{\left[\mu\ \frac{\partial u}{\partial y}\right]_{wa}}{\left[\mu\ \frac{\partial u}{\partial y}\right]_{wp}}$$

$$= \delta_p/\delta_a\ .$$

/Hence ...

Hence

$$\frac{\tau_{wa}}{\tau_{wp}} = \frac{\theta_p}{\theta_a} . \qquad \ldots\ldots\ldots\ldots(25)$$

Equation (23) can therefore be written

$$(r_o\theta_a)' + (r_o\theta_a)\left[(H_p+2)\frac{u_1'}{u_1} + \frac{\rho_1'}{\rho_1}\right] = \frac{\tau_{wp}\theta_p r_o}{\theta_a . \rho_1 u_1^2} \qquad \ldots\ldots(26)$$

Now, consider the transformation

$$x_p = \frac{1}{d^2}\int_o^{x_a} r_o^2\, dx_a , \qquad \ldots\ldots\ldots\ldots(27)$$

where d is some reference length. Equation (26) can then be written

$$\frac{d}{dx_p}\left[\frac{r_o\theta_a}{d}\right] + \left(\frac{r_o\theta_a}{d}\right)\left[\frac{(H_p+2)}{u_1}\frac{du_1}{dx_p} + \frac{1}{\rho_1}\frac{d\rho_1}{dx_p}\right] = \frac{\theta_p . d}{r_o\theta_a} . \frac{\tau_{wp}}{\rho_1 u_1^2} .$$

$$\ldots\ldots\ldots\ldots(28)$$

Comparing this equation with equation (22) we see that they are identical if

$$\theta_p = \frac{r_o\theta_a}{d} , \qquad \ldots\ldots\ldots\ldots(29)$$

and if x_p is the coordinate parallel to the surface in the two dimensional flow, i.e. corresponding stations are related by the transformation (27). Summarising the argument we see that with the assumption of similar velocity profiles for the flows past a two dimensional and axi-symmetric shape, such that corresponding stations are related by equation (27), and with the same values of the main stream velocity at corresponding points, then the values of the boundary layer momentum thicknesses are related by equation (29). Further, the frictional stresses at the wall at corresponding points are related by (from equations 25 and 29)

$$\frac{\tau_{wa}}{\tau_{wp}} = \frac{\theta_p}{\theta_a} = \frac{r_o}{d} . \qquad \ldots\ldots\ldots\ldots(30)$$

If we define local Reynolds numbers R_{xa} and R_{xp} by

$$R_{xa} = \frac{u_1 x_a \rho_1}{\mu_1} , \quad R_{xp} = \frac{u_1 x_p \rho_1}{\mu_1} ,$$

then

$$\frac{\tau_{wp}}{\tau_{wa}}\left(\frac{R_{xp}}{R_{xa}}\right)^{\frac{1}{2}} = \frac{d}{r_o}\left(\frac{x_p}{x_a}\right)^{\frac{1}{2}} = \left\{\frac{\int_o^{x_a} r_o^2 dx_a}{r_o^2 . x_a}\right\}^{\frac{1}{2}} \ldots\ldots(31)$$

/This relation ...

This relation was derived by Mangler, who then deduced that in the particular case of the flow past a circular cone at zero incidence in an otherwise uniform supersonic stream

$$c_{fp} / c_{fa} = 1 / \sqrt{3} . \qquad \qquad (32)$$

In this case, of course, the corresponding two dimensional flow is that past a flat plate at zero incidence with the same constant pressure, velocity, etc. just outside the boundary layer as that behind the shock wave attached to the cone tip. Equation (32) refers to the local skin friction coefficients at points the same distance downstream from the cone nose and the plate leading edge, in terms of the density and velocity just outside the boundary layer in the two cases.

There is nothing in the above argument that refers explicitly to the laminar boundary layer and it might at first be thought that it could be applied with equal justification to the turbulent boundary layer. We note, however, that the initial assumption of the existence of similar velocity profiles leads to equation (30) viz.

$$\frac{\tau_{wa}}{\tau_{wp}} = \frac{\theta_p}{\theta_a} = \frac{r_o}{d} .$$

Now consider the body of revolution to be a circular cylinder of large radius r_o. Then in the limit as r_o tends to infinity the flow past the body must tend to that past a flat plate, and hence the relation between τ_{wp} and θ_p must tend to that between τ_{wa} and θ_a. But we have accepted for incompressible flow on both theoretical and experimental grounds a relation between τ_{wp} and θ_p of the form given by equation (6), i.e. for the case considered

$$\tau_{wp} = \text{const.} \ \theta_p^{-\frac{1}{n-1}} ,$$

where for laminar flow $n = 2$, but for turbulent flow $n = 6$. Hence for large r_o

$$\tau_{wa} = \text{const.} \ (\theta_a)^{-\frac{1}{n-1}} ,$$

and therefore

$$\frac{\tau_{wp}}{\tau_{wa}} = \left(\frac{\theta_p}{\theta_a}\right)^{-\frac{1}{n-1}} .$$

We see that this relation is not consistent with equation (30)

/unless ...

unless n=2, i.e. unless the boundary layer is laminar. It follows, therefore, that whilst the assumption that similarity of velocity profiles exist between the two dimensional and axi-symmetric flows does not lead to inconsistencies in the case of the laminar boundary layer it does so in the case of the turbulent boundary layer and the argument based on it is then invalid. A different approach must therefore be used when we consider the turbulent boundary layer.

Reverting, however, for the moment to the axi-symmetric laminar boundary layer case and dropping the suffix a, it follows from equations (1A), (27) and (29) that

$$\left[\rho_1\, r_o^2\, \theta^2\right]_{x_1} = \frac{4}{Rfu_1{}^g}\int_o^{x_1} \rho_1 u_1{}^{g-1}\, r_o^2\, dx, \quad \ldots\ldots\ldots (33)$$

where we have put $d = \ell$, the reference length used in R.

