REPORT NO. 74

MAY, 1953

THE COLLEGE OF AERONAUTICS

CRANFIELD

The Numerical Solution of Certain Differential Equations occurring in Crocco's Theory of the Laminar Boundary Layer

-by-

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SUMMARY

A numerical method is described for the solution of certain differential equations which result from the application of Crocco's transformation to the laminar boundary layer equations appropriate to high supersonic Mach numbers. (i.e. at hypersonic speeds).

Solution is obtained by continuous application of a rapidly convergent relaxation process to a pair of simultaneous differential equations, for which one of boundary conditions is a first derivative. The Prandtl number occurs as a parameter.

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1. INTRODUCTION

When considering the boundary layer equations for laminar flow over a flat plate in a hypersonic stream, it is convenient to apply Crocco's transformation in order to separate the variables so that the velocity in the boundary layer is the new, and only, independent variable. The details of such a method are described by Nonweiler[1] who shows that the boundary layer equations may be expressed as a pair of simultaneous non-linear ordinary differential equations of the second order: one of first degree and the other of second degree. The corresponding boundary conditions, which are of jury type, involve a first derivative.

For numerical analysis, it is convenient to express the differential equations as difference equations and obtain a solution by relaxation. The process is in fact rapidly convergent.

Although the equations considered apply strictly to hypersonic boundary layers, and for that reason have certain inherent simplicity, the method of numerical solution is applicable equally to the corresponding equations for flow at other speeds.

2. The Equations

From reference 1 we have to solve the pair of simultaneous total differential equations:

$$ZZ'' + f/\sqrt{Y} = 0 \qquad \text{............} \quad (2.1)$$

$$Y'' + (1-\sigma)Y'Z'/Z + 2 = 0 \qquad \text{............} \quad (2.2)$$

subject to the boundary conditions

$$Y = Z' = 0 \text{ at } f = 0 \qquad \text{............} \quad (2.3)$$

$$Y = Z = 0 \text{ at } f = 1 \qquad \text{............} \quad (2.4)$$

where a dash denotes differentiation with respect to f.

In the above equations, the independent variable f is a non-dimensional form of the total gas velocity, Y is a non-dimensional form of the total heat (or enthalpy), Z is a non-dimensional form of the shear stress, and σ is the Prandtl number.

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The differential equations can be expressed approximately as finite difference equations in the form

$$Z.\delta^2 Z + h^2 f/\sqrt{Y} + Z\Delta_1 = 0 \quad \ldots\ldots\ldots\ldots (2.5)$$

$$\delta^2 Y + (1-\sigma).\mu\delta Y.\mu\delta Z/Z + 2h^2 + \Delta_2 = 0 \quad \ldots. (2.6)$$

where Δ_1 and Δ_2 are difference corrections which include all but the dominant differences when derivatives are expressed in terms of central differences, and where h is the constant interval between successive pivotal points.

We denote functional values of Y and Z corresponding to f_o, $f_o + h, \ldots, f_o + nh$, by suffices $0, 1, \ldots, n$ and express differences in terms of functional values according to the relations

$$\left.\begin{aligned} \mu\delta Y_o &= \tfrac{1}{2}(Y_1 - Y_{-1}) \\ \mu\delta Z_o &= \tfrac{1}{2}(Z_1 - Z_{-1}) \\ \delta^2 Y_o &= (Y_1 - 2Y_o + Y_{-1}) \\ \delta^2 Z_o &= (Z_1 - 2Z_o + Z_{-1}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2.7)$$

Thus equations (2.5) and (2.6) may be written

$$Z_o(Z_1 - 2Z_o + Z_{-1}) + h^2 f_o/\sqrt{Y_o} + Z_o\Delta_1 = R_1 \quad \ldots\ldots\ldots (2.8)$$

$$(Y_1 - 2Y_o + Y_{-1}) + (1-\sigma)(Y_1 - Y_{-1})(Z_1 - Z_{-1})/4Z_o + 2h^2 + \Delta_2 = R_2 \quad \ldots\ldots\ldots (2.9)$$

where R_1 and R_2 are residuals.

