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FLUID FLOW ANALYSIS USING THE BOUNDARY ELEMENT METHOD

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TO MY PARENTS

Abstract

An Attempt at development of theory and computer program for the solution of potential and viscous flow problems using the Boundary Element Methods is introduced in this work. A program for potential flow analysis of cascades is developed and properly validated. Theory and programs for boundary element analysis of viscous flow using vorticity-velocity formulation and pressure-velocity formulation along with the penalty function approach were developed. Many tests were carried out on the programs, and case studies of Poiseuille, Couette and driven-cavity flows were analysed and results were compared with existing solutions. The developed vorticity-velocity algortithms converge well for low Reynolds number, but the program based on the penalty function approach has limited sucess only for cases without domain integrations.

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Nomenclature

Symbols	Definitions
b	Body force term given in Eq. (3.1).
C_i	Coefficient appearing in all BIEs which is defined in Appendix B.
$\mathbf{F}$	Body force term in the elasticity analysis discussed in Appendix G.
F	Generic field function defined to be used in Eq. (6.26) and others related to singular domain integration.
F_{kl}	Fundamental solution for traction defined by Eq. (5.9).
$\mathbf{f}$	Body force in the Navier-Stokes equation given by Eq. (4.2).
G_{kl}	Fundamental solution for velocity given by Eq. (5.8).
g	Parameter defined in Chapter 6, which is also given by Eq. (G.19).
h	Total head used in Eq. (4.14) and defined in Chapter 4.
IR	Residual of the numerical solution defined in Chapter 6.
$J(\xi)$, $J(\xi,\eta)$	Jacobians.
ℓ	Direction cosine.
m	Direction cosine.
$N(\xi)$, $N(\xi,\eta)$	Interpolation functions used to describe the field function within the boundary element and domain cell, respectively.
P^*	Generic kernel used in Eq. (6.26) .
p	Static pressure.
q	Normal derivative of the field variable u in Chapter 3.
q^*	Normal derivative of the fundamental solution u^* in Chapter 3.

$\vec{q}$	Displacement vector used in the elasticity analysis.
R_Ω	Residual of the domain solution used in Chapter 3.
R_Γ	Residual of the boundary solution used in Chapter 3.
$\vec{r}$	The position vector.
r	Magnitude of the position vector that in general represents the distance between the source point and the field point.
S^*	Generic kernel used in the Section 6.2.
T_x, T_y	Traction components in x- and y-directions, respectively.
u^*	Fundamental solution for Poisson's equation used in Chapters 3 and 4.
u	General scalar variable adopted in Chapter 3 or an array with the solution over the boundary for the velocity and traction components used in Chapter 6.
$\tilde{u}$	Approximate solution in Eq. (3.4).
$\vec{v}$	Velocity vector.
v	Magnitude of the velocity vector.
v_n, v_t	Normal and tangential velocity components, respectively.
v_x, v_y	Velocity components in x- and y-directions, respectively.
v_∞	Free stream velocity.
W	Relaxation factor.

Greek Letters

α_i	Corner angle given in Figure B.1.
Γ	Boundary of the solution domain.

δ	Dirac delta function appearing in Eq. (3.8).
ε	Value of the residual below of which the solution is considered converged.
η	Intrinsic coordinate.
θ	Generic field function adopted in section 6.2.
λ	Penalty function parameter.
λ'	Lame's constant.
μ'	Lame's constant.
ν	Kinematic viscosity.
$\bar{\nu}$	Parameter defined by Eq. (5.10).
ξ	Intrinsic Coordinate.
ρ	Density.
ϕ	Velocity potential variable.
ψ	Stream function variable in Chapter 3.
$\vec{\psi}$	Vector potential function used in Chapter 4.
Ω	Define the solution domain.
$\vec{\omega}$	Vorticity vector.
ω	Vorticity component in two-dimensional problems.

Table simbols

DWN	Normal derivative of vorticity.
I	Boundary node number in the first set of results and domain node number in the second set (Tables 7.1 to 7.5).

VX	Velocity component in x-direction.
VY	Velocity component in y-direction.
W	Vorticity.
WP	Vorticity on the boundary.
WX	Derivative of vorticity in x-direction.
WY	Derivative of vorticity in y-direction.
X	X-coordinate of the node I.
Y	Y-coordinate of the node I.

Abbreviations

BEM	Boundary Element Method.
BIE	Boundary Integral Equation.
BIEM	Boundary Integral Equation Method.
CFD	Computational Fluid Dynamics.
FDM	Finite Difference Method.
FEM	Finite Element Method.
FVM	Finite Volume Method.

Chapter 1

Introduction

1.1- OVERVIEW

The set of differential equations governing the flow of a viscous fluid, the so-called Navier-Stokes equations, were first proposed at the beginning of this century. These equations along with the principle of conservation of mass, known as the Continuity equation, are the basic equations used in fluid flow analysis. When, temperature is also a variable, the Energy Transport equation has also to be considered. Together, these form a set of partial nonlinear differential equations that, unless some additional assumptions are introduced, mean that a complete analytical solution is impossible. Some simplifications are usually employed in the mathematical modelling of any real flow, and in order to obtain an analytical solution further simplifications may be required, which leads to the fact that acceptable analytical solution, are only available for very limited applications. Prior to the computer era, experimental analysis was much more important than theoretical methods and was the main tool adopted to provide solutions to engineering problems and the analysis of complex fluid phenomena. Only after the introduction of computers, did people start to look back to the basic governing equations in the emergence of computational numerical analysis techniques.

Today, Computational Fluid Dynamics (CFD) is indispensable for engineering and encompasses several methods. In the wake of the increasingly powerful computers many numerical techniques for fluid flow analysis have emerged. For example, the Method of Characteristics, the Panel Method, Finite Deference

Methods, Finite Element Methods and many others. Most of these are also suitable for the analysis of heat transfer phenomena which are dominated by the convective process. The ideas behind some of those methods were available before the computer era, but only with the advent of computers did they start to be exploited and studied more seriously. The important point is that the computational solution began to compete with the experimental approach after the 60's, when the most important and powerful techniques adopted nowadays in CFD were established. With realistic engineering problems, solutions have evolved which are progressively more able to cope with the complexity introduced through geometry and nonlinearities due to turbulence, heat transfer, shock waves, etc. Favourable to computational analyses, is that computing expenditure is now, no longer a problem, since computers have become more capable and much lower priced. Conversely, the cost of experimental solutions constantly increases. However, although nowadays the numerical simulation is a much more important tool to the solution of engineering problems, experimental simulation is still very important and has played an important role in validating numerical solutions. With more accessible computers, the tendency is clearly toward greater reliance on computer based predictions in engineering design. This work is interested in the numerical methods adopted in CFD.

Nowadays, there are lots of numerical techniques used in CFD, most of them with a particular field of application, such as external or internal flows, subsonic or supersonic flows, heat transfer, and so on. Therefore, there are many peculiarities in each numerical technique adopted in CFD. An alternative is to define an area of interest, like aeronautics, turbomachinery, etc. in order to concentrate the research. The aim of this work is to explore the techniques adopted in the solution of internal subsonic flow using the Navier-Stokes equations. The Finite Difference Method (FDM) and the Finite Element Method

(FEM) are without doubt the most important so far used to solve this class of problems. In fact, these methods have a much broader range of applications than any other in CFD. In addition, they are also applied in other fields of engineering, including structural analysis from which they were originated.

The Finite Difference Method (FDM) is a numerical technique for the solution of ordinary and partial differential equations. The domain of the given problem is discretized point-wise into a grid of nodes where the unknown parameters of the problem are located. Expressing the derivatives of the field function in terms of the values of the function at proximate nodes, a differential equation can be reduced to an algebraic linear equation. The unknowns in this equation are the values of the variable of the problem at the nodes of the mesh and the coefficients are geometric parameters involving the distance between the nodes. In the beginning, truncated Taylor expansions to represent the derivatives, which results in the so-called central, forward and backward differencing schemes, were mainly adopted. The conventional FDM approach applied in CFD is discussed in many references, mainly in Roache (1982), who analyze the flow in terms of vorticity and stream-function as dependent variables. In the early stages, the treatment given to flow problems was basically the same as the one used to solve problems in other areas. However, in early 70's new concepts were introduced with the so-called Finite Volume Method (FVM) or Control Volume Method, Patankar & Spalding (1972), that gave a new impetus to the numerical fluid flow analysis. This approach is normally regarded as a hybrid, lying between the classical FDM and the FEM. But, no distinction is made between the FVM and the traditional FDM throughout this work. The important point is the fact that after this, many new algorithms emerged as a consequence of this new approach.

In the solution of the Navier-Stokes equations, having pressure and velocity as dependent variables, basically only the artificial compressibility method discussed in Chorin (1967) and the Marker-and-Cell (MAC) given in Harlow & Welch (1965) were available. These approaches are applied to unsteady state analysis, however, a number of algorithms were proposed later to couple the pressure to the velocity. The most widely adopted Semi-Implicit Method for Pressure-Linked Equations (SIMPLE) of Patankar & Spalding (1972) was the first. Others followed such as the SNIP (Start with New Integration for Pressure) of Pun & Spalding (1977), SIMPLER (SIMPLE Revised) of Patankar (1980), SIMPLEC (SIMPLE Consistent) and so on. These different algorithms proposed can be regarded as attempts to improve convergence.

Also, it was very soon realized that the stability of the algorithm depended largely on the accuracy in the evaluation of the convective and nonlinear terms of the Navier-Stokes equations. Central differencing was an alternative to representing the convective term at the early stages, but that gives good results only for low Reynolds number (Re) flow. For high Re a new way of representing the convective term had to be worked out. New ideas were proposed, among the most widely adopted is the Upwind interpolation scheme. However, more flexible schemes like the Hybrid, Spalding (1972) and Power-Law differencing scheme, Patankar (1980), were later proposed. Recently, some new schemes, classified as high-order, for example, QUICK (Quadratic Upstream Interpolation for Convective Kinematic), Leonard (1979), have been proposed for reducing the artificial viscosity problem.

The classical inconvenient of the FDM related with the limitation on discretizing complex geometry was eliminated also with the introduction of new techniques to generate the mesh. The most important technique is the so-called Body Fitted Coordinate, Thompson *et al.* (1982)

The FDM, or the FVM, is nowadays a very powerful numerical tool to solve fluid flow problems and very popular. In fact, there are many commercial codes based on this technique, worldwide adopted in engineering design, such as: PHOENIX, FUENT and FLOW3.

The Finite Element Method (FEM) is also a numerical technique to solve differential equations. The first step is to obtain the corresponding integral formulation of the differential equations by using Variational or Weight-Residual Methods. Then, the domain of the problem is discretized in a piece-wise way into a number of small subdomains, or finite elements, where the governing equations for each domain are obtained. If the variation of the field function inside of the subdomains is described in terms of an interpolation function and nodal values of the field functions, the governing equations are transformed using one of the techniques mentioned above into an algebraic equation. Finally, the algebraic equations corresponding to each finite element can be assembled together to generate a simple system of algebraic equations, whose solution gives the distribution of the field function at the nodes in the domain.

Very few fluid dynamics problems can be expressed in a variational form. Thus, most finite element applications in fluid flow analysis have adopted the Weighted-Residual approach using the Galerkin method to solve the weighted-residual expressions.

The first indication that the FEM could be also applied to solve fluid flow problems was given by Zienkiewicz & Cheung (1965). After that a great amount of research has been carried out using FEM to solve fluid problems. In the 70's many works were published on this subject, for example, Taylor & Hood (1973) and Oden & Wellford (1972). That decade was also important because the first conference on FEM in fluid flow was held, Oden *et al.*(1974), and the potential of the FEM begun to be exploited. The FEM entered the 80's as a well established numerical tool to solve fluid flow problems where books specializing on this subject were also published. The application of traditional FEM to fluid mechanics is treated more recently by Baker (1983).

In the beginning, the solution using pressure and velocity as dependent variables presented problems related with numerical instability, as happened with the FDM. In this period, the vorticity-stream function approach was mostly adopted to avoid this problem, Baker (1983). Eventually, the problem was overcome with the introduction of special interpolation functions to represent the convective term including a similar effect to the Upwind scheme.

The Penalty Function approach was also adopted in the solution of incompressible flow. In this approach, the term containing the pressure is replaced by another one given in terms of the penalty function parameters. It was soon realized that in order to improve the convergence this term had to be integrated using the so-called Reduced Integration technique. This approach is reviewed in Hughes *et al.* (1979). The advantage of the Penalty Function is that the incompressibility condition is satisfied and the pressure is obtained as a post-processing.

The Vorticity-Stream Function formulation approach in fact, was where the FEM succeeded initially, for the solution of two-dimensional incompressible flow, for example, Taylor & Hood (1973). The main reason was because the algorithms are more stable, along with the advantage of having the pressure removed from the equations. The main disadvantage is that this formulation is not adequate to deal with three-dimensional flow.

Nowadays, there are some variations of the conventional FEM that are making relative success. The most important of them seems to be the Spectral Method. This method adopts orthogonal functions for interpolating and weighting functions used in the Galerkin approach, Fletcher (1984).

These two methods briefly summarized above, are already well-established for solutions of CFD problems. They are basically at the same level in terms of the complexity of applications and the accuracy of results. Considering the number of works published, it seems that the FDM is preferred to FEM. The fact that FEM is more complex, from the mathematical point of view, than the FDM can be given as one reason. While the FEM is nowadays mathematically well understood, the achievements of the FDM, on the other hand, were obtained with improvements based on a physical understanding of the phenomena. This characteristic may have helped the FDM to become more popular. It seems also that due to its simplicity, the FDM is more flexible in terms of algorithms as indicated by the great number of alternative schemes and techniques found in the literature.

One problem that affects these methods, regarded as domain methods, is the problem of the discretization of the potential and boundary layer regions of the flow. In most applications, a considerable amount of the domain can be treated as potential flow, which is a simplification that reduces the difficulties of the solution. With domain methods, these regions are treated using the same approach, which means that unnecessary calculations are carried out. The problem is even more serious for high Reynolds number flow, since the boundary layer region is smaller. Some techniques, such as the multi-grid method used in FDM, have been used in order to restrict the use of a fine mesh only to the boundary layer region. The potential region is discretized using a coarse mesh, which means less nodes and a system of algebraic equations with less equations. In the case of external flow, a vast region around the body is normally discretized in order to allow the imposition of the free-stream condition. Again, unnecessary calculations have to be performed. When the potential flow is considered, these domain methods are not appropriate because the domain has to be discretized. In the case, for example, of potential flow through cascade of blades of compressors, fans, etc. other methods are sometimes adopted, like the Panel Method. In spite of this drawback, domain methods are still adopted very often. As an example see Baskharone & Hamed (1981).

In this work, the Boundary Element Method (BEM) applied to a fluid flow solution is investigated. This technique is relatively recent and may overcome the problems with domain methods mentioned above. It can be considered as well-established in the field of structural analysis, where it was first applied, and has now already achieved a certain level of maturity in the field of fluid flow solution. The fact that this method has been given recently special attention by those working with CFD, is indicated by the growing number of the publications in this field. However, although the results presented so far are encouraging, the application of this method to solve complex engineering problems should be regarded as a matter of academic interest only.

The BEM has been under investigation by the Finite Element Group of the School of Mechanical Engineering at Cranfield for about 10 years, with a primary interest in the structural analysis and heat conduction areas. This work is, therefore, the first investigation of fluid flow using the BEM at Cranfield.

1.2- OBJECTIVES OF AND MOTIVATIONS FOR THE PRESENT WORK

The main objective of this work is the development of a theory and computer programs to solve potential and viscous flow problems using the Boundary Element Method.

The initial aim of this work was the development of a program for viscous flow analysis using the BEM. The decision to develop also a program for potential flow analysis was taken later. The main motivation for the development of a simple program came after it has been realized that there are many situations in fluid mechanics where a potential model can be adopted to investigate the phenomena. The flow through a cascade of turbomachinery blades is an important example of a situation where in most cases the potential analysis can at least give a good indication of the behaviour of the flow. Hence, the development of a program intended to solve potential flow in cascades formed by turbomachinery blades was included in the objectives.

The analysis of viscous flow using the BEM is the main interest of this work. Unfortunately, unlike potential analysis, the problem in this case is nonlinear and the flow domain needs also to be discretized, along with the boundary. The main advantage of the BEM over domain methods is thus lost. However, some formulations adopted in connection with the BEM have the characteristic of limiting the solution to the viscous region of the flow only. Thus, in this case, only the viscous region of the flow needs to be discretized. This seems to be the best effective way available in order to reduce the effect of the discretization of the domain in this situation. The vorticity-velocity formulation is the most important of those included in this category.

An essential objective of the work, is the development of a program based on the vorticity-velocity formulation, where a new set of equations is attempted. The program is aimed, in the first case, at the solution of simple cases for the validation purpose, because of the difficulties involved in the development of a suitable program.

Because of the difficulties faced during the development of the vorticity-velocity formulation program, it was decided at a certain stage of this work, to start in parallel the development of another program, based on a different formulation. The penalty function approach was chosen for the main reason that a program could be generated based on a program for stress analysis purposes. This seemed to be the best option at that time, considering the short time left to carry out this task. Thus, a program based on the penalty function approach for solving viscous flow was also included as an objective of this research.

1.3- OUTLINE OF THE THESIS

The idea behind the BEM is discussed in Chapter 2, where a comprehensive literature survey of the method applied to fluid flow solution is also presented. However, more emphasis was given to the discussion of the literature on the vorticity-velocity formulation and on the most important formulations derived from the Navier-Stokes equations given in terms of primitive variables (pressure and velocity). The penalty function approach is included in this later category

In Chapter 3, the boundary integral equation (BIE) corresponding to Poisson's equation is derived. The BIE relative to Laplace's equation is given as a particular case. Finally, a discussion of the equations to solve potential flow is presented.

The boundary integral equations corresponding to vorticity-velocity formulation are derived in Chapter 4. The differential equations based on vorticity and velocity are derived initially and it is shown that the analysis can be split into two parts: kinematics and kinetics. Afterward, the kinematic equation is transformed into BIE to give an origin to an integro-differential approach. Finally, the kinetic equation is also given in integral form in order to obtain a fully integral approach. The two different ways of representing the integral equations for the kinetics are given. In parallel, new integral equations to represent the kinematics and kinetics are proposed.

The penalty function approach is discussed in Chapter 5 where the formulation in differential form, based on the Navier-Stokes equations, is given first. Then, using the analogy with elasticity analysis, the corresponding boundary integral equations are derived.

Chapter 6 is entirely dedicated to the discussion of the numerical aspects of the application of the BEM to solve the BIEs presented in the previous Chapters. The way the equations were discretized are discussed and the discretized equations are presented. The approaches adopted to deal with singular integrals are discussed, along with the corner treatment adopted. The steps of the algorithms adopted in the programs to solve the BIE corresponding to the potential analysis and to solve the set of BIEs corresponding to vorticity-velocity and penalty function formulations are discussed.

In Chapter 7, typical case studies of internal laminar incompressible flow are considered using the programs developed and the results obtained assessed against analytical or numerical solutions from published works.

Finally, conclusions and recommendations for future works are summarized in Chapter 8.

Chapter 2

Literature Review

2.1- INTRODUCTION

Engineering problems has reached a level of great complexity to a point where there is no alternative but to solve many of them using numerical methods for the determination of an approximate solution. Nowadays, the importance of numerical methods in engineering calculations is evident, but their success is mainly associated with the emergence of powerful digital computers. Among these methods are at least two important techniques widely adopted to solve engineering problems namely the Finite Element Method (FEM) and the Finite Difference Method (FDM). However, it is true that there is no single technique that can solve efficiently every problem. Thus, searching for improvement in already well established techniques and even for the introduction of new methods is continuous. The Boundary Element Method (BEM) emerged few years ago as a result of this process and represents another alternative to engineers and scientists. This relatively new technique has become recognized as a powerful numerical tool to solve engineering problems. In some classes of problems the BEM, presents enormous advantages over traditional domain numerical methods, for example, Finite Element and Finite Difference methods. This document is turned to the discussion of the BEM applied to fluid mechanics.

2.2- BACKGROUND

The boundary element method (BEM), or boundary integral equation method (BIEM), is based on the integral representations for the solution of the partial differential equations. The derivation of various boundary integral formulations, which may be called boundary reduction, has a long history. One can appreciate that, for instance, more than a century has passed since the first application of integral equation methods. In these early stages, due to the difficulty in finding analytical solutions, theoretical investigations were mainly carried out by mathematicians. However, the motivation for early boundary reduction was in constructing an expression for the solution, especially for problems with an infinite or semi-infinite domain, so as to establish the existence of the solution for certain boundary value problems but not for the purpose of numerical calculation. Even in the classical works of Kellogg, Muskhelishvili, Mikhlin, and Kupradze, mentioned in many books on BEM, for example, Brebbia *et al.* (1984), one finds that this method is analytical rather than numerical. The methods developed by Trefftz and Prager in 1917 and 1928, respectively, mentioned in Beskos (1987), for solving integral equations in potential fluid flow theory might be considered the precursors of modern boundary integral techniques, even though such methods are really impractical without the use of a computer. Not until the 60's has the method been gradually considered a numerical technique, a period that was characterized by the widespread use of computers. This combines the theoretical results of classical integral equations and the practical techniques of the FEM discretization and is the consequence of extensive use of digital computers. The BEM showed a considerable expansion and development during the 70's, in step with the rapid improvements in computers, and approached a level of maturity during the first half of the 80's.

The term "Boundary Element Method" emerged in the 70's to indicate the surface discretization character of the method, and this name has been universally acknowledged, in preference to the former term "boundary integral equation method". However, the latter term is still being used by some authors. These terms are sometimes misleading since they give the idea that the integral equations used contain only boundary integrals, and so only the boundary needs to be discretized. However, non-linear problems normally include domain integrals in the integral equations and the domain has thus to be discretized as well. Hence, a new term "Boundary-Domain Integral Method" was introduced very recently, Zagar *et al.* (1990), especially for this situation, but it is still adopted by a few authors. The term BEM is likely to continue in use for all cases.

The BEM has come now to mean both the procedure of reducing the governing differential equations into integral equations on the surface of the domain and the numerical solution procedures for the integral equations. According to the way by which the boundary integral equation is formed and to the meaning of its unknowns, boundary element methods can be classified into two groups: the direct and the indirect approaches. The direct approach, indicates the fact that unknown functions appearing in the integral equation are the actual physical variables of the problem. In this case, the integral formulations are obtained with the help of certain fundamental integral theorems and connect directly the unknowns with the known boundary quantities. Conversely, the indirect approach can be used, which means that the unknowns, usually called density functions, have no direct physical meaning. In other words, the discretized integral equations are first solved for the density of singular solutions over the boundary surface. Then the remaining boundary quantities are computed in terms of these densities. Even though it has been shown by some authors that the indirect and direct BEM are formally equivalent, more emphasis is usually given to the direct BEM because they are more appealing to scientists and engineers.

Because of its generality with respect to geometry, its simplicity of the input data required, and its numerical accuracy, the boundary element method is applied to some extent in engineering and scientific calculations, mainly to solve problems with infinite or semi-infinite domains. Nowadays the BEM has a wide acceptance. The main advantage of the method is to reduce the domain of calculation by one dimension. Once the values of variables in the nodes on the boundary, such as the specific densities, potential and flux, displacement and traction, have been obtained, one can calculate readily the related physical quantities at any point in the domain under consideration. But one of the most significant advantages of the BEM is the capacity for transforming a differential equation into a full boundary integral equation (all integrals are performed on the boundary). Unfortunately, this is only possible for linear and a few classes of nonlinear problems. Many problems of engineering interest can not be represented by full boundary integral equations. In this case, the BEM loses part of its striking advantages over domain methods since domain integrals appear in the integral formulation. The numerical solution of the integral equations now requires that the domain be discretized as well. Researchers have been trying to find means to eliminate those domain integrals or at least to transform them into boundary integrals. Until now, however, the only effective way possible is when the integrands are constant or linear, which may occur only in very few practical situations.

The mathematical difficulty involved with the BEM is a factor that has prevented the increase in its popularity among engineers, together with the fact that the BEM is less forgivable to approximations made in the elaboration of the program. Also, most engineers are more familiar with the FEM and the FDM because they are still the methods adopted to solve the majority of engineering problems. One can not, therefore, expect that they will change their minds overnight. However, the particular mathematical problems, such as domain integrals, that appear in the formulation, have been under investigation and became a major target for many researchers in this area. It is, therefore, expected that solutions for the current difficulties will be proposed within a few years.

One of the most difficult mathematical tasks that one has to carry out when using the BEM is the derivation of the fundamental solution of the governing differential equation. That is the primary assumption required in the boundary reduction. The use of the fundamental equation makes the method rely on linear characteristic equations and means less versatility for the problems of variable coefficients and non-linearities. Nevertheless, the applicability of the boundary element method to non-linear problems, such as elastoplasticity, viscoplasticity, large deflection analysis and viscous flow, has been demonstrated in the areas of solid mechanics and fluid mechanics.

The first practical application of the BEM in engineering was in the area of solid mechanics. Since then, the BEM has been successfully applied to the solution of a wide variety of problems in this area. Applications in solid mechanics were responsible for most progress made in the use of this technique, as has happened to the FEM and FDM. Until recently, the BEM was a technique turned to solve problems only in this area. Nowadays, however, one can see that its applications have spread to other areas of engineering, such as acoustics, fluid mechanics, heat transfer, corrosion, geomechanics, etc. There are some very good textbooks on the BEM, such as Banerjee & Butterfield (1981) and Brebbia *et al.* (1984), which contain chapters dedicated to the discussion of the BEM applied to many fields of engineering science. However, the BEM literature is recently becoming richer and richer in textbooks devoted to specialized subjects. In particular, the BEM has recently received much attention by people involved with fluid mechanics as indicated by the increasing number of publications in this area.

Included is the first book entirely dedicated to viscous flow solution, namely, Kitagawa (1990). In fact, it has already gained the confidence of some researchers in this area since the results are promising. There is, however, a long way to go before it reaches the status of conventional methods, such as the FEM or FDM, so as to be used on a large scale in solving engineering problems of fluid mechanics.

The method developed by Trefftz in 1917 for solving integral equations in potential fluid flow theory can be considered as the first fluid mechanics problem solved by the BEM (the numerical value of the contraction coefficient of a round jet issuing from an infinite tank). Also, it can be considered the first axisymmetric problem in which the fundamental solution was expressed by an elliptic integral, and the first nonlinear free surface problem. Nevertheless, the first important application of the BEM in fluid mechanics area was made in the field of aerodynamics through the pioneering work of Hess in the late 50's and early 60's. In these early stages, the term "boundary elements" was not established yet and the integral equation technique proposed by Hess was called the Singularity Method or Panel Method. But, using the new nomenclature, his method is classified as an indirect boundary elements method. However, many people still prefer to adopt the former nomenclature. This method has been successfully applied to inviscid flow analysis for more than 20 years and is recognized as a powerful numerical technique. Besides, it offers all the advantages of the BEM since it is applicable to potential flow (linear) and so the integral formulation includes only boundary integrals. In fact, it is still one of the most reliable techniques adopted by the aircraft industry in aerodynamic calculations.