Further, from (3A) and (30)

$$c_f = \frac{(\Lambda + 12)u_1}{3\, Rf\, \theta} \quad \ldots\ldots\ldots\ldots (34)$$

where $\Lambda = R\left[u_1'\, \theta^2\, f^2\, \rho_1\, \mu_w\right]$.

In dealing with the turbulent boundary layer, we bear in mind the fact that for large r_o the axi-symmetric case must tend to the two dimensional case. This leads us to make the assumption that equation (7) holds for axi-symmetric flow as well as for two dimensional flow. The momentum equation (equation 23) for axi-symmetric flow can then be written

$$(r_o\theta)' + (r_o\theta)\, F(x) = C_2 R^{-\frac{1}{n-1}}\, (r_o\theta)^{-\frac{1}{n-1}}\, G(x).\, r_o^{\frac{n}{n-1}} \quad \ldots (35)$$

where $F(x)$ and $G(x)$ are the functions already defined in equation (8) when considering the two dimensional case. Like equation (8) equation (35) is readily integrated to give

$$(r_o\theta)_{x_1}^{\frac{n}{n-1}}\left[\exp.\int_T^{x_1} F(x).\, \frac{n}{n-1}.\, dx\right] - (r_o\theta)_T^{\frac{n}{n-1}}$$

$$= \frac{n}{n-1}\, .\, C_2\, R^{-\frac{1}{n-1}}\int_T^{x_1} G(x)\, r_o^{\frac{n}{n-1}}.\, \exp\left[\int_T^{x} F(x).\, \frac{n}{n-1}\, dx\right].\, dx$$

$$\ldots\ldots\ldots\ldots (36)$$

Having determined $r_o\theta$ as a function of x by means of equation (36)

/we can ...

we can obtain the skin friction distribution from equation (7), thus.-

$$\left.\begin{aligned} c_f &= 2\,C_2 R^{-\frac{1}{n-1}} \rho_1 u_1^2 \left(\frac{u_1 r_o \theta}{\nu_1}\right)^{-\frac{1}{n-1}} .\ r_o^{\frac{1}{n-1}} .\ h(M_1) \\ &= .01756\ \rho_1 u_1^2 \left(\frac{u_1 r_o \theta\, R}{\nu_1}\right)^{-\frac{1}{n-1}} .\ r_o^{\frac{1}{n-1}} .\ h(M_1)\ . \end{aligned}\right\} \quad \ldots\ldots(37)$$

3. <u>The interaction of the boundary layer and the external flow</u>

3.1. <u>The effective displacement of the surface due to the boundary layer (not in the neighbourhood of shock waves)</u>

It is usual to assume, as in incompressible flow, that the effect of the boundary layer on the external flow is equivalent to a displacement of the surface equal to the displacement thickness (or area) of the boundary layer. As far as the author is aware, however, there has been as yet no published justification of this assumption for compressible flow. The following discussion follows in essentials the lines of the argument developed for incompressible flow by Preston.[12]

Consider first two dimensional flow. The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0,$$

and hence, integrating with respect to y through the boundary layer, we have

$$\Big[\rho v\Big]_o^{\delta} = -\int_o^{\delta} \frac{\partial}{\partial x}(\rho u).\ dy$$

$$= -\frac{\partial}{\partial x}\int_o^{\delta} \rho u.\ dy + \rho_1 u_1 \frac{d\delta}{dx}$$

$$= \frac{d}{dx}\Big[\rho_1 u_1 (\delta^* - \delta)\Big] + \rho_1 u_1 \frac{d\delta}{dx}$$

Or $$\psi \approx \frac{v_1}{u_1} = \frac{\delta^* - \delta}{\rho_1 u_1} .\ \frac{d}{dx}(\rho_1 u_1) + \frac{d\delta^*}{dx} \quad \ldots\ldots\ldots\ldots(38)$$

where ψ is the angle, assumed small, that the streamline at the outer edge of the boundary layer (i.e. at $y = \delta$) makes with the

/direction $y=0$.

direction y=0.

Now suppose the surface to be displaced outwards by a small distance δ^*, and consider an inviscid flow past this surface, keeping the coordinate system the same as in the original flow. The equation of continuity is unchanged and again, by integrating with respect to y from 0 to δ we have

$$\left[\rho v\right]_\delta = \frac{d}{dx}\left[\int_0^\delta \rho_1 u_1 \left(1 - \frac{\rho u}{\rho_1 u_1}\right) . \, dy - \rho_1 u_1 \delta\right] + \rho_1 u_1 \frac{d\delta}{dx} .$$

But for $y < \delta^*$, $u = 0$, and for $y > \delta^*$, $\rho u = \rho_1 u_1$, hence

$$\left[\rho v\right]_\delta = \frac{d}{dx}\left[\rho_1 u_1 (\delta^* - \delta)\right] + \rho_1 u_1 \frac{d\delta}{dx} ,$$

and therefore

$$\frac{\left[\rho v\right]_\delta}{\rho_1 u_1} = \frac{(\delta^* - \delta)}{\rho_1 u_1} \frac{d}{dx}(\rho_1 u_1) + \frac{d\delta^*}{dx}$$

i.e. the direction of the streamlines at $y = \delta$ in the inviscid flow with the displaced surface is the same as the direction in the actual viscous flow at $y = \delta$ if we can assume $\rho_1 u_1$ is the same in both cases. This fact does not by itself prove that the two flows for $y > \delta$ are identical. However, it follows from the above analysis that any other displacement of the surface other than δ^* leads to flow conditions in the inviscid case at $y = \delta$ differing from those in the actual case. It follows that if there exists an effective displacement of the surface to yield an equivalent boundary in inviscid flow, it must be a displacement equal to the displacement thickness of the boundary layer. A proof that such an effective displacement exists in the general case does not seem easy, but there are ample grounds for accepting the existence of such a displacement, as a working hypothesis, particularly when the equivalence of the boundary layer to an effective source distribution is borne in mind. It should be noted, however, that in the above discussion the usual assumptions of boundary layer theory are implicit, hence the deductions would not necessarily be valid if the associated limitations required on boundary layer thickness and curvature of the streamlines and surface did not apply. Thus, it is doubtful whether a displacement equal to the displacement thickness, as normally defined, will produce an effect on the outside inviscid flow equivalent to that of the boundary layer in regions of high curvature of flow or surface. It is possible, however, that by modifying the

/definition of ...

definition of the displacement thickness on the lines suggested in Ref. 12 it could still be shown to be equal to the effective displacement of the surface in such cases. But regions of high flow curvature on aerofoils or bodies generally result from the influence of shock waves on the boundary layer, and such cases require special treatment.

