The Diffusion of Loads in non-rigid Circular Frames

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SUMMARY

This report extends the work of W.J. Goodey and gives numerical examples of the shear distribution around a frame subjected to a single concentrated radial load for variations in the parameters, such as, frame stiffness, skin thickness, stringer spacing, etc.

It also indicates when the beam theory distribution of shear can be used with a reasonable degree of accuracy.

The report contains a number of curves, figs. 5-17 giving the shear distribution around a frame for a single concentrated radial load of 1000 lb. The parameters chosen are those common to aircraft design, and it is possible to obtain a reasonably accurate shear distribution around a frame from the data supplied, without doing the actual shear calculation.

The case chosen is that of a long cylinder with a closed end, or that of a long cylinder where the portion aft of the loaded frame has a restraining effect upon the forward section, see fig. 1.

The appendix gives the method of obtaining the shear load due to a tangential load and moment from the radial load expressions.

Fig. (1)
NOTATION

The notation used is:

- \( R \) = the radius of the frame at the skin line, ins.
- \( t \) = the actual skin thickness ins.
- \( I_f \) = the moment of inertia of the adjacent frames about an axis parallel to the skin line ins.
- \( rI_f \) = the moment of inertia of the loaded frame about an axis parallel to the skin line ins.
- \( L \) = frame spacing ins.
- \( v \) = Poisson's ratio, value taken as 0.3.
- \( N_x \) = shear per ins in the skin.
- \( m \) = \( \frac{R^2L}{E} \)
- \( e \) = \( 3m^2 \left( \frac{2k_x(1 + v) - v}{2} \right) \)
- \( \varphi \) = \( \frac{R^2k_x t}{I_f} \)
- \( c \) = \( 6m^2 \varphi \)
- \( k_x \) = ratio of equivalent skin thickness to actual skin thickness, see Fig. (2)

\[ k_x = \begin{cases} \frac{(A_0 + bt)}{bt} & \text{if the skin is not buckled} \\ \frac{(A_0 + b't)}{bt} & \text{if the skin is buckled and the effective width of skin is } b' \end{cases} \]
INTRODUCTION

Goodey in his work considered two cases, namely (a) a long cylinder with the loaded frame at the free end,
(b) a long cylinder with the loaded frame at a distance from the free end.

The most practical case is that of (b) above, which was investigated by Goodey by considering a combination of the cases of a long cylinder loaded in the middle and a long cylinder loaded at its free end. See fig. (2).

SOLUTIONS

The solutions contained in this report are for case (b) above the expression for the shear in the skin at the loaded frame, due to a radially applied load \( W \) is given by.

\[
N_{x0} = \frac{W}{\pi R} \sum_{n=1}^{\infty} \left[ \frac{\left(\lambda_2+1\right)\left(\lambda_1-1\right)^2 - \left(\lambda_1+1\right)\left(\lambda_2-1\right)^2}{\lambda_1} \right] \frac{n \sin n \theta}{a_2} \left(\frac{\left(\lambda_1^2 + 2\lambda_1 a + a^2\right) - \left(\lambda_1^2 + 1\right)\left(\lambda_2^2 + 1\right) - a^2(\lambda_2^2 + 1) - 2a(1+a^2)\cos \theta}{2\pi R} \right)
+ \frac{W}{\pi R} \sin \theta. \quad (1)
\]

or

\[
N_{x0} = \frac{W}{\pi R} \sum_{n=1}^{\infty} \left[ \frac{a_n^2 \sin 2\lambda_1 a_n^2 - 2a \sin \lambda_1 a_n^2 \sin n \theta}{\lambda_1^2 + 2\lambda_1 a + a^2 - r(a^2 + 1)} - \frac{n \sin n \theta}{a_2} \frac{2a(1+a^2) \cos \theta}{2\pi R} \right] + \frac{W}{\pi R} \sin \theta. \quad (2)
\]

The above expressions for \( N_{x0} \) are given by Goodey, where for the loaded frame \( \lambda_2 = 1 \).

In expression (1) above \( \lambda_1 \) and \( \lambda_2 \) are obtained as shown below.