Nevertheless, most fluid phenomena that appear in nature can not be simplified to an inviscid case, the problem having to be solved considering the effects of viscosity. This kind of flow is governed by the Navier-Stokes equations. It was only recently that applications of the BEM to the solution of viscous problem were investigated, because of the difficulty related to dealing with the non-linearity of the governing equations. As mentioned before, this is a kind of problem where domain integrals appear in the integral formulation and so difficulties are encountered. The main advantage of the BEM (no need of domain discretization) is lost in this case. Unfortunately, there is, as yet, no effective way to transform those domain integrals into boundary integrals. Some proposed approaches, however, have the advantage of confining the calculations over the domain to the viscous, or vortical, region of the flow. This region is in general small and a great amount of computational time can be saved. These approaches are in fact more effective for external flows (especially flows around streamlined bodies) than for internal flows. In extreme cases of recirculating flow (driven-cavity flow, for example) it becomes ineffective. However, nowadays these approaches seem to be the only way available in order to reduce the harmful effect caused by the appearance of domain integrals in the formulation. In the field of viscous flows, several formulations and approaches employing different dependent variables have been presented in the literature.

2.3- LITERATURE SURVEY

2.3.1- Vorticity Based Formulations

Searching for a more efficient numerical approach to solve high Reynolds number external viscous flows led J. C. Wu and his co-workers at Georgia Institute of Technology (U.S.A.) through a series of publications to initiate a research programme with the goal of removing some of the difficulties experienced in computing such flows. As a result, they have begun to use the BEM, along with a formulation involving vorticity (ω) and velocity (v) as dependent variables, to solve viscous flow problems, Wu & Thompson (1973). The idea of recasting the Navier-Stokes equations in terms of velocity and vorticity as dependent variables is not new and was suggested by Lighthill in 1963. In this way, it is possible to separate the set of equations into a kinetic part, which deals with the change of the vorticity field with time (equation of transport of vorticity), and a kinematic part, which relates the velocity field at any instant of time to the vorticity at that instant. Regarding the BEM, this formulation is the first and best known in recent years, among several types of formulation available. Wu & Thompson (1973) developed initially an integro-differential formulation, where the BEM is adopted only in connection with the kinematic part of the problem, while the kinetic part is solved by FDM. The main advantage of using an integral representation for the kinematic part is that it permits the explicit point-by-point computation of the velocity. As a result, the calculation can be concentrated to the vortical region of the flow, as will be shown later. Also, this formulation allows the flow field to be segmented. Each segment can be solved independently from the others, Wu *et al.* (1974). This technique of segmentation can be used to drastically reduce the amount of computation required. It also permits, for example, the boundary layer region of the flow to be treated separately from the detached viscous region, Wu & Gulcat (1981). The ability to treat the boundary layer and the detached regions of the Navier-Stokes flows separately is of particular importance to the computation of high Reynolds number flows. This is because the problem related to length scale of different regions of flow field is removed.

In this interval, Wu & Wahbah (1976) solved steady enclosed flow problems using this formulation. The important step, however, given in this work, was that they managed to produce an integral representation for the kinetic part of the flow as well. Both kinematic and kinetic parts were, therefore, represented by integral equations. They indeed took advantage of the fact that the equations for the kinematic and kinetic parts are, in the case of steady-state flow, similar (elliptic equations) to derive the integral representation for kinetic part. Additionally, the kinetic integral equation derived contains a boundary integral involving the total head, requiring the calculation of the boundary pressure (p) distribution. The important point is that the property of concentrating the calculation to the vortical region of flow field was preserved. A brief review of Wu's research programme at that time and some discussion about the future work in this subject is presented by Wu (1977).

The previous works deal only with laminar flow. Wu *et al.* (1977), however, managed to extend this formulation to consider turbulent flow as well. It is shown that the numerical procedures previously established for steady and unsteady laminar flows move in a straightforward manner to turbulent flow. Although this work discusses some aspects related to unsteady flow problem, it is mainly dedicated to the steady flow case. Later, El-Refaee *et al.* (1982) proposed a formulation to deal with compressible flow.

An integral representation for the time-dependent vorticity transport equation was proposed by Wu (1982). An algorithm to solve time-dependent laminar incompressible viscous flow problems, using integral representations for both kinematic and kinetic aspects of flow, is presented. The derivation of the time-dependent kinetic integral representation is more involved than the steady case since now the vorticity transport equation is parabolic. In Wu *et al.* (1984), again the solution of time-dependent Navier-Stokes flow is discussed. But this time they analyse a hybrid procedure where the integral representation is used only in the solution of the flow regions adjacent to the solid boundary. In regions far from the solid boundary the FDM is used to solve the flow. Further discussions about BEM in fluids using this formulation was presented by Wu (1984 and 1985). Recently, Wu (1987) used the viscous flow problem to describe physical perceptions associated with the BEM, where the procedure presented in Wu &Wahbah (1976) is discussed in more detail.

In the field of viscous flow, the research programme carried out by J. C. Wu and his co-workers seems to be the first important step in order to solve this kind of flow using the BEM. Most of their analysis so far has been mainly limited to external flow since they are more interested in the solution of problems in the field of aerodynamics. However, they pointed out, and gave an example that, this formulation can be applied to internal flow, as well. They indeed managed to solve problems considering many factors such as time-dependence, turbulence, compressibility and three-dimensionality. Although some authors have criticized their approach, mainly due to the difficulty of dealing with three-dimensional and compressible flows, it is certainly promising. The main advantage of the ω-v formulation is the fact it allows the calculation of the velocity components to be carried out explicitly. As a result, the analysis can be concentrated on the vortical region of the flow only.

Bharadvaj *et al.* (1987) proposed an approach based on vorticity and vector potential to analyse incompressible unsteady viscous flow that has some similarity with Wu's approach. In their approach, the phenomenon is studied as a limit of a sequence of infinite accelerations followed by intervals when diffusion occurs. The equation for velocity is obtained using Helmholtz decomposition and it is shown that the correct boundary condition to be imposed in this case is the normal component of velocity both for inviscid and viscous flow. At the same time, the tangential velocity boundary condition is automatically satisfied. On this point they disagreed with Wu's approach where both normal and tangential velocities are specified as boundary conditions in order to create the conditions to calculate the vorticity distribution on the boundary using the kinematic equation. They also claimed some advantages for their approach in comparison with Wu's approach, namely simplicity when dealing with three-dimensional flows and compressible flows. It is worth mentioning that the vorticity transport equation was solved by the FDM. Their approach seems interesting, but it is too early to assess its applicability since there are just few published paper on this subject.

In the mid 80's, P. Skerget and his co-workers began to employ basically the same vorticity-velocity formulation as J. C. Wu, associated with the BEM, with the objective of analysing internal viscous flow. Accordingly, new issues were introduced by them. In one of their earliest works of a series, Skerget *et al.* (1984) presented solutions of the steady Navier-Stokes equations for some simple laminar internal flow cases (Couette, Poiseuille and driven-cavity flows), using the ω-v formulation. Their work is mainly based on Wu's proposal presented above. However, it is valuable to point out that they preferred to adopt a different integral representation for the kinetic part, that does not consider the pressure distribution on the boundary. Thus, the problem is solved for ω and v alone, but the normal derivative of vorticity is introduced in place of the pressure. The pressure in this case is obtained as a post-processing with the solution of a Poisson-type equation for pressure.

However, attention was paid also to the formulation involving the pressure in the same fashion as Wu's approach in the works of Skerget *et al.* (1985a, 1985b, 1986a, 1986b and 1987) and Skerget, Alujevic, Kuhn and Brebbia (1987). They applied this procedure to solve some test cases. For example, flow in an open cavity (used by Wu & Wahbah (1976), as well), flow in a closed cavity with rotating cylinder inside, Couette flow and flow over a backward facing step. It is pointed out that the inclusion of the pressure in the algorithm provides more stable results. However, only in Skerget *et al.* (1986a) are there results for the pressure distribution around the boundary. Pressure distribution inside the domain was not obtained.

An important contribution was made by Skerget *et al.* (1986a) who managed to produce an integral representation for the Energy equation of the flow, and so the temperature was included as a dependent variable along with vorticity, pressure and velocity. The ω-v formulation including the integral representation for the Energy equation is used to analyse the mixed-convection flow problems in Skerget *et al.* (1987) and in Skerget *et al.* (1988), where the thermally driven cavity flow is adopted as a test case. Also, the integral representation for the Energy equation corresponding to natural convection flow was derived and used in connection with the ω-v-p and ω-v formulations by Skerget, Alujevic, Kuhn and Brebbia (1987). Here the thermally driven cavity flow was analysed. Results for mixed convection flow in a horizontal concentric annulus with a heated rotating inner cylinder were also presented.

The time dependent ω-v-T formulation was used to analyse two-dimensional natural convection flow inside a thermally driven cavity and inside a horizontal channel (Bernad's cellular flow) by Skerget, Kuhn, Alujevic and Brebbia (1989). The importance of this work, apart the fact that time-dependence was introduced, is that it was carefully elaborated and more detail about the formulation is given. It can be considered as a good review of the derivations concerned with the vorticity-velocity formulation applied to two-dimensional flow. The time dependent laminar recirculating flow was analysed by Skerget *et al.* (1989), where the flow inside a channel with a square obstacle at various Reynolds number was used as the study case. The results show good agreement with those obtained using FEM.

Relevant derivations to the extension of this formulation to time dependent three-dimensional flow was presented in Skerget, Brebbia and Kuhn (1988) and an application is found in Kuhn *et al.* (1989), where the external natural convection flow around inclined cylinders is analysed. It is of note that due to the difficulty of handling three-dimensional flow, the subdomain technique had to be adopted in that work in order to reduce the computer time and memory demands. Later on, the time-dependent three-dimensional laminar flow, corresponding to the combined forced and free convection in the entrance region of an inclined tube, was analysed by Zagar *et al.* (1990). Isotherm contours and velocity fields are given for three different positions of the tube, however no quantitative comparisons with other methods were presented and the validation relied on a qualitative analysis of the results. They mentioned that proper validation could not be carried out because of the fact that the coarse mesh used was insufficient to describe the flow near the wall. Another case for time dependent three-dimensional free convection was analysed by Skerget *et al.* (1990), where the thermally driven flow inside a unit cube is solved. The fluid motion in this case is due to heating on the front wall, where the temperature is prescribed, while the temperature on the rear wall is kept equal to zero. All other boundaries are assumed to be adiabatic.

A review of derivations concerning time dependent three-dimensional laminar flow is presented in Zagar, Skerget and Alujevic (1990). The importance of this work is in the fact they managed to extent the formulation to include turbulence. The final boundary integral equations based on time average differential equation is given. However, only the results for two-dimensional laminar flow corresponding to free convection around a cylinder and the natural convection inside a square cavity with water, due to temperature difference from the left to right side, are presented. More recently, Rek *et al.* (1991) solved a simple case for turbulent flow. They analysed the Poiseuille flow using a purely algebraic effective viscosity. One conclusion is that a much finer mesh should be adopted in order to achieve satisfactory results. At the same time, Alujevic *et al.* (1991) made an important contribution to the analysis of turbulent flow using the BEM. The derivations of all equations are discussed and the two-dimensional turbulent flow inside a duct analysed using an algebraic model of turbulent viscosity based on Prandtl's mixing length theory. The results obtained using two different models to determine the mixing length (due to Nikuradse and Van Driest) are not so different to the results given by the FEM. Results for the case of laminar natural convection flow inside a square cavity with water, investigated by Zagar, Skerget and Alujevic (1990) were also presented. It is worth mentioning that again the subdomain technique was adopted.

The research programme carried out by P. Skerget and his co-workers employed basically the same formulation proposed by J. C. Wu, but has been concentrated on the analysis of internal flows. The results they have achieved so far are promising and, as a contribution, they managed to extent this formulation to consider temperature in the analysis. They also extended it to solve three-dimensional problems, but it seems this formulation is less attractive in this case.

The investigations carried out by both J. C. Wu and P. Skerget and their co-workers, individually, have been a significant contribution to the development of the ω-v formulation. It has, therefore, reached a reasonable level of maturity and, in spite of some continuing difficulties, it seems promising.

A vorticity-stream function integral approach was developed by Onishi *et al.* (1984), for the time-dependent Navier-Stokes equations, where the kinetics are represented by a vorticity transport equation similar to the one proposed by J. C. Wu. A Poisson equation relates the stream function and vorticity. They used this formulation to analyse some simple internal flow cases, such as, Poiseuille flow, flow over a backward facing enlargement and driven-cavity flow. Their analysis, however, was limited to low Reynolds number flow. This formulation was extended to consider temperature as a dependent variable in Onishi *et al.* (1985), where some applications are presented that include the analysis of natural convection flow in a closed compartment. An integro-differential scheme involving stream function, vorticity and temperature was developed by Onishi (1986) and applied to analyse natural convection flow in a compartment. Tosaka & Onishi (1986) employed three different formulations to analyse steady and unsteady state natural convection in a closed cavity. The vorticity-stream function formulation is included along with a new formulation that considers only the stream function and temperature as unknowns.

Camp & Gipson (1987 and 1989) solved the nonlinear biharmonic equation describing steady two-dimensional viscous flow of an incompressible fluid at low Reynolds number. The governing equations for the flow field is reformulated into a set of coupled nonlinear Poisson-type boundary integral equations, in terms of vorticity and stream function. In order to achieve this, fundamental solutions for the biharmonic and Laplacian operators were derived. An important contribution is the way they discretize the domain. Linear isoparametric elements were used on the boundary while the domain was divided into a series of triangular areas, each formed implicity by a set of three vertices. Two are the end nodes of each boundary element and the other the source point under consideration. Each of the elemental triangular regions is divided into a series of smaller triangular areas.

The effect is to concentrate quadrature points in a region close to the source point where the fundamental solution is singular, and so to "fan" the region about the point in question. Relaxation had to be used in order to get convergence of the iterative solution process. Two examples of flow were presented : the circular moving-wall problem (where the flow is completely enclosed and generated by the moving wall) and flow through an array of impermeable cylindrical fibres.

The vorticity-stream function formulation is not generally applicable, since very seldom will boundary conditions be known in terms of vorticity, stream function and their normal derivatives. Very recently, Rodriguez-Prada *et al.* (1990) proposed the use of a new set of fundamental solutions that provide a complete coupling between the stream function and vorticity equations, this avoids a special treatment of boundaries in which there are two specifications for the stream function (stream function and its normal derivatives) and none for the vorticity. They applied their procedure to analyse the flow inside a square cavity whose upper lid is moving at a constant velocity. Although the results presented are good, the algorithm did not converge for Reynolds numbers larger than 300. It is worth mentioning that the boundary element equations were solved by the regular boundary elements method.

For the sake of completeness concerning this formulation, an alternative integral formulation for the solution of the biharmonic equation governing steady two-dimensional viscous flow of an incompressible Newtonian fluid (considering Navier-Stokes equations without the inertial terms) was derived by M. A. Kelmanson, using stream function and vorticity as dependent variables. His approach is employed in the analysis of several problems, such as, steady flow in a rectangular cavity with a sliding wall and in an infinite channel containing a symmetrical constriction in the form of a step, Kelmanson (1983a), and free surface flow, Kelmanson (1983b). An extension of this method was applied by Ingham & Kelmanson (1983) to a problem in lubrication technology.

The success of the vorticity-stream function formulation has been much associated with the solution of enclosed flows (cavity flow, for example) and low Reynolds flows, which can be represented by harmonic and biharmonic equations. It does not, therefore, seem to be a valid choice for high Reynolds number flow inside a open channel.

2.3.2- Formulations Using Primtive Variables

Formulations involving pressure and velocity as dependent variables are more familiar to us because the Navier-Stokes equations, in their differential representation, are normally recast in a form that contains these variables. In terms of integral equation representation, this formulation is earlier than any other because integral equations governing flow of incompressible viscous fluid have been available since the early part of this century, due to the pioneering work of Oseen (1927). He developed exact integral representations, along with the corresponding infinite space fundamental solutions, for both two- and three-dimensional flows. However, unless approximations are introduced, solutions can only be obtained for the simplest geometries and boundary conditions, even under steady state conditions (see Dargush & Banerjee (1990)). Youngreen & Acrivos (1975) applied numerical integration along with the method of collocation to these integral equations to examine steady Stokes flow (steady slow incompressible viscous flow - creeping flow) past arbitrary shaped objects, basically three-dimensional and axisymmetric problems. The advantage of this process is that the fundamental solutions for this problem are equivalent to the corresponding ones in linear elasticity (the well known Kelvin solutions) with

Poisson's ratio equal to 0.5. This initial application of the direct boundary element method to viscous flow utilised constant surface elements. The next major step was taken by Bush & Tanner (1983), who included nonlinear terms in the two-dimensional integral formulation as presented by Oseen, in the solution of a number of interesting problems at low Reynolds number. For example, two-dimensional flow in a converging channel (Hamel flow), drag experienced by a sphere and axisymmetric free jet flow. The formulations proposed were also based on the Navier equations of elasticity, using the concept of "pseudo-forces". Once again constant boundary elements were used, but now three-node linear cells were introduced throughout the domain of interest.

After the initial stage, the formulation involving pressure and velocity as dependent variables were focused by some researchers, mainly in Japan, which brought a new impetus to this formulation. As a result, a series of interesting works paper have been published. Kakuda &Tosaka (see Tosaka and Onishi (1985)) published two papers (in Japanese) where they used this formulation along with the penalty function approach, usually adopted in FEM, to solve some cases of internal flow. This approach was extended to analyse the unsteady state convection flow inside a square cavity by Kuroki *et al.* (1985), where the time derivative in the equation of motion is approximated by finite differences schemes. Linear elements were used to discretize the boundary, while the domain was divided into triangular linear cells. Simple iteration was used to solve the nonlinear equations and the size of the time increment was limited by the finite difference scheme. They pointed out that computer programs for the elastodynamics can be used to solve viscous flow with minimum modifications.

The use of the penalty function concept to solve the Navier-Stokes equations, in terms of integral representation, was initiated with the pioneering work discussed above. Later, this approach has been investigated by another group of researchers. Kitagawa, Brebbia, Wrobel and Tanaka (1986) solved the steady Navier-Stokes equations for incompressible fluids using a formulation involving a pseudo-body force and employing the penalty function to eliminate the pressure. It is shown that since the resulting equation is very similar to the so-called Navier equation of elastostatics, the viscous flow problem can be analysed in a similar manner. The integral equations for this case are derived straightforwardly from the elastostatics, since the same fundamental solutions are applied. Conversely, however, viscous flow problems are nonlinear due to the convective term (body forces), and so a different procedure is adopted for the numerical solution. One objective of their work was to investigate ways to calculate the derivatives appearing in the convective terms. These include a finite difference scheme (employing both upwind and central approximations) and a boundary integral equation. In order to evaluate the domain integral that contains the convective term, both rectangular constant cells and triangular linear cells were employed. They applied this formulation to solve the Hagen-Poiseuille flow and driven-cavity flow problems. The general conclusion was that the boundary integral formulation of the convective term is more accurate than the finite difference schemes.

This proposed approach was also extended to the thermal convection flow problem by Kitagawa *et al.* (1986), where the convective and buoyancy force terms in Navier-Stokes equations are considered as body forces. The Energy equation is considered as a Poisson type equation, and so the derivation of its integral representation follows the normal procedure presented in many textbooks. They analysed the natural convection flow inside a rectangular cavity and inside an annular cavity. They point out that, in order to achieve convergence, an under-relaxation technique has to be used. Kitagawa *et al.*

(1987) enhanced this approach with the introduction of a self-adaptive coordinate transformation technique to obtain accurate results near the boundary. The isothermal step flow and the natural convection flow inside a square cavity were analysed.

Considering also the recent achievements, their approach is used to analyse the natural convection flow inside a square cavity by Tanaka *et al.* (1988). Kitagawa *et al.* (1988) briefly reviewed this formulation and made a comparison of the results obtained by using the quadrilateral elements and the linear triangular elements as internal cells. They use the natural convection flow inside a square cavity as a test case and conclude that the use of quadratic quadrilateral elements has some advantages, such as the improvement in accuracy and of the convergence of the results. They also examine the effect of the penalty function parameter and conclude, considering linear triangular elements, that they have little effect on the values of results except for a low value of this parameter. A range for the value of this parameter is also recommended. In addition, the evaluation of the pressure field is discussed and some results are presented. They show that the pressure distribution can be calculated easily by post-processing.

Very recently, Kitagawa (1990) published the first book entirely dedicated to viscous flow solutions using the BEM. His book is a review of the papers referred to above, and the extension of this technique to three-dimensional incompressible flow is discussed.

The approach based on the penalty function can be considered as having achieved a reasonable level of maturity, due mainly to the expertise borrowed from the progresses made in the elastodynamic field. The results achieved are promising, however, most of the applications made so far refer to enclosed flows. The problem is that their approach produces an implicit solution procedure, which can present problems related to the need for having to deal with the whole flow field (viscous and inviscid regions) pointed out in the discussion of Wu's approach. However, the main advantage of this approach is the fact that a program can easily be developed based on the one for elasticity analysis.

Tosaka & Onishi (1985) used the Weighted-Residual Method to transform the Navier-Stokes equations in two- and three-dimensions, for steady state and incompressible flow into integral equations. The fundamental solutions of the Stokes approximate equations are used as the weight functions for the velocity equation while the fundamental solution of the Laplacian is used as weight functions for the pressure equation. Two-dimensional flow in a driven-cavity was solved in order to show the workability of that approach. Although they presented details of the mathematical derivation of the fundamental solutions, very little is said about the numerical procedure. However, the numerical solution procedure employed by them and other numerical examples on Couette, Poiseuille and channel flows were presented in Tosaka *et al.* (1985). They adopted an iterative solution procedure in which the nonlinear terms were treated as the known forcing functions.

Tosaka & Kakuda (1986a) enhanced the implementation by utilising a Newton-Raphson algorithm for the iterative solution of the set of nonlinear equations, instead of adopting a simple iterative scheme. The efficiency of the method is demonstrated by the application to several two-dimensional incompressible internal flow problems, such as flow in a driven-cavity, flow through a sudden enlargement and flow past a step. It is useful to note that solutions for the flow in a driven-cavity are obtained over a broad range of Reynolds numbers.

In Tosaka & Onishi (1986a), this formulation is extended to unsteady state cases, where the first order time derivatives are approximated by a one-step finite difference. The mathematical details of the fundamental solution derivation are presented, but no results are shown. Tosaka & Kakuda (1986b), presented details of the numerical procedure concerning the solution of unsteady state flow. The Newton-Raphson method is used to solve the set of equations for each time step. In order to show the effectiveness of the method, some results for axisymmetric flow through a sudden enlargement and flow past a step are presented. The same numerical procedure related to a pressure-velocity formulation is adopted as before. New applications of this methodology to solve steady state and unsteady state flow were presented in Tosaka & Kakuda (1986c). In order to demonstrate the accuracy and versatility of the technique, the flow past a step and flow past double steps are solved as test cases. A review of this approach to solve steady- and unsteady-state incompressible viscous flow is given in Tosaka (1989).

The Energy equation for the flow was considered in terms of integral representation by Tosaka (1986), where the steady and unsteady formulations for Navier-Stokes equations are reviewed. He analysed the steady state natural convection flow inside a rectangular cavity, and the unsteady state axisymmetric flow through a sudden enlargement. In these cases, the uncoupled problem was considered, where the Energy equation is solved separately. Tosaka & Onishi (1986b) used the methodology proposed in previous papers to show that one can derive systematically the integral equations in the same way, corresponding to steady and unsteady natural convection flow, based on three different formulations: the pressure-velocity-temperature, the stream function-vorticity-temperature and the stream function-temperature. They also present some results for the analysis of driven-cavity flow. The steady state natural convection flow inside cavities of different geometries (rectangular, non-rectangular and circular annulus enclosures) is analysed in Tosaka & Fukushima (1986). The unsteady state natural convection flow inside rectangular cavities was again analysed by Tosaka & Fukushima (1987), using the same numerical procedure presented before. Further results are presented in Tosaka & Fukushima (1988). A review of their previous analysis of natural convection flow is presented in Tosaka & Fukushima (1992).

An important contribution was made by Tosaka & Kakuda (1987), who extended the pressure-velocity formulation to consider turbulent flows. The numerical procedures are basically the same as for laminar flow cases. Some test cases were solved but only the plane channel flow case considers turbulence, the other cases refer to laminar flow.

Previous works referring to this formulation consider only two-dimensional flow, although some indication is given that extension to three-dimensional flow is not a difficult task. Tosaka *et al.* (1990) effectively extended the analysis to consider three-dimensional incompressible flow. The numerical procedure is very similar to the previous work, except that a simple iterative process was adopted. The flow inside a cubic cavity driven by a lid sliding at a uniform velocity was used as a test case and the results are presented for Reynolds numbers of 1, 10 and 20.

The approach that has been under investigation by Tosaka and his co-workers, which appeared just a few years ago, can already be considered as an important alternative for viscous flow solutions. His group has been publishing a series of interesting papers in order to demonstrate the validity of this technique. Included in their analysis are factors such as time-dependence, turbulence and three-dimensionality. Although the results achieved are good, their approach leads to an implicit solution procedure like the Kitagawa's approach.

2.3.3- Alternative Formulations

A quite different analysis of unsteady incompressible viscous and inviscid flows is presented in Piva *et al.* (1987), where the formulation is in terms of dynamic pressure and has the advantage of limiting the field integral to the vortical region. Both viscous and inviscid flows were solved and compared in order to see how the presence of viscosity affects the flow. Their proposed algorithm for high Reynolds number flow is based on the difference between the viscous and potential equations. This they claim, can lead to a reduction of the difficulties encountered in solving the original equations directly. Their idea is interesting because the calculations can be concentrated on the vortical region of the flow field. However, it is too early to assess this technique since just a few papers have so far been published.

A different approach proposed by F. K. Hebeker which considers a formulation involving pressure and velocity as dependent variables to solve viscous flow, has also been under investigation. However this approach is mathematically more involved, and is, therefore, less attractive to engineers. Besides, results for very few applications are presented. Hebeker (1985) used a point collocation type of BEM to solve the three-dimensional viscous flow represented by Stokes equations. Borichers & Hebeker (1986) attempted an hybrid approach by connecting the BEM with a Spectral Method to analyse unsteady Stokes problem. Unsteady Navier-Stokes flow are analysed in Hebeker (1987) where a Lagrangean approach is adopted to handle the convective term of the equation. This approach was extended to consider compressible Navier-Stokes flow by Hebeker (1988) and new issues about it are addressed in Hebeker (1989). Hebeker's approach seems to be in the early stages of development, and only a few results have been published.

Most of the works reviewed until now refer to low subsonic flow. However, for additional information only, the BEM applied to transonic and high subsonic flow has been systematically under investigation by some researchers including, for instance, Zuosheng (1985,1986 and 1987) and Ogana (1989a and 1989b).

In order to solve any boundary integral equation using the BEM the source point can be located anywhere, but researchers usually consider only to place source point inside the domain and over the boundary. However, there are some advantages in placing the source point outside the domain. Mainly, performing singular integrals on the boundary is avoided and the corners of the domain can be treated easily. This technique is called "Regular Boundary Element Method". Despite these advantages, it seems that this technique has not interested researchers in many areas. In fluid mechanics, for instance, there are only a few published works dealing with this technique. Most are due to a group of researchers in UK. In Patterson & Sheikh (1982 and 1983), for example, this technique is used to solve two-dimensional, steady state, inviscid, laminar flow in channels considering different shapes of obstacle, where a formulation involving the stream function is employed. They also investigated the possibility of using higher order weighting functions and a better position of the source point outside the domain. The regular BEM is a valid alternative to deal with some types of problem. However, this is still under investigation since, in some cases, it introduces behaviour in the results that are not completely understood. To author's knowledge, this technique has hardly been applied to viscous flows.