In the case of axi-symmetric flow we note that the equation of continuity can be written

$$\frac{\partial}{\partial x}(\rho r u) + \frac{\partial}{\partial y}(\rho r v) = 0,$$

where x and y are the usual curvilinear coordinates parallel to and normal to the meridian profile of the body considered, and r is measured from the axis of symmetry. Again integrating with respect to y between the limits 0 and δ, multiplying by 2π we get

$$2\pi(r_o + \delta\cos\vartheta)\,\rho_1 v_1 = -\frac{\partial}{\partial x}\left[\int_0^\delta 2\pi(r_o + y\cos\vartheta)\,\rho u\,dy\right] + \rho_1 u_1\,2\pi(r_o + \delta\cos\vartheta).\frac{d\delta}{dx}, \quad \ldots.(39)$$

where r_o is the radius of cross-section of the body, and ϑ is the angle between the tangent to the meridian profile and the axis.

We now define the displacement area $A_{\delta*}$ by

$$A_{\delta*} = 2\pi r_o.\delta^* = \frac{1}{\rho_1 u_1}\int_0^\delta 2\pi(r_o + y\cos\vartheta)(\rho_1 u_1 - \rho u).dy,$$

and we define the boundary layer area A_δ by

$$A_\delta = \int_0^\delta 2\pi(r_o + y\cos\vartheta)\,dy.$$

It follows from equation (39) that

$$\begin{aligned}\rho_1 v_1\,2\pi(r_o + \delta\cos\vartheta) &= \frac{\partial}{\partial x}\left\{\rho_1 u_1(A_{\delta*} - A_\delta)\right\} + \rho_1 u_1\frac{dA_\delta}{dx}\\ &= \rho_1 u_1\left\{\frac{(A_{\delta*} - A_\delta)}{\rho_1 u_1}\frac{d}{dx}(\rho_1 u_1) + \frac{dA_{\delta*}}{dx}\right\} \quad (40)\end{aligned}$$

We now consider the surface displaced outwards normal to itself a small distance ε, such that the area A_ε traversed in the displacement normal to the surface is equal to $A_{\delta*}$, and we

/consider inviscid ...

consider inviscid flow past this displaced surface but as before keeping the coordinate system undisplaced. From the equation of continuity we again obtain at a distance δ from the surface that

$$2\pi\,(r_o + \delta\,\cos\vartheta).(\rho v)_\delta = -\frac{\partial}{\partial x}\left[\int_o^\delta 2\pi(r_o + y\cos\vartheta)\rho u\,dy\right] + \rho_1 u_1\,\frac{dA_\delta}{dx}\,.$$

But for $y < \varepsilon$, $u = 0$, and for $y > \varepsilon$, $u = u_1$, and hence

$$2\pi\,(r_o + \delta\,\cos\vartheta)(\rho v)_\delta = -\frac{\partial}{\partial x}\left[\rho_1 u_1 \int_\varepsilon^\delta 2\pi(r_o + \delta\,\cos\vartheta)dy\right] + \rho_1 u_1\,\frac{dA_\delta}{dx}$$

$$= -\frac{\partial}{\partial x}\left[\rho_1 u_1\,(A_\delta - A_\varepsilon)\right] + \rho_1 u_1\,\frac{dA_\delta}{dx}$$

$$= \rho_1 u_1\left\{\frac{(A_\delta * - A_\delta)}{\rho_1 u_1}\,\frac{d}{dx}\,(\rho_1 u_1) + \frac{dA_\delta *}{dx}\right\}$$

$$(\text{since } A_\delta * = A_\varepsilon)\,.$$

Hence, comparing this equation with equation (40) we see, as in the two dimensional case, that the directions of the streamlines at $y = \delta$ in the inviscid flow considered are the same as in the viscous flow if $\rho_1 u_1$ is the same. With the same argument and provisos as before we deduce that the required effective displacement of the surface equivalent in effect on the external flow to the boundary layer is ε, i.e. is such that the area normal to the surface traversed in the displacement is equal to the displacement area.

In regions other than in the neighbourhood of shock waves, we can allow then for the effect of the boundary layer on the external flow by assessing the effect of the equivalent surface displacement. Strictly the boundary layer development should then be recalculated using the modified external flow, but the calculations that have been made to date indicate that at least in the case of two dimensional flow such further recalculations are unnecessary.

It is clear that the effective displacement of the surface will result in an increase of pressure over the whole

/surface of ...

surface of the wing or body, the effective slope of the surface relative to some datum being increased by $\frac{d\delta^*}{dx}$ in the case of two dimensional flow and by $\frac{d\varepsilon}{dx}$ in the case of axi-symmetric flow.

To estimate the effect of this increase of slope in two dimensional flow it is assumed that it is sufficiently accurate to consider the flow outside the boundary layer as a simple wave flow. It then follows that the pressure increase due to the pressure of the boundary layer is

$$\Delta p = \frac{\gamma p_1 M_1^2}{\sqrt{M_1^2 - 1}} \cdot \frac{d\delta^*}{dx} = \frac{\rho_1 u_1^2}{\sqrt{M_1^2 - 1}} \cdot \frac{d\delta^*}{dx} \quad \ldots\ldots\ldots (41)$$

From equation (41) the resulting changes in drag, lift and pitching moment can be readily calculated. It will be clear that with the boundary layer turbulent the increment in pressure Δp is generally greater than when the boundary layer is laminar, and this effect will become increasingly marked with increase of Reynolds number. Consequently, it is possible for the resulting change in drag to be negative when transition occurs at about 0.5c, although with a fully laminar or fully turbulent boundary layer the integrated effect of this pressure increment on drag is generally positive. These remarks are illustrated by the results of some specimen calculations shown in Fig. 5.

