THE COLLEGE OF AERONAUTICS,

CRANFIELD

THE PROFILE DRAG OF YAWED WINGS OF INFINITE SPAN [*]

-by-

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SUMMARY

A method is developed for calculating the profile drag of a yawed wing of infinite span, based on the assumption that the form of the spanwise distribution of velocity in the boundary layer, whether laminar or turbulent, is insensitive to the chordwise pressure distribution. The form is assumed to be the same as that accepted for the boundary layer on an unyawed plate with zero external pressure gradient. Experimental evidence indicates that these assumptions are reasonable in this context. The method is applied to a flat plate and the NACA 64-012 section at zero incidence for a range of Reynolds numbers between 10^6 and 10^8, angles of yaw up to $45^°$, and a range of transition point positions. It is shown that the drag coefficient of a flat plate varies with yaw as $\cos^2 \Lambda$ (where Λ is the angle of yaw) if the boundary layer is completely laminar, and it varies as $\cos^{4/5} \Lambda$ if the boundary layer is completely turbulent. The drag coefficient of the NACA 64-012 section, however, varies very closely as $\cos \Lambda$ for transition point positions between 0 and $0 \cdot 5$ c. Further calculations on wing sections of other shapes and thicknesses and more detailed experimental checks of the basic assumptions at higher Reynolds numbers are desirable.

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[*] Part of this paper appeared in the Thesis presented by T.B. Booth for the College Diploma, 1949.

YSB.

x, y — curvilinear orthogonal coordinates on the surface of a body (y is in spanwise direction for yawed infinite span wing). } See para. 2.1, and Fig. 4.

z — normal distance from the body

u, v, w — velocity components in the boundary layer in the x, y, z directions, respectively.

U_1, V_1 — velocity components just outside the boundary layer in x, y directions, respectively.

$\underline{v}$ — velocity vector $= (u, v, w)$.

$\underline{\omega}$ — curl $\underline{v}$.

Q_0 — undisturbed stream velocity.

U_0, V_0 — components of Q_0 in direction of chord and span respectively, of yawed infinite wing.

$\bigwedge$ — angle of yaw of wing.

u', v', w' — turbulent fluctuation components of velocity in x, y, z directions respectively.

c — chord of wing, measured normal to span.

p — pressure.

ρ — density.

μ — coefficient of viscosity.

ν — kinematic viscosity $= \mu/\rho$.

D — drag of unit span of the wing.

X — component of D in chordwise direction.

Y — component of D in spanwise direction.

$C_D[\alpha, \beta]$ — drag coefficient at a Reynolds number α and an angle of yaw β.

R — Reynolds number $= Q_0 c/\nu$.

δ — boundary layer thickness.

δ_x^{*} — $\displaystyle\int_0^{\delta} \left(1 - \frac{u}{U_1}\right)\,dz,$

δ_y^{*} — $\displaystyle\int_0^{\delta} \left(1 - \frac{v}{V_1}\right)\,dz,$

θ_{xx} — $\displaystyle\int_0^{\delta} \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right)dz,$

θ_{yy} — $\displaystyle\int_0^{\delta} \frac{v}{V_1}\left(1 - \frac{v}{V_1}\right)dz,$

θ_{xy} $\qquad \displaystyle\int_0^\delta \frac{v}{U_1}\left(1 - \frac{u}{U_1}\right)dz,$

θ_{yx} $\qquad \displaystyle\int_0^\delta \frac{u}{V_1}\left(1 - \frac{v}{V_1}\right)dz,$

H_x $\qquad \delta_x^*/\theta_{xx}.$

H_y $\qquad \delta_y^*/\theta_{yy}.$

τ_{xz} $\qquad$ component of shear stress in chordwise (x) direction.

τ_{yz} $\qquad$ component of shear stress in spanwise (y) direction.

$\left(\tau_{xz}\right)_0$ $\qquad$ value of τ_{xz} at surface.

$\left(\tau_{yz}\right)_0$ $\qquad$ value of τ_{yz} at surface.

K_x, K_y $\qquad$ curvature of lines y = const. x = const. respectively.

n $\qquad$ $(H_x + 5)/2$ at the trailing edge.

K $\qquad$ $\dfrac{V_1}{U_1}\,\dfrac{\theta_{yx}}{\theta_{xx}}.$

λ $\qquad$ $\dfrac{-\delta^2}{\mu U_1}\cdot\dfrac{\partial p}{\partial x}.$

The letters T.E. refer to the trailing edge, L.E. to the leading edge.

1. Introduction and Review.

The increasing use of sweepback to delay compressibility troubles has aroused interest in the behaviour of boundary layers on yawed and swept back wings, for we require an understanding of this behaviour if we are to predict and control satisfactorily the characteristics of swept back wings. This paper is concerned only with a limited and relatively simple aspect of the problem, viz. the profile drag of a yawed wing of infinite span. It is hoped, however, that apart from the intrinsic interest of the results, the discussion will add to the basic store of data and ideas needed to tackle the more general problem.

The treatment begins with the development of the boundary layer equations and the momentum integral equations for laminar and turbulent flow. The development of the laminar boundary layer equations is not new, having in essentials been given elsewhere,[1,2,3,4,] it is however included here for completeness. It is shown that the chordwise flow is independent of the spanwise flow and hence the corresponding boundary layer equation can be solved by any of the accepted methods. A method for solving the spanwise flow equation is suggested which is approximate but sufficiently accurate for the purpose in mind. For a flat plate at zero incidence it is found that the spanwise velocity profile is strictly similar at all points to that in the chordwise direction, and the resultant direction of flow is everywhere that of the free stream. The skin friction coefficient at any point, as well as the overall drag coefficient, is reduced by the factor $\cos^{\frac{1}{2}} \Lambda$ due to yaw, and thus the skin friction at a point is the same as that on an unyawed plate at the same distance measured parallel to the undisturbed stream direction downstream from the leading edge. This result is peculiar to laminar flow in the boundary layer.

The turbulent boundary layer equations also lead to the conclusion that the chordwise component of flow in the boundary layer can be treated as independent of the spanwise component. The momentum integral equation of the former component can be solved on accepted lines to yield the component of the drag per unit span in the chordwise direction. The drag of unit span of the wing is the resultant of this component and the spanwise component. To obtain the latter it is assumed that the form of the spanwise velocity distribution in the boundary layer is independent of the chordwise pressure distribution and everywhere satisfies a power law, of the type normally assumed to hold in the turbulent boundary layer on an unyawed flat plate in a zero external pressure gradient. This assumption, which might be based somewhat loosely if intuitively on the argument that the spanwise conditions of flow correspond in some respects to the simplifying conditions usually assumed in the theory from which the power law is derived, is shown by experiments to be reasonable. For the calculations discussed in this paper the law assumed is the 1/7th. power law. The spanwise momentum thickness at the trailing edge is then derived in terms of the

chordwise momentum thickness and readily yields the
spanwise component of the drag.   An expression for
the resultant drag coefficient is then presented in
a simple and concise form.   For a fully turbulent
boundary layer on a flat plate at zero incidence,
with the velocity distributions in both spanwise and
chordwise directions satisfying the 1/7th. power law,
the drag coefficient is reduced by yaw by the factor
$\cos^{4/5}\wedge$.   Here we may note that although the flow
is everywhere parallel to the undisturbed stream
direction the skin friction at a point is not the
same as that on an unyawed plate at the same distance
measured parallel to that direction from the leading
edge.   Were it so, the drag coefficient would change
with yaw as the factor $\cos^{1/5}\wedge$.