2.4- CONCLUSIONS OF THE PREVIOUS WORKS

The works discussed above demonstrate the potentiality of the BEM in the field of fluid mechanics with relatively simple examples. However, the BEM is still considered a technique in development for the solution of viscous flow problems of interest in engineering, because some issues need to be addressed in terms of realistic problems, such as accuracy, efficiency, convergence and so on. Certainly, the application of the BEM to solve complex problems is nowadays under investigation, but just a few results have been published. It is expected that as a consequence of those studies improvements will be introduced. Only after this stage of refinement can the technique be applied as a reliable technique for solving fluid mechanics problems. Nowadays, engineering tasks using the BEM for the solution of viscous flow problems, especially those governed by the Navier-Stokes equations, must be seen as a matter of academic interest, since the applicability of the technique to solve this class of problem is still under investigation.

Other conclusions can be drawn from the works reviewed. The Vorticity-Velocity, Penalty function and Tosaka's approach are the most important formulations adopted to solve viscous flow using the BEM. The Vorticity-Velocity Formulation has the advantage over others with relation to the fact that the calculation can be limited to the vortical region of the flow. However, its extension to deal with compressible or three-dimensional flow has not yet proved to be efficient. The Penalty Function formulation has the advantage of being easy with respect of the development of the program, provided that a program to elasticity problems is available. This formulation, however, suffers the same problem as Tosaka's approach, due to the fact that they lead to an implicit solution. That is, the whole domain has to be solved. Besides, the approach has been attached to the analogy between fluid mechanics and elasticity formulations, and so some advantages may be lost if extension to other situations, like turbulence, is tried. The Tosaka's approach is more involved from the mathematical point of view and may require more skill than the others. However, in terms of applications, compared with the alternatives, Tosaka's approach solves the most difficult test cases. The main problem is still the fact that this approach leads to an implicit solution scheme.

Chapter 3

Potential Problems

3.1 - INTRODUCTION

In this Chapter the analysis of the potential problems described by the Laplace's or Poisson's equations is considered. This class of problems are described by very simple governing differential equations that at the same time are among the most useful of all the partial differential equations occurring in physics and engineering.

The significance of the Laplace's and Poisson's equations lie in the diversity of physical phenomena that they govern. This class of potential problem includes, for example, potential fluid flow, heat flow, shaft torsion, and many others.

In the BIE applied to potential problems, the domain integral that appears through the non-homogeneous term of Poisson's equation, creates no problem since, in general, it contains no unknown in the integrand. The problem becomes even easier when the Laplace's equation is considered since no domain integral is present, and the BIE retains the characteristic of being represented only by boundary integrals. In other words, the dimensionality of the problem is reduced. In this case one does not need to discretize the domain. These characteristics help to reduce the work involved in the elaboration of programs and input files, and also to reduce the computer memory required. This is a example of a case where the advantages of the BEM is fully exploited. These features are usually misunderstood by people not directly involved with the BEM, who assume that the BEM is applied only for potential problems or that the discretization of the domain can be always avoided.

3.2- DIFFERENTIAL EQUATION

As was mentioned before, many engineering problems are governed by an equation of type

$$\nabla^2 u + b = 0 \tag{3.1}$$

applied to a simply connected domain Ω bounded by a closed smooth surface Γ and subjected to boundary conditions

$$u = \bar{u} \qquad \text{on } \Gamma_1 \tag{3.2}$$

$$q = \frac{\partial u}{\partial n} = \bar{q} \qquad \text{on } \Gamma_2 \tag{3.3}$$

according to Figure 3.1. In the general case, the boundary conditions are of mixed or Cauchy type, in that they combine Dirichlet and Neumann type boundary conditions given by Eqs. (3.2) and (3.3), respectively. Note that b, sometimes regarded as a "body force" term, in Eq. (3.1) represents any known function, otherwise the problem would be non-linear and an iterative process would be required to solve it. Eq. (3.1) is the Poisson equation upon which the following analysis on boundary integral equation will be based. The Laplace equation results when the term b vanishes, and so is treated as a special case.

3.3- BOUNDARY INTEGRAL EQUATION (BIE)

The first task related to the application of the BEM is the transformation of the differential equation of the problem into a corresponding BIE. At least for potential analysis, the derivation of the BIE is shown in every textbook on BEM and so only the main steps are given here. Firstly, however, one has to decide which approach should be adopted in order to transform the differential equations into BIE. In the direct BEM there are two principal but different ways that can be adopted. The one normally used is that based on the Weighted-Residual Method, which is closely related to the FEM. However, one may prefer to adopt the process based on the use of Green's identities. The former is adopted throughout this work because among other advantages, it is more general and has been used in connection with other approaches, like FEM.

3.3.1- Weighted-Residual Statement

The Weighted Residual Method is discussed in several textbooks on FEM and BEM. Briefly, this technique is based on a theorem that states that if there is a solution $\tilde{u}$ to the problem defined by any differential equation $D(u) = g(x,y)$, where D is a differential operator and $g(x,y)$ the non-homogeneous term, and if

$$\iint_{\Omega} u^* R_{\Omega} dx\,dy + \oint_{\Gamma} v^* R_{\Gamma} d\Gamma = 0$$

is satisfied for any arbitrary sets of weighting functions u^* and v^*, where R_{Ω} and R_{Γ} represent the residuals (or error) due to the use of the solution $\tilde{u}$ in the differential equation and boundary conditions, respectively, then

$$R_{\Omega} = 0,$$
$$R_{\Gamma} = 0$$

and $\tilde{u}$ represents an exact solution. This theorem is used in connection with many numerical techniques, like the point collocation method, that differ from each other mainly in the criterion used to select the weighting functions. These methods are classified as weighted-residual methods.

Considering the problem described by Eqs. (3.1) to (3.3), if u is approximated by another function $\tilde{u}$, then residuals are formed by use of this approximate function in the following manner:

$$\nabla^2 \tilde{u} + b = R_\Omega$$

$$\tilde{u} - \bar{u} = R_{\Gamma_1}$$

$$\tilde{q} - \bar{q} = R_{\Gamma_2}$$

For such cases, a generalized weighted-residual expression may be written as follows:

$$\begin{aligned} &\iint_\Omega u^* \nabla^2 \tilde{u}\, dx dy + \iint_\Omega u^* b\, dx dy \\ &+ \oint_{\Gamma_1} v^* (\tilde{u} - \bar{u})\, d\Gamma + \oint_{\Gamma_2} w^* (\tilde{q} - \bar{q})\, d\Gamma = 0 \end{aligned} \tag{3.4}$$

where u^*,v^* and w^* are arbitrary weighting functions. This expression is sometimes called the weak weighted-residual statement. The terms containing the boundary integrals would disappear if the solution $\tilde{u}$ satisfied the given boundary conditions. Since it is not always possible to verify this constraint a general expression like Eq. (3.4) must be used.

This expression can be manipulated further in order to obtain the so-called inverse weighted-residual expression. First, using Green's identities or integration-by-parts theorem given in Appendix B, the first domain integral term in Eq.(3.4) can be transformed into:

$$\iint_{\Omega} u^*(\nabla^2\tilde{u})\,dx\,dy = \oint_{\Gamma} u^*\tilde{q}\,d\Gamma - \oint_{\Gamma} q^*\tilde{u}\,d\Gamma + \iint_{\Omega} (\nabla^2 u^*)\tilde{u}\,dx\,dy \quad (3.5)$$

In order to investigate the error introduced by the fact that the solution $\tilde{u}$ in general does not satisfy exactly the boundary conditions, let us define u and q such that they satisfy exactly the give boundary conditions, that is

$$u = \bar{u} \quad \text{on } \Gamma_1\,, \quad u = \tilde{u} \quad \text{on } \Gamma_2 \text{ or in } \Omega$$
$$q = \tilde{q} \quad \text{on } \Gamma_1\,, \quad q = \bar{q} \quad \text{on } \Gamma_2$$

and so, it can be shown that

$$\oint_{\Gamma} q^*\tilde{u}\,d\Gamma = \oint_{\Gamma} q^* u\,d\Gamma + \oint_{\Gamma_1} q^*(\tilde{u} - \bar{u})\,d\Gamma$$

and

$$\oint_{\Gamma} u^*\tilde{q}\,d\Gamma = \oint_{\Gamma} u^* q\,d\Gamma + \oint_{\Gamma_2} u^*(\tilde{q} - \bar{q})\,d\Gamma$$

Substituting the above equations into Eq. (3.5) and then introducing the final expression into Eq. (3.4), a generalized inverse weighted-residual expression is obtained as follows:

$$\iint_{\Omega} (\nabla^2 u^*)u\,dx\,dy + \iint_{\Omega} u^* b\,dx\,dy + \oint_{\Gamma} u^* q\,d\Gamma - \oint_{\Gamma} q^* u\,d\Gamma + B = 0 \quad (3.6)$$

where B is given by:

$$B = \oint_{\Gamma} (v^* - q^*)(\tilde{u} - \bar{u}) d\Gamma + \oint_{\Gamma} (w^* + u^*)(\tilde{q} - \bar{q}) d\Gamma$$

and represents the boundary conditions weighted error.

Such a weighted residual expression may be simplified if a relationship between the weighting functions exists. As discussed by El-Zafrany (1993), an optimum choice for the weighting functions is the one which minimizes the boundary condition error. This can be achieved by selecting them such that B=0, for any prescribed values of u and q. Hence,

$$v^* = q^* \ , \quad w^* = -u^*$$

and the final inverse weighted-residual expression is given by:

$$\iint_{\Omega} (\nabla^2 u^*) u \, dx dy + \iint_{\Omega} u^* b \, dx dy + \oint_{\Gamma} u^* q \, d\Gamma - \oint_{\Gamma} q^* u \, d\Gamma = 0 \quad (3.7)$$

This same expression is obtained in many textbooks considering directly the relationship between the weighting functions above into the weak weighetd-residual expression. Note in Eq. (3.7) that the weighting function u^* now needs to be continuous up to the second derivatives and u needs to have continuity of the function itself in the domain.

3.3.2- Fundamental Solution

Note that if there is a weighting function that satisfies a singular Poisson's equation, then the domain integral that containing the unknown can be simplified, and so the only remaining unknowns are inside the boundary integrals. This problem corresponds to an infinity domain where a concentrated unit potential is applied in one point. The solution for this particular problem is the so-called fundamental solution. The derivation of the fundamental solution is, in fact, one of the most important steps towards obtaining the BIE in the application of the BEM. Sometimes, however, the derivation is not simple.

The fundamental solution corresponding to the Laplace's equation was probably the first one to be obtained and also one of the simplest. It is discussed in every textbook on BEM. In El-Zafrany (1993), for example, the fundamental solution for this case is derived in two ways: mathematically and based on the heat conduction approach to provide a physical understanding behind this concept. It will not therefore be derived here, only the final expression is given.

With the help of the Dirac delta function to represent the concentrated unit potential at any point "i" in the domain, the equation for the particular problem described above is given by:

$$\nabla^2 u^* + \delta(x-x_i, y-y_i) = 0 \tag{3.8}$$

where the fundamental solution is a function of two points, as shown in the Figure 3.2: the source point (x_i,y_i) at which the singularity of delta function is located and also where the concentrated unit potential is applied; and the reference point (x,y), also called the field point. The fundamental solution in this case is:

$$u^* = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right) \tag{3.9}$$

where r is the distance between the source and reference points, given by:

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2} \tag{3.10}$$

3.3.3- Boundary Integral Equation (BIE)

The role of the fundamental solution in the derivation of the BIE is now considered. Substituting Eq. (3.8) into the appropriate domain integral of Eq. (3.7), and also using the property of the Dirac delta function given in Appendix B, the domain integral containing the unknown being transformed is given by the following:

$$-\iint_{\Omega} \delta(x-x_i, y-y_i)u\,dx\,dy = -C_i u(x_i, y_i)$$

where the constant C_i is defined in Appendix B. Hence, substituting the above result into Eq. (3.7) gives the following BIE:

$$C_i u(x_i, y_i) + \oint_{\Gamma} u q^* d\Gamma = \oint_{\Gamma} u^* q d\Gamma + \iint_{\Omega} u^* b\,dx\,dy \qquad (3.11)$$

which is the BIE corresponding to the specified point "i" in the domain considered. This equation can be applied to any internal or boundary points of the domain, provided that the appropriate value for the constant C_i is adopted.

In some problems, it may be necessary to calculate the derivatives in the x- and y- directions of the potential u, inside the domain. In this case expressions suitable for the calculation can be obtained deriving Eq. (3.11) with respect to the coordinates of any arbitrary source point, x_i and y_i. This can be carried out because Eq. (3.11) is regarded as an expression that can give the value of u continually at any source point inside the domain. Therefore, the following BIE for the derivatives of u can be obtained:

$$\frac{\partial u}{\partial x}\bigg|_{(x_i,y_i)} = \oint_{\Gamma} u \frac{\partial q^*}{\partial x} d\Gamma - \oint_{\Gamma} q \frac{\partial u^*}{\partial x} d\Gamma - \iint_{\Omega} b \frac{\partial u^*}{\partial x} dx\,dy \tag{3.12}$$

and

$$\frac{\partial u}{\partial y}\bigg|_{(x_i,y_i)} = \oint_{\Gamma} u \frac{\partial q^*}{\partial y} d\Gamma - \oint_{\Gamma} q \frac{\partial u^*}{\partial y} d\Gamma - \iint_{\Omega} b \frac{\partial u^*}{\partial y} dx\,dy \tag{3.13}$$

for derivatives in x- and y-directions, respectively. Note that in order to obtain these expressions, the derivatives of the fundamental solution and its normal derivative, with respect to x_i and y_i, were replaced by derivatives with respect to x and y using the following relations, given in El-Zafrany (1993):

$$\frac{\partial f(r)}{\partial x_i} = \frac{df(r)}{dr}\frac{\partial r}{\partial x_i} \equiv - \frac{\partial f(r)}{\partial x} \tag{3.14}$$

and

$$\frac{\partial f(r)}{\partial y_i} = \frac{df(r)}{dr}\frac{\partial r}{\partial y_i} \equiv - \frac{\partial f(r)}{\partial y} \tag{3.15}$$

where f(r) is any function of r.

3.4- POTENTIAL FLOW ANALYSIS

In spite of the fact that in all fluid flow phenomena viscosity effects are present, a great number of cases can be conveniently treated as inviscid and irrotational, i.e. the so-called potential flow. This kind of approximation is particularly important to solve aerodynamic problems. The potential approach is employed in this case to calculate, for example, the velocity and pressure distributions around the body in order to evaluate the lift with good accuracy. However, this analysis cannot predict drag accurately, for example, since this a phenomenon closely linked to viscous effects. One important application in the field of internal flow solutions, is the analysis of the flow through cascades of turbomachinery like fans,compressors and pumps, especially axial ones. This simplified model is very often adopted in order to evaluated the pressure distribution on the surface of cascade blades and the flow angles, parameters very important for the design of the turbomachinery. There are a great quantity of works on this subject in the literature, where the potential flow in cascades is solved using the domain methods, see for example Thompson (1973) and Baskharone & Hamed (1981). The problem of these methods is that they require the domain discretization. They are thus inefficient if compared with the BEM, which needs, in this case, that only the boundary is discretized. The Panel Method, which is regarded as a type of BEM, is sometimes adopted for solving this problem. The analysis here, however, adopts a type of BEM that uses directly the physical variables of the problem, the so-called direct BEM. It seems that the first work in this area adopting this type of BEM was the one due to Carte (1992). His work gives a good example of the potentiality of the BEM in the turbomachinery area. However, he fails to present any comparison of his results to assess the accuracy of the method. Also, although qualitatively his results seem convincing there are some points in his work that may be opened to criticism, mainly with regard to the imposition of the trailing edge condition.

Another advantage of the BEM is that it allows the continuous calculation at interior points of the domain, which could be interesting to analyse certain regions of the flow in more detail. For example, an extra program was adopted by Miller & Serovy (1975), which adopt the FDM in their analysis, to magnify the trailing edge region of the flow in order to analyse in detail the flow in this region. This is another example where the BEM can replace domain methods with advantages.

The potential flow of an incompressible fluid is governed by Laplace's equation in terms of either the velocity potential,ϕ, or the stream-function,ψ, parameter. Both velocity potential and stream-function satisfy the continuity equation identically. The adoption of one or another variable depends mainly on the problem being considered. For example, in a three-dimensional analysis, the velocity potential should be used in place of the stream-function, which in this case has many components. On the other hand, the velocity potential is strictly valid only for irrotational flow, while stream-function can be used for both rotational and irrotational flow. In addition, depending on the geometry of the problem, it may be convenient to use one or another variable. In a domain formed by the blade-to-blade channel of cascades, for example, the stream-function is normally used in order to deal with the periodic boundaries. On the other hand, if the domain in this case is one that includes one blade in the middle, use of the velocity potential is more appropriate.

If the flow is irrotational, the velocity components, v_x and v_y, in x- and y-directions, respectively, are expressed in terms of the velocity potential as follows:

$$v_x = \frac{\partial \phi}{\partial x} \; ; \; v_y = \frac{\partial \phi}{\partial y}$$

and in the case of incompressible flow from the Continuity equation, Eq.(A.1) of Appendix A:

$$\nabla^2 \phi = 0$$

The boundary conditions consist of two parts: on one part of the boundary, the value of the velocity potential is prescribed, ϕ_b, while on the rest of the boundary the normal derivative of the velocity potential is prescribed as equal to the normal component of the velocity, v_n. That is,

$$\phi = \phi_b \qquad \text{on } \Gamma_1$$
$$\frac{\partial \phi}{\partial n} = v_n \qquad \text{on } \Gamma_2$$

On the other hand, the velocity components may also be represented in terms of the stream-function, as follows:

$$v_x = \frac{\partial \psi}{\partial y} \; ; \; v_y = - \frac{\partial \psi}{\partial x}$$

and from the condition of irrotationality (vorticity equal to zero), the vorticity relationship given in Appendix A, gives:

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$$

and

$$\nabla^2 \psi = 0$$

Once again the boundary condition is assumed to consist of two parts: on part of the boundary, the value of the stream-function ψ_b is prescribed, and on the rest of the boundary, the tangential component of velocity, v_t, is prescribed, such as

$$\psi = \psi_b \qquad \text{on } \Gamma_1$$

$$\frac{\partial \psi}{\partial n} = - v_t \qquad \text{on } \Gamma_2$$

Special subroutine has to be introduced in the program to deal with the periodicity conditions, since in this case no boundary conditions is prescribed. The additional information to solve this case comes out of the relationship between the distributions of the normal derivative of potential and the potential itself on the periodic boundaries.

The potential flow analysis can be carried out by the BEM through Eq. (3.11) using either the velocity potential or the stream-function approach. Eqs. (3.12) and (3.13) are also important in this analysis in order to evaluated the velocity components at internal points of the domain. Note that in this case, the remaining domain integral appearing in these equations vanish because the equations are homogeneous.

The Navier-Stokes equations, discussed in Appendix A, are reduced in this case to the Bernoulli's equation, given by:

$$p + \frac{1}{2} \rho v^2 = K$$

where p is the static pressure, ρ is the density of the fluid and the constant K assumes one value for each streamline in rotational flow. There is therefore, one equation for each streamline. On the other hand, if the flow is irrotational, as is the case for potential flow, K assumes one value which is the same throughout the flow, i.e., it has the same value in every streamline. This expression is used to obtain the pressure results from the solution for velocity.

Chapter 4

Vorticity-Velocity Formulation

4.1- INTRODUCTION

Most of the analysis of the fluid flow problems using conventional methods are based on the formulation involving the so-called primitive variables, that is pressure and velocity. However, sometimes the mathematical statement of flow problems may be conveniently formulated using vorticity as a major dependent variable, since along with some numerical advantages, the vorticity itself plays an important role in fluid mechanics analysis. For example, turbulence phenomena and flow separation. Also some flow configurations are most readily understood through the vorticity parameter. The main advantage from the numerical point of view, is the fact that the pressure is eliminated from the main solution procedure, and the incompressibility constraint, that is the Continuity equation, is satisfied by definition. This makes the calculation easier. Thus, the vorticity-related formulations are an important alternative to the analysis of flow problems that has recently been increasing in importance.

Among the existing formulations using vorticity, the vorticity-stream function formulation is already well established for the solution of two-dimensional, compressible and incompressible, internal and external flows. The main advantage of such a formulation is associated with the fact that the number of unknowns is less than the formulation in terms of primitive variables. However, this kind of formulation is not attractive for solving three-dimensional flow because the number of unknown increases, since in this case both vorticity and stream function have three components. An alternative in this situation is to adopt the velocity potential instead of the stream function. However, a further difficulty related to the implementation of the boundary conditions appears, especially for internal flows. Even the traditional formulation involving primitive variable presents some problems in imposing boundary conditions for the pressure equation in this situation. The vorticity-velocity formulation has, therefore, become an increasingly attractive alternative because of the easier treatment of the boundary conditions, even though the number of equations increases. In other words, it seems to be much easier to impose boundary conditions to velocity than to variables like velocity potential and stream function. It is for this reason that, the formulation involving vorticity and velocity as dependent variables has been investigated for the solution of two- and three-dimensional flows using basically domain methods like Finite Elements and Finite Differences. However, the question of why one would want to use a vorticity-velocity formulation is not easy to answer yet, in spite of the popularity which this formulation seems to be achieving in engineering circles. This is pointed out by Gunzburger *et al.* (1990), who presents a comprehensive discussion on this formulation. Some justification for problems using non-inertial frames of reference is given is Speziale (1987). The fact is that this formulation has only recently been investigated if compared with others and little comparative data with other solutions exists, and so just a little is known about it. Consequently, its potentiality has not sufficiently been exploited yet.

Conversely, those formulations involving vorticity suffer from the classical problem of imposing the boundary conditions for vorticity and so the accuracy of the results of this parameter is not generally good. This is a serious drawback even if the one is not interested in the vorticity parameter itself, since any known method which recovers the pressure from vorticity and stream function, for example, will yield pressure approximations which are never of better accuracy than are the vorticity approximations. The same conclusion seems to be true when the vorticity-velocity formulation is being considered.

Another problem associated with the use of domain methods is the fact that in order to deal with high Reynolds number flows a very fine mesh is required, a problem which affects indistinctly all those formulations.

The integral formulation proposed by J. C. Wu mentioned previously, based on the Boundary Element method, which adopts the vorticity and velocity as dependent variables, presents none of these problems. In fact, the integral formulations generated in connection with the BEM has become a focus of interest for many aerodynamicists, since they seem promising. As mentioned previously, this kind of integral formulation is one of those adopted in this research. Since the integral equations are generated from the differential formulation, it is convenient first to derive the differential equations corresponding to the vorticity-velocity formulation.

4.2 - DIFFERENTIAL FORMULATION

The motion of a viscous fluid is governed by the law of mass conservation and Newton's second law of motion. The mathematical statement of these two laws are familiarly expressed in differential form and known as the Continuity and Navier-Stokes equations, respectively. These equations are represented for a steady-state and incompressible flow by Eqs. (4.1) and (4.2), respectively,

$$\nabla \cdot \vec{v} = 0 \tag{4.1}$$

and

$$(\vec{v} \cdot \nabla)\vec{v} = \frac{\vec{f}}{\rho} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} \tag{4.2}$$

where $\vec{v}$ is the velocity; p the static pressure, $\vec{f}$ are the conservative body forces; ρ the mass density and ν the kinematic viscosity of the fluid. Equation (4.2) is derived in Appendix A, where only the most important steps are presented and the meaning of each term is discussed.

Equations (4.1) and (4.2) are, in principle, sufficient for the determination of $\vec{v}$ and p, known as the primitive variables of the problem, provided that adequate boundary conditions for the velocity vector are prescribed. Non-penetrating and non-slip boundary conditions at the surface of a body are most frequently considered.

The vorticity of a fluid in motion is defined in Appendix A as:

$$\vec{\omega} = \nabla \wedge \vec{v} \tag{4.3}$$

this applies at every point within the fluid domain.

The existence of vorticity generally indicates that viscous effects are important. This is because fluid particles can only be set into rotation by an unbalanced shear stress. Therefore, vorticity dynamics, roughly speaking, offers a method to separate a flow into viscous and inviscid effects. Also, it allows the problem to be split into the kinetic and kinematic aspects. These concepts are discussed below.

The stress-strain relation, which differentiates the kinetic behaviour of a solid from that of a fluid, is part of Eq. (4.2). Consequently, it is valid only in the fluid region. Equation (4.1), however, is a kinematic equation, which is valid for both the fluid and solid regions. These remarks are in general not important, but they are used here to introduce two terms not usually adopted in the literature on fluid flows: the kinetic and the kinematic aspect of the flow. According to these concepts, the kinetic term refers to any analysis in which the forces are involved, and in this case fluids and solids behaves very differently. Conversely, the kinematics refers to any analysis where only the geometry of the motion is considered. The forces causing the motion are ignored. This term applies to any parameter closely related to the velocity field of the fluid, for example, vorticity. The reason for introducing this terminology here is that an integral formulation based on vorticity and velocity is normally split into two parts referred to in the literature as the kinetic and the kinematic parts, that seems appropriate in order to understand the problem and to produce a better solution procedure.

The kinetic aspect of the flow problem is concerned with various processes like convection, diffusion, etc. These govern the distribution of field variables such as the momentum, energy, vorticity, etc. Kinetic aspects of the problem are normally described by a differential equation known as the transport equation. For the present analysis, the principal variable of the flow is vorticity, whose transport equation can be obtained for incompressible flow by taking the curl of each term of the equation (4.2). Thus, for each term we have

(i) $\nabla \wedge [(\vec{v}\,.\,\nabla)\vec{v}] = -\nabla \wedge [\vec{v} \wedge (\nabla \wedge \vec{v})]$

according to the vector property (B.11) in Appendix B;

(ii) $\nabla \wedge \left(\frac{\vec{f}}{\rho} \right) = 0$

because $\vec{f}$, as discussed in Appendix A, can be considered conservative and in this case, irrotational;

(iii) $\nabla \wedge \left(\frac{1}{\rho} \nabla p \right) = 0$

according to (B.1); and

(iv) $\nabla \wedge (\nu \nabla^2 \vec{v}) = \nu \nabla^2 \vec{\omega}$

that can be obtained using the vector property (B.9) twice, along with (B.2) and Eq. (4.1).

Hence, the curl of the Eq. (4.2) can be written in terms of $\vec{v}$ and $\vec{\omega}$ as follows:

$$- \nabla \wedge (\vec{v} \wedge \vec{\omega}) = \nu \nabla^2 \vec{\omega} \tag{4.4}$$

or using (B.12) and (B.2) the term on the left-hand side of Eq. (4.4) can be represented by

$$\nabla \wedge (\vec{v} \wedge \vec{\omega}) = (\vec{\omega} . \nabla) \vec{v} - \vec{\omega} (\nabla . \vec{v}) - (\vec{v} . \nabla) \vec{\omega}$$

If the Continuity equation is also applied, it makes the second term on the right-hand side of the previous equation equal to zero, and thus

$$\nabla \wedge (\vec{v} \wedge \vec{\omega}) = (\vec{\omega} . \nabla) \vec{v} - (\vec{v} . \nabla) \vec{\omega} \tag{4.5}$$

Substituting Eq. (4.5) into (4.4), gives:

$$(\vec{v}\,.\,\nabla)\,\vec{\omega}\;-\;(\vec{\omega}\,.\,\nabla)\,\vec{v}\;=\;\nu\,\nabla^2\,\vec{\omega} \tag{4.6}$$

However, the main interest in this work is the analysis of plane problems. In this case the vorticity vector has just one component, perpendicular to the plane of the flow. Thus, due to the orthogonality between $\vec{v}$ and $\vec{\omega}$, the second term on the left-hand side of Eq. (4.6) vanishes, and so

$$\nabla^2\,\vec{\omega}\;=\;\frac{1}{\nu}\,(\vec{v}\,.\,\nabla)\,\vec{\omega} \tag{4.7}$$

This is the diffusion-convection equation for vorticity.