Solve the equation

\[
x^2 + UX + V = 0 \quad (3)
\]

where

\[
U = \left(\frac{n^2(n^2-1)}{a^2} \frac{(n^2-2a)}{4} - 1\right) \quad (4)
\]
The roots of equation (3) are $X_1$ and $X_2$.

Next solve the equations

$$\lambda + 1 = X_1 \quad \text{and} \quad \frac{1}{\lambda} = X_2$$

The roots of equations (6) are $\lambda_1$ and $\lambda_2$, and $\lambda_3$ and $\lambda_4$ respectively, where $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{\lambda_3}$.

The values of $\lambda$ used in the expression (1) for $N_{x0}$ are those less than unity.

If the roots $X_1$ and $X_2$ of equation (3) are less than 2.0 then we use expression (2) for $N_{x0}$.

"a" and "U" are obtained as shown below.

$$Y'' - (V+4)Y + U^2 = 0$$

where $U$ and $V$ are given by expressions 4 and 5.

The roots of equation (7) are $Y_1$ and $Y_2$, where one root is greater than 4.0 and one less than 4.0.

Let $Y_1$ be the root greater than 4.0 and $Y_2$ the root less than 4.0, we then solve the equations

$$(a + 1) = +\sqrt{Y_1} \quad \text{and} \quad (2 \cos \phi)^2 = Y_2$$

The values of "a" found are $a_1$ and $a_2$ where $a_1 > a_2$ and the value of "a" used in the expression (2) for $N_{x0}$ is that which is less than unity.

The value of "V" used in the expression (2) for $N_{x0}$ is the value given by $Y_2$ which is of opposite sign to "U" in expression (4). This is so, because

$$- U = 2(a+\frac{1}{\lambda}) \cos \phi$$

the term (a+\frac{1}{\lambda}) is always +ve hence $\cos \phi$ must be -ve, i.e. the sign of "U" must be opposite to that of "V".

The expression for $N_{x0}$ can be written in the form

$$N_{x0} = \sum_{n=1}^{\infty} F(\lambda_1 \lambda_2 k, r) n \sin n\phi + \frac{W}{2\pi} \sin \phi$$

or

$$N_{x0} = \sum_{n=1}^{\infty} F(a, \phi, k, r) n \sin n\phi + \frac{W}{2\pi} \sin \phi$$

In both expressions 9 and 10 the value of $F(\cdots)$ $n \sin n\phi$ for $n=1$ is equal to $\sin \phi$.

For practical purposes it is sufficiently accurate to take the $\sum_{n=1}^{\infty}$ term up to a value of $n=6$, hence the expression for shear can be written as
The curves of figs. 5-20 were obtained by putting numerical values in the expression (11) above. The parameters for each particular case are given in the accompanying curves.

I am indebted to Dr. Kirkby and the computing section of the Aerodynamics Department for the valuable aid given in the computation of the numerical examples.

Conclusions

1) The distribution of shear around the frame is not sensitive to variations in values of \( k_x \), the results show a maximum increase of 7% in value of \( N_x \) at \( 30^\circ \) (i.e., maximum value) for a 100% increase in value of \( k_x \). See figs. 15, 16 and 17.

2) The distribution of shear around the frame is not sensitive to variations in skin thickness \( t \). In aircraft structures going from one gauge to the next is an increase of 30% and for this increase in skin thickness the average percentage decrease in maximum value of \( N_x \) is 4%. This is assuming that \( k_x \) remains constant.

3) The distribution of shear is not sensitive for reasonably large variations in the moment of inertia of the adjacent frames, and to take a mean value of \( I_f \) is sufficiently accurate for the results. The effect can be obtained from fig. (5) where it is seen that increasing the stiffness of the adjacent frames, the loaded frame remaining constant increases the maximum value of \( N_x \). For the parameters chosen it is seen that the average percentage increase in value of \( N_x \) at \( 30^\circ \) is 0.12% for a 1.0% increase in value of \( I_f \).