In the axi-symmetric case the problem of estimating the effect of the equivalent displacement is perhaps less simple. Over the forebody any of the simple methods for calculating the pressure distribution developed by Bolton Shaw and Zienkiewicz[13] can be readily applied with adequate accuracy. Over the rear the method of characteristics or the second order method of Van Dyke[14] is always available, but it is hoped to develop simpler methods of adequate accuracy for the problem in mind. Such methods are under investigation.

3.2. Shock wave - boundary layer interaction effects

A review of current knowledge on the complex nature and problems of shock wave - boundary layer interaction effects has been given by Zienkiewicz[15], and only a brief summary of the main points relevant to the problem under consideration as well as of further information that has become available since Ref.15 was written need be given here.

It is clear from the available evidence[16,17] that in

/practice the ...

practice the conditions at the leading edge of a sharp nosed aerofoil or body may differ markedly from those predicted by inviscid flow theory, and these differences are due to interaction effects between the leading edge shock waves and the boundary layer and in some measure to the fact that the leading edge of a wing is never truly sharp. Nevertheless, these differences appear to have relatively little effect on the pressure distribution except very close to the leading edge, and the pressure distribution obtained from inviscid flow theory corrected for the displacement thickness effect of the boundary layer gives very close agreement with experiment except perhaps within about 0.02c of the leading edge. At the leading edge the rate of growth of the displacement thickness becomes infinite according to theory and the corresponding correction to the pressure according to equation (41) becomes inacceptable. However, a plausible extrapolation of the pressure distribution forward from 0.02c can readily be made and is recommended, the resulting possible error in overall characteristics being small.

It is near the trailing edge of a wing, and also presumably at the rear of a body, that the interaction between shock waves and boundary layer can have a most profound effect on the pressure distribution and hence on the overall aerodynamic characteristics. It is of course well known that in a region of interaction between a shock wave and a boundary layer, the pressure rise across the shock is diffused upstream and downstream in the layer, and both boundary layer and shock are to some extent modified by the interaction. Thus, the boundary layer on an aerofoil section near the trailing edge will be subjected to a positive pressure gradient due to the shock wave springing from the rear of the wing; the boundary layer will thicken in that region as a result and may separate before reaching the trailing edge. The effective shape of the wing, however, will then be such as to modify the shock pattern and strength at the trailing edge. The resulting effect on the pressure distribution and hence on the lift, drag and pitching moment may be considerable if the separation is extensive. In the main region of separation the pressure is generally nearly constant, and our problem therefore reduces to that of determining the surface pressure and extent of the separated region in any given case. The subject is still largely unexplored, and the data required for a simple empirical approach is not yet available. However, there are indications discussed below that the problem may be resolved with the accumulation of sufficient experimental data for which a relatively modest experimental programme may suffice. The work

/of Holder and Gadd ...

of Holder and Gadd[18,19] Drougge[20] and Lighthill[21] are valuable contributions in this connection.

It is known that with the boundary layer laminar approaching the shock separation of the layer is likely to occur even with relatively weak shocks, both theory and experiment suggest that shocks with a pressure rise ratio of the order of 1.05 - 1.1 will cause separation. On the other hand with the boundary layer turbulent much stronger shocks with pressure rise ratios of the order of 1.7 - 1.8 are required for separation to occur. This latter fact indicates that for a wide range of practical cases with the boundary layers turbulent ahead of the trailing edge separation will not occur. This is fortunate from the point of view of the problem discussed in this paper because it appears that the thickening of the turbulent boundary layer associated with the shock wave is confined, in the absence of separation, to a distance ahead of the trailing edge less than the boundary layer thickness, and therefore the effects of such thickening can be ignored. In the first instance it is proposed to consider, in the main, cases for computation based on the methods described in this paper where transition occurs in the boundary layer ahead of the trailing edge, and where the trailing edge shock pressure rise ratio is less than about 1.8, consequently the problems introduced by separation effects will not arise in such cases.

Nevertheless, some of the facts relating to conditions when boundary layer separation has occurred are worth reviewing. Zienkiewicz[15] has analysed available data obtained with a ten per cent thick biconvex section and a nine per cent thick symmetrical section with a 4^{o} trailing edge angle tested at the N.P.L. The tests referred to were made at Reynolds numbers in the region of 0.5 to 1.0 x 10^{6} and the boundary layers approaching the trailing edge were laminar but became turbulent after separation. The Mach numbers covered varied from about 1.6 to 2.5. Zienkiewicz showed that for these results the values of the ratio $p_{sep.}/p_o$ (where $p_{sep.}$ is the nearly constant surface pressure in the region of separated flow and p_o is the undisturbed stream static pressure) fell reasonably close to a single curve as a function of wing incidence up to quite large angles of incidence, the variations in trailing edge angle and Mach number covered appearing to have little significant effect (Fig. 14, Ref. 15). It is almost certain that if a larger range of Reynolds number had been covered scale effect would have been revealed, thus the results of Gadd and Holder's[18,19] work on

/shock wave ...