The method has been applied to calculate
the drag coefficient of a flat plate and of the
NACA 64-012 section at zero incidence for various
transition positions, Reynolds numbers and angles
of yaw.   In the case of the flat plate (Fig. 1b)
the factor describing the variation of the drag co-
efficient with yaw changes steadily from $\cos^{4/5}\wedge$
to $\cos^{1/2}\wedge$ as the transition point moves back from
the leading to the trailing edge, and there is no
noticeable effect of Reynolds number on this factor
within the range 10^{6} to 10^{8}.   In the case of the
12 percent thick wing section (Fig. 1a) the factor
remains very closely $\cos^{1/2}\wedge$ for transition positions
from the leading edge to 0.5 c, and the effect of
Reynolds number on this factor is very slight.

Preliminary experiments made on a flat
plate at a fairly low Reynolds number (about 10^{6})
tend to confirm the assumptions and conclusions of
the theory for angles of yaw up to 45°, but further
experimental work at higher Reynolds numbers is
desirable.

2.   Boundary Layer Equations.

2.1.   Laminar boundary layer

The equations of motion of a steady viscous, incompressible fluid with negligible body forces are in the vector notation of Ref. 5,

$$- \underline{v} \times \underline{\omega} = - \text{grad} \left(\frac{p}{\rho} + \tfrac{1}{2} \underline{v}^2 \right) - \nu \, \text{curl} \, \underline{\omega}. \qquad \dots\dots (1)$$

The equation of continuity is

$$\text{div } \underline{v} = 0. \qquad \dots\dots (2)$$

Consider the flow past a surface and take x,y,z as orthogonal curvilinear coordinates, such that z is the normal distance from the surface and the lines x = const., y = const., define an orthogonal network on the surface.   We will now make the usual assumptions for the flow inside the boundary layer, viz. that the rates of change of the velocity components normal to the surface are large compared with the corresponding rates of change parallel to the surface and that the boundary layer thickness is small compared with a representative linear dimension of the surface.   We will further assume that, if K_x and K_y are the curvatures of the lines y = const. and x = const., then the quantities $K_x \delta$, $K_y \delta$, $\delta^2 \frac{\partial K_x}{\partial x}$,

$$\delta^2 \frac{\partial K_y}{\partial y}, \quad \delta^2 \frac{\partial K_x}{\partial y}, \quad \delta^2 \frac{\partial K_y}{\partial x}, \quad \delta^3 \frac{\partial^2 K_x}{\partial y^2},$$

$$\delta^3 \frac{\partial^2 K_y}{\partial x^2}, \quad \delta^3 \frac{\partial^2 K_x}{\partial x \partial y}, \quad \text{and } \delta^3 \frac{\partial^2 K_y}{\partial x \partial y} \text{ are all small compared}$$

with unity, where δ is the boundary layer thickness.

With these assumptions the equations of motion yield the boundary layer equations:-

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}, \qquad \dots\dots (3)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}. \qquad \dots\dots (4)$$

The equation of motion in the z direction leads to the result that the pressure may be taken as constant across the boundary layer.   The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \dots\dots (5)$$

For the case of the yawed wing of infinite span, we will adopt the notation illustrated in Fig.4, viz. x is measured along a section profile from the leading edge in a plane normal to the span and y is measured along the span.   In this case the rate of

change of any quantity with respect to y is zero, and hence equations (3), (4) and (5) become

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \, , \qquad \dots \dots (6)$$

$$u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} = \nu \frac{\partial^2 v}{\partial z^2} \, , \qquad \dots \dots (7)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \qquad \dots \dots (8)$$

At the surface $u = v = w = 0$.

Just outside the boundary layer $u = U_1, v = V_1$,

$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$, and the flow is irrotational, consequently

$$U_1 \frac{\partial U_1}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \, , \qquad \dots \dots (9)$$

and

$$\frac{\partial U_1}{\partial y} - \frac{\partial V_1}{\partial x} = 0. \qquad \dots \dots (10)$$

Since, however, $\frac{\partial U_1}{\partial y} = 0$ it follows that $\frac{\partial V_1}{\partial x} = 0$, i.e. $V_1 = \text{const.} = V_o$, where V_o is the component of the undisturbed stream velocity along the y axis.

Equations (6), (8) and (9) and the corresponding boundary conditions are identical with those for the boundary layer over the wing in an unyawed flow and with a main stream velocity $= U_o = Q_o \cos \Lambda$. Hence, the chordwise flow in the boundary layer is independent of the spanwise flow and may be derived by any of the well-established methods. An approximate method for solving for the spanwise flow will be discussed in para. 3.

2. 2.   Turbulent Boundary Layer.

With the usual boundary layer assumptions the equations of motion for the turbulent boundary layer in three dimensions become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left(\overline{u'^2}\right) \frac{\partial}{\partial y}\left(\overline{u'v'}\right)$$

$$- \frac{\partial}{\partial z}\left(\overline{u'w'}\right), \qquad \dots \dots (11)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} - \frac{\partial}{\partial x}\left(\overline{u'v'}\right) - \frac{\partial}{\partial y}\left(\overline{v'^2}\right)$$

$$- \frac{\partial}{\partial z}\left(\overline{v'w'}\right), \qquad \dots \dots (12)$$

/where ....

where undashed letters now refer to time means, the dashes denote turbulent fluctuations and a bar denotes a time mean. The third equation of motion may again be interpreted as indicating that the variation of pressure across the boundary layer can be neglected. The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad \ldots\ldots (13)$$

For the yawed wing of infinite span, these equations reduce to

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left(\overline{u'^2}\right) - \frac{\partial}{\partial z}\left(\overline{u'w'}\right), \ldots (14)$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} - \frac{\partial}{\partial x}\left(\overline{u'v'}\right) - \frac{\partial}{\partial z}\left(\overline{v'w'}\right), \qquad \ldots\ldots (15)$$

and $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0. \qquad\qquad\qquad \ldots\ldots (16)$

Again we note from equations (14), (15) and (16) that the chordwise flow is independent of the spanwise flow and can be calculated as for the unyawed wing in the stream component normal to its span.

2. 3. Momentum Integral Equations.

It is usual to assume that the rates of change of the mean turbulent stresses in directions parallel to the surface can be neglected compared with the rates of change normal to the surface. Both the laminar and turbulent boundary layer equations for general three-dimensional flow can then be written

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}, \qquad \ldots\ldots (17)$$

and

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z}, \qquad \ldots\ldots (18)$$

where $\tau_{xz} = \mu\left(\dfrac{\partial u}{\partial z}\right)$, $\tau_{yz} = \mu\left(\dfrac{\partial v}{\partial z}\right)$, in laminar flow, and

$$\tau_{xz} = \mu\left(\frac{\partial u}{\partial z}\right) - \rho\,\overline{u'w'}, \quad \tau_{yz} = \mu\left(\frac{\partial v}{\partial z}\right) - \rho\,\overline{v'w'}, \text{ in}$$
turbulent flow.