The first and second terms on the left-hand side of Eq. (4.6) represent, respectively, the convection of vorticity with the fluid and the amplification and rotation of vorticity by the strain rate. The term on the right-hand side of this equation represents the diffusion of vorticity through viscous action. These kinetic processes redistribute the vorticity in the fluid flow. One of the most important things about the vorticity transport equation is that the pressure does not appear.

The usefulness of vorticity in interpreting fluid flow problems is that it tracks only the effect of viscous forces; pressure and gravity forces will not directly change it. The physical reason behind this has to do with the fact that vorticity is an indicator of solid-body rotation. Pressure forces and gravity forces act through the centre of mass of a particle and cannot produce rotation. On the other hand, shear stresses act tangentially at the surface of a particle and, if they are unbalanced, will generate vorticity. The intimate connection between unbalanced shear stresses, or viscous actions, and vorticity is made even clearer by noting that the viscous term in Equation (4.2) can be written as

$$\nu \nabla^2 \vec{v} = - \nu \nabla \wedge \vec{\omega}$$

using the relation (B.9) of Appendix B. An unbalanced shear stress can only exist when the vorticity is non-zero. As a general rule the existence of vorticity means that a particle is, or at least in its recent history, was subject to net viscous forces.

The kinematic aspect of the problem relates the vorticity distribution at any instant of time to the velocity distribution at that instant and vice-versa. The equations representing this part of the problem fall into three main categories: (a) those that deal directly with the lower order kinematic equation in differential form, (b) those that introduce a vector Poisson equation for the velocity, and (c) those that utilise an integral formulation of the kinematic equations.

In the first category of kinematic equations are the Continuity and the vorticity definition equations, given by (4.1) and (4.3), respectively. With a known velocity distribution, the corresponding vorticity distribution is uniquely determined by the vorticity definition equation. On the other hand, with a known vorticity distribution, the unique determination of the corresponding velocity distribution requires the solution of Eqs. (4.1) and (4.3), subject to appropriated boundary conditions for velocity. There is an analogy between these two equations and the Maxwell's equations relating the magnetic field and the steady field of electricity current. Thus, the well established techniques for magnetostatics may be utilized to treat the kinematic aspects of incompressible flow problem under consideration, Wu & Thompson (1973). Equations (4.1) and (4.3) are valid for incompressible flow, steady- and unsteady-state flows, turbulent and laminar flows, and internal and external flows.

The set of equations (4.1), (4.3) and (4.7), with $\vec{\omega}$ and $\vec{v}$ as dependent variables, replaces the set (4.1) and (4.2) in which $\vec{v}$ and p are the dependent variables. Gatski *et al.* (1985) appear to be among the first to solve this set of equations.

Nevertheless, the usual method of evaluating the vorticity employs a Poisson type equation, which can be derived by taking first the curl of the vorticity definition equation, given by Eq. (4.3),

$$\nabla \wedge \vec{\omega} = \nabla \wedge (\nabla \wedge \vec{v})$$

and then, considering the relation (B.9) given in Appendix B, another representation for the curl of vorticity can be obtained from the above equation,

$$\nabla \wedge \vec{\omega} = \nabla (\nabla . \vec{v}) - \nabla^2 \vec{v}$$

With this equation and using the Continuity equation, Eq. (4.1), it is possible to obtain a vector Poisson's equation for $\vec{v}$ in the form

$$\nabla^2 \vec{v} = - \nabla \wedge \vec{\omega} \tag{4.8}$$

which is frequently used to replace the vorticity definition equation. The above equation, together with the Continuity equation, is included in the second category of kinematic equations. They are adopted in most works, as for example, Farouk & Fusegi (1985), Guj & Stella (1988) and very recently Guvremont *et al.* (1990).

However, Wu & Thompson (1973) are among the first that have questioned the validity of this formula to properly represent the kinematics of the flow. They pointed out that the solution of Eqs. (4.1) and (4.3) for $\vec{v}$ is unique if either the tangential or normal components of $\vec{v}$ are prescribed over the boundary, but the solution of Eq. (4.8) is unique only if both components of the velocity are prescribed on the boundary. They meant that while the solutions of Eqs. (4.1) and (4.3) with prescribed normal or tangential velocity components satisfy Eq. (4.8), the solutions of Eq. (4.8) with velocity prescribed on the boundary do not necessarily satisfy Eqs. (4.1) and (4.3). In other words, Eq. (4.8) admits solutions where neither Eq. (4.1) nor Eq. (4.3) is satisfied. This problem is also discussed in Gresho (1991). In his opinion, based on the analysis of previous works, if the distribution of vorticity is not special, that is, includes a proper-amount of vorticity on the boundary to enforce the non-slip condition, Eqs. (4.1) and (4.3) will typically yield a solution satisfying the normal component of velocity but not the tangential component: Rather, a vortex sheet will be present, which according to him, would not be accounted for by Eq. (4.8) if solved with the two velocity components as boundary conditions.

The problem related to the choice of the proper boundary conditions, including the boundary condition for vorticity, seems not to have been solved yet, especially when the Eq. (4.8) is adopted. However, if on one hand this equation presents some problems regarding the boundary conditions, it is the equation usually adopted to represent the kinematic aspect of the flow. The fact is that the results obtained so far have been good. Some aspects of the formulation, however, have not yet been well understood. This could be considered as an evidence that the numerical experimentalist managed to sort the problem out first. The first steps to understand these formulation were given only recently with some interesting studies. Among those, is worth mentioning the work of Gunzburger *et al.* (1990), where the vorticity-velocity formulation is discussed in great depth, including the boundary conditions issue.

Finally, the kinematic aspect of the flow can also be represented by an integral equation relating $\vec{v}$ and $\vec{\omega}$. Thus, such an integral equation along with the vorticity transport equation forms an integro-differential formulation. This kind of approach normally solves the differential equation corresponding to the vorticity transport using the domain methods, such as Finite Difference and Finite

Element, but requires a special technique to solve the integral equation, normally the Boundary Element method, depending on the algorithm adopted. Hence, the integro-differential formulation will be given special attention in this document because its kinematic part will be used later in connection with the derivation of the integral formulation, which is the main aim of this work.

4.3 - INTEGRO-DIFFERENTIAL FORMULATION

The use of vorticity to describe the flow phenomena allows the flow field to be split into two main regions: the irrotational and the viscous or vortical regions. The vortical region of an incompressible flow is the only region where viscous effects are important, since vorticity is absent in the potential region. There is therefore, no need to compute the vorticity in the inviscid (or potential) region of the flow. However, the solution of the vorticity transport equation requires the knowledge of the velocity in the flow field, since vorticity and velocity fields are related kinematically. The problem is that the solution of kinematic equations represented by Eqs. (4.1) and (4.3), or Eq. (4.8), using conventional methods, like FDM or FEM, requires that both potential and vortical regions of the flow are solved, because such methods normally leads to an implicit algorithm. Therefore, if a numerical procedure is found which permits the kinematic computation of the velocity field to be confined to the vortical region of the flow, then the solution of any viscous problem can be confined to the viscous region, where most important phenomena, like boundary layers, flow separation, etc, normally exist. The important feature of a procedure like this is therefore, a drastic reduction in the region of actual computation and a corresponding reduction in computer time and data storage requirement.

There are some representations of the kinematic aspects of the flow in terms of an integral equation that allows the calculation of the velocity field, and hence of other variables explicitly point-by-point using the information of the boundary condition and vorticity field. Hence, this kind of equation possesses the distinguishing ability of confining the solution field to the viscous region.

The first integral equation with such a characteristic is the well-known Biot-Savart law (sometimes called the velocity induction law). This basically inverts the vorticity definition equation, Eq. (4.3), using the Continuity equation, so that a velocity can be computed from a given vorticity field. For two-dimensional external flow problems, it is represented by the following equation :

$$\vec{v}(x_i,y_i) = \frac{1}{2\pi} \iint_{\Omega} \frac{\vec{r} \wedge \vec{\omega}}{r^2} \, dx\,dy + \vec{v}_\infty \tag{4.9}$$

where $\vec{r}$ is the position vector between any point where the velocity is to be calculated and the points within the region where the vorticity distribution is known, and its magnitude, r, is given by

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2}$$

and $\vec{v}_\infty$ represents the free stream velocity. This relationship between vorticity and velocity is very important because it allows the effect of the vorticity distribution on the velocity at any point over a determined region of flow to be estimated explicitly, provided the vorticity field is known a priori.

Eq. (4.9), or any equation equivalent to that, forms the basis of many numerical methods, such as the ones discussed in, for example, Hung & Kinney (1988) and Bharadvaj *et al.* (1986). However, consideration will be given later in this document to the integral equation proposed by Wu & Thompson (1973) for three main reasons. Firstly, the integral equation they have proposed can be regarded as an extension of the Bio-Savart law to internal flows, and is, therefore, expected to be more adequate for this work. Secondly, their integral equation can be derived using the technique associated with the BEM. Finally, more recently, Wu and his co-workers managed to transform the vorticity transport equation into integral form. This will be seen later using the BEM. Thus eventually, a fully integral formulation based on vorticity and velocity is obtained.

The first integral equation used to represent the kinematic aspects of the flow developed by Wu & Thompson (1973) was derived from Equation (4.8). This equation is a vector Poisson type equation for the velocity that can easily be transformed into an integral equation using the same derivation presented in the Chapter 3 for potential problem. Hence, if u is replaced by the velocity components and the domain load b by $(\nabla \wedge \vec{\omega})$ in Eq. (3.11), by analogy the following integral equation results:

$$C_i \vec{v}(x_i, y_i) + \oint_{\Gamma} \vec{v} \frac{\partial u^*}{\partial n} d\Gamma = \oint_{\Gamma} \frac{\partial \vec{v}}{\partial n} u^* d\Gamma + \iint_{\Omega} (\nabla \wedge \vec{\omega}) u^* dx dy \tag{4.10}$$

where u^* is the fundamental solution for the Laplacian operator given by Eq. (3.9), x_i and y_i are the coordinates of the source point, C_i is a coefficient whose value depends on the position of the source point and the geometry of the boundary as shown in Appendix B, and n is the coordinate component normal to the boundary.

Wu & Thompson (1973), pointed out some difficulties related to the boundary conditions required by Eq. (4.10), apart the problem previously mentioned relating to the boundary conditions of Eq. (4.8), from which Eq. (4.10) was derived. The problem is that in order to solve this equation it is necessary to specify both Dirichlet's and Newmann's boundary conditions for the velocity. However, either one or the other, or a linear combination of both, are normally prescribed on the entire boundary in order to solve the problem. They also noted that this difficulty could be overcome only for certain problems, using, for example, Green's functions, but that these are impractical for all but a few problems involving very simple geometry. A better alternative is to try to recover the missing information on the boundary, using the same equation, provided that the vorticity field is known to allow the solution of the domain integral. Unfortunately, in most situations one does not know a priori the distribution of such a parameter. Nevertheless, Wu & Thompson (1973) pointed out that in the important problem of a flow past a solid body on the surface of which the components of the velocity are zero, a specialized expression involving only a domain integral containing vorticity in the integrand, similar to Bio-Savart law, results from (4.10). In such a case Eq. (4.10) could be adopted without difficulty. They suggested, however, that for more general situations, where normally the boundary integrals do not disappear (as for example, in internal flows), the expression given below is more convenient:

$$C_i\,\vec{v}(x_i,y_i) + \oint_\Gamma (\vec{v}\,.\,\vec{n})\,\nabla u^*\,d\Gamma = \oint_\Gamma (\vec{v}\wedge\vec{n})\wedge\nabla u^*\,d\Gamma + \iint_\Omega \vec{\omega}\wedge\nabla u^*\,dx\,dy \tag{4.11}$$

where the same definitions apply as before. Such an expression is derived in Appendix D using both the concept of a fundamental solution and Green's theorem for vectors. Nevertheless, the important point in the derivation is that it was obtained working with the vorticity definition equation and a vector potential function $\vec{\psi}$, rather than Eqs. (4.1) and (4.3) directly. This can be postulated because of Eq. (4.1), such that

$$\nabla\wedge\vec{\psi} = \vec{v}$$

since the velocity field is solenoidal, and as the latter equation does not uniquely define $\vec{\psi}$, the assumption may be made that it is solenoidal as well. Therefore,

$$\nabla\,.\,\vec{\psi} = 0$$

Equation (4.11) constitutes the entirety of the kinematic aspect of the problem. That is, it is completely equivalent to the differential equations (4.1) and (4.3) subject to the appropriated boundary conditions to velocity. It expresses the velocity field as the sum of three contributions, each of them represented by an integral and is a velocity field by itself. The integral on the left-hand side of Eq. (4.11) represents the contribution of the normal component of the velocity boundary conditions to the velocity field. Similarly, the boundary integral on the right-hand side of this equation represents the contribution of the tangential velocity component. The domain integral represents the contribution of the vorticity field to the velocity field. This integral can regarded as a generalized statement of the well-known Biot-Savart law.

It is worth mentioning that according to Wu (1987) the contributions of the normal and tangential velocity components are equivalent to that of a sheet of concentrated source of strength $(\vec{v}.\vec{n})$ and that of a vortex sheet of strength $(\vec{v} \wedge \vec{n})$ lying on the boundary, respectively. In other words, the potential flow is incorporated into the boundary integrals of Eq. (4.11), while the domain integral gives the effect of the vorticity field in the development of the velocity field. Thus, the potential flow is represented by a BIE containing only boundary integrals.

This equation was first derived by Wu & Thompson (1973) and later in other works like Brebbia & Wrobel (1986) and Skerget *et al.* (1984). The derivation presented in Appendix D is a little different from the derivations included in the previous works, since the Dirac delta function was employed in the derivation to obtain the first term of Eq. (4.11) readily.

Eq. (4.11) is a vectorial representation of the kinematic equations that can be applied for both internal and external problems. For two-dimensional internal problems, this equation can be represented in terms of its two components in the x and y directions, according to the derivations presented in Appendix F:

$$
\begin{aligned}
&C_i\, v_x(x_i,y_i) + \oint_\Gamma v_n \frac{\partial u^*}{\partial x}\, d\Gamma \\
&= \oint_\Gamma v_t \frac{\partial u^*}{\partial y}\, d\Gamma - \iint_\Omega \omega \frac{\partial u^*}{\partial y}\, dx\, dy
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
&C_i\, v_y(x_i,y_i) + \oint_\Gamma v_n \frac{\partial u^*}{\partial y}\, d\Gamma \\
&= -\oint_\Gamma v_t \frac{\partial u^*}{\partial x}\, d\Gamma + \iint_\Omega \omega \frac{\partial u^*}{\partial x}\, dx\, dy
\end{aligned}
\tag{4.13}
$$

where v_n and v_t are the normal and tangential velocities, respectively, and v_x and v_y are the velocity components in x and y directions, respectively. The other parameters stand as before. Equations (4.12) and (4.13) could also be represented in terms of v_x and v_y in place of v_n and v_t. These equations will represent the kinematic aspect of the flow discussed throughout this work.

Note that the use of equations (4.12) and (4.13) for the evaluation of the velocity requires the knowledge of both the tangential and normal components of the velocity over the boundary. This is, in fact, admissible and does not overspecify the problem, provided both boundary conditions are compatible with each other. In other words, one of them is identical to the value obtained from the solution of (4.1) and (4.3) using the other as prescribed by the boundary conditions.

Like Eq. (4.10), Eq. (4.11) can also be reduced to Eq. (4.9) for cases of external fluid flow problems. In this case the domain tends to infinity and the boundary is divided into a part on which the non-slip and non-penetrating conditions applies and another at infinity for which the free stream velocity condition applies. Wu & Thompson (1973), however, suggested that the use of (4.11) is generally correct for more general problems, including for example cases where a part of the boundary is inside the fluid domain.

Wu and his co-worker have successfully applied the integro-differential formulation using Eq. (4.11) to solve a great variety of problems, mainly external flows. They have also managed to produce a new algorithm where both the kinematic and kinetic aspect of the problem are represented in terms of integral equations. This integral procedure is an extension of the integro-differential procedure discussed previously and introduces some additional advantages while preserving the advantages of its predecessor. This is the so-called integral formulation which is discussed next.

4.4 - INTEGRAL FORMULATION

The ability of the integro-differential formulation to confine the solution field to the viscous region of the flow, where the vorticity is non-zero, is present only because the kinematic aspect of the problem is treated by an integral method whose distinguishing feature is that of permitting the explicit, point-by-point, computation of the velocity. On the other hand, any suitable method, not necessary an integral one, may be used for the kinetic aspect without destroying the ability to confine the solution field to the vortical region of the flow. However, only an integral representation, similar to the one used to represent the kinematic aspect of the flow, possesses an inherent flexibility in locating nodes and in accommodating complex boundary shapes. Wu (1976) was the first to suggest that it was possible to utilise the concept of a fundamental solution to obtain an integral representation for the kinetic aspect of the flow. Thus, both the kinetic and kinematic aspects of the flow would be represented in terms of integral equations. Such an approach was discussed in Wu & Wahbah (1976) for steady flow and by Wu (1982) for unsteady flow cases. The latter has also been investigated by other authors, principally by P. Skerget and his co-workers. The integral approaches presented in these works are now well-known as the BEM.

For steady-state flows, the kinetic and kinematic parts of the problem are both described by elliptic differential equations. It is therefore, relatively straightforward to extend the integral representation procedures for the kinematics of the problem, previously presented, to the kinetic part. On the other hand, for unsteady-state flows, the kinetic part of the problem is described by parabolic differential equations, and so its integral representation is dissimilar to that for the kinematic part. In other words, while the kinematic aspects of both steady and unsteady flows are basically represented by almost the same integral representation, the integral equations corresponding to the kinetic aspect of steady and unsteady flows are different. By virtue of the fact that this work is concerned with the analysis of steady flow, only the integral equation corresponding to the kinetic aspect of steady flow will be discussed.

In reality, there are two basic integral equations in the literature to represent the kinetic part of the flow. The first one, due to Wu (1976), and later discussed by Wu & Wahbah (1976) and in many others publications. The second was first presented by Skerget *et al.* (1984). The main difference between them is that the latter includes explicitly the pressure distribution on the boundary. Both equations are discussed in this document.

4.4.1- Wu's Approach

In order to derive the integral equation for the kinetic aspect of the flow due to Wu and his co-workers one needs first to substitute some of the terms of Eq. (4.2) with the corresponding terms of vorticity. Thus, considering the relation (B.9) of Appendix B and the fact that the velocity field is solenoidal, the diffusion term can be represented by

$$\nu \nabla^2 \vec{v} = - \nu (\nabla \wedge \vec{\omega})$$

and the convective term becomes

$$(\vec{v}.\nabla)\vec{v} = - \vec{v} \wedge \vec{\omega} + \frac{1}{2} \nabla(\vec{v}.\vec{v})$$

using Equation (B.8) of Appendix B.

Hence, with the concept of vorticity introduced through the expressions above, Equation (4.2) can be expressed as

$$- \vec{v} \wedge \vec{\omega} + \frac{1}{2} \nabla(\vec{v}.\vec{v}) = - \frac{1}{\rho} \nabla p - \nu (\nabla \wedge \vec{\omega})$$

Note that the term corresponding to the body force, which can be regarded as irrotational, is not considered in this equation. This can be represented by

$$- \vec{v} \wedge \vec{\omega} = - \nu (\nabla \wedge \vec{\omega}) - \nabla h \tag{4.14}$$

where h is the total head defined by

$$h = \frac{p}{\rho} + \frac{1}{2} v^2$$

Finally, Eq. (4.14) can be manipulated to produce the following equation :

$$\nabla \wedge \vec{\omega} = \frac{1}{\nu} (\vec{v} \wedge \vec{\omega} - \nabla h) \tag{4.15}$$

and since $\vec{\omega}$ is defined as the curl of $\vec{v}$, the divergence of $\vec{\omega}$ is identically zero, that is

$$\nabla . \vec{\omega} = 0 \tag{4.16}$$

Equations (4.15) and (4.16) for vorticity are analogous to equations (4.3) and (4.1) for velocity, respectively. Therefore, it is possible to obtain an integral expression for the kinetic part of the flow simply by replacing $\vec{v}$ by $\vec{\omega}$ and $\vec{\omega}$ by $(\vec{v} \wedge \vec{\omega} - \nabla h)/\nu$ in the equation (4.11). Thus, the following equation results:

$$\begin{aligned} C_i \vec{\omega}(x_i,y_i) + \oint_\Gamma (\vec{\omega} . \vec{n}) \nabla u^* \, d\Gamma &= \oint_\Gamma (\vec{\omega} \wedge \vec{n}) \wedge \nabla u^* \, d\Gamma \\ &+ \frac{1}{\nu} \iint_\Omega (\vec{v} \wedge \vec{\omega} - \nabla h) \wedge \nabla u^* \, dx dy \end{aligned} \tag{4.17}$$

This contains all the terms of a similar expression adopted for three-dimensional flow analysis. However, for two-dimensional problems, the vorticity vector has only one component, perpendicular to the domain of the problem, and so $\vec{\omega} . \vec{n} = 0$. As a result the boundary integral on the left-hand side of Eq. (4.17) vanishes. In addition, one can manipulate further the domain integral, since

$$\nabla \wedge (h \nabla u^*) = \nabla h \wedge \nabla u^*$$

from property (B.4) and (B.1) of Appendix B, and from Gauss' divergence theorem,

$$\iint_\Omega \nabla \wedge (h \nabla u^*) \, d\Omega = \oint_\Gamma \vec{n} \wedge (h \nabla u^*) \, d\Gamma$$

Introducing these modifications, Eq. (4.17) can be rewritten as

$$\begin{aligned} C_i \vec{\omega}(x_i,y_i) - \oint_\Gamma (\vec{\omega} \wedge \vec{n}) \wedge \nabla u^* \, d\Gamma &= \frac{1}{\nu} \oint_\Gamma h \nabla u^* \wedge \vec{n} \, d\Gamma \\ &+ \frac{1}{\nu} \iint_\Omega (\vec{v} \wedge \vec{\omega}) \wedge \nabla u^* \, dx dy \end{aligned} \tag{4.18}$$

This equation is the vorticity transport equation in integral form, and so should take in consideration some of the several process related with the vorticity transport phenomenon. The domain integral represents the contribution of the convective process to the vorticity field. The form of this integral shows that its contribution is similar to the one given by the Biot-Savart law. In other words, the effect of the quantity $(\vec{v} \wedge \vec{\omega})$ on the vorticity field in a steady flow is similar to the effect of $\vec{\omega}$ on the velocity field in Eq. (4.9). The boundary integral on the left-hand side is the contribution of the vorticity boundary conditions to the vorticity field. The other boundary integral represents the contribution of total head to the vorticity field. Note that the boundary integral containing $(\vec{\omega}.\vec{n})$, which appears to three-dimensional problems, represents also another contribution of the vorticity boundary condition to the vorticity field.

Eq. (4.18) is given in terms of components, by:

$$C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\,d\Gamma = -\frac{1}{\nu}\oint_\Gamma h \frac{\partial u^*}{\partial t}\,d\Gamma + \frac{1}{\nu}\iint_\Omega \omega \left(v_x \frac{\partial u^*}{\partial y} + v_y \frac{\partial u^*}{\partial x} \right) dx\,dy \tag{4.19}$$

where $\partial u^*/\partial n$ and $\partial u^*/\partial t$ are the derivatives of the fundamental solution in the normal and tangential directions. Note that the arrow above ω was dropped out to indicate that this is the only component of vorticity vector.

It is worth mentioning that Skerget *et al.* (1987) manipulated the equation (4.18) further to obtain an equation that, at least in vector representation, is slightly different. Working with some of the boundary integral terms in the Eq. (4.18), the one on left-hand side and the one involving $(\vec{\omega}.\vec{n})$, which was dropped out from Eq. (4.17), and using the property (B.5) twice, they managed to obtain the following equation:

$$C_i\,\vec{\omega}(x_i,y_i) + \oint_\Gamma (\nabla u^*.\vec{n})\,\vec{\omega}\,d\Gamma = \oint_\Gamma (\nabla u^* \wedge \vec{n}) \wedge \vec{\omega}\,d\Gamma + \frac{1}{\nu}\oint_\Gamma h\,\nabla u^* \wedge \vec{n}\,d\Gamma + \frac{1}{\nu}\iint_\Omega (\vec{v}\wedge\vec{\omega}) \wedge \nabla u^*\,dx\,dy \tag{4.20}$$

Eq. (4.20) is a general one since its representation is similar to that one applied for three-dimensional cases. Again, the mathematical description of the plane flow is much simpler, since the vorticity vector has only one component. Therefore, the first term on the right-hand side of Eq. (4.20) vanishes, and so

$$C_i\,\vec{\omega}(x_i,y_i) + \oint_\Gamma (\nabla u^*.\vec{n})\,\vec{\omega}\,d\Gamma = \frac{1}{\nu}\oint_\Gamma h\,\nabla u^* \wedge \vec{n}\,d\Gamma + \frac{1}{\nu}\iint_\Omega (\vec{v}\wedge\vec{\omega}) \wedge \nabla u^*\,dx\,dy \tag{4.21}$$

which, if represented in terms of the components, would give Eq. (4.19) exactly.

Wu's approach includes integral Eqs. (4.12) and (4.13) for the kinematic and Eq. (4.19) for the kinetic aspect of the flow. These form a set of equations that has to be solved iteratively. They are usually called simply the ω-v-p formulation, which are the dependent variables of the problem. Note, however, that only the values of velocity on the boundary and in the viscous region of the flow are needed for the calculation of vorticity, and so the solution field may be confined to the viscous region of the flow. Equation (4.19) is specially important when one needs to calculate the pressure distribution on the boundary. Note also that in order to deal with three-dimensional problems, the whole of Eq. (4.17), must be used in place of Eq. (4.19).

4.4.2- Skerget's Approach

Skerget *et al.* (1984) derived a different integral representation for the kinetic aspect of the flow that does not include the pressure. Their equation along with Eqs. (4.12) and (4.13) for the kinematics of the problem is the so-called ω-v formulation. This new representation for the kinetic part of the flow is derived next for two-dimensional problems only, since for three-dimensional cases, a different expression is required whose derivation is much more involved.