shock wave boundary layer interaction on a flat plate suggest that $\frac{p_{sep} - p_o}{p_o} \propto \frac{1}{R^{1/3}}$ approx. It is also possible that there is a wing thickness effect in addition and further data are clearly needed. Nevertheless, the simple nature of this result is encouraging. Further, Zienkiewicz found that the extent of the separated region of flow also agreed fairly closely with a single curve when plotted as a function of incidence for the same ranges of Mach number and wing shape (Fig. 16, Ref. 15). In this connection Gadd and Holder's results suggest a scale effect such that the separation distance is approximately inversely proportional to the Reynolds number. It will be readily appreciated that if a set of such simple relations can be established for a practical range of Mach numbers, Reynolds numbers and wing shapes it should be possible to predict not only the effects of viscosity on drag, but its effects on lift, pitching moment and aerodynamic centre, etc., over a wide range of incidence. It is one of the consolations of the study of purely supersonic flow that the pressure distribution and the overall aerodynamic characteristics are not markedly more sensitive to conditions at the trailing edge than elsewhere, and no greater accuracy is needed in determining the boundary layer development for the purposes of estimating lift and pitching moment than is needed for determining the drag. In marked contrast we may note that for subsonic flow the lift and pitching moment but not necessarily the drag can only be adequately estimated when the extension of the Kutta-Joukowski condition at the trailing edge for viscous flow, first formulated by Preston[12], has been properly applied. This is a process involving considerable computation, and it has so far only been applied in the absence of separation.

In the case of the turbulent boundary layer when the pressure ratio across the shock is large enough to cause separation, the available data indicates that scale effects are very small, but no comprehensive information on other effects is yet available. However, Gadd[22] has suggested a simple if approximate hypothesis for determining the pressure in the region of separation that gives values in reasonable agreement with available experimental data. His argument is that except very close to the wall the effects of pressure gradient on the flow in the boundary layer are much greater than the effects of friction, and it is assumed that the lower limit of the region in which this is so is where the velocity is 0.6 of the velocity just outside the boundary layer. This corresponds to the 'shoulder' of the zero pressure gradient boundary layer velocity distribution. It is then argued

/accordingly ...

accordingly that the separation pressure is that required to bring the air at this 'shoulder' to rest isentropically, this leads to values of $p_{sep.}/p_1$ of 1.84 and 2.51 at values of M_1 of 2.0 and 3.0, respectively, where p_1 and M_1 refer to the pressure and Mach number ahead of the shock. The corresponding values of the separation pressure coefficient $c_{ps} = (p_s - p_1) / \frac{1}{2}\rho_1 u_1^2$ are 0.3 and 0.24.

The available data for the extent of the separated region cannot however be so readily generalised. Gadd and Holder[18,19] have investigated the interaction of oblique shock waves of various strengths and the boundary layer on a flat plate over a range of Mach numbers from 1.5 to 4.0 and their results show that with increasing shock strength the separation distance increases. Expressing the pressure rise across the shock as a coefficient c_p in terms of $\frac{1}{2}\rho_1 u_1^2$ then with $c_p = 0.5$ the separation distance is about $35\delta^*$ whilst with $c_p = 1.0$ the separation distance is about $140\delta^*$ for the Mach numbers tested. However, Drougge[20] has investigated the flow in corners less than 180° where the shock is generated by the corner and the flow bears more direct similarity to that at the rear of an aerofoil than do the cases investigated by Gadd and Holder. His separation distances are considerably less than those of Gadd and Holder for a given value of c_p, thus for $c_p = 0.5$ Drougge obtains a separation distance of the order of $10\delta^*$ and for $c_p = 1.0$ the separation distance is about $25\delta^*$. The two types of experiment exemplified in the work of Gadd and Holder on the one hand and Drougge on the other differ in important respects, and it is not altogether surprising that their results do not agree. However it is clear that without further experimental evidence one could not confidently apply the results of either set of experiments to the problem of the separation distance on an aerofoil, although it is likely that the results of Drougge will provide a closer estimate.