Outside the boundary layer

$$U_1\frac{\partial U_1}{\partial x} + V_1\frac{\partial U_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \ldots\ldots (19)$$

and

$$U_1\frac{\partial V_1}{\partial x} + V_1\frac{\partial V_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \qquad \ldots\ldots (20)$$

If we now integrate equations (17) and (18) with respect to z from z = 0 to z = δ, and make use of equations (13), (19) and (20) we get eventually

$$\Big/\tau_{xz_0} \quad \ldots$$

$$\left(\tau_{xz}\right)_o = \frac{\partial}{\partial x}\left(\rho\, U_1{}^2 \theta_{xx}\right) + \frac{\partial}{\partial y}\left(\rho\, U_1{}^2 \theta_{xy}\right) + \frac{\partial U_1}{\partial x}\left(\rho\, U_1 \delta_x^{\textasteriskcentered}\right)$$

$$+ \frac{\partial U_1}{\partial y}\left(\rho\, V_1 \delta_y^{\textasteriskcentered}\right), \qquad \ldots\ldots (21)$$

and

$$\left(\tau_{yz}\right)_o = \frac{\partial}{\partial y}\left(\rho\, V_1{}^2 \theta_{yy}\right) + \frac{\partial}{\partial x}\left(\rho\, V_1{}^2 \theta_{yx}\right) + \frac{\partial V_1}{\partial y}\left(\rho\, V_1 \delta_y^{\textasteriskcentered}\right)$$

$$+ \frac{\partial V_1}{\partial x}\left(\rho\, U_1 \delta_x^{\textasteriskcentered}\right), \qquad \ldots\ldots (22)$$

where $\left(\tau_{xz}\right)_o$ and $\left(\tau_{yz}\right)_o$ are the values of τ_{xz} and τ_{yz}, respectively, at the surface, and

$$\delta_x^{\textasteriskcentered} = \int_0^\delta \left(1 - \frac{u}{U_1}\right)dz, \qquad \delta_y^{\textasteriskcentered} = \int_0^\delta \left(1 - \frac{v}{V_1}\right)dz,$$

$$\theta_{xx} = \int_0^\delta \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right)dz, \qquad \theta_{yy} = \int_0^\delta \frac{v}{V_1}\left(1 - \frac{v}{V_1}\right)dz,$$

$$\theta_{xy} = \int_0^\delta \frac{v}{U_1}\left(1 - \frac{u}{U_1}\right)dz, \qquad \theta_{yx} = \int_0^\delta \frac{u}{V_1}\left(1 - \frac{v}{V_1}\right)dz.$$

For the case of the yawed wing of infinite span, these equations reduce to

$$\left(\tau_{xz}\right)_o = \frac{\partial}{\partial x}\left(\rho\, U_1{}^2 \theta_{xx}\right) + \frac{\partial U_1}{\partial x}\left(\rho\, U_1 \delta_x^{\textasteriskcentered}\right), \qquad \ldots\ldots (23)$$

and

$$\left(\tau_{yz}\right)_o = \frac{\partial}{\partial x}\left(\rho\, V_1{}^2 \theta_{yx}\right). \qquad \ldots\ldots (24)$$

Writing $H_x = \delta_x^{\textasteriskcentered}/\theta_{xx}$, equation (23) can be cast in the familiar form

$$\frac{\left(\tau_{xz}\right)_o}{\rho\, U_1{}^2} = \frac{\partial \theta_{xx}}{\partial x} + \frac{1}{U_1}\cdot\frac{\partial U_1}{\partial x}\left[H_x + 2\right]\theta_{xx}. \qquad \ldots\ldots (23\text{A})$$

3.   The Calculation of the Profile Drag of a Yawed Wing.

3. 1.   The drag components

We can resolve the drag (D) of unit span of the wing into components along the x and y axis, X and Y.   From the above it follows that X is the drag of unit span of the wing when unyawed but with the same transition position in a stream of velocity $U_o = Q_o \cos\Lambda$, where Q_o is the resultant undisturbed

stream velocity (see Fig.4).   Hence X can be calculated
by any of the established methods (see for example,
Refs. 6,7 and 8).   It remains then to determine the
spanwise component of the drag, Y.   To do this, we
note that, since the undisturbed stream is uniform
and the wing section is constant along its infinite
span, Y has no form drag component but is solely the
resultant of the frictional stress $\left(\tau_{yz}\right)_o$.   Thus, per

unit span of wing, and for each surface

$$Y = \int_0^c \left(\tau_{yz}\right)_o \, dx$$

$$= \int_0^c \rho V_0^2 \frac{\partial}{\partial x} (\theta_{yx}) \, dx, \text{ from (24)},$$

$$= \rho V_0^2 \left[\theta_{yx}\right]_{L.E.}^{T.E.}$$

$$= \rho V_0^2 \left[\theta_{yx}\right]_{T.E.}, \qquad\qquad \dots\dots (25)$$

where L.E. and T.E. refer to the leading edge and
trailing edge, respectively.   Here we have assumed
that the momentum thickness θ_{yx} is zero at the

leading edge and is continuous at the transition
point, it being accepted that the stress $\left[\tau_{yz}\right]_o$

could not become infinite.   Our problem then re-
duces to the determination of $\left[\theta_{yx}\right]_{T.E.}$.

Now

$$\theta_{yx} = \int_0^\delta \frac{u}{V_1}\left(1 - \frac{v}{V_1}\right) dz$$

$$= \frac{U_1}{V_1} \int_0^\delta \frac{u}{U_1}\left(1 - \frac{v}{V_1}\right) dz.$$

$$= \frac{U_1}{V_1} \cdot K \theta_{xx}, \text{ say,}$$

$$\qquad\qquad\qquad\qquad \left.\begin{array}{r}\\\\\\\\\\\end{array}\right\} \dots (26)$$

$$\text{where } K = \int_0^\delta \frac{u}{U_1}\left(1 - \frac{v}{V_1}\right) dz \Big/ \int_0^\delta \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right) dz.$$

Let us suppose for the moment that we know the value
of K at the trailing edge of the wing.   From equa-
tions (25) and (26)

$$Y = \rho U_{T.E.} V_0 K_{T.E.} \theta_{xx} .$$

/ But ....