4) The distribution of shear is not sensitive for reasonably large variations in the moment of inertia of the loaded frame, and a mean value is sufficiently accurate for the results. This effect can be obtained from figs. 12, 13, 14, 16, 19 and 20.

5) The distribution of shear is not sensitive for moderate variations in frame spacing. For a frame radius up to 40" the average increase in value of \( N_x \) at \( 30^\circ \) is 0.4% for a 1.3% increase in value of \( "m" \) and for a radius of 50" the increase in value of \( N_x \) at \( 30^\circ \) is 0.22% for a 1.0% increase in value of \( "m" \). This effect can be obtained from figs. 16, 19, and 20.

6) The beam theory distribution of shear is sufficiently accurate if the value of \( r_f \) lies above the line of fig. 21. This curve is obtained by extrapolation of the curves figs.12, 13 and 14, and is intended to give the lowest value of \( r_f \) for which the beam theory is a reasonable approximation.
7) The shear loading on frames adjacent to the loaded frame can be obtained with sufficient accuracy for practical purposes by interpolation. The shear distribution on a frame 2 frame spacings distant from the loaded frame can be taken as that given by the beam theory and intermediate frames can be obtained by a straight line interpolation between the frames, see fig. 4.

Fig. (4)

Reference

APPENDIX A

Derivation of the shear load due to tangential load and moment from the radial load expression.

Goodey in his work indicates a method by which the shear load for a radial load and moment may be obtained from the expression for a tangentially applied load. Here the complete solution is given for obtaining the shear load due to a tangential load and moment from the radial load expressions.

Let \( (N_x)_r = \text{shear load due to a tangentially applied load} \ T \)

This will be a function of \( T \) and \( \theta \).

\[
(N_x)_r = -Tf(\theta - \alpha) = -Tf(\theta) + T\alpha f'(\theta)
\]

when \( \alpha \) is small

\( Ta = W \) and \( T = T \cos \alpha \).

\[
\therefore \Delta (N_x)_r = T\alpha f'(\theta) = Wf'(\theta) = (N_x)_w
\]

\[
\begin{align*}
(N_x)_w &= \frac{d}{d\theta} W = \frac{d}{d\theta} (N_x)_r \\
(N_x)_r &= \left\{ (N_x)_w \right\}_{\theta=0} + \frac{T}{\pi} \int_0^\theta (N_x)_w \, d\theta
\end{align*}
\]

where the term \( (N_x)_w \) is a constant of integration and can be evaluated by considering the equilibrium of the frame.

\[
\begin{align*}
TR &= \int_0^{2\pi} (N_x)_r R^2 \, d\theta \\
&= \left\{ (N_x)_r \right\}_{\theta=0} 2\pi R^2 + \frac{T}{\pi} \int_0^{2\pi} (N_x)_w(\theta) R^2 \, d\theta \\
&= 2\pi R^2 \left\{ (N_x)_r \right\}_{\theta=0} + \frac{T}{\pi} \int_0^{2\pi} (2\pi - \theta)(N_x)_w(\theta) R^2 \, d\theta
\end{align*}
\]
-7-

\[ R^2 \left\{ \left( \frac{\partial}{\partial R} \right)^2 \left( \frac{N_{x \theta}}{R} \right) \right\} + \frac{\partial}{\partial R} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta = \int_0^R \left( \int_0^R \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \right) R \sin \theta \, dR \]

\[ \frac{\partial^2}{\partial R^2} \left( \frac{N_{x \theta}}{R} \right) + \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{N_{x \theta}}{R} \right) + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \left( \frac{N_{x \theta}}{R} \right) = \int_0^R \left( \int_0^R \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \right) R \sin \theta \, dR \]

\[ \left( \frac{N_{x \theta}}{R} \right) \bigg|_{\theta=0} = \frac{T}{2\pi R} + \frac{1}{2\pi R} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

which gives the constant of integration for (12) above.

\[ \left( \frac{N_{x \theta}}{R} \right) \bigg|_{\theta=\pi} = \frac{T}{2\pi R} + \frac{1}{2\pi R} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