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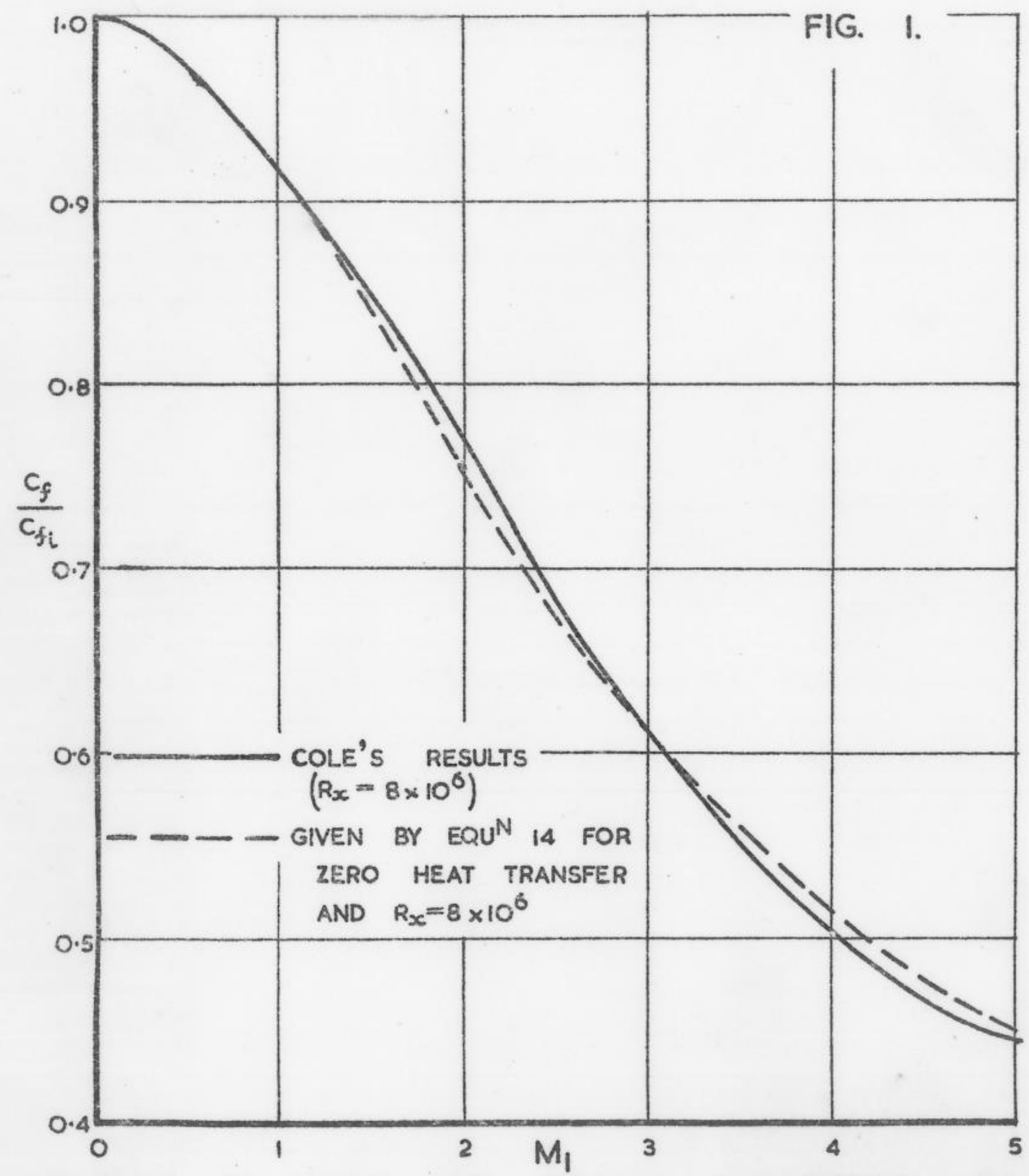
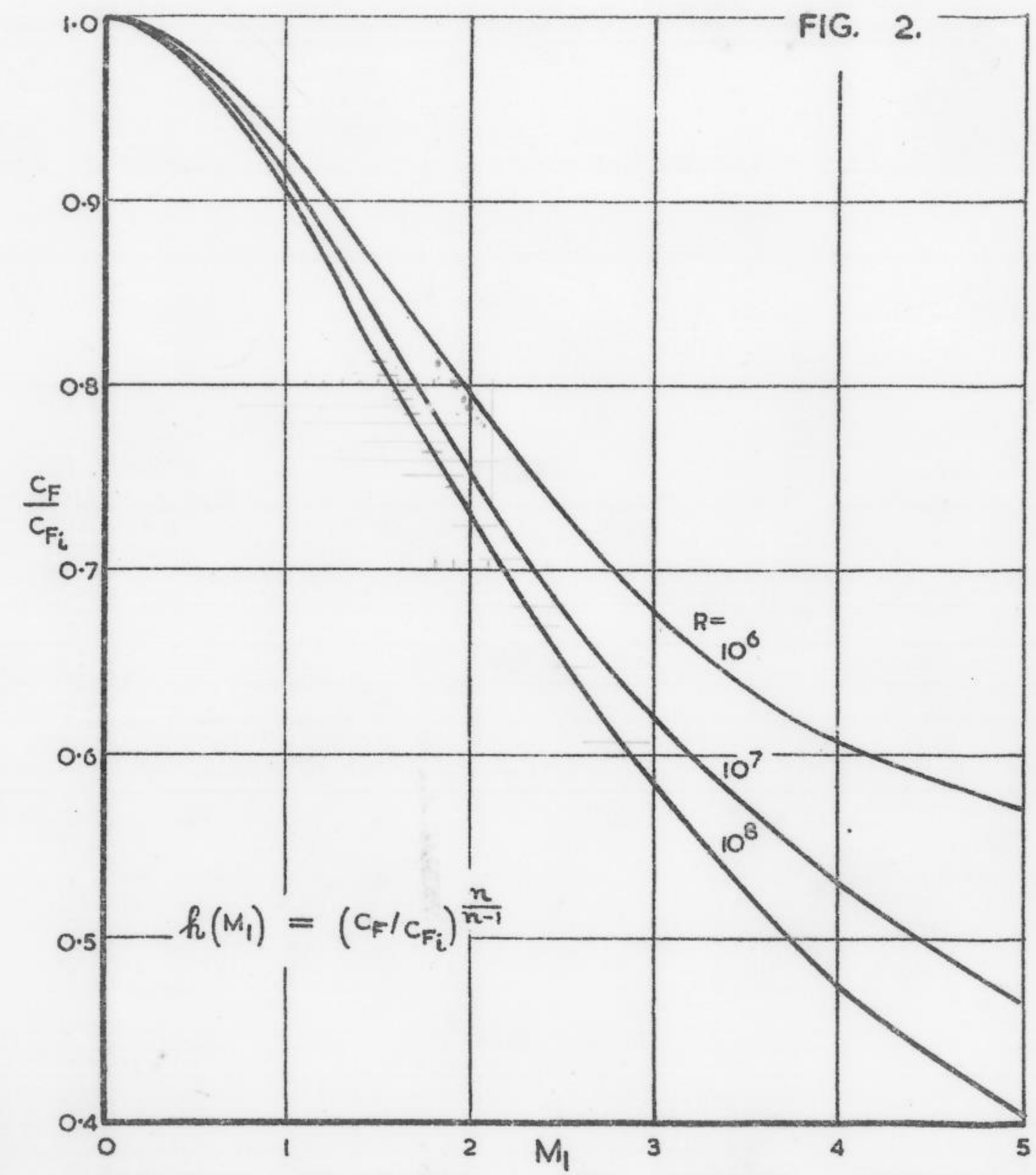
22.	Gadd, G.E.	On the Interaction with a Completely Laminar Boundary Layer of a Shock Wave Generated in the Main Stream. A.R.C. Rep. No. 15,100. Aug. 1952.

FIG. 1.