But (see Ref.6)

$$2 \frac{\theta_{xx}}{c} = \frac{X}{\frac{1}{2}\rho U_0^2 c} \left(\frac{U_0}{U_{T.E.}}\right)^n ,$$

where $n = \left[\left(H_x\right)_{T.E.} + 5\right]/2.$

Hence

$$Y = \frac{Sin \wedge}{Cos \wedge} \cdot K_{T.E.} \left(\frac{U_0}{U_{T.E.}}\right)^{n-1} \cdot X.$$

The resultant drag D is given by

$$D = X \, Cos \wedge + Y \, Sin \wedge,$$

and therefore

$$C_D\left[R,\wedge\right] = \frac{D}{\frac{1}{2}\rho Q_0^2 c} = \frac{X}{\frac{1}{2}\rho U_0^2 c} Cos^3 \wedge + \frac{Y}{\frac{1}{2}\rho U_0^2 c} \cdot Cos^2 \wedge \cdot Sin \wedge$$

$$= \frac{X}{\frac{1}{2}\rho U_0^2 c}\left\{Cos^3 \wedge + Cos \wedge \cdot Sin^2 \wedge \, K_{T.E.} \, \frac{U_0}{U_{T.E.}}^{n-1}\right\}$$

$$= C_D\left[R \, Cos \wedge, 0\right] \cdot Cos \wedge \left\{Cos^2 \wedge + Sin^2 \wedge \, K_{T.E.}\left(\frac{U_0}{U_{T.E.}}\right)^{n-1}\right\}.$$

$$\dots\dots\dots (27)$$

Here $C_D\left[\alpha,\beta\right]$ denotes the drag coefficient of the wing at a Reynolds number α and an angle of yaw β. Equation (27) enables us to calculate the drag co-efficient of the yawed wing in terms of the existing results for the drag coefficient of the unyawed wing, the angle of yaw, the chordwise component of velocity just outside the boundary layer at the trailing edge and the quantities n and K at the trailing edge. The quantity n is known from the chordwise calculations, our problem is therefore solved when we have determined $K_{T.E.}$.

3. 2. The Spanwise Flow in the Laminar Boundary Layer.

For our purpose it is sufficiently accurate to treat the chordwise flow in the boundary layer by the Karman-Pohlhausen method or by one of the variants based on it (see, for example, Ref.9). Thus we assume

$$\frac{u}{U_1} = a_1\left(\frac{z}{\delta}\right) + b_1\left(\frac{z}{\delta}\right)^2 + c_1\left(\frac{z}{\delta}\right)^3 + d_1\left(\frac{z}{\delta}\right)^4 ,$$

/ and if ,.....

and if we satisfy the boundary conditions

$$\nu \frac{\partial^2 u}{\partial z^2} = - \frac{1}{\rho} \frac{\partial p}{\partial x}, \text{ at } z = 0, \text{ and } u = U_1, \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial z^2} = 0,$$

at $z = \delta$, we find

$$\frac{u}{U_1} = F\left(\frac{z}{\delta}\right) + G\left(\frac{z}{\delta}\right),$$

where

$$F\left(\frac{z}{\delta}\right) = 2\left(\frac{z}{\delta}\right) - 2\left(\frac{z}{\delta}\right)^3 + \left(\frac{z}{\delta}\right)^4, G\left(\frac{z}{\delta}\right) = \frac{z}{6\delta}\left(1 - \frac{z}{\delta}\right)^3,$$

and

$$\lambda = - \frac{\delta^2}{\mu U_1} \cdot \frac{\partial p}{\partial x} \cdot$$

It then follows that

$$\theta_{xx} = \delta\left[\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right].$$

If we likewise assume that

$$\frac{v}{V_1} = a_2\left(\frac{z}{\delta}\right) + b_2\left(\frac{z}{\delta}\right)^2 + c_2\left(\frac{z}{\delta}\right)^3 + d_2\left(\frac{z}{\delta}\right)^4,$$

and we satisfy the boundary conditions $\nu \frac{\partial^2 v}{\partial z^2} = 0$, at $z = 0$, and

$$v = V_1, \frac{\partial v}{\partial z} = \frac{\partial^2 v}{\partial z^2} = 0, \text{ at } z = \delta, \text{ we find}$$

$$\frac{v}{V_1} = F\left(\frac{z}{\delta}\right),$$

i.e. the form of the spanwise velocity distribution is independent of the chordwise pressure distribution.[*]
Hence, from (26)

$$K = \delta \int_0^1 \left[F(\eta) + \lambda G(\eta)\right]\left[1 - F(\eta)\right] d\eta / \theta_{xx}$$

$$= \left[\frac{37}{315} + \frac{11\lambda}{3024}\right] \Bigg/ \left[\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right]$$

$$= K(\lambda), \text{ say.} \qquad \dots\dots\dots (28)$$

[*] Sears[2] has calculated by a more accurate method the spanwise velocity profiles for a simple case of an external chordwise pressure gradient leading to chordwise separation. The spanwise profiles show small changes of form up to a point near the separation, after which the change of form becomes more marked, however these changes are much less severe than the corresponding changes of form undergone by the chordwise profiles.

The value of H_x is given by

$$H_x = \left(\frac{3}{10} - \frac{\lambda}{120} \right) \bigg/ \left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} \right) \qquad \cdots\cdots\cdots (29)$$

The usual methods of solution for the chordwise flow yield λ as a function of x, and hence we can determine the value of K and H_x (and therefore n) at the trailing edge from equations (28) and (29), assuming that the flow remained laminar to the trailing edge and was not too near separation anywhere for the above method to be seriously in error. In applying equation (27) we are implicitly assuming that the solution of the momentum integral equation for the wake developing from completely laminar boundary layers follows exactly the lines of the solution for the case when the boundary layers are turbulent at the trailing edge. On the face of it this assumption seems plausible enough, but in any case the problem of the wing with completely laminar boundary layers is at present of little more than academic interest.

We note that for a flat plate at zero incidence with completely laminar boundary layers the above leads to $\dfrac{v}{V_o} = \dfrac{u}{U_o}$. or K = 1. Since $U_{T.E.} = U_o$, it follows from (27) that

$$\begin{aligned}
C_D[R, \wedge] &= \mathrm{Cos}\wedge. \; C_D[R\,\mathrm{Cos}\wedge, 0] \\
&= \mathrm{Cos}^{\frac{1}{2}}\wedge \; C_D[R, 0] \qquad \cdots\cdots\cdots (30) \\
&= C_D[R/\mathrm{Cos}\wedge, 0].
\end{aligned}$$

These results also follow directly from exact solutions of the boundary layer equations as can be seen from the following argument. The equations of motion are now

$$\left.\begin{aligned}
u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} &= \nu\,\frac{\partial^2 u}{\partial z^2} \; , \\[2mm]
u\,\frac{\partial v}{\partial x} + w\,\frac{\partial v}{\partial z} &= \nu\,\frac{\partial^2 v}{\partial z^2} \; ,
\end{aligned}\right\} \qquad \cdots\cdots\cdots (31)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

Proceeding along classic lines, we infer from the equation of continuity the existence of a stream function ψ such that $u = \dfrac{\partial \psi}{\partial z}$, $w = -\dfrac{\partial \psi}{\partial x}$, and assuming that $\psi = \sqrt{\nu x U_o}.f(\eta)$, where $\eta = z\sqrt{U_o/\nu x}$, we derive the Blasius equation for $f(\eta)$ from the first equation of (31), viz.