The vorticity vector in this case can be regarded as a scalar parameter since it has only one component. Thus, one can easily obtain the integral equation for the vorticity steady-state diffusion-convection equation by simply considering the term on the right-hand side of equation (4.7) and the vorticity to be equivalent to the "body force" term, b, and the scalar function u of Eq. (3.11) given in Chapter 3, respectively. Hence, by analogy the following results:

$$C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\,d\Gamma = \oint_\Gamma \frac{\partial \omega}{\partial n} u^*\,d\Gamma - \frac{1}{\nu}\iint_\Omega (\vec{v}\,.\,\nabla)\,\omega\, u^*\,dx\,dy$$

this is the vorticity transport equation proposed by Skerget and his co-workers. In this equation, the two boundary integrals represent the contribution of the vorticity boundary conditions and of the normal derivative of vorticity to the velocity field. Conversely, the domain integral represents the contribution of the convective term. In terms of the components, this equation is represented by

$$C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\,d\Gamma = \oint_\Gamma \frac{\partial \omega}{\partial n} u^*\,d\Gamma - \frac{1}{\nu}\iint_\Omega \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y}\right) u^*\,dx\,dy \tag{4.22}$$

However, they found convenient to eliminate the derivatives of vorticity that appears in the integrand of the domain integral. This can be done using Gauss' divergence theorem. Thus, first using the relations (B.4), the integrand of the integral is transformed into

$$\vec{v}\,.\,\nabla\omega\, u^* = \nabla.(\omega u^* \vec{v}) - \vec{v}\,.\,\nabla u^*\, \omega - (\nabla.\vec{v})$$

and finally, using the Gauss' divergence theorem and considering the velocity field as solenoidal, the following equation results:

$$\begin{aligned} C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\, d\Gamma &= \oint_\Gamma \frac{\partial \omega}{\partial n} u^*\, d\Gamma \\ &- \frac{1}{\nu}\oint_\Gamma \omega\,(\vec{v}\,.\,\vec{n})\, u^*\, d\Gamma + \frac{1}{\nu}\iint_\Omega \omega\,(\vec{v}\,.\,\nabla u^*)\, dx\, dy \end{aligned} \tag{4.23}$$

where the same definitions apply as before, and the arrow upon ω was dropped out since the vorticity can be treated as a scalar parameter in plane problems. Note that the pressure does not appear explicitly in the above equation. In this equation, the domain integral is still related to the convective effects. The additional boundary integral gives the vorticity convective flux from the boundary, which vanishes on solid boundaries. In terms of its components, Eq. (4.23) can be represented by:

$$\begin{aligned} C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\, d\Gamma &= \oint_\Gamma \frac{\partial \omega}{\partial n} u^*\, d\Gamma \\ &- \frac{1}{\nu}\oint_\Gamma \omega\, v_n u^*\, d\Gamma + \frac{1}{\nu}\iint_\Omega \omega\,(v_x \frac{\partial u^*}{\partial x} + v_y \frac{\partial u^*}{\partial y})\, dx\, dy \end{aligned} \tag{4.24}$$

Equations (4.12), (4.13) and (4.24) constitute the set of equation used in Skerget's approach. These have to be solved iteratively. In their approach, the pressure distribution on the boundary is recovered with post-processing using equation (4.19) or a Poisson's equation for the pressure, after the results for vorticity and velocity fields have first been obtained.

4.4.3- Vorticity-Velocity Formulation's Proposed Approach

The two fully integral approaches discussed previously have successfully been applied to the solution of both internal and external fluid flow problems by the two research groups, with different interests. Wu and his co-workers, for example, are more interested in external aerodynamic fields. They have however, also published some applications to internal flow as well. Skerget and his co-workers, however, have been mainly concerned with the solution of internal flows, which is also the aim of this work. Thus, Skerget and his co-workers are responsible to a great extend for the development of techniques applied to internal flows.

According to the literature, these two approaches give a very stable solution due, mainly, to the type of fundamental solution adopted, which is of diffusion type since it satisfies the Poisson's equation. Additionally, the literature shows some evidence that the Wu's approach is more stable than that due to Skerget and his co-workers. It seems, however, that results of investigations which compare these two approaches have not been brought advanced further. Furthermore, the literature is not clear about problems of instability arising from these two approaches, nor are some crucial points of a numerical nature, discussed. For example, corner treatment and singular integral calculation are not discussed in great detail. As a consequence one does not know for certain, how serious the difficulties involved in each approach are.

In reality, there is not much difference between those two approaches. The main difference is in the kinetic equation. While Wu's approach includes the boundary pressure distribution, Skerget's approach uses instead the normal derivative of vorticity. This is basically the only important novelty introduced in the vorticity related formulation after the original proposal by J. C. Wu. The similarity between these two approaches produces computational procedures which are quite similar. However, considering the fact that this technique is relatively recent, it can be assumed that several alternative representations to the equations which would produce different algorithms, have not yet been investigated.

In this work, it was decided to analyse internal flow problems adopting a new approach based on a formulation derived from the original work developed by Skerget and his co-workers. As a result a new algorithm is proposed, which introduces some novelty to the subject. The main motive for deciding to base this study on the Skerget's formulation was because it concerns the advancement of solution to internal flow problems.

The origin of this new approach was a consequence of the modification introduced into Eqs. (4.12) and (4.13), corresponding to the kinematic part of the problem, used in both Wu's and Skerget's approaches. In both cases, the first step in the solution procedure calculates the vorticity distribution on the boundary extracting it from the existing domain integral. The boundary vorticity in this case has to be calculated implicity by solving the resulting set of algebraic equations. Doing this does not seem to introduce much difficulty in the algorithm. This seems, however, to present some sort of problem provoked by the fact that the kernel in the domain integrals are given by a weak operator, as mentioned in, for example, Skerget *et al.* (1990). In fact, Eqs. (4.12) and (4.13) result in two sets of algebraic equations that are linearly dependent. Only one set has therefore, to be used to calculate the vorticity distribution on the boundary. In order to generate such a set of equations, Skerget and his co-workers, for example, preferred to work with a projection of the components of Eq. (4.11), in the x- and y- directions, represented by Eqs. (4.12) and (4.13), respectively, in either the tangential or normal direction. However, the reasons behind this technique to avoid the problem have not been completely explained. That step is certainly the key point of the approach, since any inaccuracy in this part of the calculation will strongly affect the other parts of the algorithm.

In the proposed approach, it was decided to overcome this point by integrating equations (4.12) and (4.13) by parts, using the properties (B.13) and (B.14) of Appendix B. This results in a boundary integral which includes the vorticity parameter so that the boundary vorticity can be isolated. The resulting equations corresponding to the kinematic part of the problem are given as follows, for the x- and y- direction components, respectively:

$$\begin{aligned} C_i v_x(x_i,y_i) &+ \oint_\Gamma v_n \frac{\partial u^*}{\partial x} d\Gamma - \oint_\Gamma v_t \frac{\partial u^*}{\partial y} d\Gamma \\ &+ \oint_\Gamma \omega u^* m \, d\Gamma = \iint_\Omega u^* \frac{\partial \omega}{\partial y} dx\,dy \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} C_i v_y(x_i,y_i) &+ \oint_\Gamma v_n \frac{\partial u^*}{\partial y} d\Gamma + \oint_\Gamma v_t \frac{\partial u^*}{\partial x} d\Gamma \\ &- \oint_\Gamma \omega u^* \ell \, d\Gamma = - \iint_\Omega u^* \frac{\partial \omega}{\partial x} dx\,dy \end{aligned} \tag{4.26}$$

where v_n and v_t are the normal and tangential velocities, respectively, and ℓ and m are the direction cosines. These equations are a new form of equations (4.12) and (4.13), respectively. Thus, the vorticity was transferred to a boundary integral which was expected to make boundary vorticity calculations easier than in the previous procedure. More importantly, the singularity contained in the domain integral seems to be easy to deal with. Hence, it was expected to generate a set of algebraic equations to obtain the boundary vorticity distribution more reliably. Another advantage is in the fact that one then does not need to connect the nodes of both boundary and domain meshes. Thus, isoparametric elements to discretize the domain as is the case in the other procedures are not necessary. It is then possible to adopt constant elements to discretize both boundary and domain, even though this type of element is not normally used in flow analysis.

Unfortunately, the domain integrals now contain derivatives of vorticity requiring an additional procedure to evaluate them. This is the major drawback of the proposed solution procedure. At least three different ways can be used to tackle the problem. The derivatives may be computed explicitly by derivation of equation (4.22), for example, corresponding to the kinetic part of the problem, with respect to the source point coordinates, in a way similar to the one used to obtain Eqs. (3.12) and (3.13). These may be solved by finite difference schemes and finally, these can be derived in a similar manner as in FEM by means of interpolation functions. Here the first alternative was adopted to evaluate the derivatives of vorticity on the domain in order to maintain the use only of the BEM. Thus, the equations that allow the calculation of the derivatives of vorticity are obtained deriving the vorticity transport equation with relation to the source point coordinates. In this case, two options arise since both equations (4.22) and (4.24) can be adopted. On one hand, from Eq. (4.22) the following equations result :

$$\left(\frac{\partial \omega}{\partial x}\right)_{(x_i,y_i)} = - \oint_{\Gamma} \frac{\partial \omega}{\partial n} \frac{\partial u^*}{\partial x} d\Gamma + \oint_{\Gamma} \omega \frac{\partial^2 u^*}{\partial x \, \partial n} d\Gamma + \frac{1}{\nu} \iint_{\Omega} \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right) \frac{\partial u^*}{\partial x} dx\,dy \tag{4.27}$$

and

$$\left(\frac{\partial \omega}{\partial y}\right)_{(x_i,y_i)} = - \oint_{\Gamma} \frac{\partial \omega}{\partial n} \frac{\partial u^*}{\partial y} d\Gamma + \oint_{\Gamma} \omega \frac{\partial^2 u^*}{\partial y \, \partial n} d\Gamma + \frac{1}{\nu} \iint_{\Omega} \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right) \frac{\partial u^*}{\partial y} dx\,dy \tag{4.28}$$

On the other hand, from Eq. (4.24), the following equations results:

$$\left(\frac{\partial \omega}{\partial x}\right)_{(x_i,y_i)} = - \oint_{\Gamma} \frac{\partial \omega}{\partial n} \frac{\partial u^*}{\partial x} d\Gamma + \oint_{\Gamma} \omega \frac{\partial^2 u^*}{\partial x \, \partial n} d\Gamma + \frac{1}{\nu} \oint_{\Gamma} \omega \, v_n \frac{\partial u^*}{\partial x} d\Gamma - \frac{1}{\nu} \iint_{\Omega} \left[(\omega \, v_x) \frac{\partial^2 u^*}{\partial x^2} + (\omega \, v_y) \frac{\partial^2 u^*}{\partial x \, \partial y} \right] dx\,dy \tag{4.29}$$

and

$$\left(\frac{\partial \omega}{\partial y}\right)_{(x_i,y_i)} = - \oint_{\Gamma} \frac{\partial \omega}{\partial n} \frac{\partial u^*}{\partial y} d\Gamma + \oint_{\Gamma} \omega \frac{\partial^2 u^*}{\partial y \, \partial n} d\Gamma + \frac{1}{\nu} \oint_{\Gamma} \omega \, v_n \frac{\partial u^*}{\partial y} d\Gamma - \frac{1}{\nu} \iint_{\Omega} \left[(\omega \, v_y) \frac{\partial^2 u^*}{\partial y^2} + (\omega \, v_x) \frac{\partial^2 u^*}{\partial x \, \partial y} \right] dx\,dy \tag{4.30}$$

Analysis shows that Eqs. (4.27) and (4.28) introduce some advantages over Eqs. (4.29) and (4.40) since they contain one term less. In addition, the singularities in the domain integral are not so strong. Note, however, that none of these equations should be used to calculate the derivatives of vorticity on the boundary because they are expected to give very inaccurate results due to singular integral related problems. In this work, therefore, a different alternative that calculates the vorticity derivatives from the distributions of normal derivative of vorticity and vorticity itself on the boundary was adopted. This is discussed in Appendix K.

In conclusion, two similar but alternative approaches were devised in this work. The first one is based on Eqs. (4.25), (4.26), (4.22), (4.27) and (4.28). The second uses Eqs. (4.25), (4.26), (4.24), (4.29) and (4.40). Therefore, the only difference between them is in the kinetic equations. These equations are summarised below:

Kinematic Equations

$$\begin{aligned} C_i v_x(x_i,y_i) &+ \oint_\Gamma v_n \frac{\partial u^*}{\partial x}\, d\Gamma - \oint_\Gamma v_t \frac{\partial u^*}{\partial y}\, d\Gamma \\ &+ \oint_\Gamma \omega\, u^*\, m\, d\Gamma = \iint_\Omega u^* \frac{\partial \omega}{\partial y}\, dx\, dy \end{aligned} \tag{4.25}$$

$$\begin{aligned} C_i v_y(x_i,y_i) &+ \oint_\Gamma v_n \frac{\partial u^*}{\partial y}\, d\Gamma + \oint_\Gamma v_t \frac{\partial u^*}{\partial x}\, d\Gamma \\ &- \oint_\Gamma \omega\, u^*\, \ell\, d\Gamma = -\iint_\Omega u^* \frac{\partial \omega}{\partial x}\, dx\, dy \end{aligned} \tag{4.26}$$

Option 1 :

$$C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\,d\Gamma = \oint_\Gamma \frac{\partial \omega}{\partial n} u^*\,d\Gamma - \frac{1}{\nu}\iint_\Omega \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right) u^*\,dx\,dy \tag{4.22}$$

$$\left(\frac{\partial \omega}{\partial x}\right)_{(x_i,y_i)} = -\oint_\Gamma \frac{\partial \omega}{\partial n}\frac{\partial u^*}{\partial x}\,d\Gamma + \oint_\Gamma \omega \frac{\partial^2 u^*}{\partial x\,\partial n}\,d\Gamma + \frac{1}{\nu}\iint_\Omega \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right) \frac{\partial u^*}{\partial x}\,dx\,dy \tag{4.27}$$

and

$$\left(\frac{\partial \omega}{\partial y}\right)_{(x_i,y_i)} = -\oint_\Gamma \frac{\partial \omega}{\partial n}\frac{\partial u^*}{\partial y}\,d\Gamma + \oint_\Gamma \omega \frac{\partial^2 u^*}{\partial y\,\partial n}\,d\Gamma + \frac{1}{\nu}\iint_\Omega \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right) \frac{\partial u^*}{\partial y}\,dx\,dy \tag{4.28}$$

Option 2 :

$$C_i\,\omega(x_i,y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n}\,d\Gamma = \oint_\Gamma \frac{\partial \omega}{\partial n} u^*\,d\Gamma - \frac{1}{\nu}\oint_\Gamma \omega\, v_n u^*\,d\Gamma + \frac{1}{\nu}\iint_\Omega \omega\,(v_x \frac{\partial u^*}{\partial x} + v_y \frac{\partial u^*}{\partial y})\,dx\,dy \tag{4.24}$$

$$\left(\frac{\partial \omega}{\partial x}\right)_{(x_i,y_i)} = -\oint_{\Gamma} \frac{\partial \omega}{\partial n} \frac{\partial u^*}{\partial x} d\Gamma + \oint_{\Gamma} \omega \frac{\partial^2 u^*}{\partial x\, \partial n} d\Gamma + \frac{1}{\nu} \oint_{\Gamma} \omega\, v_n \frac{\partial u^*}{\partial x} d\Gamma - \frac{1}{\nu} \iint_{\Omega} \left[(\omega\, v_x) \frac{\partial^2 u^*}{\partial x^2} + (\omega\, v_y) \frac{\partial^2 u^*}{\partial x\, \partial y} \right] dx\, dy \tag{4.29}$$

and

$$\left(\frac{\partial \omega}{\partial y}\right)_{(x_i,y_i)} = -\oint_{\Gamma} \frac{\partial \omega}{\partial n} \frac{\partial u^*}{\partial y} d\Gamma + \oint_{\Gamma} \omega \frac{\partial^2 u^*}{\partial y\, \partial n} d\Gamma + \frac{1}{\nu} \oint_{\Gamma} \omega\, v_n \frac{\partial u^*}{\partial y} d\Gamma - \frac{1}{\nu} \iint_{\Omega} \left[(\omega\, v_y) \frac{\partial^2 u^*}{\partial y^2} + (\omega\, v_x) \frac{\partial^2 u^*}{\partial x\, \partial y} \right] dx\, dy \tag{4.30}$$

Chapter 5

Penalty Function Formulation

5.1 INTRODUCTION

One of the most crucial strategies in solving incompressible Navier-Stokes equations is to find a pressure distribution in the flow such that a divergence free velocity distribution (or solenoidal field) is ensured. Various numerical methods using the primitive variables have been developed to solve this problem. Among them there is the widely used penalty function approach, whose concept was first introduced in the area of variational calculus by Courant. This approach, however, has been mainly adopted in connection with the FEM and it was first applied to solve elasticity problems. More recently, it was used in incompressible flow analysis, due to the close correspondence between the formulations of those two problems. This technique, like vorticity-related formulations, uncouples the solution of pressure from the velocity field. In other words the velocity field is obtained first and the pressure solution comes normally as a post-processing.

Nowadays, the use of the penalty function concept to solve incompressible viscous flow through the FEM is already well-established. However, only very recently has this technique been adopted in connection with the BEM. This has probably followed the relative success of the BEM in solving elasticity problems, and also because of the correspondence between the two formulations, like what happened with the Finite Element Method. It seems that the first work using the penalty function approach and BEM in fluid flow analysis was due to Kakuda & Tosaka (in Japanese) published in 1984. This work is briefly reviewed in Tosaka

& Onishi (1985). Subsequently, many works appeared in the literature. The real credit for the development of this approach should be given to the works of Kitagawa and his co-workers. In a series of papers they reproduced early incompressible flow solutions and managed to extend the analysis to include temperature and time dependence. Recently, Kitagawa (1990) published the first book on this subject summarizing the results published in his papers. Even so the results presented so far show that there is a lot to be done, mainly with regard to the numerical treatment necessary to extend the range of the application of this technique. Such problems, of course, have mostly to do with the use of the BEM itself. Many people working in the past with the FEM faced numerical problems as well in dealing with viscous flow using the penalty function approximation, even though the FEM used in solving elasticity problems during that same period was very successful.

In this Chapter the integral formulation for steady, incompressible and isothermal flow is discussed. The differential formulation is first presented since it forms the basis of the integral formulation. The derivations of the BIE for incompressible flow analysis, however, based on the penalty function technique will not be more time consuming than the previous ones, since all the development experience available from elasticity analysis is going to be used here. Besides, the Navier-Stokes equations in the common form similar to one given by Eq. (4.2) is used.

5.2 DIFFERENTIAL FORMULATION

The equations governing viscous flow are Continuity and the Navier-Stokes equations, given by Eqs. (4.1) and (4.2) and previously described in section 4.1. However, for convenience we will consider the Navier-Stokes equations in the form given by Eq. (A.12) of Appendix A, as follows:

$$(\vec{v}.\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu\,[\nabla^2\vec{v} + \nabla(\nabla.\vec{v})] \tag{5.1}$$

where the last term on the right-hand side of this equation is retained. The body force term, $\vec{f}$, is not considered relevant for reasons already given and articulated in Appendix A.

One of the main difficulties in the numerical solution of these equations is due to the presence of the incompressibility constraint given by the Continuity equation, Eq. (4.1). In this work, the penalty function approach is used to handle this problem. This consists of introducing a small pressure perturbation into Eq. (4.1) as follows:

$$\nabla.\vec{v} + \frac{p}{\lambda} = 0 \tag{5.2}$$

where λ is the so-called penalty parameter. Since the pressure has a finite value, this approximation can be justified because the Continuity equation is satisfied approximately as $\lambda \to \infty$. One can consider this perturbation as the introduction of a slight artificial compressibility since, in the actual numerical calculations, λ must be large but have a finite value. The λ must be assigned a large positive value in order to approximate incompressibility closely. Eq. (5.2) is sometimes seen as a pseudo-equation of state but it can be better understood as describing a field of a distributed mass source and sink, in which the local positive pressure is proportional to the rate of mass destruction. Locally, negative pressure is proportional to the rate of mass creation. In other words, it is the mass conservation equation which is being approximated and then associated errors amount to net fluid loss or gain.

Replacing Continuity equation by a relation given by Eq. (5.2), and substituting into Eq. (5.1), the following equation governing viscous flow is obtained :

$$\nu\,\nabla^2\vec{v}_\lambda + \left(\frac{\lambda}{\rho} + \nu\right)\nabla(\nabla.\vec{v}_\lambda) = (\vec{v}_\lambda.\nabla)\,\vec{v}_\lambda \tag{5.3}$$

Equation (5.3) represents the so-called penalty function formulation for fluid analysis, where the subscript "λ" was introduced to indicate that this is an approximate expression, which tends to the exact one when $\lambda \rightarrow \infty$. This approximation exists because an incompressible problem is being treated by an expression that will eventually accommodate a slight compressibility. However, in order to avoid the proliferation of indices the symbol "λ" will not be used throughout this work. A close examination of the term containing the coefficients in parentheses in Eq. (5.3) reveals that the additional term included into Eq. (5.1), which should disappear because of the Continuity equation, seems to have very small effect on the final results. This is because the term containing λ is much greater than the viscosity. In reality, it seems that the only role of this term is to keep a total similarity with the corresponding equation used to solve elasticity problems.

One advantage of the penalty function formulation is that the additional unknown, pressure, p, is eliminated. Also, the satisfaction of the incompressibility condition is, from a numerical point of view, a considerable simplification.

5.3- ANALOGY WITH ELASTICITY

The equations for the plane-strain elasticity problem is briefly discussed in Appendix G. It can be seen that Eq. (G.9) has the same form as Eq. (5.3) discussed above. This establishes the analogy between the elasticity and viscous flow problems. In reality, this analogy has been exploited since the penalty function was introduced into flow solutions in order to take advantage of success achieved in the solution to the elasticity field. The correspondence between the variables appearing in these equations is summarised in Table 5.1.

The most important observation is that the body force of the elasticity equation is replaced by the convective term of equation (5.3), which is non-linear. This fact is responsible for the additional difficulty introduced in the solution of this equation leading to the need for an iterative approach.

Table 5.1 : Correspondence Between the Variables of the Two Problems.

VISCOUS FLOW ANALYSIS		ELASTICITY ANALYSIS	
PARAMETER	SYMBOL	PARAMETER	SYMBOL
Velocity	$\vec{v}$	Displacement	$\vec{q}$
Penalty Function/Density	λ / ρ	Lamé's Constant	λ'
Kinematic Viscosity	ν	Lamé's Constant	μ'
Convective Term	$-(\vec{v}.\nabla)\vec{v}$	Body Force	$\vec{F}$

5.4- INTEGRAL FORMULATION

One important consequence of the analogy between the differential formulation for fluid flow and elasticity problems is that the time-consuming task of deriving the boundary integral equation for this problem is avoided. Accordingly, the fact that the analogy exists, means it is possible to use exactly the same integral expressions presented in Appendix G, provided that the correspondence between the variables and the parameters of each problem is observed. Hence, Eqs. (G.26) and (G.27) can be used to produce the boundary integral equations for the velocity components, v_x and v_y, as follows:

$$C_i v_x(x_i,y_i) + \oint_\Gamma (F_{11} v_x + F_{21} v_y)\, d\Gamma = \oint_\Gamma (G_{11} T_x + G_{21} T_y)\, d\Gamma + V_x(x_i,y_i) \tag{5.4}$$

and

$$C_i v_y(x_i,y_i) + \oint_\Gamma (F_{12} v_x + F_{22} v_y)\, d\Gamma = \oint_\Gamma (G_{12} T_x + G_{22} T_y)\, d\Gamma + V_y(x_i,y_i) \tag{5.5}$$

where:

$$V_x(x_i,y_i) = - \iint_\Omega \left[G_{11}(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}) + G_{21}(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}) \right] dx\, dy \tag{5.6}$$

and

$$V_y(x_i,y_i) = - \iint_\Omega \left[G_{12}(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}) + G_{22}(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}) \right] dx\, dy \tag{5.7}$$

Eqs. (5.6) and (5.7) represent the "domain load" terms. The fundamental solution parameters for velocity and traction, G_{kl} and F_{kl}, respectively, with $k = 1,2$ and l = 1,2, appearing in these integrals are given by the following expression, obtained from Eqs. (G.21) and (G.24):

$$G_{kl} = \frac{1}{8\pi\nu(1-\bar{\nu})}\left[\{(3-4\bar{\nu})\left(\ln\left(\frac{1}{r}\right)-1\right)-0.5\}\delta_{kl} + \frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\right] \quad (5.8)$$

and

$$F_{kl} = -\frac{1}{4\pi(1-\bar{\nu})r}\left[2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l} + (1-2\bar{\nu})\left(\ell_l\frac{\partial r}{\partial x_k} - \ell_k\frac{\partial r}{\partial x_l} + \frac{\partial r}{\partial n}\delta_{kl}\right)\right] \quad (5.9)$$

where:

$$\bar{\nu} = \frac{\lambda}{2(\lambda + \nu\rho)} \quad (5.10)$$

This is the equivalent of Poisson's ratio relation for elasticity given by Eq. (G.8) where λ represents the penalty function parameter. Note also that r is the distance between the source and field points and ℓ_1 and ℓ_2 are the direction cosines ℓ and m, respectively. δ_{kl} is the delta of Kronecker.

It should further be noted that since the traction components, T_x and T_y, were included into Eqs. (5.4) and (5.5) the boundary conditions in this case are in terms the velocity and traction components, while the boundary conditions for Eq. (5.3) involve only the velocity components.

Some elasticity problems can be represented by a boundary integral equation where the variables of the problem appear only in the integrand of the contour integrals. This situation occurs when dealing with linear elasticity or when the loading state given by the body force is known. It can be seen from Table 5.1 that in viscous flow analysis the correspondence of the body force is the convective term of the Eq. (5.3). The integral representation for this term is given above by Eqs. (5.6) and (5.7). Hence, in this case, the integral representation also includes the domain integral with the variable of the problem appearing in the integrand. This fact introduces additional difficulties for the numerical solution of the fluid problem. Firstly, the domain Ω has also to be discretized. In this case one of the most important advantages of the BEM is lost, that is, the need of discretizing only the boundary. Another important point is that since the boundary integral equation is non-linear, it has to be solved iteratively. Accordingly, some difficulties related with divergence of the results may appear particularly for high Reynolds number flows. It is recognized that in these cases, the convective term has to be evaluated accurately, otherwise the results may diverge. Finally, the convective term included in the domain integrals contains derivatives of velocity. An additional approach has, therefore, to be devised in order to evaluate these parameters so that very accurate results are achieved for the calculation of the convective term represented by Eqs. (5.6) and (5.7).

According to the discussion already presented in Chapters 3 and 4, there are alternative approaches that could be adopted to evaluate the derivatives of any variable of the problem. In that discussion, it was concluded that the use of an approach based on the boundary integral equation retained an analysis based on the BEM. In this analysis, the same approach will be adopted. This means that the equations to calculate derivatives of velocity will be obtained by derivation of Eqs. (5.4) and (5.5) with respect to the source point coordinates, x_i and y_i. This is undertaken in Appendix G and given by Eq. (G.31) in tensorial form. Using this same equation for the present problem and replacing appropriately variables and parameters according to their correspondence in the two analogous problems given in Table 5.1, results in the following equations:

$$\frac{\partial v_l}{\partial x_m}(x_i,y_i) + \oint_\Gamma v_k \frac{\partial F_{kl}}{\partial x_m} d\Gamma = \oint_\Gamma T_k \frac{\partial G_{kl}}{\partial x_m} d\Gamma + \frac{\partial V_l}{\partial x_m}(x_i,y_i) \tag{5.11}$$

where:

$$\frac{\partial V_l}{\partial x_m}(x_i,y_i) = -\iint_\Omega v_j \frac{\partial v_k}{\partial x_j} \frac{\partial G_{kl}}{\partial x_m} dx\,dy \tag{5.12}$$

$$\frac{\partial G_{kl}}{\partial x_m} = \frac{1}{8\pi\nu(1-\bar{\nu})r}\left[2\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} + (3-4\bar{\nu})\frac{\partial r}{\partial x_m}\delta_{kl} - \frac{\partial r}{\partial x_k}\delta_{lm} - \frac{\partial r}{\partial x_l}\delta_{mk}\right] \tag{5.13}$$

and

$$\begin{aligned}\frac{\partial F_{kl}}{\partial x_m} = \frac{-1}{4\pi(1-\bar{\nu})r^2}\Bigg[&2\frac{\partial r}{\partial n}\{4\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} + (1-2\bar{\nu})\frac{\partial r}{\partial x_m}\delta_{kl} - \frac{\partial r}{\partial x_k}\delta_{lm} - \frac{\partial r}{\partial x_l}\delta_{km}\}\\ &+ 2\{(1-2\bar{\nu})(\ell_k\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} - \ell_l\frac{\partial r}{\partial x_m}\frac{\partial r}{\partial x_k}) - \ell_m\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\}\\ &- (1-2\bar{\nu})(\ell_k\delta_{lm} - \ell_l\delta_{mk} + \ell_m\delta_{kl})\Bigg]\end{aligned} \tag{5.14}$$

where the tensorial indices k, l, m and j can assume the values 1 or 2, and the other parameters apply as before. One can see from these equations the difficulty in evaluating the integrals in Eq. (5.11), mainly because their integrand contains singularities. Note, for example, that some kernels appearing in the boundary integrals of the Eq. (5.11) contain a strong singularity caused by the term $1/r^2$. Thus, it was avoided locating the source point on the boundary nodes in this case. Special relations were used to evaluate the derivatives of velocity on the boundary. That problem is mentioned in Appendix G but the technique adopted is only discussed in Appendix K.