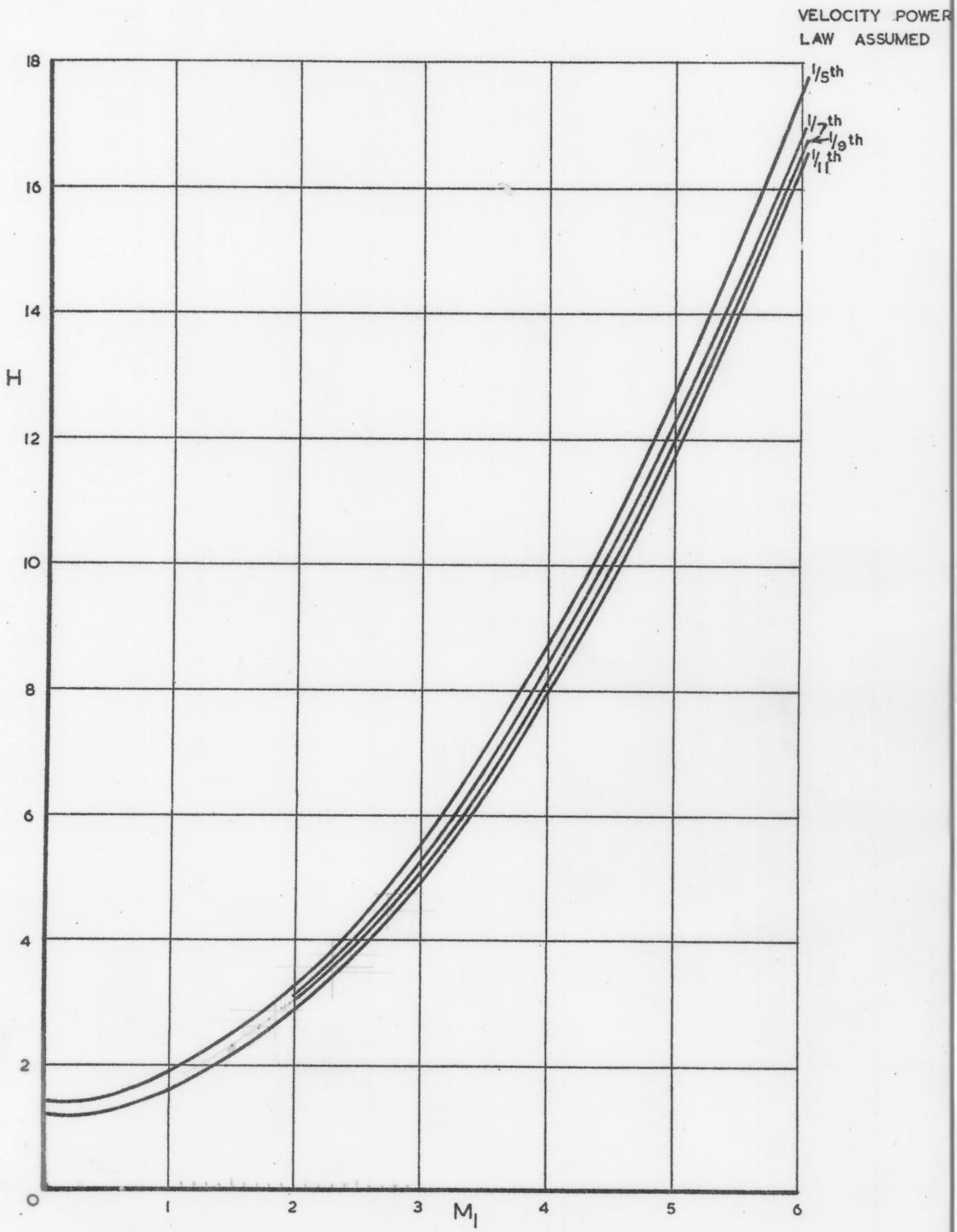
VARIATION OF LOCAL SKIN FRICTION COEFFICIENT WITH MACH NUMBER AS GIVEN BY THEORY (EQU$^{\underline{N}}$ 14) & EXPERIMENT (REF 3)

FIG. 2.



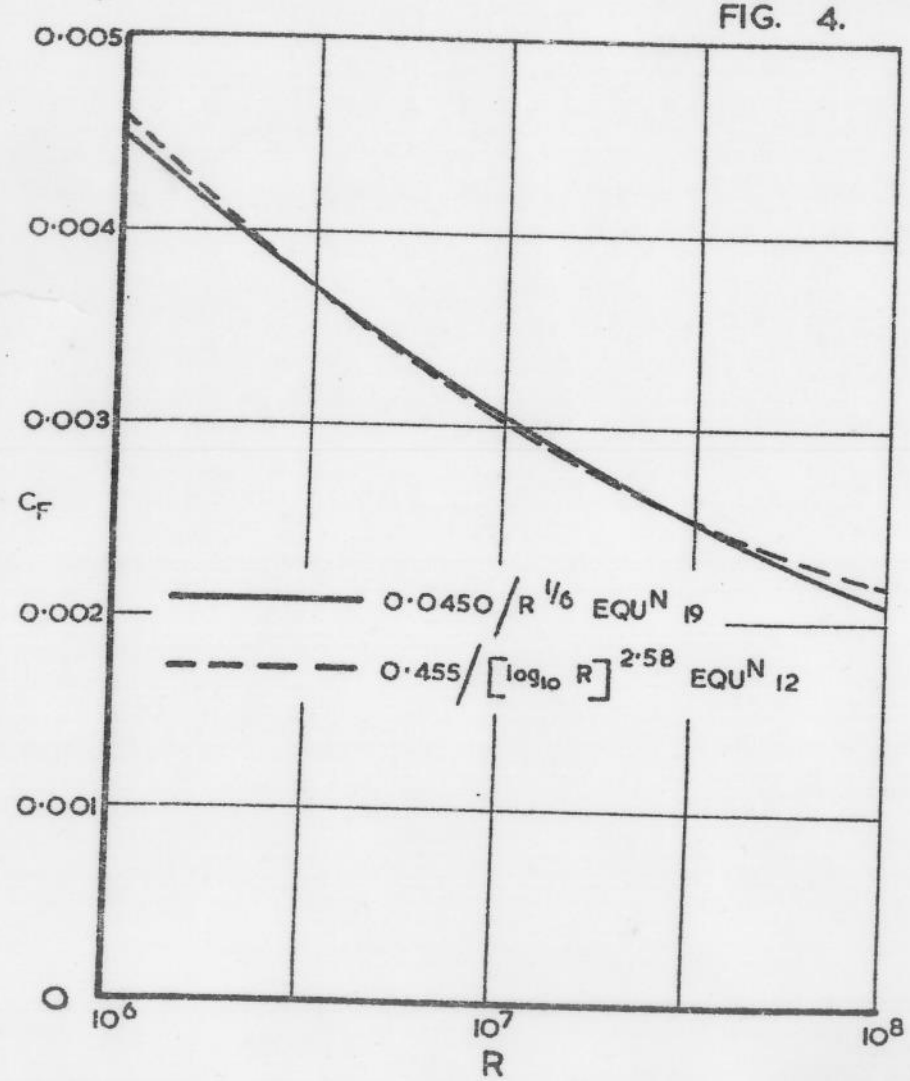
VARIATION OF OVERALL SKIN FRICTION COEFFICIENT WITH MACH NUMBER & REYNOLDS NUMBER AS GIVEN BY EQU$^{\underline{N}}$S 15 & 16 ($\sigma = 0{\cdot}72$, $\omega = 0{\cdot}89$)