$$f\,f'' + 2f''' = 0. \qquad \cdots\cdots\cdots (32)$$

We now write $v = \dfrac{\partial \phi}{\partial z}$, and assume ϕ is of the form $\phi = \sqrt{\nu x U_o} \cdot g(\eta)$. The second equation of (31) then reduces to

$$fg'' + 2g''' = 0,$$
$$\text{with } v = U_o g'.$$

$$\ldots\ldots\ldots\ldots (33)$$

From (32) and (33) we see that

$$f'''/f'' = g'''/g'' ,$$

and hence $f' = \text{const. } g' + \text{const.}$
But $f' = u/U_o$, and $g' = v/U_o$, and since $u = 0$,
$v = 0$, when $z = 0$, and $u = U_o$, $v = V_o$, when $z = \infty$,
we obtain finally

$$\frac{u}{U_o} = \frac{v}{V_o} .$$

As already noted, it follows that in this case the resultant flow in the boundary layer is everywhere parallel to the undisturbed flow. Equation (30) indicates that the drag coefficient is the same as that of an unyawed wing with chord equal to the distance from leading edge to trailing edge measured parallel to the direction of the undisturbed flow (i.e. $c/\cos\Lambda$) in a stream of velocity Q_o. This result is peculiar to laminar flow in the boundary layer and follows from the fact that for an unyawed $C_D \propto 1/R^{\frac{1}{2}}$. Any other C_D, R relation would not lead to this result even with the flow in the boundary layer everywhere parallel to the undisturbed stream flow.

3. 3. The Spanwise Flow in the Turbulent Boundary Layer.

To obtain the value of $(K)_{T.E.}$ when the boundary layer is partly or wholly turbulent the following assumptions will be made:-

(1) The distribution of v/V_o is the same for all x where the boundary layer is turbulent and is independent of the chordwise pressure gradient and therefore of the distribution of u/U_1,

(2) The distribution of v/V_o is then given by $v/V_o = (z/\delta)^m$, where m will be taken to be $1/7$.

It is highly unlikely that these assumptions describe exactly what happens in practice, but experimental evidence tending to indicate that they are sufficiently close to the truth for our purposes will be discussed in a subsequent paragraph (para. 5). We may note, however, that in the usual argument underlying the derivation of the power or 'log' laws for the velocity distribution in the turbulent boundary layer on a flat plate the development of the boundary layer with distance from the leading edge plays no direct part.

From this point of view, the argument might be expected to apply rather more readily to the span-wise flow in the boundary layer of a yawed wing than in the accepted case of the chordwise flow in the boundary layer of an unyawed wing.

With these assumptions we can write

$$\frac{\theta_{yx}}{\theta_{xx}} = \frac{U_1}{V_0} \int_0^{\delta/\theta_{xx}} \frac{u}{U_1} \left[1 - \left(\frac{z}{\theta_{xx}}\right)^{1/7} \left(\frac{\theta_{xx}}{\delta}\right)^{1/7} \right] d\left(\frac{z}{\theta_{xx}}\right).$$

From the evidence marshalled in Ref. 7 we may assume that $\frac{u}{U_1}$ is a function of z/θ_{xx} and H_x, only, whilst θ_{xx}/δ is a function of H_x, only. Hence we can write

$$\frac{\theta_{yx}}{\theta_{xx}} = \frac{U_0}{V_0} \left(\frac{U_1}{U_0}\right) K,$$

where K is now a function of H_x, only, given by

$$K = K(H_x) = \int_0^{\delta/\theta_{xx}} \frac{u}{U_1} \left[1 - \left(\frac{z}{\theta_{xx}}\right)^{1/7} \left(\frac{\theta_{xx}}{\delta}\right)^{1/7} \right] d\left(\frac{z}{\theta_{xx}}\right). \ldots \ldots \ldots (34)$$

The function $K(H_x)$ can be readily determined using the curves of Ref. 7. The value of H_x can either be assumed to be a constant over the wing as in the method of profile drag calculation described in Ref. 6, or it can be calculated as part of the chordwise profile drag calculations as discussed in Ref. 7. Hence the value of H_x at the trailing edge can be determined, and therefore we can calculate $K_{T.E.}$. Sample values of K are as follows:-

$H_x = 1.29$, $K=1$; $H_x=1.4$, $K=0.910$; $H_x = 1.5$, $K=0.777$.

Having determined K and H_x (and therefore $n=(H_x+2)/5$) at the trailing edge, the drag coefficient can be determined from equation (27).

For a flat plate at zero incidence it is reasonable to assume $\frac{u}{U_0} = \frac{v}{V_0}$, and hence K = 1. In this case (27) reduces to

$$C_D \left[R, \backslash \right] = C_D \left[R \cos \Lambda, 0 \right] . \cos \Lambda. \qquad \ldots \ldots \ldots (35)$$

For a fully turbulent boundary layer on the plate the 1/7th power law leads to

$$C_D \left[R, 0\right] = \text{const.} /R^{1/5}.$$

Hence

$$C_D \left[R, \Lambda\right] = C_D \left[R, 0\right] . \cos^{4/5} \Lambda. \qquad \ldots \ldots \ldots (36)$$

To illustrate the remarks made at the end of para. 3.2 we note that in this case the drag coefficient of an unyawed flat plate with chord $= c/\cos \Lambda$ would be

$$C_D \left[R/\cos \Lambda, 0\right] = C_D \left[R, 0\right] . \cos^{1/5} \Lambda$$

$$\neq C_D \left[R, \Lambda\right] ,$$

in spite of the fact that the flow is everywhere parallel to the undisturbed stream direction.

4. Specimen Calculations and Results

The method described above has been used to calculate the profile drag coefficient of a yawed wing of section NACA 64-012 at zero incidences for angles of yaw up to 45°, for Reynolds numbers of 10^6, 10^7 and 10^8 and for various transition positions from the leading edge to 0.5 c. Similar calculations, but covering a range of transition positions from the leading edge to the trailing edge, have also been made for a flat plate at zero incidence. For the aerofoil the calculations were made on the assumption that H_x was constant over the section and equal to 1.4. The results are shown in Fig. 1 (a) and (b).

It is of interest to note that although the variation with yaw of the drag coefficient of the flat plate is described by the factor $\cos^{4/5} \Lambda$ when the boundary layer is fully turbulent, and changes to the factor $\cos^{1/2} \Lambda$ as the transition moves back to the trailing edge, the corresponding factor for the wing section remains very closely $\cos^{1/2} \Lambda$ for transition positions between 0 and 0.5 c. The latter factor shows only slight variation with Reynolds numbers. It appears that for a wing, where there is usually a small positive pressure at the trailing edge helping to reduce the chordwise drag, the factor $\left(U_o/U_{T.E.}\right)^{n-1}$ in equation (27) weights the spanwise drag component as compared with the chordwise drag component, so that the resultant factor $C_D \left[R, \Lambda\right] / C_D \left[R, 0\right]$ is somewhat higher than that for a flat plate. More extensive calculations are required, however, to throw additional light on the effect of section shape and thickness.

5. Some Experimental Results

A few preliminary experiments have been made to check some of the assumptions of the above theory and the results. The experiments were made on a flat plate of 19" chord extending across an open jet tunnel (jet dimensions $3'6" \times 2'6"$) of top speed about 125 f.p.s. giving a Reynolds number of about 1.2×10^6. It was intended to check (a) whether the flow in the turbulent boundary layer of the yawed plate at zero incidence was everywhere parallel to the undisturbed stream direction,

(b) whether with the plate at incidence the spanwise
velocity distribution was independent of the chord-
wise pressure gradient, and (c) whether the drag
coefficient at zero incidence with a fully turbulent
boundary layer obeyed the law given by equation (35).