In the ensuing solution we obtain a first approximation to the dependent variables by neglecting the difference corrections Δ_1, Δ_2 and applying the method of relaxation to obtain zero residuals R_1, R_2. More accurate numerical representation of the dependent variables is obtained by differencing the above values and including approximate difference corrections before continuing the relaxation.

Leading terms in the difference corrections are

$$\Delta_1 = -\delta^4 Z_o/12 + \delta^6 Z_o/90 - \ldots \quad \ldots\ldots\ldots\ldots (2.10)$$

$$\Delta_2 = -\delta^4 Y_o/12 + \delta^6 Y_o/90 - \ldots \quad \ldots\ldots\ldots\ldots (2.11)$$

$$+ (1-\sigma)\left\{(\mu\delta Y_o - \mu\delta^3 Y_o/6 + \ldots)(\mu\delta Z_o - \mu\delta^3 Z_o/6 + \ldots) - \mu\delta Y_o . \mu\delta Z_o\right\}/Z_o$$

3. The Boundary Conditions

In view of the boundary conditions (2.3) and (2.4), the relaxation procedure is straightforward at $f = 1$ and for $Y = 0$ at $f = 0$. It is necessary, however, to pay special attention to the condition $Z' = 0$ at $f = 0$.

In general we should use a numerical differentiation formula for the first derivative at $f = 0$. Such a formula is

$$Z'(0) = (1/3)\left\{4Z(h) - Z(2h)\right\} - Z(0), \quad \ldots\ldots\ldots \quad (3.1)$$

and to satisfy the boundary condition $Z'(0) = 0$ we should then require

$$Z(0) = (1/3)\left\{4Z(h) - Z(2h)\right\}. \quad \ldots\ldots\ldots\ldots \quad (3.2)$$

Hence we should relax Z only over the range $h \leq f \leq 1$ and then evaluate $Z(0)$ from equation (3.2).

Formulae of greater accuracy than the three point formula quoted in equation (3.1) may, of course, be used and are listed by Bickley[2].

For the present problem a slightly more accurate representation of the condition $Z'(0) = 0$ than is given by equation (3.1) was obtained by noticing that equation (3.1) is equivalent to assuming

$$Z = Z(0) + \text{const.}\ f^2 \quad \ldots\ldots\ldots\ldots \quad (3.3)$$

in the neighbourhood of $f = 0$.

However, examination of equation (2.1) suggests, for the present problem, that we may express

$$Z = Z(0) + \alpha(\sigma)\ f^{5/2} \quad \ldots\ldots\ldots\ldots \quad (3.4)$$

for all σ, where α is a function of σ only. When the value of α in equation (3.4) is expressed in terms of the values of Z at $f = 0$, h and $2h$ we have, instead of equation (3.2), and in view of the boundary conditions (2.3),

$$Z(0) = \frac{2^N Z(h) - Z(2h)}{2^N - 1} \quad \ldots\ldots\ldots\ldots \quad (3.5)$$

where $N = 5/2$.

This modification has been incorporated in the present work since second and higher derivatives of Z with respect to f are singular at $f = 0$. These further conditions were not represented by equations (3.1) - (3.3).

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4. Starting Values for the Solution

It will be seen that equation (2.2) has a closed analytical solution when $\sigma = 1$. For in this special case

$$Y'' + 2 = 0 \qquad \text{............} \quad (4.1)$$

and so, in view of the boundary conditions (2.3) and (2.4), we obtain

$$Y = f(1-f). \qquad \text{............} \quad (4.2)$$

Approximate values of Z at $f = 0$ and 0.5 may now be obtained from equations (2.4), (2.8), (3.5), and (4.2), for neglecting Δ_1 and R_1 in equation (2.8), we have approximately

$$Z_o(Z_1 - 2Z_o + Z_{-1})\sqrt{Y_o} + h^2 f_o = 0. \qquad \text{............} \quad (4.3)$$