Chapter 6

Numerical Solution Procedures

6.1- INTRODUCTION

The BEM is the numerical technique adopted to solve the governing differential equations for the fluid flow problems discussed throughout this work. As a first step towards the application of the BEM, the partial differential equations were transformed into BIEs according to the discussions presented in the previous three Chapters for problems governed by potential flow and viscous flow equations. In this Chapter the next step concerning the application of the BEM is discussed. This refers basically to the discretization technique applied in order to make the solution of the BIEs possible, since they themselves, like their differential counterpart, are not solvable analytically. The objective of this discretization process is to transform each BIE into a set of algebraic linear equations, similar to most numerical techniques for solving engineering problems, whose solution gives the distribution of the sought parameter at the nodes of the discretized boundary, as well as over the domain nodes for problems that include unknowns in the integrand of a domain integral. The solution procedures and the corresponding computer programs are also briefly discussed, along with the most important numerical features included in the programs.

The basic steps in order to solve numerically a given BIE are:

1- discretization of the boundary Γ into small boundary elements, each one containing a number of nodes;

2- representation of each variable of the problem inside of the boundary elements in terms of interpolation functions, using for example the ones adopted in the FEM, sometimes called shape functions, and nodal values of the variables;

3- transformation of the BIE into an algebraic linear equation using the representation of variables in terms of interpolation functions and nodal unknowns. The coefficients of this equation are given in terms of geometrical parameters. In fact, they are boundary integrals having the fundamental solution or its derivatives as the kernel and that also contain the interpolation functions as part of the integrand;

4- evaluation of these integrals using normally Gaussian quadrature to perform the regular integrals and a suitable technique to deal with singular ones, since it is not always possible to perform them analytically;

Steps 1-4 transform a BIE into a algebraic linear equation that is valid for one specific source point position and has as many unknowns as the number of nodes of the mesh times the number of unknown parameters in the problem. Therefore, a system of algebraic equations has to be generated with as many equations as the number of unknowns. This can most conveniently be done if the source points are located at the boundary nodes created with the discretization process, although theoretically they can placed anywhere.

5- application of steps 3 and 4 to obtain algebraic equations corresponding to source points located at boundary nodes, thus generating a set of algebraic equations;

6- imposing the boundary conditions and rearrangement of the terms so as to obtain a solvable set of equations;

7- solution of this set of algebraic equations in order to obtain the distribution of the variables at the nodes of the mesh; and finally

8- using discretized BIE to find the values of the unknowns at any internal point inside the domain.

This is basically the procedure adopted for solving potential problems where the domain integral that may appear contains no unknowns. The solution can be obtained in this case in just one go. Unfortunately, non-linear problems require an iterative procedure in order to obtain the solution. Besides, most of them, such as viscous flow phenomenon, contains domain integrals with unknowns in the integrand. The presence of domain integrals not only destroys the most important advantage of the BEM over domain methods, the discretization only of the boundary, since it also brings a great deal of difficulty that sometimes has the effect of preventing the iterative procedure from working properly. As a result, divergence of results may occur. Apart from the problems of domain integral and divergence of results, there are in general three difficulties associated with the application of the BEM that affect indistinctively both linear and non-linear formulations, they are:

- Solution of singular integrals;

- Corner problems; and

- Ill-conditioning of the system of algebraic equations.

Other problems may appear, such as inaccurate results when the source points get close to the boundary. This is normally caused by the fact the boundary integrals may become quasi-singular. All these difficulties are of numerical origin and make the application of BEM a very difficult task. This kind of problems and the fact that the derivations of BIE are mathematically more involved than most conventional techniques have in part been responsible for preventing the widespread use of the BEM among engineering circles.

6.2- DISCRETIZATION

Having derived the formulations represented in terms of integral equations in the previous Chapter, it is possible now to apply the piecewise-discretization concept similar to the one used in FEM, in order to discretize both the boundary and domain and genèrate the discretized equations corresponding to the BIEs. In order to do that, the boundary Γ has to be divided in N_e boundary elements while the domain Ω, if necessary, is divided in N_c cells connected to each other, as shown in the Figure 6.1, where

$$\Gamma = \bigcup_{e=1}^{N_e} \Gamma_e$$

$$\Omega = \bigcup_{c=1}^{N_c} \Omega_c$$

Accordingly, each integral appearing in a BIE can be represented by the summation of independent integrals performed on each boundary element, for boundary-type integrals, and on each domain cell, for domain-type integrals. That is, provided that θ is a continuous function of x and y, defined over Γ and Ω, then it can be shown that

$$\oint_{\Gamma} \theta \, d\Gamma = \sum_{e=1}^{N_e} \int_{\Gamma_e} \theta \, d\Gamma$$

and

$$\iint_{\Omega} \theta \, dx dy = \sum_{c=1}^{N_c} \iint_{\Omega_c} \theta \, dx dy$$

Applying this discretization technique to typical integrals appearing in a BIE results in:

$$\oint_{\Gamma} S^*(x-x_i, y-y_i) \, \theta \, d\Gamma = \sum_{e=1}^{N_e} \left[\int_{\Gamma_e} S^*(x-x_i, y-y_i) \, \theta(\Gamma_e) \, d\Gamma \right] \tag{6.1}$$

and

$$\iint_{\Omega} S^*(x-x_i, y-y_i) \, \theta \, dxdy = \sum_{c=1}^{N_c} \left[\iint_{\Omega_c} S^*(x-x_i, y-y_i) \, \theta(\Omega_c) \, dxdy \right] \tag{6.2}$$

for the boundary and domain integral, respectively, where $S^*(x-x_i,y-y_i)$ represents the kernel of these integrals which contains fundamental solution parameters with respect to the source point (x_i,y_i). This approximation is not sufficient to solve the integrals because the function θ is normally unknown. One way of solving them approximately is to define some nodes on each boundary element and domain cell, and approximate the function θ by means of interpolating expressions in terms of its value at those nodes, similar to FEM. Replacing the new representation for θ into the integrals, it is possible to see that the only unknowns are those nodal values of θ, which can be removed from the integrand. In this way, the integrals will be in terms of known functions and can be evaluated.

The expressions that represent the variation of θ on the boundary elements and cells using suitable interpolation functions and nodal values of θ assume different forms depending on the type of the boundary element and cell adopted in the discretization of the boundary and domain, respectively. Thus, the interpolation functions are dependent on the geometry and number of the nodes of the elements used to discretize the boundary and domain. In general, these interpolation functions are conveniently built using the so-called intrinsic coordinates of the element, instead of using the global coordinates that describe both boundary and domain with relation to the x- and y-axis, which are the independent variables of the problem. Hence, if each boundary element Γ_e contains n_e nodes and the domain cell Ω_c contains n_c nodes, the variable θ, for example, can be represented by

$$\theta(\xi)=\sum_{j=1}^{n_e} \theta_j \, N_j(\xi) \tag{6.3}$$

and

$$\theta(\xi,\eta) = \sum_{j=1}^{n_c} \theta_j \, N_j(\xi,\eta) \tag{6.4}$$

on a boundary element and over a domain cell, respectively, where $N_j(\xi)$ and $N_j(\xi,\eta)$ are interpolation functions, and (ξ) and (ξ,η) are intrinsic coordinates for the boundary elements and domain cells, respectively. The coordinates ξ and η are local coordinates of the transformed boundary element and cell, which in this work their values vary from 0 to 1. Substituting Eqs. (6.3) and (6.4) into Eqs. (6.1) and (6.2), respectively, results in:

$$\oint_\Gamma S^*(x-x_i,y-y_i)\,\theta\, d\Gamma = \sum_{e=1}^{N_e}\left[\sum_{j=1}^{n_e} \overline{S}_j^{(e)}(x_i,y_i)\,\theta_j\right] \tag{6.5}$$

and

$$\iint_{\Omega} S^{*}(x-x_i,y-y_i)\,\theta\,dxdy = \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\overline{S}_j^{(c)}(x_i,y_i)\,\theta_j\right] \qquad (6.6)$$

where:

$$\overline{S}_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi)\,S^{*}(x-x_i,y-y_i)\,J(\xi)\,d\xi$$

and

$$\overline{S}_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta)\,S^{*}(x-x_i,y-y_i)\,J(\xi,\eta)\,d\xi d\eta$$

and $J(\xi)$ and $J(\xi,\eta)$ are the Jacobians corresponding to the change of variable in the integrals, such as

$$d\Gamma = J(\xi)\,d\xi$$

$$dxdy = J(\xi,\eta)\,d\xi d\eta$$

for boundary and domain integrals, respectively.

Employing the discretization technique presented above, discretized equations corresponding to BIEs can easily be generated. However, before doing so it may be interesting to represent the boundary integral expression in a more compact form. This may be convenient for programming. In this case, the coefficients corresponding to the same nodal value of θ are added, which is the case for nodes connecting every two boundary elements. As a result, Eq. (6.5) for example, can be recast into another expression that contains only the summation of the product of nodal values of θ and their coefficients. Thus, if TA (e,j) is a topological array to represent the global number J of the j^{th} local node on the e^{th}

boundary element, then Eq. (6.5) can be rewritten as

$$\oint_{\Gamma} S^*(x_i,y_i)\,\theta\, d\Gamma = \sum_{J=1}^{m_\Gamma} S_{I,J}\theta_J \tag{6.7}$$

where m_Γ is the total number of nodes on the boundary Γ, I refers to the source point in question and $S_{I,J}$ is given by:

$$S_{I,J} \equiv \sum_{e}^{?} \overline{S}_j^{(e)}(x_i,y_i)$$

where the summation is only carried out on all elements (one or two) which contain the node J.

A similar procedure can be applied to the domain integral given by Eq. (6.6). Thus, defining a topologic array CTA(c,j) to represents the J global node corresponding to the j^{th} local node on the c^{th} domain cell, it results in:

$$\iint_{\Omega} S^*(x_i,y_i)\,\theta\, dxdy = \sum_{J=1}^{m_\Omega} S_{I,J}^{(c)}\theta_J \tag{6.8}$$

where m_Ω is the total number of nodes over the domain Ω including the boundary nodes, I refers to the source point and $S_{I,J}$ is given for an expression similar to the one given above.

6.2.1- Type of Elements

There are several types of boundary elements and cells, depending on the position and number of nodes and geometry of the element. In this analysis, the isoparametric-type elements with only a few nodes are the only ones considered to represent both boundary elements and cells. For the simple reason that if on the one hand constant elements (containing only one node in the middle of element) are hardly adopted in fluid problems, on the other hand highly elaborated elements seeing in FEM analysis, which are described by several nodes, seem to be unnecessary in BEM analysis. Eventually, one has to comprise between the accuracy of results and size of the system of equations to be solved, which are dependent on the number of nodes of the mesh. In this work only linear and quadratic elements are adopted to discretize the boundary, while the domain is discretized using either the three-node triangular elements, with nodes placed at the vertices of the triangle, or the eight-node rectangular element, with one node at each corner and one at the midpoint of each side of the element. The decision towards these types of elements was made taking into consideration the advices available in the literature, where the combination of the triangular element on the domain with linear element on the boundary is used very often. Note, also, that the nodes of the two meshes, the one on the boundary and the other on the domain, are normally related. Thus, according to the type of elements adopted, two options to combine the meshes were used in this work: a linear element on the boundary with triangular element on the domain, and a quadratic element on the boundary with quadrangular element on the domain. However, the later was adopted in most cases because it gives more accurate results. Appendix H presents all the information with regard to these types of elements, such as, Jacobians, interpolation functions, and so on.

The mesh generation does not represent a major concern in BEM as it does, for example, in FVM (also regarded as FDM) analysis since it is a relatively easy task to generate a mesh on the boundary. The mesh over the domain, when necessary, can be generated using the techniques largely available for FEM, that is, the same mesh generator may be used in connection with BEM. Meshes similar to the ones used in FEM seem to be adequate for BEM analysis, according to the results presented in the literature. However, there are also in the literature some suggestions of meshes built especially to suit the special feature of BEM which may introduce some benefits. One of the most important of these is the technique suggested by Camp & Gipson (1987 and 1989). Although the domain is divided into cells their technique, in fact, does not use a fixed mesh, since a different subdivision of domain is carried out for each source point. This mesh has the main advantage of being of variable size, where the finest part of the mesh is placed around the source point and the coarsest away from the source point. They claim that in this way the accuracy of the domain integrations is enhanced without an exaggerated increase in the number of nodes of the mesh. An ordinary mesh, similar to the one used in FEM analysis, but with the capability of refinement in the region of the corner is adopted in this work.

6.2.2- Discretized Equations

Before presenting the discretized equations for the formulations discussed in the previous three Chapters, it is interesting to discuss some important points related to the way the boundary and domain integrals appearing in the equations were discretized in this work. The main question is whether the integrals in the discretized equations should be represented in the way given by Eqs. (6.7) and (6.8), where the value of the variable in a globally numbered node is made explicit, or whether they should be represented in the general form given by Eqs. (6.5) and (6.6), where the nodal value of the variable is made explicit locally within each element.

All the formulations previously discussed, that include domain integrals, are solved iteratively. In this case the domain integrals corresponding to each source point is solved using the distribution of the variable appearing in the integrand obtained in the previous iteration or assumed as an initial guess. Considering all source points chosen, this would produce an array with the value of the domain integral calculation for each source point. This is possible because the distribution on the domain of the variables appearing in the integrand of the domain integrals, is obtained explicitly node-by-node using specific equations, instead of the implicity solution of the BIEs using the variable in the integrand as unknowns. This means that it is not necessary to arrange the domain integral discretization in the same way given by the Eq. (6.8), where nodal values of the variable on the domain are made explicit and the coefficients corresponding to each node are added. In this way, the array with the result of the domain integral calculation is the result of the product of a matrix, containing the coefficient of influence involving the integrals of the type given in Eq. (6.6), and the vector with the nodal values of the variable on the domain. This form of representation of domain integrals is especially advantageous since this matrix needs to be calculated just once and stored to be used in the next iteration, since the elements of this matrix normally contains only geometrical information of the domain. This would reduce the CPU time but may create a problem of space in computer memory, especially for small computers.

A similar situation appears in the discretization of the boundary integrals. When the source point is located inside the domain, in order to calculate the distribution of any variable on the domain, the way the boundary integrals are discretized does not matter, since both ways can be used. However, when the distribution of the variable on the boundary is sought, the boundary integrals must be discretized like the compact form given by Eq. (6.7).

In this work, the equations were discretized like the form given by Eqs. (6.5) and (6.6) whenever possible for the sake of the computer memory space, since in this case it is not necessary to store the matrices with the coefficients of influence. In the programs, only in the subroutines that calculate the distribution of the variables on the boundary the way of discretizing the boundary integral given by Eq. (6.7) was used, and so the coefficients of influence were stored.

The BIEs presented in the previous chapters were discretized in this work based on the way give by Eqs. (6.5) and (6.6), as follows:

a) <u>Potential Flow Formulation</u>

The equation used to calculate the distribution of either the stream-function or the potential of velocity, Eq. (3.11), is given in discretized form by:

$$C_i u_i + \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n} \left(h_j^{(e)}(x_i,y_i)\, u_j - g_j^{(e)}(x_i,y_i) \left(\frac{\partial u}{\partial n} \right)_j \right) \right] = 0 \tag{6.9}$$

Since the derivatives of parameter u are normally required in order to calculate the distribution of velocity inside the domain, Eqs. (3.12) and (3.13) are also important. Their discretized form are, respectively, given by:

$$\left(\frac{\partial u}{\partial x} \right)_i = \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n} \left(hxn_j^{(e)}(x_i,y_i) u_j - hx_j^{(e)}(x_i,y_i) \left(\frac{\partial u}{\partial n} \right)_j \right) \right] \tag{6.10}$$

and

$$\left(\frac{\partial u}{\partial y} \right)_i = \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n} \left(hxn_j^{(e)}(x_i,y_i) u_j - hy_j^{(e)}(x_i,y_i) \left(\frac{\partial u}{\partial n} \right)_j \right) \right] \tag{6.11}$$

b) Vorticity-Velocity Formulation

The discretized form of the kinematic equations are given by:

$$C_i (v_x)_i + \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n_e} \left(hx_j^{(e)}(x_i,y_i)(v_n)_j - hy_j^{(e)}(x_i,y_i)(v_t)_j + gm_j^{(e)}(x_i,y_i)(w_b)_j \right) \right] = \sum_{c=1}^{N_c} \left[\sum_{j=1}^{n_c} gc_j^{(c)}(x_i,y_i) \left(\frac{\partial \omega}{\partial y} \right)_j \right] \tag{6.12}$$

and

$$C_i (v_y)_i + \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n_e} \left(hx_j^{(e)}(x_i,y_i)(v_t)_j + hy_j^{(e)}(x_i,y_i)(v_n)_j - gl_j^{(e)}(x_i,y_i)(w_b)_j \right) \right] = - \sum_{c=1}^{N_c} \left[\sum_{j=1}^{n_c} gc_j^{(c)}(x_i,y_i) \left(\frac{\partial \omega}{\partial x} \right)_j \right] \tag{6.13}$$

With regard the kinetic equations, including the equations for the derivatives of vorticity, two options were investigated. The equations for the first option include Eqs. (4.22), (4.27) and (4.28), which in discretized form are given by:

$$C_i \omega_i + \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n_e} \left(h_j^{(e)}(x_i,y_i)\, \omega_j - g_j^{(e)}(x_i,y_i) \left(\frac{\partial \omega}{\partial n} \right)_j \right) \right] = - \frac{1}{\nu} \sum_{c=1}^{N_c} \left[\sum_{j=1}^{n_c} gc_j^{(c)}(x_i,y_i) \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right)_j \right] \tag{6.14}$$

$$\left(\frac{\partial \omega}{\partial x} \right)_i = \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n_e} \left(- hx_j^{(e)}(x_i,y_i) \left(\frac{\partial \omega}{\partial n} \right)_j + hxn_j^{(e)}(x_i,y_i)\, \omega_j \right) \right] + \frac{1}{\nu} \sum_{c=1}^{N_c} \left[\sum_{j=1}^{n_c} hxc_j^{(c)}(x_i,y_i) \left(v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} \right)_j \right] \tag{6.15}$$

and

$$\left(\frac{\partial\omega}{\partial y}\right)_i = \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(- hy_j^{(c)}(x_i,y_i)\left(\frac{\partial\omega}{\partial n}\right)_j + hyn_j^{(c)}(x_i,y_i)\,\omega_j\right)\right] + \frac{1}{\nu}\sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c} hyc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial\omega}{\partial x} + v_y\frac{\partial\omega}{\partial y}\right)_j\right] \tag{6.16}$$

The equations for the second option, Eqs. (4.24), (4.29) and (4.30), are in discretized form given by, respectively:

$$C_i\,\omega_i + \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(h_j^{(c)}(x_i,y_i)\,\omega_j - g_j^{(c)}(x_i,y_i)\left(\frac{\partial\omega}{\partial n}\right)_j + \frac{1}{\nu}\,g_j^{(c)}(x_i,y_i)\,(\omega\,v_n)_j\right)\right] = \frac{1}{\nu}\sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(hxc_j^{(c)}(x_i,y_i)\,(v_x\,\omega)_j + hyc_j^{(c)}(x_i,y_i)\,(v_y\,\omega)_j\right)\right] \tag{6.17}$$

$$-\left(\frac{\partial\omega}{\partial x}\right)_i = \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(hx_j^{(c)}(x_i,y_i)\left(\frac{\partial\omega}{\partial n}\right)_j - hxn_j^{(c)}(x_i,y_i)\,\omega_j - \frac{1}{\nu}\,hx_j^{(c)}(x_i,y_i)\,(\omega\,v_n)_j\right)\right] + \frac{1}{\nu}\sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(hxxc_j^{(c)}(x_i,y_i)\,(\omega\,v_x)_j + hxyc_j^{(c)}(x_i,y_i)\,(\omega\,v_y)_j\right)\right] \tag{6.18}$$

and

$$-\left(\frac{\partial \omega}{\partial y}\right)_i = \sum_{e=1}^{N_e}\left[\sum_{j=1}^{n_e}\left(hy_j^{(e)}(x_i,y_i)\left(\frac{\partial \omega}{\partial n}\right)_j - hyn_j^{(e)}(x_i,y_i)\,\omega_j\right.\right.$$

$$\left.\left. - \frac{1}{\nu} hy_j^{(e)}(x_i,y_i)\,(\omega\, v_n)_j\right)\right] \tag{6.19}$$

$$+ \frac{1}{\nu}\sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(hxyc_j^{(c)}(x_i,y_i)\,(\omega\, v_x)_j + hyyc_j^{(c)}(x_i,y_i)\,(\omega\, v_y)_j\right)\right]$$

C) <u>Penalty Function Formulation</u>

In order to calculate the velocity components Eqs. (5.4) and (5.5) are used, which are given in discretized form by, respectively:

$$C_i\,(v_x)_i + \sum_{e=1}^{N_e}\left[\sum_{j=1}^{n_e}\left(fxx_j^{(e)}(x_i,y_i)\,(v_x)_j + fyx_j^{(e)}(x_i,y_i)\,(v_y)_j\right.\right.$$

$$\left.\left. - gxx_j^{(e)}(x_i,y_i)\,(T_x)_j - gyx_j^{(e)}(x_i,y_i)\,(T_y)_j\right)\right]$$

$$= -\sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(gxxc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right)_j\right.\right. \tag{6.20}$$

$$\left.\left. + gyxc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right)_j\right)\right]$$

and

$$
\begin{aligned}
C_i (v_y)_i &+ \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n_e} \left(fxy_j^{(e)}(x_i,y_i)(v_x)_j + fyy_j^{(e)}(x_i,y_i)(v_y)_j \right.\right. \\
&\left.\left. - gxy_j^{(e)}(x_i,y_i)(T_x)_j - gyy_j^{(e)}(x_i,y_i)(T_y)_j \right) \right] \\
&= - \sum_{c=1}^{N_c} \left[\sum_{j=1}^{n_c} \left(gxyc_j^{(c)}(x_i,y_i) \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right)_j \right.\right. \\
&\left.\left. + gyyc_j^{(c)}(x_i,y_i) \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right)_j \right) \right]
\end{aligned}
\tag{6.21}
$$

The derivatives of the velocity components are evaluated at any internal node by the Eq. (5.2), that in expanded and discretized form are given by:

$$
\begin{aligned}
\left(\frac{\partial v_x}{\partial x} \right)_i &= - \sum_{e=1}^{N_e} \left[\sum_{j=1}^{n_e} \left(fxxx_j^{(e)}(x_i,y_i)(v_x)_j + fyxx_j^{(e)}(x_i,y_i)(v_y)_j \right.\right. \\
&\left.\left. - gxxx_j^{(e)}(x_i,y_i)(T_x)_j - gyxx_j^{(e)}(x_i,y_i)(T_y)_j \right) \right] \\
&- \sum_{c=1}^{N_c} \left[\sum_{j=1}^{n_c} \left(gxxxc_j^{(c)}(x_i,y_i) \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right)_j \right.\right. \\
&\left.\left. + gyxxc_j^{(c)}(x_i,y_i) \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right)_j \right) \right]
\end{aligned}
\tag{6.22}
$$

$$\left(\frac{\partial v_x}{\partial y}\right)_i = -\sum_{e=1}^{N_e}\left[\sum_{j=1}^{n_e}\Big(fxxy_j^{(e)}(x_i,y_i)(v_x)_j + fyxy_j^{(e)}(x_i,y_i)(v_y)_j - gxxy_j^{(e)}(x_i,y_i)(T_x)_j - gyxy_j^{(e)}(x_i,y_i)(T_y)_j\Big)\right] - \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(gxxyc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right)_j + gyxyc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right)_j\right)\right] \tag{6.23}$$

$$\left(\frac{\partial v_y}{\partial x}\right)_i = -\sum_{e=1}^{N_e}\left[\sum_{j=1}^{n_e}\Big(fxyx_j^{(e)}(x_i,y_i)(v_x)_j + fyyx_j^{(e)}(x_i,y_i)(v_y)_j - gxyx_j^{(e)}(x_i,y_i)(T_x)_j - gyyx_j^{(e)}(x_i,y_i)(T_y)_j\Big)\right] - \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(gxyxc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right)_j + gyyxc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right)_j\right)\right] \tag{6.24}$$

and

$$\left(\frac{\partial v_y}{\partial y}\right)_i = -\sum_{e=1}^{N_e}\left[\sum_{j=1}^{n_e}\left(fxyy_j^{(e)}(x_i,y_i)(v_x)_j + fyyy_j^{(e)}(x_i,y_i)(v_y)_j\right.\right.$$

$$\left.\left. - gxyy_j^{(e)}(x_i,y_i)(T_x)_j - gyyy_j^{(e)}(x_i,y_i)(T_y)_j\right)\right]$$

$$- \sum_{c=1}^{N_c}\left[\sum_{j=1}^{n_c}\left(gxyyc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right)_j\right.\right. \tag{6.25}$$

$$\left.\left. + gyyyc_j^{(c)}(x_i,y_i)\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right)_j\right)\right]$$

The coefficients appearing in the discretized equations, Eqs. (6.9) to (6.25), are presented in Appendix C.

The algebraic equations given by Eqs. (6.9) to (6.25) are valid for a generic source point (x_i,y_i). After the introduction of the boundary conditions, these equations will contain as many unknowns on the boundary as m_Γ times the number of unknown parameters of the problem. Hence, in order to solve the problem, a set of algebraic equations with as many equations as the number of unknowns has to be generated. This can be done by choosing a suitable number source points, normally placed conveniently at every boundary nodes. Thus, equations corresponding to each source point are obtained. Note, however, that this set can only be solved if the existing domain integrals are evaluated first. In general, when the unknowns in question are located on the boundary nodes of the domain, the problem is solved implicitly since a system of equations has to be solved. That is the case of Eq. (6.9) , for instance. On the other hand, results for internal nodes (sometimes including the boundary nodes as well) can be obtained explicitly node-by-node using the equations in the form given previously, which may be advantageous for the computer memory point of view. That is the case of Eqs. (6.10), (6.11), (6.15), (6.16), (6.18), (6.19) and (6.22) to (6.25). Some equations are used twice to obtain the solutions on the boundary and domain as well. Eqs. (6.14) and (6.17), for example, are used to generate a system of equations to be solved implicity in order to obtain the distribution of normal derivative of vorticity on the boundary and afterwards they are used to obtain explicitly the distribution of vorticity at internal nodes of domain. Thus, from those equations above matrices with the coefficients of influence may have to be assembled or not, depending on the situation.