FIG. 3.



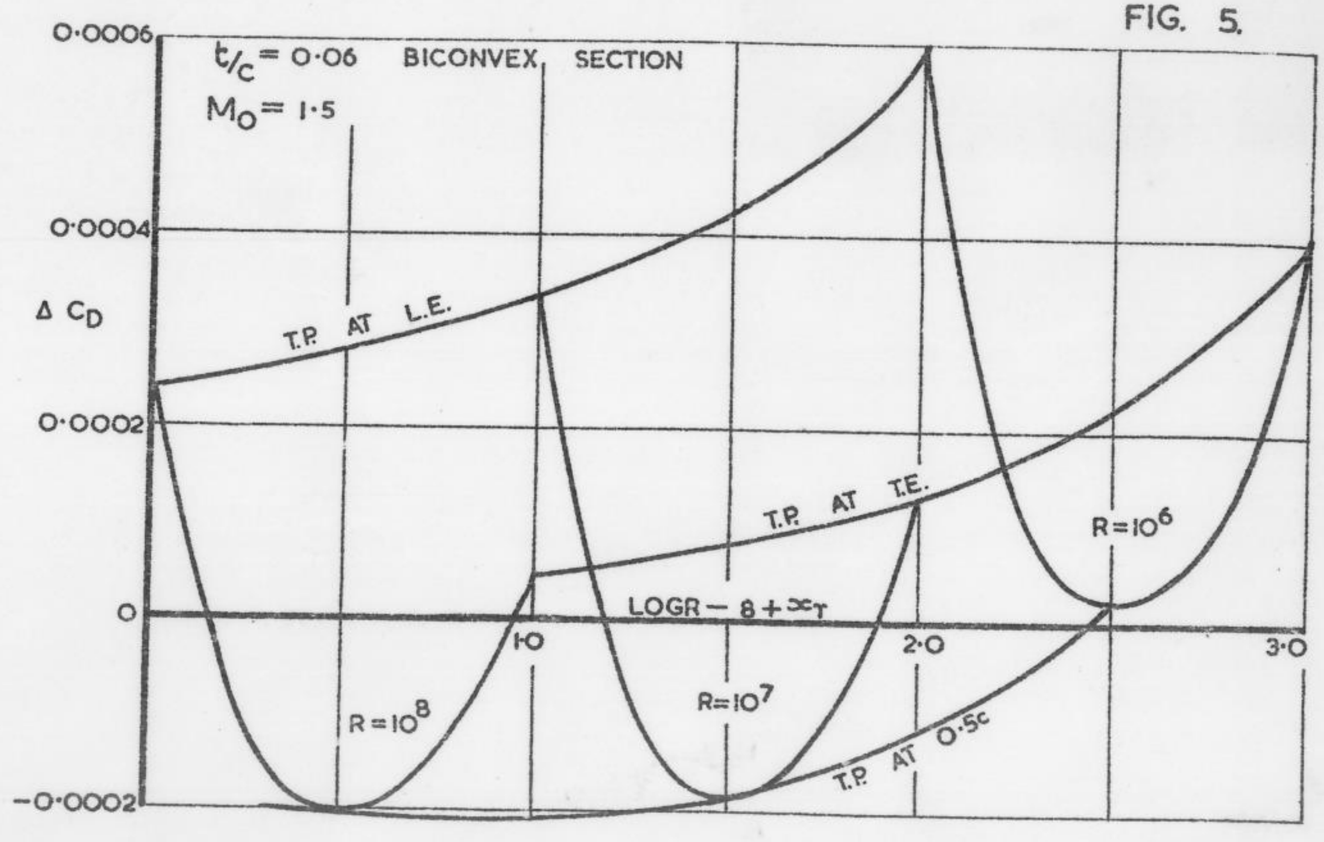
VARIATION OF H WITH M_1 FOR VARIOUS VELOCITY POWER LAW RELATIONS, CALCULATED ON THE ASSUMPTION THAT THE TOTAL ENERGY IS CONSTANT ACROSS THE BOUNDARY LAYER.

FIG. 4.



COMPARISON OF SUGGESTED POWER LAW RELATION FOR FLAT PLATE SKIN FRICTION WITH PRANDTL-SCHLICHTING RELATION.

FIG. 5.



CHANGE IN WAVE DRAG DUE TO EFFECTIVE DISPLACEMENT OF SURFACE CAUSED BY THE BOUNDARY LAYER AS FUNCTION OF REYNOLDS NUMBER AND TRANSITION POSITION.