Unfortunately, at the low Reynolds numbers
of the tests it was impossible to secure a fully
turbulent boundary layer or a sharp transition with-
out the aid of a turbulence wire of such a size that
its drag coefficient was an appreciable and not
accurately known fraction of that of the plate.
The experiments were made with the plate at zero
incidence and at $3\frac{1}{2}^{o}$ incidence with various angles
of yaw up to 45°, and consisted of velocity and
yawmeter traverses through the boundary layer at
various stations aft of the leading edge.   From
these traverses the spanwise and chordwise velocity
traverses were deduced as well as the drag coeffi-
cient of the part of the plate ahead of the traverse
station

The yawmeter traverses made at zero inci-
dence with various angles of yaw showed no measurable
deviation anywhere in flow direction from that of the
undisturbed stream, thus checking the assumption that
at zero incidence the chordwise and spanwise velocity
distributions in the boundary layer were similar.
Some of the spanwise and chordwise velocity distribu-
tions measured at $x/c = 0.855$ at 45° yaw and 0° and
$3\frac{1}{2}^{o}$ incidence are shown in Fig. 2.   It will be seen
that whilst the chordwise distribution shows a marked
change of shape with change of incidence the spanwise
distribution shows comparatively little change.   The
latter falls between a 1/5th. and 1/7th. power law
for regions in the boundary layer near the surface
but fits a somewhat smaller power than 1/7th. in the
outer part of the layer.   Because of the lack of
measurements very near the surface it is difficult
to estimate the corresponding values of $H_y = (\delta_y^*/\theta_{yy})$
with satisfactory accuracy but the value appears
to be about 1.4.   In this connection we may note
that similar velocity traverses were made in the
turbulent boundary layer of a yawed wing of finite
span and are reported in Ref. 10.   The wing was of
elliptical plan form and of aspect ratio about 5.4,
it was tested for an angle of yaw of 25° and at
angles of incidence of 12° and 14°, at which latter
incidence part of the wing was stalled.   Neverthe-
less, even under these extreme conditions the
velocity profiles of the spanwise flow showed
themselves to be very nearly independent of the
chordwise pressure gradients.   An indication of
this is the fact that the value of H_y for the
spanwise profiles rarely deviated from a value
within the range 1.3 to 1.5, whilst the corresponding
values of H_x ranged from about 1.7 to 6.7.   The
evidence therefore tends to support the assumption
that the form of the spanwise velocity distribution
is relatively insensitive to the chordwise pressure
gradients.

The measured ratios of the drag coefficients $C_D[R,\wedge]/C_D[R,0]$ are shown in Fig. 1(b) for comparison with the theory. It will be seen that the experimental results show an even greater reduction of drag coefficient with yaw than is given by the theoretical $\text{Cos}^{4/5}\wedge$ law. This may well have been due to the drag contribution of the transition wire, this drag is almost entirely form drag and would vary as $\text{Cos}^3\wedge$.

It is of interest to note that the theory as developed in this paper leads to the following result for the thickness at a given chordwise position of the fully turbulent boundary layer:-
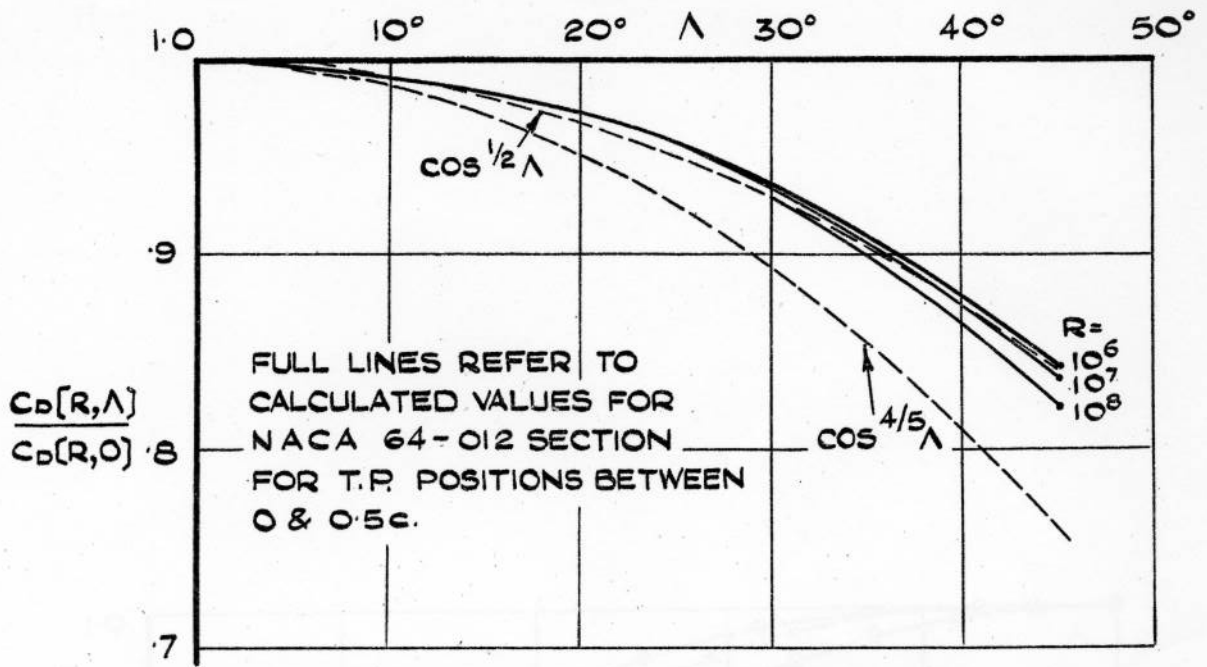
$$\frac{\delta[R,\wedge]}{\delta[R,0]} = \text{Cos}^{-1/5}\wedge,$$