Let $f_{-1} = 0$, $f_o = 0.5$, $f_1 = 1$ so that $h = 0.5, \sqrt{Y_o} = 0.5$ by equation (4.2), the boundary condition $Z_1 = 0$ by, (2.4) and $Z_{-1} = 4\sqrt{2}\,Z_o/(4\sqrt{2} - 1)$ by equation (3.5), so that equation (4.3) may be written

$$Z_o\left(\frac{4\sqrt{2}\,Z_o}{4\sqrt{2} - 1} - 2Z_o\right)\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = 0$$

from which we obtain

$$Z_o = 0.564$$

and hence $Z_{-1} = 0.685$.

The corresponding variation of Z with f, as given in equation (3.4)*, is

$$Z = 0.685\,(1 - f^{5/2}) \qquad \text{...........} \quad (4.4)$$

which may be used as an interpolation formula to estimate Z in the range $0 \leqslant f \leqslant 1$ for the special case $\sigma = 1$.

This special solution, for $\sigma = 1$, has limited physical significance since for real fluids the Prandtl number σ is less than unity, say about 0.7.

However, to obtain numerical solutions for real fluids, it is convenient to use values given by the special solution, equations (4.2) and (4.4), as initial values for the relaxation method described below.

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* On the assumption that Z varies parabolically with f, one obtains $Z = 0.685 + 0.202f - 0.888f^2$.

5. Solution by Relaxation

To calculate values of the dependent variables Y and Z for values of σ less than unity the following procedure is suggested.

First calculate $Y\big|_{\sigma=1} = f(1-f)$, from equation (4.2) and $Z\big|_{\sigma=1}$, from equation (4.4), for a range of equally spaced values of f in the range (0,1). Then, neglecting the difference corrections in equations (2.8) and (2.9), follow the regular sequence of operations:-

(i) relax equation (2.8) with $Y = Y\big|_{\sigma=1}$ to obtain $Z = Z\big|_{\sigma=1}$

(ii) substitute $Z = Z\big|_{\sigma=1}$ in equation (2.9) with the given value of σ and relax to obtain a first approximation $Y = Y\big|_{\sigma}^{(1)}$

(iii) substitute $Y = Y\big|_{\sigma}^{(1)}$ in equation (2.8) and relax to obtain a first approximation $Z = Z\big|_{\sigma}^{(1)}$

(iv) substitute $Z = Z\big|_{\sigma}^{(1)}$ in equation (2.9) and obtain a second approximation $Y\big|_{\sigma}^{(2)}$ as in (ii) and a second approximation $Z\big|_{\sigma}^{(2)}$ as in (iii).

Continue the above relaxation procedure until satisfactory values of $Y\big|_{\sigma}$ and $Z\big|_{\sigma}$ are obtained for the dependent variables.

Next difference the values obtained for $Y\big|_{\sigma}$ and $Z\big|_{\sigma}$ and estimate the difference corrections Δ_1 and Δ_2 given by equations (2.10) and (2.11) respectively. Insert these corrections in equation (2.8) and (2.9) respectively and obtain more accurate values of the dependent variables as follows:-

(v) relax equation (2.8) to improve $Z\big|_{\sigma}$

(vi) substitute $Z\big|_{\sigma}$ in equation (2.9) and relax to improve $Y\big|_{\sigma}$

(vii) substitute the improved values of $Y\big|_{\sigma}$ in equation (2.8) and relax to obtain $Z = Z\big|_{\sigma}^{*}$

(viii) substitute $Z\big|_{\sigma}^{*}$ in equation (2.9) and relax to obtain $Y = Y\big|_{\sigma}^{*}$

If necessary, continue the above sequence of operations until satisfactory values have been obtained for $Y\big|_{\sigma}^{*}$ and $Z\big|_{\sigma}^{*}$.

Difference the values of $Y\big|_{\sigma}^{*}$ and $Z\big|_{\sigma}^{*}$ and re-estimate the difference corrections. If necessary, repeat operations (v) to (viii) until satisfactory final values are obtained for the

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dependent variables.