For the case considered, of a loaded frame along a cylinder

\[ \left( N_{x \theta} \right) = \frac{W}{\pi R} \sum_{n=1}^{\infty} F\left( \frac{\pi R}{\lambda_n} \right) \sin n \theta \]

\[ = \frac{W}{\pi R} \sin \theta + \frac{W}{\pi R} \sum_{n=2}^{\infty} F\left( \frac{\pi R}{\lambda_n} \right) \sin n \theta \]

substituting in equation (13)

\[ \left( N_{x \theta} \right) = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

\[ = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

\[ = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

\[ = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

\[ = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

\[ = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]

\[ = \frac{T}{2\pi R} - \frac{1}{2\pi R} \sum_{n=2}^{\infty} \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{N_{x \theta}}{R} \right) R \sin \theta \, d\theta \]
\[
\begin{align*}
- \theta &= - \frac{T}{2\pi R} - \frac{T}{\pi R} \sum_{n=2}^{\infty} F(\frac{\pi}{n}) - \frac{T}{\pi R} \cos \theta + \frac{T}{\pi R} \sum_{n=2}^{\infty} \frac{6}{n^2} F(\frac{\pi}{n}) \cos n\theta - 1 \\
&= - \frac{T}{2\pi R} - \frac{T}{\pi R} \sum_{n=2}^{\infty} F(\frac{\pi}{n}) - \frac{T}{\pi R} \cos \theta + \frac{T}{\pi R} \sum_{n=2}^{\infty} \frac{6}{n^2} F(\frac{\pi}{n}) \cos n\theta \\
&= \frac{T}{2\pi R} - \frac{T}{\pi R} \cos \theta - \frac{T}{\pi R} \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \\
&= \frac{T}{2\pi R} - \frac{T}{\pi R} \cos \theta - \frac{T}{\pi R} \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
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&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
&= \left[ \frac{1}{2} - \cos \theta - \sum_{n=2}^{\infty} F(\frac{\pi}{n}) \cos n\theta \right] \\
\end{align*}
\]
VALUES OF SHEAR/INS FOR VARYING STIFFNESS OF THE ADJACENT FRAME, THE LOADED FRAME HAVING CONSTANT STIFFNESS.
RADIAL APPLIED LOAD OF 1000 LBS.

\[ \Psi^2 = \frac{4343 \text{ IN}^2}{155} = 8.666 \]
\[ \Gamma = 1 \text{ IN} \]
\[ R = 24.25 \text{ IN} \]
\[ K = 155 \]
\[ t = 0.028 \text{ IN} \]

- \( \Gamma_f = 0.05 \text{ IN}^4 \) \( R = 8.666 \)
- \( \Gamma_f = 0.10 \) \( R = 4.343 \)
- \( \Gamma_f = 0.20 \) \( R = 2.172 \)
- \( \Gamma_f = 0.40 \) \( R = 1.086 \)

1000 lb.
VALUES OF SHEAR IN INS FOR VARYING RADIUS OF FRAME. FLANGE, SPACING CONSTANT AT 15 INS. RADIUS, APPLIED LOAD OF 1000 LIS.

<table>
<thead>
<tr>
<th>R</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.2 INS</td>
<td>m = 1.62</td>
</tr>
<tr>
<td>12.4 INS</td>
<td>m = 2.67</td>
</tr>
<tr>
<td>8.0 INS</td>
<td>m = 4.0</td>
</tr>
<tr>
<td>5.0 INS</td>
<td>m = 6.0</td>
</tr>
</tbody>
</table>

FIG. 6
VALUES OF SHEAR/INS FOR VARYING RADIUS OF FRAMES AND CONSTANT FRAME SPACING.

RADIUS APPLIED LOAD OF 1000 lb.

L = 150 IN
I = 0.05 IN^4
r = 1.0
k = 1.53
L = 0.25 IN

- R = 162 IN    m = 1.08
+ R = 40 IN    m = 2.67
O R = 60 IN    m = 4.0
VALUES OF SHEAR/IN. FOR VARYING RADIalli 
AND FRAME SPACING, 
RADIAL APPLIED LOAD OF 1000 LB.

\[ R = 16.2 \text{ in.} \]
\[ R = 40.0 \text{ in.} \]
\[ \theta = 60^\circ \]
VALUES OF SHEAR/INS FOR VARYING RADII OF FRAMES
AND FRAME SPACING.