where $\delta[\alpha,\beta]$ is the boundary layer thickness at the specified position for a Reynolds number equal to α and a yaw of angle β. This relation is shown in Fig. 3 for comparison with the measured values. The very close agreement between theory and experiment in this instance is probably in part fortuitous, but it is clear that the assumptions of the theory cannot be seriously wrong. For comparison the factor $\text{Cos}^{-4/5}\wedge$ is also shown, this defines what the ratio $\delta[R,\wedge]/\delta[R,0]$ would be if the development of the boundary layer on the plate depended solely on the distance from the leading edge measured in the direction parallel to the undisturbed stream and was otherwise unaffected by yaw.
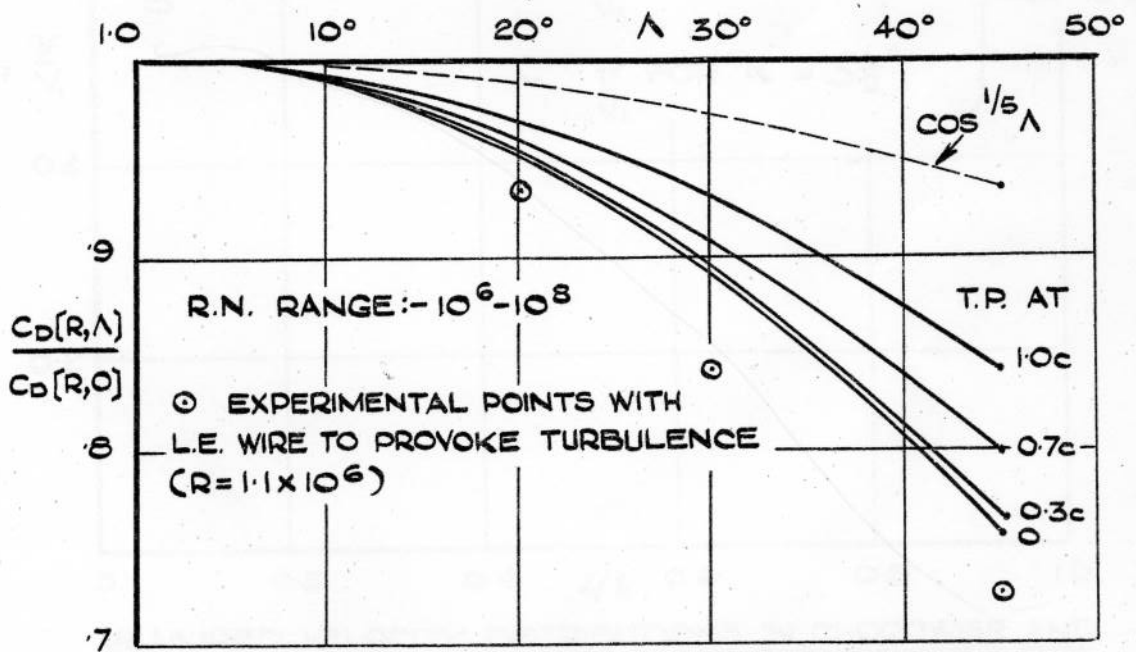
These experiments cannot be regarded as completely conclusive, and further work is required particularly at higher Reynolds numbers, but the results indicate that the assumptions made in the theory developed here are unlikely to lead to serious errors.

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R E F E R E N C E S
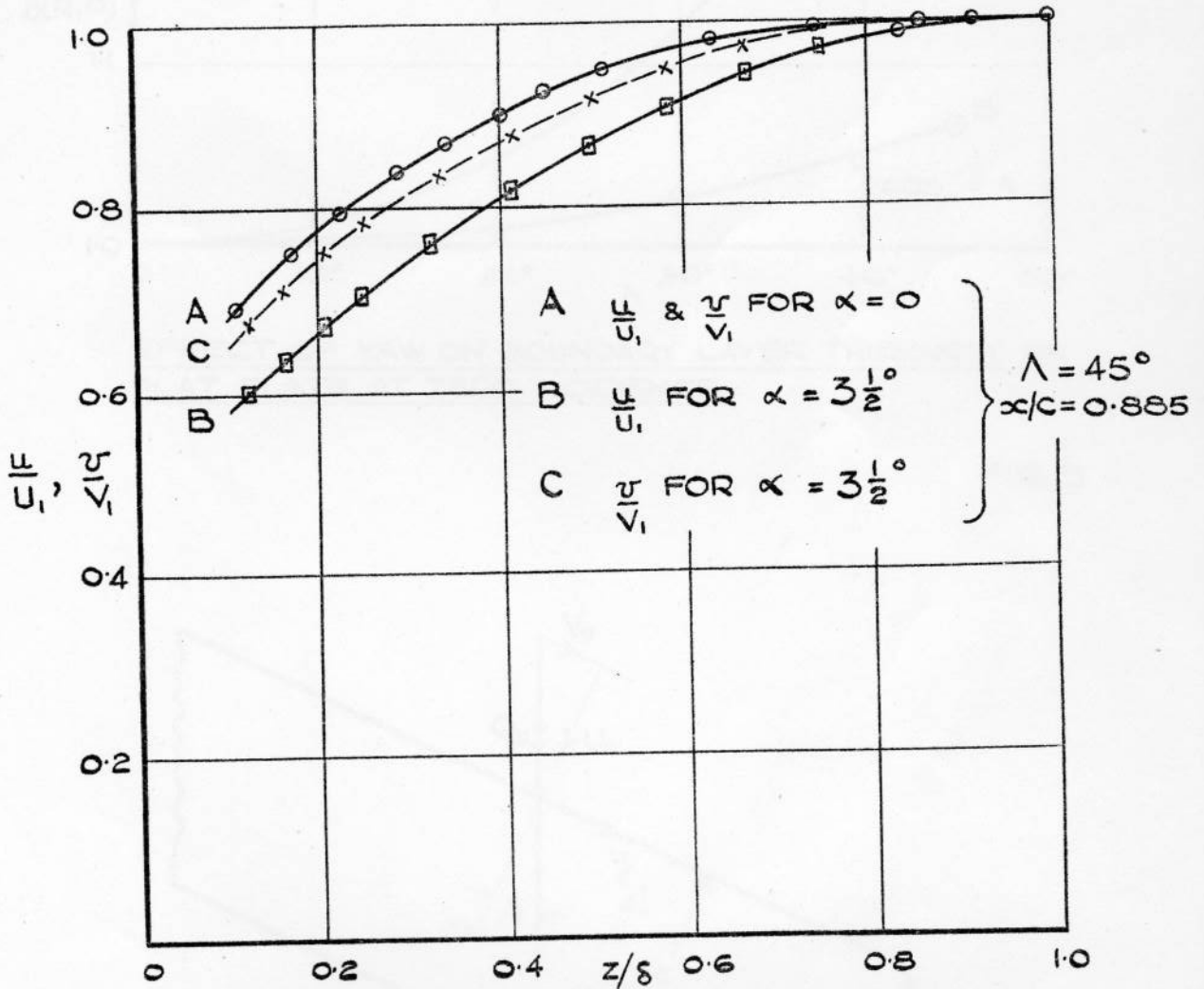
1.   Prandtl, L.   On boundary layers in
                                     three dimensional flow.
                                     (Volkenrohde R and T No.64,
                                     File No. B.I.G.S.84, 1946.

2.   Sears, W.R.   Boundary layer of yawed
                                     cylinders.   Jnl. Aero. Scs.
                                     Vol.15, Jan. 1948. pp.49-52.

3.   Wild, J.M.   Boundary layer of yawed
                                     infinite wings.   Jnl. Aero.
                                     Scs. , Vol.16, Jan.1949.
                                     pp. 41-45.

4.   Jones, R.T.   Effect of sweepback on
                                     boundary layer and separa-
                                     tion.   NACA T.N. No.1402.
                                                        (1947).

5.   Goldstein (Ed)   Modern Developments in
                                     Fluid Dynamics.   Vol.I,
                                     Ch.III (Clarendon Press,
                                     1938).

6.   Squire and Young   The calculation of the
                                     profile drag of aerofoils.
                                     A.R.C. R and M 1838 (1937).