6. The Relaxation Pattern

Since the given differential equations (2.1) and (2.2) are non-linear, special attention has to be paid to the relaxation pattern which is as follows.

When relaxing equation (2.8) to obtain values of Z corresponding to given values of Y we note that if we vary Z_o by ε then we must change the residual R_1 at f_{-1}, f_o, and f_1 by

$$(R_1)_{-1} = \varepsilon Z_{-1}$$

$$(R_1)_o = \varepsilon(Z_1 - 4Z_o + Z_{-1} - 2\varepsilon)$$

and $$(R_1)_1 = \varepsilon Z_1$$

respectively.

Similarly, when relaxing equation (2.9) to obtain values of Y, if we vary Y by ε the corresponding changes in the residual R_2 at f_{-1}, f_o, and f_1 are

$$(R_2)_{-1} = \varepsilon \left\{1 + (1-\sigma)(Z_o - Z_{-2}) / 4Z_{-1}\right\}$$

$$(R_2)_o = -2\varepsilon$$

and $$(R_2)_1 = \varepsilon \left\{1 - (1-\sigma)(Z_2 - Z_o) / 4Z_1\right\}$$

respectively.

Relaxation could be effected simultaneously in both variables but the above procedure is preferred since, for real fluids, the value of Y only differs slightly from the values calculated for the special solution (4.2).

7. Computational Procedure

In practice it has proved satisfactory to use six or eleven equally spaced pivotal points in the range $0 \leq f \leq 1$.

Values of $Y = f(1-f)$ and $Z = 0.685(1-f^{5/2})$ are calculated from equations (4.2) and (4.4) of the special solution for $\sigma = 1$.

Residuals R_1, R_2 are then calculated from equations (2.8) and (2.9), with difference corrections neglected, for each pivotal point except $f = 0$ and $f = 1$.

Relaxation proceeds as described in section 5 using the pattern given in section 6.

Improved approximations are obtained by differencing calculated values of Y and Z to third and fourth differences respectively; substituting in equations (2.8) and (2.9) values for the difference corrections given by equations (2.10) and (2.11) respectively; and repeating the relaxation. One repetition has been found sufficient to guarantee four figure accuracy.

Results are tabulated (i) to three decimals for six pivotal points and (ii) to four decimals for eleven points.

References

1. Nonweiler, T. The two-dimensional laminar boundary layer of hypersonic speeds. College of Aeronautics Rep. No. 67 (1953).

2. Bickley, W.G. Formulae for numerical differentiation. Math. Gaz. XXV, 263, 19-27 (1941).

TABLE 1

Six points. Three decimals

f	Y			Z		
	$\sigma = 1$	0.8	0.6	1	0.8	0.6
0.0	0.00	0.000	0.000	0.703	0.692	0.686
.2	.16	.163	.167	.695	.684	.679
.4	.24	.247	.255	.660	.649	.645
.6	.24	.252	.265	.573	.563	.559
.8	.16	.175	.192	.394	.386	.381
1.0	0.00	0.000	0.000	0.000	0.000	0.000

TABLE 2

Eleven points. Four decimals

f	Y			Z		
	$\sigma = 1$	$\sigma = 0.8$	$\sigma = 0.6$	$\sigma = 1$	$\sigma = 0.8$	$\sigma = 0.6$
0.0	0.00	0.0000	0.0000	0.7200	0.7114	0.6966
.1	.09	.0922	.0949	.7188	.7100	.6954
.2	.16	.1646	.1700	.7131	.7037	.6898
.3	.21	.2172	.2255	.7004	.6904	.6772
.4	.24	.2500	.2615	.6783	.6679	.6552
.5	.25	.2630	.2779	.6439	.6335	.6211
.6	.24	.2560	.2744	.5940	.5837	.5717
.7	.21	.2286	.2502	.5232	.5133	.5019
.8	.16	.1799	.2034	.4223	.4136	.4031
.9	.09	.1073	.1284	.2703	.2643	.2563
1.0	0.00	0.0000	0.0000	0.0000	0.0000	0.0000