Searching for High Density Material in Cargo Containers Using Gravity Gradiometry

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Introduction

Imaging cargo containers at ports in this country is an important task. Gravity gradiometry is a sensing technique which does not rely on using radiation to find fissile material, but instead uses only density. This makes it possible to detect suspect dense materials including any shielding that may be hidden inside cargo containers. Gravity gradiometry can provide better resolution than straightforward gravity readings, with the trade-off being less penetration power. Once measurements have been made external to the container, a density distribution must then be reconstructed.

A level set method uses a level set function to define two distinct regions based on the sign of the function at each point, the boundary being where the function is zero. It then uses a gradient-based iterative method to allow the shape to deform to better fit the data.

Genetic algorithms are methods which draw inspiration from the process of natural selection using steps such as crossover and mutation. By limiting the population size and by use of a reparameterisation the algorithm can work at the speed required for the time-constraints we have. Both methods have their strengths and weaknesses when applied to this real-life problem.

The Level Set Model

In our problem the whole domain is a cargo container, which we will denote by Ω. Our hope is to find a small region of high density material within Ω if there is such a region to be found. Denote this region as $D \subset \Omega$. The unknowns of the problem are the density distribution of the cargo container, denoted by ρ, and the domain D.

We could simplify the problem and assume we know the density in the two regions, and that it is a constant in both, ρ_i in D and ρ_e in $\Omega \backslash D$. So now the unknown is simply the domain D. Next the function $\phi : \Omega \rightarrow \mathbb{R}$

$$\phi(\boldsymbol{x}) = \begin{cases} \phi(\boldsymbol{x}) \leq 0 & \text{for all } \boldsymbol{x} \in D \\ \phi(\boldsymbol{x}) > 0 & \text{for all } \boldsymbol{x} \in \Omega \backslash D \end{cases} \tag{1}$$

is called the level set representation of D.

Gradient Based Inversion

An array of sensors will be used on the outer edge of the cargo container in the form of a sensor gate the cargo container passes through which can measure the gradient of gravity in all three coordinate directions. The gravitational potential of a point source is known to be

$$\Phi = \frac{\hat{g}m}{|\boldsymbol{r} - \boldsymbol{r_0}|} \tag{2}$$

where m is the mass of the point source and $\hat{g} = 6.672 \times 10^{-8} m^3/g/s^2$ is the gravitational constant. Calculating the second differential of Φ we store the measurements in a vector $\boldsymbol{d}$.

Next we use a finite element approximation of the cargo container, and divide it up into discrete cuboid elements known as voxels. The gravitational effect of each voxel is stored in a matrix G. The residual between calculated and measured data is

$$\mathcal{R}(\boldsymbol{\rho}) = G\boldsymbol{\rho} - \boldsymbol{d}. \tag{3}$$

A suitable measure of the size of the residual is known as the cost functional

$$\mathcal{J}(\boldsymbol{\rho}) = \frac{1}{2}||\mathcal{R}(\boldsymbol{\rho})||^2_{Da} = \frac{1}{2}\left\langle \mathcal{R}(\boldsymbol{\rho}), \mathcal{R}(\boldsymbol{\rho}) \right\rangle_{Da}. \tag{4}$$

An iterative method is then employed

$$\phi^{(n+1)} = \phi^{(n)} + \tau\delta\phi \quad \text{on} \quad \partial D \tag{5}$$

where $\delta\phi$ is found using the steepest descent method and τ is a chosen step length which suitably reduces the cost functional. Smoothing can also be used on the update.