7.   Doenhoff and   Determination of general
      Tetervin   relations for the behaviour
                                     of turbulent boundary layers.
                                     NACA T.R. No. 772 (1943).

8.   Nitzberg, G.E.   A concise theoretical
                                     method for profile drag cal-
                                     culation.   NACA ARC No.4805
                                                        (1944).

9.   Young, A.D.   Skin friction in the
                                     laminar boundary layer in
                                     compressible flow. Aero-
                                     nautical Quarterly, Vol.1
                                     Aug. 1949, pp. 137-164.

10.   Kuethe, McKee   Measurements in the
      and Curry   boundary layer of a yawed
                                     wing.   NACA T.N. No. 1946
                                                        (1949).

(a) CALCULATED VALUES OF $C_D(R,\Lambda)/C_D(R,0)$ FOR NACA 64-012 SECTION AT ZERO INCIDENCE.

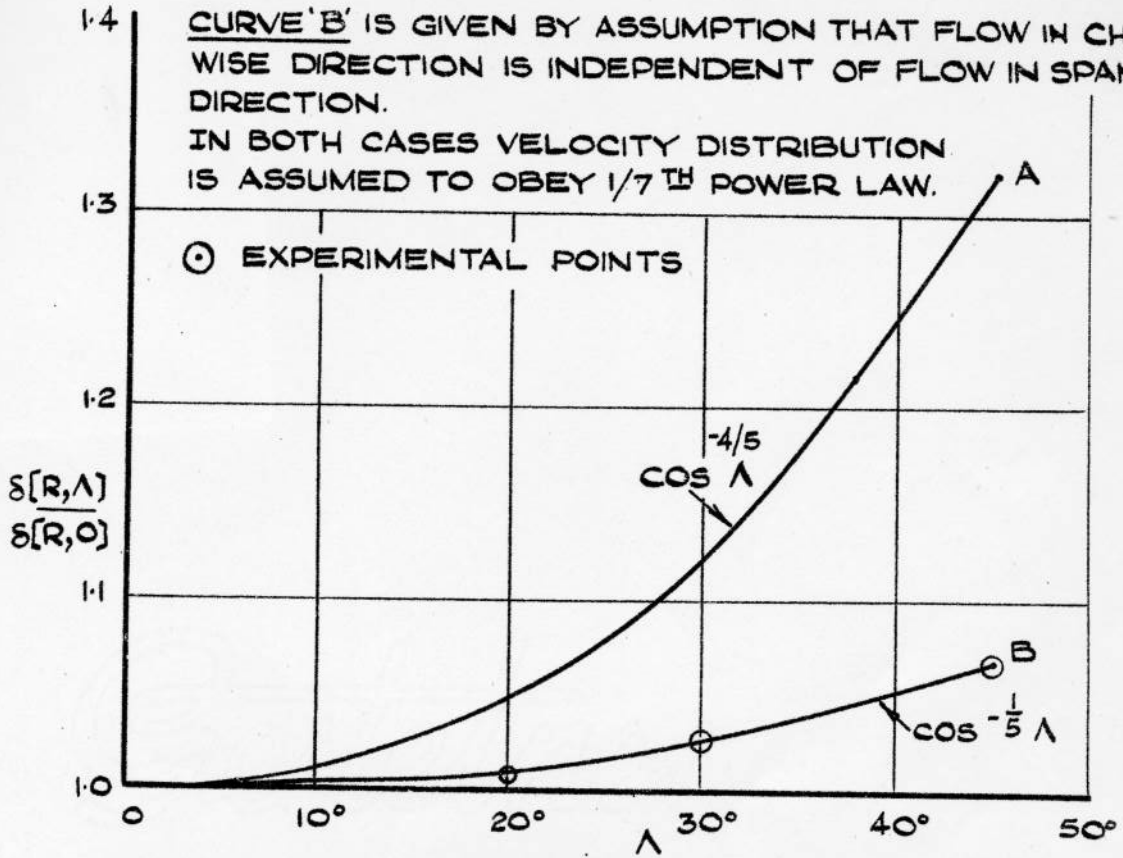(b) CALCULATED VALUES OF $C_D(R,\Lambda)/C_D(R,0)$ FOR FLAT PLATE.

FIG. 1

MEASURED VELOCITY DISTRIBUTIONS IN CHORDWISE AND
SPANWISE PLANES OF BOUNDARY LAYER OF YAWED
FLAT PLATE AT ZERO INCIDENCE AND AT INCIDENCE
OF $3\frac{1}{2}°$:-

FIG. 2.

CURVE 'A' IS GIVEN BY ASSUMPTION THAT FLOW
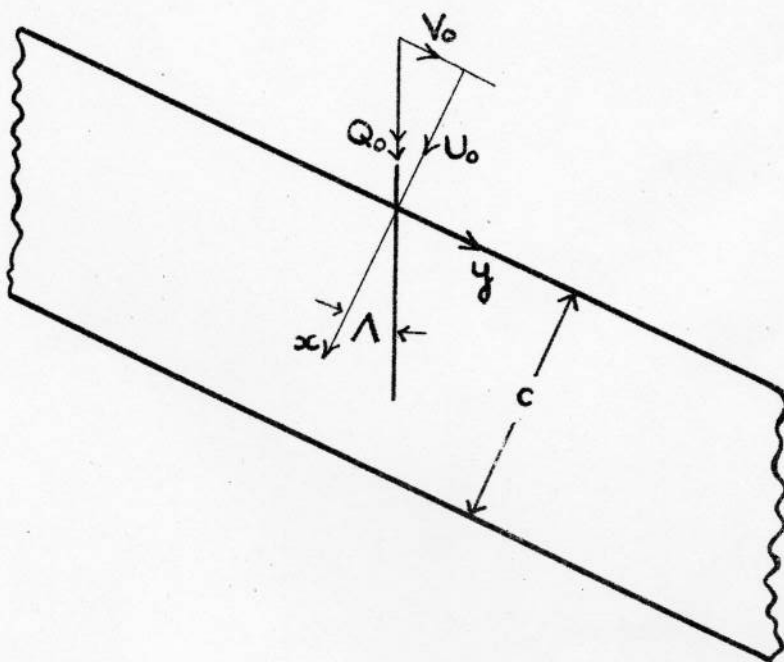IS SAME AS THAT OVER UNYAWED WING OF CHORD=C SEC Λ

CURVE 'B' IS GIVEN BY ASSUMPTION THAT FLOW IN CHORD
WISE DIRECTION IS INDEPENDENT OF FLOW IN SPANWISE
DIRECTION.
IN BOTH CASES VELOCITY DISTRIBUTION
IS ASSUMED TO OBEY 1/7 TH POWER LAW.

⊙ EXPERIMENTAL POINTS

$$\frac{\delta[R,\Lambda]}{\delta[R,O]}$$

$\cos \Lambda^{-4/5}$

$\cos^{-\frac{1}{5}} \Lambda$

EFFECT OF YAW ON BOUNDARY LAYER THICKNESS ON
FLAT PLATE AT ZERO INCIDENCE.

FIG. 3.

SKETCH ILLUSTRATING NOTATION

FIG. 4.