Before discussing the solution procedures to solve the equations corresponding to the three formulations, presented previously in discretized form, it is convenient to discuss the following two topics which are extremely important for the success of the numerical solution of the discretized equations namely, integration procedures and corner treatment.

6.3- INTEGRATION PROCEDURES

To obtain the coefficients appearing in the discretized equations given in the previous section requires the solution of integrals on the boundary and the domain. It is recognized that most of computational effort of a BEM program is spent performing integrations in order to calculate these influence coefficients. Apart this fact, this task would not require considerable attention if most of these integrals were not singular. Unfortunately, the problem is that the kernel of most integrals appearing in the BIE has a singularity at the source point, and so the integration on the boundary elements or cells where the source point is placed must be performed through special techniques. To make matters worse, the coefficients arising from singular integrals lie on the leading diagonals of the BEM matrices. They therefore, have a strong effect on the solution. It is therefore, very important to adopt a technique which allows the calculation of singular integrals with good accuracy.

This subject has received the attention it deserves from the many researchers in this area, as is indicated by the large number of publications addressing the problem. As a result, there are now several ways to deal with singular integrals. It seems, however, that a technique that can be used to deal efficiently with different singular integrals normally appearing in the use of BEM has not yet been published. Thus, different approaches have to be considered in a program to deal with different types of singularity appearing in the integrand. Sometimes one may not find an approach in the literature to deal with a special singularity and so an adaptation has to be introduced in the original approach, which can affect the accuracy of the results.

The singular integral can be performed analytically but only in a limited number of cases. For the isoparametric-type of boundary element and cell used in this work, the interpolation functions deliver an integrand which is very difficult to integrate analytically. Approximate techniques have, therefore, to be used. These techniques are in general based on transformations, Taylor series expansions and special integration formulae. Sometimes, the singularity involved in the boundary integrals are so difficult to remove, that the technique of locating the source point outside the domain has to be considered. This technique is referred to as a regular Boundary Element Method. Also, when an interior solution is to be obtained, including also the boundary, it is very common to avoid locating the source point on the boundary nodes, if any boundary integral exists whose kernel contains a strong singularity. In this case, the results on the boundary nodes are obtained using different techniques. In general, the distribution of the parameter on the boundary is already known from the previous steps of the algorithm and so there is no need to calculate it again.

The fact is that dealing with singular integrals demands a lot of skill. In consequence, with the success of the BEM and its increasingly expanding range of applications, the need for techniques that allow a rapid, accurate and consistent evaluation of integrals became very crucial for the continued development of the BEM. This was so important that the solution of singular integrals became a major topic of research itself.

In this work, no attempt was made to investigate the performance of the many different techniques for evaluation of the integrals before deciding on one nor is a new technique proposed. It was decided, instead, to use an approach suggested in the literature that seems to give good accuracy for the purpose of this work.

The regular integrals are relatively easy to perform, and the usual Gaussian quadrature is normally adopted. In this work the modified Gaussian quadrature, which uses an interval of integration from 0 to 1, as described in El-Zafrany (1993), was adopted to calculate the regular boundary and domain integrals. With regard the singular integrals, different approaches were adopted depending on the type of the kernel appearing in the integral. Thus, starting from boundary integrals, the integrals given previously are discussed.

Fortunately, most singularity occurring in the boundary integrals could be avoided, such as

$$\frac{\partial u^*}{\partial x} = -\frac{1}{2\pi}\frac{(x-x_i)}{r^2}$$

and
$$\frac{\partial u^*}{\partial y} = -\frac{1}{2\pi}\frac{(y-y_i)}{r^2}$$

These are represented by a rational function and no special technique is necessary, in this case, to perform the integrals. In fact, the singularity exists but was ignored since it will be corrected by means of jump functions or rigid translation techniques, that will be discussed later in connection with corner treatment. Other kernels derived from these will be dealt in a similar way, like

$$\frac{\partial u^*}{\partial n} = -\frac{1}{2\pi r^2}\left[(x-x_i)\ell + (y-y_i)m\right]$$

There are also other kernels appearing in the previous integrals which are represented by rational functions. These require no special technique to deal with the singularity. They are:

$$G_{12} = G_{21} = \frac{1}{8\pi\nu(1-\bar{\nu})}\left(\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\right)$$

$$F_{11} = -\frac{1}{4\pi(1-\bar{\nu})r}\frac{\partial r}{\partial n}\left[2\left(\frac{\partial r}{\partial x}\right)^2 + (1-2\bar{\nu})\right]$$

$$F_{12} = -\frac{1}{4\pi(1-\bar{\nu})r}\left[2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} + (1-2\bar{\nu})\left(m\frac{\partial r}{\partial x} - \ell\frac{\partial r}{\partial y}\right)\right]$$

$$F_{21} = -\frac{1}{4\pi(1-\bar{v})r}\left[2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} + (1-2\bar{v})\left(\ell\frac{\partial r}{\partial y} - m\frac{\partial r}{\partial x}\right)\right]$$

and

$$F_{22} = -\frac{1}{4\pi(1-\bar{v})r}\frac{\partial r}{\partial n}\left[2\left(\frac{\partial r}{\partial y}\right)^2 + (1-2\bar{v})\right]$$

where:

$$\frac{\partial r}{\partial x} = \frac{(x-x_i)}{r}$$

$$\frac{\partial r}{\partial y} = \frac{(y-y_i)}{r}$$

and

$$\frac{\partial r}{\partial n} = \ell\frac{\partial r}{\partial x} + m\frac{\partial r}{\partial y}$$

On the other hand, kernels given below possess a strong singularity which are very difficult to deal with due the presence of $1/r^2$:

$$\frac{\partial^2 u^*}{\partial y\,\partial n} = -\frac{1}{2\pi r^2}\left(m - 2\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\right)$$

and

$$\frac{\partial^2 u^*}{\partial x\,\partial n} = -\frac{1}{2\pi r^2}\left(\ell - 2\frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\right)$$

These appear in the Eqs.(6.15), (6.16), (6.18) and (6.19) which are used to calculate the derivatives of vorticity inside the domain, including the boundary. In this work, it was decided to avoid placing the source point on the boundary, in this case, to prevent problems in dealing with the boundary integrals containing these kernels. Instead, an expression was derived to calculate the derivatives of vorticity on the boundary analytically, taking into consideration the information of the distribution of the normal derivative of vorticity. The technique of using an auxiliary expression to calculate the distribution of the parameter on the boundary instead of using the BIE is normally adopted in such a situation to avoid the singularity. An additional advantage of doing so is that the algorithm becomes faster. A similar approach had to be used in order to solve Eqs. (6.22) to (6.25) since the kernels given by the derivatives of the fundamental solutions components F_{11}, F_{12}, F_{21} and F_{22}, defined by Eq. (5.14), contain a strong singularity due a term $1/r^2$. The derivatives of the fundamental solution components G_{11}, G_{12}, G_{21} and G_{22}, defined by Eq. (5.13), appearing also in the integrals given in these equations, are represented by terms containing only rational functions. These represented no problem also since the singular integrations on the boundary were avoided.

The boundary integrals whose kernel contains ln r were in fact the only ones where a special technique had to be adopted in order to perform them. The kernels included in this category are:

$$u^* = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$$

$$G_{11} = \frac{1}{8\pi\nu(1-\bar{\nu})}\left[(3-4\bar{\nu})\ln\left(\frac{1}{r}\right) + \left(\frac{\partial r}{\partial x}\right)^2\right]$$

and

$$G_{22} = \frac{1}{8\pi\nu(1-\bar{\nu})}\left[(3-4\bar{\nu})\ln\left(\frac{1}{r}\right) + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

The approach presented in the section I.1 of Appendix I was adopted, in order to overcome the problem. This approach comprises basically the introduction of a change of variable in order to split the integrand into two parts: one singular and the other regular. They can then be calculated using the usual quadrature formulae available in the literature since the singular part can, in this case, be evaluated in this way.

With regard to the domain integrals, the situation is a little bit more complicated because all domain integrals given in Appendix C require a special approach to deal with the singularity. In addition, most domain integrals are closely related to the convective term of the transport equations governing the motion of fluids. According to the literature a good accuracy in the evaluation of these domain integrals has to be obtained otherwise divergence of the results my occur, Kitagawa *et al.* (1986).

The technique presented in the section I.2 of Appendix I was adopted, with some minor adaptations from one case to another to overcome most of the singularities appearing in the domain integrals. This technique consists of enforcing that the domain integral with the singular kernel tends to zero at the source point. The only remaining term at the source point is a boundary integral which can be solved using the approaches discussed previously. The technique obeys the general expression given by Eq. (I.8) of Appendix I, where if F is a function of the field variable, then the domain integral can conveniently be transformed into:

$$\iint_{\Omega} F \frac{\partial P^*}{\partial n_o} = \iint_{\Omega} (F - F_i) \frac{\partial P^*}{\partial n_o} dxdy + F_i \oint (\vec{n}_o . \vec{n}) P^* d\Gamma \qquad (6.26)$$

where P^* is the kernel which includes the fundamental solution, F_i is the value of F at source point and $\vec{n}_o$ is the unit vector pointing in any specific direction. Note, however, that in order to adopt this technique, the domain integrals in the BIEs have to be replaced by the above equation and, as a consequence, another discretization form for the domain integrals emerges, which should replace the one given in the discretized Eqs. (6.12) to (6.25).

The main disadvantage of this approach is that, in this way, more integrals have to be solved, since an additional domain integral and a boundary integral were introduced. In consequence, this is likely to be more time consuming than other techniques. However, if the speed of calculation is of paramount importance, the matricial way of organizing a discretized equation corresponding to that given by Eq. (6.8) may be adopted. Thus, the array with the results of the domain integral at different source points may be represented in terms of the product between a matrix containing the coefficients of influence and an array with the nodal values of the function F. As was mentioned earlier the elements of this matrix contain only geometric parameter and so they need to be calculated only once. As a result, a great amount of CPU time can be saved. Considering that many source points were chosen, the discretized expression for the integral in matricial form is given by:

$$\begin{bmatrix} Z_{11} & K_{12} & K_{13} & \cdots & K_{1m_\Omega} \\ K_{21} & Z_{22} & K_{23} & \cdots & K_{2m_\Omega} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ K_{m_\Omega 1} & K_{m_\Omega 2} & K_{m_\Omega 3} & \cdots & Z_{m_\Omega m_\Omega} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_{m_\Omega} \end{bmatrix} \cong \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1m_\Omega} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2m_\Omega} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ K_{m_\Omega 1} & K_{m_\Omega 2} & K_{m_\Omega 3} & \cdots & K_{m_\Omega m_\Omega} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_{m_\Omega} \end{bmatrix}$$

$$- \begin{bmatrix} \underline{K}_{11} & 0 & 0 & \cdots & 0 \\ 0 & \underline{K}_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \underline{K}_{m_\Omega m_\Omega} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_{m_\Omega} \end{bmatrix} + \begin{bmatrix} Z_{11} & 0 & 0 & \cdots & 0 \\ 0 & Z_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & Z_{m_\Omega m_\Omega} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_{m_\Omega} \end{bmatrix}$$

where the matrix [K] on the right-hand side of the above equation represents the coefficients given by the solutions of the following integral:

$$K_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta) J(\xi,\eta) \frac{\partial P^*}{\partial n_o} d\xi d\eta$$

Note, however, that the values of the elements of [K] may need to be obtained with a summation of the coefficients given by this integral corresponding to the cells that share a particular node. The elements of the diagonal matrix [$\underline{K}$] are obtained from the solution of the following integral:

$$\underline{K}_{ii} = \iint_{\Omega} \frac{\partial P^*}{\partial n_o} dxdy$$

and the elements of the diagonal matrix [Z] are given by the solution of the following boundary integral:

$$Z_{ii} = \oint_{\Gamma} (\vec{n}_o.\vec{n}) P^* \, d\Gamma$$

Note that this representation shows clearly the contribution of the additional integrals to the leading diagonal elements of the matrix [K], where the coefficients of singular integrals are placed. It is expected roughly that as the source point is approached, the element in the leading diagonal of the matrix [K] corresponding to that particular source point cancels out with the element of the diagonal matrix $[\underline{K}]$. Its final value is thus given by the corresponding element at the leading diagonal of the matrix [Z].

On the other hand, the main advantage of the formula given by Eq. (6.26) is that it is independent of the type of the domain cell adopted to discretize the domain, which can be important if the effect of different types of cells are to be investigated. The technique that uses polar coordinates to deal with the singular domain integral, normally given by a kernel of type $1/r$, is an example of an approach that has also the advantage of allowing the matrix type representation but that has the serious disadvantage of being normally applied to triangular cells with three nodes, only.

As was mentioned previously, the matrix type representation will be avoided in this work for the sake of computer memory. The discretized expression more appropriate for this option will be given for each case, corresponding to a particular source point.

When the kernel of the domain integral is given by the derivative of the fundamental solution u^*, the domain integral is calculated by the following discretized expression:

$$\iint_\Omega F\frac{\partial u^*}{\partial x}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} hxc_j^{(c)}(x_i,y_i)\,F_j + F_i\left[\sum_{e=1}^{N_e} gl^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} hxc^{(c)}(x_i,y_i)\right]$$

and

$$\iint_\Omega F\frac{\partial u^*}{\partial y}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} hyc_j^{(c)}(x_i,y_i)\,F_j + F_i\left[\sum_{e=1}^{N_e} gm^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} hyc^{(c)}(x_i,y_i)\right]$$

for derivatives in x- and y-directions, respectively, where the coefficients $gl^{(e)}(x_i,y_i)$, $gm^{(e)}(x_i,y_i)$, $hxc^{(c)}(x_i,y_i)$ and $hyc^{(c)}(x_i,y_i)$ are integrals similar to those given in Appendix C; the only difference being that the interpolation functions do not appear. The other two coefficients are exactly the integrals defined in Appendix C . Kernels represented by the second derivatives of the fundamental solution u^* are given by:

$$\iint_\Omega F\frac{\partial^2 u^*}{\partial x^2}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} hxxc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e} hxlc^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} hxxc^{(c)}(x_i,y_i)\right]$$

$$\iint_\Omega F\frac{\partial^2 u^*}{\partial y^2}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} hyyc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e} hylc^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} hyyc^{(c)}(x_i,y_i)\right]$$

and

$$\iint_\Omega F\frac{\partial^2 u^*}{\partial x\partial y}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} hxyc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e}\left(hylc^{(e)}(x_i,y_i) + hxm^{(e)}(x_i,y_i)\right)/2 - \sum_{c=1}^{N_c} hxyc^{(c)}(x_i,y_i)\right]$$

where an average was introduced in the boundary integral representation of the latter expression in order to avoid any inaccuracy due to the change in the order of derivations. The coefficients $hxxc_j^{(c)}(x_i,y_i)$, $hyyc_j^{(c)}(x_i,y_i)$ and $hxyc_j^{(c)}(x_i,y_i)$ are defined in Appendix C and the coefficients $hxxc^{(c)}(x_i,y_i)$, $hyyc^{(c)}(x_i,y_i)$ and $hxyc^{(c)}(x_i,y_i)$ are similar to those used previously where the only difference is that they do not contain the interpolation function in the integrand. The remaining coefficients are boundary integrals defined as follows:

$$hxl^{(e)}(x_i,y_i) = \int_0^1 J(\xi)\ell\frac{\partial u^*}{\partial x}\,d\xi$$

$$hym^{(e)}(x_i,y_i) = \int_0^1 J(\xi)m\frac{\partial u^*}{\partial y}\,d\xi$$

$$hxm^{(e)}(x_i,y_i) = \int_0^1 J(\xi)m\frac{\partial u^*}{\partial x}\,d\xi$$

and

$$hyl^{(e)}(x_i,y_i) = \int_0^1 J(\xi)\ell\frac{\partial u^*}{\partial y}\,d\xi$$

On the other hand, integrals whose kernel contains the fundamental solution u^* itself, can be solved using such a technique, however, an additional derivation is necessary as shown in the section I.2.1 of Appendix I. The final expression for this case is:

$$\iint_\Omega F u^* \, dx dy = \sum_{c=1}^{N_c} \sum_{j=1}^{N_c} gc_j^{(c)}(x_i,y_i) + F_i \left[\sum_{e=1}^{N_e} ht^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} gc^{(c)}(x_i,y_i) \right]$$

where the coefficient $gc_j^{(c)}(x_i,y_i)$ is defined in Appendix C and the coefficient $gc^{(c)}(x_i,y_i)$ is similar to the previous one with the only difference that the interpolation function does not appear in the integrand. The coefficient $ht^{(e)}(x_i,y_i)$ is given by the following boundary integral:

$$ht^{(e)}(x_i,y_i) = \int_0^1 J(\xi) \frac{r}{8\pi} (2\ln r - 1) \frac{\partial r}{\partial n} d\xi$$

Note that the integrand of the above integral contains two singular terms. Fortunately, one of them is of a rational type function and is treated as mentioned previously. The other one, which involves the logarithmic function, can be overcome using the approach discussed previously to deal with this type of singular kernel.

The same method can be used to solve singular domain integrals whose kernel includes the fundamental solution G_{kl}, given by Eq. (5.8), and its derivatives. However, instead of using the fundamental solution equation in the form given in Chapter 5, it is convenient to use the representation given by Eq. (G.20) for simplicity, as shown in Appendix I. A summary of domain integrals already discretized for this case is given as follows. For a kernel involving G_{11}, G_{12}, G_{21}

and G_{22}, the domain integrals are given by:

$$\iint_\Omega FG_{11}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} gxxc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e} gxlc^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} gxxc^{(c)}(x_i,y_i)\right]$$

$$\iint_\Omega FG_{12}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} gxyc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e} gylc^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} gxyc^{(c)}(x_i,y_i)\right]$$

$$\iint_\Omega FG_{21}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} gyxc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e} gxmc^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} gyxc^{(c)}(x_i,y_i)\right]$$

and

$$\iint_\Omega FG_{22}\,dx\,dy = \sum_{c=1}^{N_c}\sum_{j=1}^{n_c} gyyc_j^{(c)}(x_i,y_i)F_j + F_i\left[\sum_{e=1}^{N_e} gymc^{(e)}(x_i,y_i) - \sum_{c=1}^{N_c} gyyc^{(c)}(x_i,y_i)\right]$$

where the coefficients $gxxc_j^{(c)}(x_i,y_i)$, $gxyc_j^{(c)}(x_i,y_i)$, $gyxc_j^{(c)}(x_i,y_i)$ and $gyyc_j^{(c)}(x_i,y_i)$ are defined in Appendix C and $gxxc^{(c)}(x_i,y_i)$, $gxyc^{(c)}(x_i,y_i)$, $gyxc^{(c)}(x_i,y_i)$ and $gyyc^{(c)}(x_i,y_i)$ are similar to the previous coefficients with the difference that the interpolation function is not included in the integrand of the integrals defining these coefficients. The remaining coefficients are defined as follows:

$$gxlc^{(e)}(x_i,y_i) = \int_0^1 J(\xi)\left(\frac{\partial g}{\partial n} - \frac{1}{2(1-\bar{v})}\ell\frac{\partial g}{\partial x}\right)d\xi$$

$$gylc^{(e)}(x_i,y_i) = \int_0^1 J(\xi)\left(- \frac{1}{2(1-\bar{v})}\ell\frac{\partial g}{\partial y}\right)d\xi$$

$$gxmc^{(e)}(x_i,y_i) = \int_0^1 J(\xi)\left(- \frac{1}{2(1-\bar{v})}m\frac{\partial g}{\partial x}\right)d\xi$$

and

$$gymc^{(e)}(x_i,y_i) = \int_0^1 J(\xi)\left(\frac{\partial g}{\partial n} - \frac{1}{2(1-\bar{v})}m\frac{\partial g}{\partial y}\right)d\xi$$

where g comes from Eq. (G.19) and is given by:

$$g = \frac{r^2}{8\pi v}\ln\left(\frac{1}{r}\right)$$

after assuming that $C_1 = -1$ and $C_2 = 0$ in that equation.

Note that the kernel of these integrals contains a singularity due to the presence of a logarithmic function, but that problem can be overcome using the technique described in the Section I.1 of Appendix I.

Usually, in interior solutions, when the source point is placed near to the boundary the accuracy of the results decrease. This setback of the BEM can appear in any kind of problem and is, in general, provoked by the so-called quasi-singular integrals. The problem occurs because when the source point approaches the boundary, the integrand of any singular integral on the nearby boundary element behaves badly. Since these integrals are not formally singular, special numerical integration is not normally used and ordinary quadrature is inaccurate. This problem is more likely to come up in the calculation of derivatives in which the order of the singularity is high. It was, in fact, noticed in the calculation of derivatives of vorticity and derivatives of velocity in the formulations based on vorticity and penalty function, respectively. Although the inaccuracy introduced in this case did not seem to create serious consequences for the final result as a whole, an attempt to overcome this problem was made by avoiding putting the source point inside a region near to the boundary. The derivatives of the parameter inside that region were calculated using a numerical approach for calculation of the derivatives. Another technique used was to consider the boundary integral as singular every time the source point laid within a certain distance from the boundary. Thus, the subroutine in the program used to perform a singular integral was activated. However, most of the tests were carried out using BIE to calculate the derivatives in every node on the domain without any special treatment, just like the conventional solution of the BIE.

6.4- CORNER TREATMENT

A special treatment is normally required to deal with corners appearing in the domain of problems investigated, which are in general introduced as a result of the discretization process of the boundary. Using corners in the BEM appears in the analysis of most engineering problems and their correct treatment is especially important for the success of a BEM code, since they are closely related to the accuracy of the results. This is, without doubt, another important subject related to the application of the BEM. Fortunately this has received the attention it deserves in the literature, as is indicated by the reasonable number of published papers discussing the matter.

The main problem with a corner is that it introduces a discontinuity in the direction of the normal to the boundary that leads the normal derivative or flux of the potential to be double-valued at the corner node. The potential itself, on the other hand, is single-valued at that node. However, the problem may not appear depending on the boundary conditions of the problem or on the discretization adopted at the boundary. When constant elements are used to discretize the whole boundary, or at least the region near to the corner, the problem does not exist even if the geometric corner does. The reason is that, in that case, there is no discontinuity in the normal derivative of potential at the nodes. When the isoparametric type of element is used, which happens in most cases, the problem may appear. In the last instance, it is the kind of boundary conditions adopted on both sides of the corner that determine whether the corner should be treated using a special technique or not. Roughly speaking, when the normal derivative of a potential is prescribed in at least one of the sides that constitute a corner, no special treatment is necessary, since there will be as many unknowns as the number of equations to be solved. In this case, no additional unknown is introduced at the corner node. On the other hand, when the normal derivatives on both sides of a corner are to be calculated, the problem of a double-valued normal derivative at the corner node becomes evident. That is, one additional unknown is introduced. In the end, a system of equations has to be solved with more unknowns than equations, since one more unknown per corner is introduced. This problem of the corner is well explained in, for example, Walker&Fenner (1989).

There are several proposals in the literature put forward to deal with this problem. In this work it was decided to use the one that adopts an auxiliary node at the corner to cope with the extra unknown for the normal derivative of the potential. Therefore, now there are two nodes at a corner, each one placed at the element on each side of the corner, and an additional equation is thereby created. The two nodes, however, are separated by a small gap to avoid having the two equations with the same coefficients. That is, the two nodes do not occupy the same position at the geometric corner. They are, in fact, moved a slight distance along either direction of the two adjacent boundary elements at the corner. Based on this idea, many variations were tried in order to improve the results. The discussion presented next refers mainly to the corner problem faced in the development of the program based on the vorticity-velocity formulation.

Most of the approaches suggested in the literature are evaluated taking as reference the accuracy of the results for a sort of potential analysis, for the sake of simplicity. In this work, however, the most serious corner problems appeared with the solution of viscous flow. The analysis of this kind of problem involves the solution of many BIEs iteratively and that contain domain integrals. The accuracy is still a parameter to be used to quantify the success of the approach in this case, since the convergence of results depends largely on it. However, the experience based on a potential solution may not be appropriate to the present analysis since the corner treatment may affect the calculation of the domain integral, whose accuracy, in its calculation, is very important for the convergence of the results. In other words, the analysis of viscous flow is much more complex than the potential case, since now the domain integral has to be considered and the isolated effects of corner treatment and domain integral calculation to the convergence of results become less evident. Divergence of results was the main problem faced in this analysis, caused partially by inaccuracies in the treatment of corners and the evaluation of domain integrals. In this work, serious problems were encountered with the corner. One of the most difficult appeared in the solution of the equation of transport of vorticity, of the vorticity-velocity formulation, used to obtain the distribution of the normal derivative of vorticity on the boundary.

The kind of approach that relies on an extra node have some disadvantages caused by the additional node. This is because, in this case, more integrations have to be performed to generate the equation corresponding to that extra node and also since it increases the order of the matrix of coefficients to be inverted. In addition, another point that should be noted is the fact that the boundary is not continuous any more, since a gap was introduced at the corner. This introduces a small error in the discretized equations to be added on top of the usual and inevitable inaccuracy introduced due to the discretization process itself and other approximations. This error can be overlooked if the gap is very small. This is not, however, the situation for most cases and a correction has to be introduced. This correction is in general introduced through the constant C_i. So far in this document, it was mentioned that the values of C_i are given in Appendix B regardless of the treatment given to the corner nodes. However, for accuracy's sake, the value of C_i should be correctly evaluated each time the double-node approach is adopted. In spite of these drawbacks, the double-node approach was adopted in this work because it can give good results and also because of its simplicity of implementation.

The value of the gap between the two corner nodes was the first parameter to be investigated in this work. However, there is not much freedom to play with the value of that parameter since it has to be kept within a narrow range given by two limits. As was mentioned before, if the gap is increased, the accuracy of the results reduces. On the other hand, if the gap is too small, ill-conditioning problem of the system of equations may appear. The value of the gap has therefore to be optimized. An attempt to eliminate one of those nodes and its corresponding equation in the system of equations was made. The result obtained for the normal derivative of the potential at the remaining corner node was also transferred to the missing node. The objective behind this technique was to simulate a situation where the values of the normal derivatives of potential at both sides of the corner are assumed equal. This measure never worked properly with the formulation based on vorticity and velocity, where it was tested. Besides, a new variable was introduced, since in this case, a decision had to be made about which of the node equations upon which the calculations should be based.

At this stage, it was decided to leave the two nodes with a small gap between them and steps were taken to improve in the calculation of C_i. At first, it was evaluated at corners using the suggestion given in Appendix B, based on the angle of the geometric corner. The same value was then assumed for both corner nodes. Following this, the idea based on the application of a constant potential around the boundary and inside the domain was adopted. This way of assessing the value of the constant in terms of physical concepts is discussed in many textbooks , for example, in El-Zafrany (1993). The result is that the constant C_i for a finite domain problem is given by the following expression:

$$C_i = - \sum_{e=1}^{N_e} \sum_{j=1}^{n_c} h_j^{(e)}(x_i,y_i) \tag{6.23}$$

where all parameters have their usual meaning. Normally in the literature, either temperature or displacement are used in the analysis to derive the expression, assuming a constant temperature field over both the boundary and the domain or by applying a rigid translation over the whole domain, respectively. Doing in this way, the results were improved but convergence of results was not achieved.