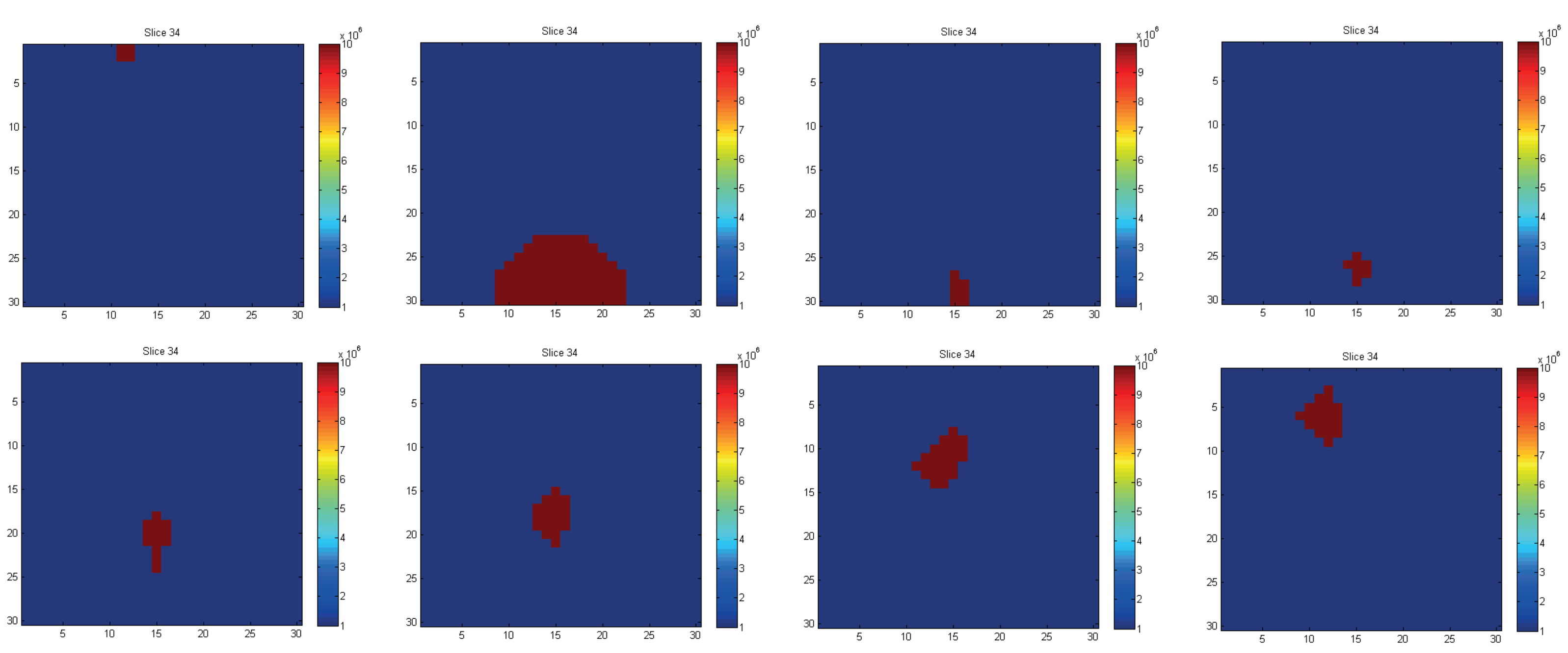

Figure 1: Level set method at various stages. Top-left is actual location. Second from left on top is the starting point and it continues right from there.

Genetic Algorithm Based Inversion

A genetic (or learning) algorithm relies on three main stages: selection, crossover and mutation, as described below and in Charbonneau [2002].

- Randomly select a density distribution $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_{nv})$ from $(0, \rho_{max})$.
- Reduce each voxel value to the interval $(0, 1)$ and concatenate a chosen amount, nd, of the digits into one string, known as a chromosome

$$\text{chromosome} = \hat{\rho}_{1,1}\, \hat{\rho}_{1,2} \,\cdots\, \hat{\rho}_{1,nd}\, \hat{\rho}_{2,1}\, \hat{\rho}_{2,2} \,\cdots \hat{\rho}_{nv,nd}. \tag{6}$$

- Create a population of chromosomes in this way and assign a fitness to each one based on how well they fit the data.
- Let $n = nv * nd$. Select two chromosomes based on fitness (higher fitness lead to higher probability of being chosen) and choose a random $k \in (1, n)$. The crossover stage involves cutting each chromosome at gene k and swapping the strings

$$\text{chromosome}_1 = c_{11}\, c_{12} \,\cdots\, c_{1k-1}\, c_{2k} \,\cdots\, c_{2n} \tag{7}$$

$$\text{chromosome}_2 = c_{21}\, c_{22} \,\cdots\, c_{2k-1}\, c_{1k} \,\cdots\, c_{1n}. \tag{8}$$

- To avoid stagnation each gene has a probability p_{mut} that it will mutate, which is chosen due to current diversity.
- When the population of the offspring matches that of the parents the method begins again. However the best fitting parent is allowed to survive and replace one offspring to aid convergence.