COLLEGE OF AERONAUTICS
REPORT No. 33.
VALUES OF SKEWNESS FACTOR FOR VARYING RADIUS OF FRAMES

\[ R = 40 \text{ IN.} \]

\[ R = 60 \text{ IN.} \]

\[ t_1 = 19.46 \text{ IN.} \]

\[ t_1 = 19.46 \text{ IN.} \]

\[ t_1 = 19.46 \text{ IN.} \]

\[ t_1 = 19.46 \text{ IN.} \]

FIGURE 11.
VALUES OF SHEAR/IN for VARYING STIFFNESS OF LOADED FRAME, FRAME RADIUS CONSTANT.
RADIAL APPLIED LOAD OF 1000 lb.

\[ I_f = 0.05 \text{ in}^4 \]

\[ R = 20 \text{ in} \]

\[ M = 2.0 \]

\[ k_c = 1.53 \]

\[ t = 0.28 \text{ in} \]

\[ r = 25 \]

\[ r = 50 \]

\[ r = 100 \]

\[ \Delta = \text{BEAM THEORY} \]
FIG. 13.
VALUES OF SHEAR/IN. FOR VARIATIONS IN VALUES OF $k_x$
RADIAL APPLIED LOAD OF 1000 lb.

<table>
<thead>
<tr>
<th>RADIUS</th>
<th>$m$</th>
<th>$I_f$</th>
<th>$r/t$</th>
<th>$N_{x0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.25 IN.</td>
<td>2.0</td>
<td>0.146 IN.$^4$</td>
<td>3.892</td>
<td>0.028 IN.</td>
</tr>
</tbody>
</table>

- $R = 24.25$ IN. $k_x = 0.5$
- $R = 24.25$ IN. $k_x = 1.0$
- $R = 24.25$ IN. $k_x = 2.0$

FIG. 15.
FIG. 17.

VALUES OF SHEAR/INS FOR VARIATIONS IN VALUES OF k_x

<table>
<thead>
<tr>
<th>k_x = 5</th>
<th>k_x = 0</th>
<th>k_x = 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>R = 60 IN</td>
<td>R = 60</td>
<td>R = 60</td>
</tr>
<tr>
<td>I = 2.4 =</td>
<td>I = 3.892</td>
<td>I = 0.25 IN</td>
</tr>
</tbody>
</table>

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VARIATIONS IN VALUES OF $N_{z0}$ AT 30° FOR VARYING VALUES OF $k_Nf$ AND $M_t$.
RADIAL APPLIED LOAD OF 1000 lb.

$R = 16.2$ INS.

$\rho = 0.05$ INS.

$\rho = 0.05$ INS.

$\rho = 0.05$ INS.

$\rho = 0.05$ INS.

$\rho = 0.05$ INS.

FIG. 18.

VARIATIONS IN VALUES OF $N_{z0}$ AT 30° FOR VARYING VALUES OF $k_Nf$, $M_t$ & $I_f$.
RADIAL APPLIED LOAD OF 1000 lb.

$R = 40$ INS.

$\rho = 0.05$ INS.

$\rho = 0.05$ INS.

$\rho = 0.05$ INS.

FIG. 19.
Variations in values of $N_x \theta$ at $\theta = 30^\circ$ for varying values of $I_f$, $m$, and $I_f$. Radial applied load of 1000 lb.

\[ k_x = 1.53 \]

\[ t = 0.28 \text{ in.} \]

\[ \begin{array}{c}
\text{o} & m = 162 \\
\text{+} & m = 20 \\
\text{x} & m = 40 \\
\end{array} \]

\[ R = 60 \text{ in.} \]

$I_f = 1946 \text{ in.}^4$

$I_f = 0.05 \text{ in.}^4$

Values of the moment of inertia of the loaded frame ($R \cdot I_f$) above which the beam theory distribution of shear gives a reasonable degree of accuracy.