An Extended Level Set Model

The genetic algorithm on a voxel basis is slow and so we must take steps to improve it. First we follow the colour level set approach, which involves multiple level set functions defining multiple domains as used in Natasha Irishina and Moscoso [2010]

- $D_1 = \{\boldsymbol{x} : \phi_1(\boldsymbol{x}) \leq 0\}$ (air/wood)
- $D_2 = \{\boldsymbol{x} : \phi_1(\boldsymbol{x}) > 0$ and $\phi_3(\boldsymbol{x}) \leq 0\}$ (plastics, some metals)
- $D_3 = \{\boldsymbol{x} : \phi_1(\boldsymbol{x}) > 0$ and $\phi_2(\boldsymbol{x}) > 0$ and $\phi_3(\boldsymbol{x}) > 0\}$ (iron, steel)
- $D_4 = \{\boldsymbol{x} : \phi_1(\boldsymbol{x}) > 0$ and $\phi_2(\boldsymbol{x}) \leq 0\}$ (lead).

Each level set function is made up of radial basis functions

$$\phi(a, \boldsymbol{b}, c, \boldsymbol{x}) = \sum_{i=1}^{m} a \exp\left(-\frac{1}{2c^2}||\boldsymbol{x} - \boldsymbol{b}||_2^2\right) \tag{9}$$

to reduce the number of unknowns.

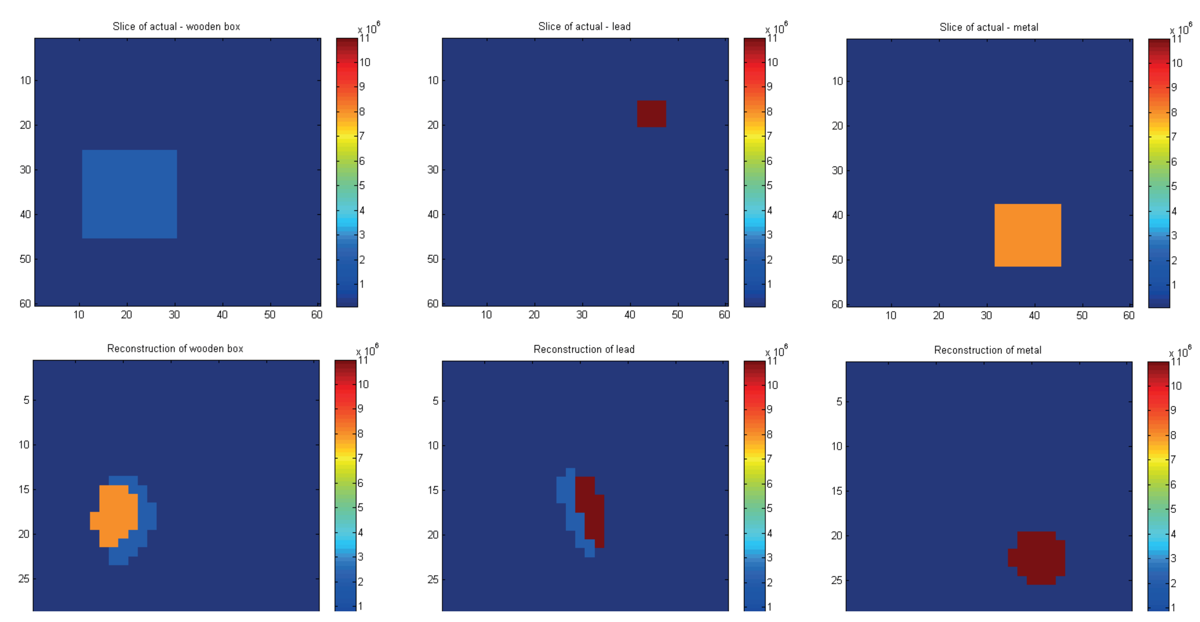

Figure 2: Top layer is actual location of objects. Bottom layer is reconstructed location.

Summary and Conclusions

Given enough time the genetic algorithm should find the global minimum. However we cannot specify how long this is since it varies even when running for the same example, and it is difficult to know when it has converged.

Gradient-based algorithms work much faster and slow down when approaching a minimum, but it is possible to find a local minimum instead. Also the weighting used to counteract the lack of penetration power can have an effect.

References

Paul Charbonneau. An introduction to genetic algorithms for numerical optimization. 2002.

Oliver Dorn Natasha Irishina, Diego Alvarez and Miguel Moscoso. Structural level set inversion for microwave breast screening. *Institute of Physics Publishing, Inverse Problems*, 26:1–26, 2010.

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