A significant breakthrough in the improvement of the results was brought about by the use of the procedure discussed in Appendix J to calculate the coefficient C_i. This procedure attempts to compensate for the error introduced with the gap between the corner nodes, in a way similar to that suggested by Ricardella (1979), using here the so-called jump functions. This can be regarded as an extension of the procedure represented by the expression above to problems where the dependent variable has two components, as in the vorticity-velocity and penalty function formulations. Thus, the terms containing the coefficients C_i of the BIEs related to these formulations were modified to take into consideration this new procedure in the way described in Appendix J.

One relatively simple technique that gave surprisingly good results was that suggested by Mitra and Ingber (1987). This consists in placing the source point corresponding to one of the corner nodes outside the domain. Thus the coefficients of the equations corresponding to the corner nodes are different and the gap can therefore be eliminated without creating problems related to ill-conditioning of system of equations. As indicated in this reference, a random choice of the source point should be avoided and the best approach is to place it near the corner. The main disadvantage of this technique is that the position of source point outside the domain may have to be optimized. Although in the first test, just one of the source points relative to the corner nodes was located outside the boundary, eventually it was found convenient in this work to locate both source points outside the domain. This avoids any possible problem related with the choice of nodes. The source points were placed on a straight line normal to each side elements that form the corner, at each node.

It is worth noting that mesh refinement in the corner region was introduced along with the early measures, but the improvement achieved was not enough to provoke convergence of results. The latest techniques, which were responsible indeed for the improvement in the results, were used without mesh refinement. Eventually, the techniques that introduce a correction in the calculation of the coefficient C_i and the one that locates the source point, corresponding to the corner nodes, outside the domain, were adopted throughout this work. The latter was used more in connection with the vorticity-velocity formulation.

6.5 - SOLUTION ALGORITHMS

The algorithms adopted in this work to solve fluid flow problems, using the formulations discussed previously, are presented here. However, a detailed discussion of all steps introduced in the programs and the listing of the programs themselves are beyond the scope of this document. The previous discussion, however, related to the numerical implementation gives a good idea about the numerical procedures used in the programs.

Although the algorithms are different from one formulation to another, many subroutines, such as those used to perform integrations, are shared by the programs. The programs also share the subroutines that calculate all the parameters related to the discretization process, such as, Jacobians, interpolation functions, direction cosines, and so on. The subroutines that read the input data of all programs are basically the same. These subroutines create the topologic arrays TA and CTA defined in Eqs. (6.7) and (6.8), respectively, and facilitate the description of the boundary elements and cells. Thus, the programs developed shared many points in common.

A program for each formulation investigated in this work was developed. Only the main version is discussed here.

6.5.1- Potential Formulation

The solution steps to solve potential problems described in the introduction of this Chapter can be used to solve potential flow, and so is not discussed again. Depending on the type of the problem to be solved, however, additional steps have to be introduced. For example, the calculation of the derivatives of velocity potential or stream function variables inside the domain are normally required in order to evaluate the velocity components on the domain. An additional step may, therefore, be required after the results for the basic algorithm have been achieved. Eqs. (6.10) and (6.11) are, in this case, adopted to calculate the derivatives of the parameter at internal nodes. However, they are not adequate to carry out the calculations at boundary nodes, because of the singularities of the boundary integrals. The best option, in this case, is to use the technique used in the vorticity-velocity formulation described in Appendix K. In this work, however, this option was not implemented in the program since, in the cases analysed, the results on the boundary given by the Eq. (6.9) were more important.

One important feature introduced in the program was a subroutine to deal with boundary conditions involving periodicity. This condition appears in general when the solution of the flow through a cascade of turbomachinery blades is required. The periodicity conditions exist when, in a domain, the values of the flux of a variable and the variable itself at the nodes on one boundary are related to the values of these parameters at the nodes on the opposite boundary. In other words, since in this situation, no explicit boundary conditions are given, additional equations have to be introduced to allow the solution of the problem.

These conditions are given by an expression relating the value of the flux of the variable and the variable itself, at opposite nodes. In the case of cascades, the values of the variable at opposite nodes are the same but the flux of the variable has opposite sign.

A program for solving Poisson's equations using the BEM available was modified to a program with these characteristic, aimed at solution of potential flow through cascade. The main modifications introduced in the original program were the subroutines to deal with the periodicity conditions and to allow interior calculations of the variable of the problem and of its derivatives.

6.5.2- Vorticity-Velocity Formulation

In this formulation two different possibilities concerning the kinetic part of the formulation were investigated. The most recent versions of the program corresponding to these two options are basically the same and so only the procedure steps of the first option, given by Eqs. (4.22), (4.25), (4.26), (4.27) and (4.28), are discussed. However, special features and other alternative solutions attempted in both options will be presented at the end.

The solution of this kind of problem, different to the potential analysis, has to be obtained iteratively. The iterative process is based on the derivatives of vorticity since they are the parameters appearing in the integrand of all domain integrals. However, the convergence criterion is placed to verify the convergence of vorticity only. In this way, after the convergence has been achieved, it is expected to obtain a low residual for the other parameters like velocity components also. Many more iterations are requested in order to obtain the derivatives of vorticity with a low residual. The solution is divided into two parts corresponding to the solution of the kinematic part, first, and the kinetics second. The main steps are:

(a) Assume an initial distribution for the derivatives of vorticity on the whole domain. Note that the distribution of these parameters can be assumed equal to zero everywhere. However, as in many simple problems, this value represents the correct answer for these parameters. It is recommended in this case, to assume a value other than zero. Otherwise, there will be no iterative process since in the first iteration, the program would give the correct answer for all parameters;

(b) Introduce the boundary conditions given by the velocities v_n and v_t, which are related to velocity components v_x and v_y, along with the conditions given in the previous step in the equations of kinematic part, Eqs. (4.25) and (4.26). The resulting system of equations are then solved to obtain the distribution of vorticity on the boundary nodes.

At this stage, the distribution of all variables are known on the boundary. However, only the boundary conditions are corrected since the vorticity on the boundary was obtained based on an initial guess for the derivatives of vorticity over the whole domain, or on the results of the previous iteration. The objective of the next steps is to update the distribution of those parameters given initially and the others taking into consideration the information now available on the boundary.

(c) Use the kinematic equations again, Eqs. (4.25) and (4.26), but with the source points located inside the domain to obtain explicitly the distribution of the velocity components, v_x and v_y, over domain nodes. The velocity components on the boundary nodes are obtained using the following equations, along with the given boundary conditions in terms of tangential and normal velocity components:

$$v_x = \ell v_n - m v_t$$

and

$$v_y = m v_n + \ell v_t$$

where ℓ and m are the direction cosines.

This concludes the calculations involving the kinematic equations. The next steps deal with the kinetic equations to obtain the distribution of vorticity and the derivatives of vorticity.

(d) Calculate the distribution of the normal derivative of the vorticity over the boundary nodes using Eq. (4.22) for the kinetics together with the previous results. The source points are located in this case on the boundary nodes and the distribution of the normal derivative of vorticity is obtained implicity with the solution of the resulting system of equations;

(e) Obtain the distribution of the vorticity over internal nodes of the domain using the kinetic equation, Eq. (4.22). The source points are located at internal nodes which allows the explicit calculation of the vorticity node-by-node. Note that here, the boundary nodes were avoided since the vorticity distribution on the boundary nodes was obtained in step (b);

(f) Calculate the derivatives of vorticity at the internal nodes by solving explicitly Eqs. (4.27) and (4.28) node-by-node. Note that the boundary node calculations are avoided since these equations contain boundary integrals with strong singularities;

(g) Calculate the derivatives of vorticity at the boundary nodes using the following equations, Eqs. (K.6) and (K.7), derived in Appendix K:

$$\frac{\partial\omega}{\partial x} = \ell\frac{\partial\omega}{\partial n} - m\frac{\partial\omega}{\partial\xi} / J(\xi)$$

and

$$\frac{\partial\omega}{\partial y} = \ell\frac{\partial\omega}{\partial\xi} / J(\xi) + m\frac{\partial\omega}{\partial n}$$

(h) Evaluate the residual of the results of vorticity using the following criterion:

$$IR = \frac{\sum_{j=1}^{n_c} (\omega_j - \omega_j|_{old})^2}{\sum_{j=1}^{n_c} \omega_j^2} \tag{6.24}$$

(i) If the residual obtained is smaller than a certain tolerance ϵ, very small, the results are considered converged. Otherwise the calculations should continue, using for the next iteration a new distribution of derivatives of vorticity, or sometimes a new distribution of vorticity also, obtained through the expressions:

$$\left(\frac{\partial \omega}{\partial x}\right) = \left(\frac{\partial \omega}{\partial x}\right)_{new} w + (1 - w)\left(\frac{\partial \omega}{\partial x}\right)_{old}$$

and

$$\left(\frac{\partial \omega}{\partial y}\right) = \left(\frac{\partial \omega}{\partial y}\right)_{new} w + (1 - w)\left(\frac{\partial \omega}{\partial y}\right)_{old}$$

where w is an under-relaxation factor;

(j) The results obtained so far are introduced in step (b) for a new iteration until convergence of the results is achieved.

The solution algorithm for option 2, given by Eqs. (4.25), (4.26), (4.24), (4.29) and (4.30), is very similar to the one presented above. The main difference is that, in this case, the vorticity appears along with the normal derivative of vorticity in the integrand of the domain integrals of the kinetic part of the problem. In this case, the vorticity has also to be under-relaxed in the iterative process and an initial distribution for it has to be guessed.

6.5.3- Penalty Function Formulation

The program developed for this formulation is based on a slightly different algorithm to the one presented by Kitagawa *et al.* (1986). The main difference is in the way the velocity components and derivatives of velocities are updated from one iteration to another. In their case, these parameters are obtained for the new value of the body force term solving the corresponding equations given previously in discretized form. Here, only part of these equations are employed to obtain the distribution of the incremental value of these parameters corresponding to an incremental variation in the body force term. This, algorithm will be explained in this document.

Initially, the body force terms are assumed equal to zero and all the equations are solved. The results obtained in this way do not, therefore, consider the effects of the body force terms, that is, domain integrals. These initial results represent the bulk of the solution, where incremental values of the velocity components and derivatives of velocity are added or subtracted in the forthcoming iterations, depending on the values of the body force terms. The incremental values of these parameters take into consideration the effects of the change in the value of the body force terms. The body force terms are initially equal to zero but later are evaluated using the results of the parameters obtained in the previous iteration. Convergence of the results is obtained when the incremental values of the velocity components becomes very small. Roughly speaking, the present algorithm can be divided into two main parts: the one that obtains the results without body forces and the other where the iterative process takes place. The main steps of the algorithm adopted are given as follows:

(a) Assume initially, that all body force terms are equal to zero. Define an array with the velocity components at the boundary nodes and do the same with the derivatives of velocities and the traction components;

(b) Assemble a system of algebraic equations formed alternately by Eqs. (6.20) and (6.21), by locating the source points at the boundary nodes. After the introduction of the boundary conditions, a system of equations results with ($2xm_\Gamma$) equations and ($2xm_\Gamma$) unknowns:

$$[A]\{u\} = \{f\}$$

where [A] represents the matrix of influence and {f} is the array containing the results due to the known parameters. {u} is the array of unknowns including velocity and traction components;

(c) Solve the previous system of equations. The velocity and traction components are thus known at every boundary node;

(d) Calculate the velocity components and the derivatives of velocities at internal nodes of the domain, explicitly node-by-node, using Eqs. (6.20) and (6.21) and Eqs. (6.22) to (6.25), respectively;

(e) Obtain the distribution of the derivatives of velocities on the boundary using the special approach discussed in Section K.2 of Appendix K;

At this stage, the distribution of all parameters has been obtained without taking into consideration the body force effect. These results are transferred to another variable, say UT, which can represent any variable of the problem and where the updated results from each iteration are stored. Next, the effect of the body force terms will be introduced in the previous results.

(f) Evaluate the domain integrals of the Eqs. (6.20) and (6.21) for the source points located at boundary nodes, using the results obtained previously and stored in UT. Then, solve the following system of equations:

$$[A]\{du\} = \{df\}$$

where {df} is the array with the information on the body force term, given by the difference between the results of this term obtained in the previous and the current iterations. [A] is the same matrix as used in the step (b). The array with the incremental results on the boundary for velocity and traction components, {du}, represents the value to be added to the corresponding solution UT obtained previously;

(g) Repeat steps (d) and (e) using the results {du} on the boundary nodes, obtained in the previous step, but taking into consideration the effect of the body force terms in step (d) evaluated using the distribution of the updated variables, UT, newly obtained;

(h) Update the velocity and traction components and the derivatives of velocities with the incremental results obtained in the steps (f) and (g). The velocity and traction components are updated using the following expression:

$$UT_{new} = UT_{old} + (du)\,w$$

where UT_{new} is the updated, or the total, value of these variable for the current iteration, UT_{old} is the result for the previous iteration and w is a relaxation factor;

(i) If the residual based only on the results for the velocity components on the boundary, obtained using the following expression:

$$IR = \frac{\sum_{j=1}^{n_e} UI_j^2}{\sum_{j=1}^{n_e} (UT_{new})_j^2}$$

is smaller than a small tolerance ϵ , convergence has been achieved. Otherwise the calculation should continue;

(j) Go to the step (f) and repeat the calculations using the new results.

Following this, it is possible to see the main difference between this algorithm and that adopted by Kitagawa. There, although in the first step, the body force terms are assumed equal to zero, the calculations in the following iterations are based on the use of the whole equations. In other words, their algorithm works always with the updated value of the variables which includes the body force effect.

6.5.4- Programming

The programs were coded in FORTRAN 77 to be used in a VAX 8650 main frame computer, but with a version using a workstation. The program for potential analysis has about 800 lines and the others have between 2500 and 3000 lines, without considering the mesh generation subroutine.

The program based on vorticity-velocity formulation was developed first. This program was initially based on a simple program available at Cranfield for potential analysis using Poisson's equation. This was, therefore, modified to carry out the calculations of steps (d) and (e) in the algorithm using the vorticity-velocity formulation. One of the main modifications introduced was due to the fact the domain integrals in the viscous flow include variable in the integrand, which requires an iterative solution. Other subroutines were developed to carry out the calculations related to the kinematic equations and derivatives of vorticity.

The same simple program to potential analysis gave origin, with slight alterations, to the program to potential flow analysis. The main modifications introduced in the program were the ones related to the subroutines to deal with periodicity conditions and to allow calculations of the derivative of the variables at internal nodes of the domain.

The program for viscous flow analysis based on the penalty function approach, was developed from a program for elasto-plastic stress analysis developed by the Finite Element group of Cranfield Institute of Technology. A great part of this program was deleted since for the fluid analysis only displacements, traction and derivatives of displacement need to be calculated. This task was very time consuming since it was necessary first to understand the structure of the program. The main modifications inserted in the program were the ones related with the introduction of the convective term of Navier-Stokes equations in place of the body force term of the original program. Subroutines to evaluate the domain integral due of the convective term had to be developed.

The solvers of the system of algebraic equations used in these programs are based on Gauss elimination with pivoting and are basically the same. Some subroutines used in the vorticity-velocity formulation program to carry out integrations on the boundary are used in the program for potential analysis, since the fundamental solution is the same. Similar subroutines are used in the program based on the penalty function formulation. The fundamental solutions are in this case however, different from the previous ones. With regard to the subroutines related to the discretization of the equations, basically the same set of subroutines is used in all programs. Roughly speaking, apart from the main algorithm with the solution procedure the subroutines included in the programs are basically the same. One difference between the programs based on vorticity-velocity and penalty functions formulations is, for example, in the subroutines which calculate the derivatives of the parameters on the boundary nodes, since they are based on different techniques.

Chapter 7

Discussion of Results and Conclusions

7.1-INTRODUCTION

In this Chapter, the results obtained by the programs developed in this work are discussed for some selected test cases. The problems of numerical nature presented by the programs are also discussed and justifications given.

Two practical case studies, whose numerical or analytical solutions are available, were adopted to validate the program for potential analysis, since this program was expected to give satisfactory results without much problem. On the other hand, it was necessary to rely on simple test cases to analyse the problems that appeared in the development phase of the programs for viscous flow solution.

The Poiseuille flow was adopted during the pre-validation phase of the program to solve viscous flow based on the vorticity-velocity formulation. The Couette and driven-cavity flows were later also attempted at the validation stage. However, only results for low Reynolds number were achieved due to divergence problems. This problem was more serious in the program based on the penalty function approach and it seems that this program still needs more improvements in order to obtain convergence of the results.

All the programs computations considered in this Chapter were performed in double precision on a VAX 8650 computer.

7.2- POTENTIAL FORMULATION

7.2.1- Introduction

An existing program for potential analysis available was modified to consider periodicity conditions, internal calculations and so on to deal with potential flow analysis of cascades. For this case a pre-validation phase was not necessary since this program does not have non-linear problems or iterative procedure. However, corner problems was expected to appear in some situations and in this case the techniques, mentioned in Chapter 6, adopted in the development of the vorticity-velocity program had to be considered. Two case studies, whose solutions are available in the literature, were considered. These are the potential flow in an L-shaped channel and through a cascade of vertical cylinders. The objective of the first test is to assess the accuracy of internal results in the region of corners and the aim of the second case analysed is to validate the program when periodicity conditions is involved and also to assess the accuracy of the solution on curved boundaries.

7.2.2- L-shaped Channel

This problem is described in Figure 7.1 and represents the flow between two plates, changing direction in a 90° angle. The flow is assumed to enter the channel with a constant x-direction velocity component and leave with a constant y-direction velocity component of equal value to that of the inlet flow. This problem was solved in Brebbia & Ferrante (1983) using both the FEM and the BEM, considering constant flow velocities with unit values and the dimensions given in Figure 7.1. In this analysis the stream-function parameter is considered as the variable of the problem. The boundary conditions in this case are: on the boundary ABC the stream-function assumes a prescribed zero value; on the boundary DEF it assumes the constant value 4; and a linear variation of stream-function from zero to 4 is assumed at the boundaries AF and CD. The objective of this analysis is to obtain the values of the stream-function parameters at some internal points located on the diagonal BE shown in Figure 7.1. First, however, the boundary results have to be obtained. Note that in this case, since the distribution of the stream-function parameter is prescribed around the whole boundary, the boundary unknowns are the values of normal derivative of the stream-function. In this case, special corner treatment is required.

In this work, the problem was discretized using 16 quadratic boundary elements with 32 nodes and 6 additional nodes at the corners. This mesh is presented in Figure 7.2.

The results for this problem are given in Figure 7.3 together with the results due to Brebbia & Ferrante (1983). It can be seen clearly that the results of this work agree very well with the results using the BEM obtained in that reference. Their results using the BEM were obtained with 24 constant boundary elements of different sizes. In their case the inaccuracy caused by corners was not a problem. The BEM solutions produce slight higher values at the region of the corner B compared with FEM results. As mentioned in Brebbia & Ferrante (1983) this is because the BEM has the characteristic of representing better rapid variations of the parameter in this region, which can be considered an advantage of the BEM. With regard to the FEM solution, their results were obtained using a mesh with 66 triangular linear elements with 46 nodes.

It is worth mentioning that the curves of Figure 7.3 were plotted based on the output of the programs given in that reference. The result at the point (1.75,1.75) on the diagonal BE, obtained in their BEM solution, was omitted in a similar figure shown in Brebbia & Ferrante (1983). This was probably because it provokes a deformation in the interpolated curve, since it lays below the FEM results. However, it was decided to include this result here since it was also obtained in this work. This may be a problem linked to the BEM itself or even a physical explanation may exist. The FEM solution did not give the value of the stream-function at this point to compare the results. To avoid problem with the adjusted curves it was decided just to join the points using straight lines.

The results obtained in this work were affected by the treatment given to the corners. Mainly the results on the diagonal BE, at the points near of the corner B. A gap between each 2 corner nodes was introduced to avoid singularity in the equations matrix, as discussed in Chapter 6. The accuracy of the results for some values for the gap, mainly 0.1, 0.01 and 0.001, were investigated. The results were assessed in terms of the symmetry of the results for the normal derivative of stream-function in relation to the diagonal BE. The rigid translation technique to evaluate the value of C_i with a gap between the corner nodes of 0.1 gave the best results. In this case, the distribution on the boundaries AB and FE were nearly symmetric in relation to the results on the boundaries BC and ED, respectively. The distribution of the stream-function parameter obtained in this work on the diagonal BE lies, in the region near to the corner B, in between the results using the BEM and the FEM obtained by Brebbia & Ferrante (1983). However, these results are closer to the BEM solution given by them. The technique of locating the source points corresponding to the corner nodes outside of the domain was also adopted. This test case showed, however, that results around the boundary are not so good, although the distribution of the stream-function on the BE diagonal is reasonable. The results obtained on this diagonal using that technique are in general higher than the results given in Figure 7.3.

This test showed also that at internal corners, like corner B, there is not much space to locate the source points outside the domain. The situation could become worse when the internal angle of the corner is reduced. Here only one of the source points corresponding to the corner nodes was located outside of the domain, on the prolongation line of the diagonal BE.

7.2.3- Cascade of Cylinders

This problem is of a potential flow around a cascade of cylinders, as illustrated in Figure 7.4. There a channel is formed between the two cylinders separated by distance t where the flow go through. The free-stream flow velocity is given by U and in this case there is no lift. This problem was solved by Thompson (1973) using the FEM and the results compared with the exact solution, which he obtained using the technique due to Merchant & Collar (1941) for the solution of potential flow around cascades of bodies without lift. Thompson (1973), in fact, selected a set of parameters to obtain the analytical solution, using that technique, for potential flow around a cascade of ovals which are very nearly circles. A table with this solution is presented in this reference for the distribution of tangential velocity on the surface of one quadrant of cylinder.

The solution domain defined by Thompson (1973) to obtain the FEM solution for this problem is a rectangle with a cylinder in the middle. This can be done since a symmetry related to the middle line of the channel exists. He discretized this domain using 195 triangular linear elements with 123 nodes. It is worth mentioning that he adopted a finer mesh around the surface of the cylinder and a coarse mesh far away from the cylinder. He solved this problem based on either stream-function or velocity potential parameters.

In this work, in order to test the subroutine to deal with periodicity conditions, the solution domain was selected that includes the whole channel, bounded by half cylinders, and is extended by a unit distance in both downstream and upstream directions. In this case, the stream-function was used as the variable of the problem. Since U was assumed to be unity, the stream-function was assumed to be zero at the boundary BC and with a prescribed value t on GF, where the parameter t was given the value 3.15. A linear variation of the stream-function from zero to t was assumed at the boundaries DE and AH. In the remaining boundaries the periodicity conditions were assumed. The boundaries HG and FE were considered to have a stream-function distribution t greater than the values on the boundaries AB and CD, respectively. The normal derivative of the stream function was assumed to have the same absolute value on these upper and lower boundaries, but with opposite signs.

The discretized boundary of this domain is shown in Figure 7.5, where 24 quadratic elements were adopted with 48 nodes and 8 additional nodes in the corners. The smallest boundary elements were located on the surface of the cylinders and the biggest ones were located at the inlet and outlet flow boundaries.

The results for this problem are shown in Figure 7.6 for one quadrant of a cylinder, which is from the middle point on the surface of the lower cylinder to the point C. It is possible to see that the results agree very well with the analytical solution due to Thompson (1973). It was also clear from the output of the program, that the results obtained on the boundary were nearly symmetric in relation to the line crossing the centre of the cylinders. The accuracy of the results very near to the corner C, for example, was not so good, but changing the value of the gap between the corner nodes the results in this region improved. In this case, the coefficients C_i were obtained using the technique of rigid translation, with the source points located on the boundary. The results obtained by Thompson (1973) using the FEM, on the other hand, agree also with analytical solution. However, they did not plot any point in the region very near to the corner.

7.2.4- Additional Comments

Following the comparisons made above between BEM and FEM solutions it can be concluded that the BEM is more adequate to solve this class of problem than domain methods. However, the BEM in general requires the use of special treatment for the corners in order to improve the accuracy of the results in regions near to them. In these cases, the technique based on the rigid translation approach to obtain C_i ,along with the optimisation of the gap between the corner nodes, gave better results. The technique based on the location of the source points corresponding to the corner nodes outside the boundary did not gave good results in the first case investigated, where it was attempted. These cases, mainly the first one, showed that the corner treatment affects the results. Hence, depending on the proximity of the corners to the region of interest in the solution domain, the corner treatment becomes another major issue inherent to the application of the BEM.

7.3- VORTICITY-VELOCITY FORMULATION

7.3.1- Introduction

The development of the vorticity-velocity formulation program represented the most difficult part of this research since novelties were been considered and many problems appeared. The main problem was that the solution of this formulation comes out after an iterative procedure arising through the nonlinearity introduced by the convective term of the kinetic equations. This term is represented by domain integrals of these equations. Also, the value of the domain integrals appearing in the kinematic equations have to be guessed initially, or evaluated using the results of the previous iteration. This iterative procedure normally requires special measures to force convergence, apart from the fact that the convective term has to be evaluated with good accuracy. To make matters worse, a new set of boundary integral equations to represent the problem was investigated, with little support from the literature in terms of numerical measures to be adopted.

The main program based on this formulation was developed using initially a program to solve the BIE corresponding to Poisson's equation. This program was used as subroutine, or block of subprograms, with the objective of calculating the distribution of normal derivative of vorticity on the boundary and the distribution of vorticity at internal nodes, using the kinetic equation. The main difference between the BIE for Poisson's equation and the kinetic equation is that, in this case, the unknowns appear in the integrand of the domain integral. These unknowns are the velocity components and the derivatives of vorticity. In order to obtain the velocity components using the kinematic equations, the derivatives of vorticity have to be known and vice-versa. Hence, due to this nonlinearity, the solution has to be obtained iteratively. Conversely, the kinematics needed a different algorithm for the solution and new subroutines were, accordingly, developed. It is not difficult to see that the program for the solution of viscous flow using the vorticity-velocity formulation has a modular characteristic, where parts of the program were elaborated to achieve the purpose explained in the algorithm given in Section 6.4 of Chapter 6.

The vorticity-velocity program was initially based on the equations involving the kinetic equations of option 2, described in Chapter 4. It was decided to start using these equations for the kinetics because they preserve some similarity with Skerget's approach. Doing it this way helps to avoid starting with a completely new set of equations and enables some support from the literature. The first test with the program was carried out with the whole program already assembled, in spite of its modular characteristic, since the main concern was at first with the behaviour of the iterative process itself. The program did not work at that stage, mainly because of divergence of results and mistakes in the development of the program. Later however, a simple test case had to be adopted to check the results of each block of the program in order to help in the debugging process, and also, to give an idea of the effect of the modifications introduced in the program. A considerable time was spent trying to make the results of the program converge.

Many tests were carried out on the program and several versions were created. The effects of some parameter changes were investigated, for instance, the corner gap, the number of integration points, the type of element and the number of nodes of the mesh and so on. Basically, two problems were detected with the analysis of the results of the simple case tested: bad behaviour in the distribution of vorticity on the boundary, observed with the iterative solution, and inaccurate results for the normal derivative of vorticity on the boundary. The main modification introduced in the original algorithm at that stage was the attempt at solving the coupled kinematic and kinetic equations. Thus, the distributions of vorticity and the normal derivative of vorticity on the boundary would be obtained with the implicit solution of a system of equations involving the kinematic and kinetic equations. This measure was intended to improve the results of these two parameters and consequently improve the convergence. However, it did not solve the problem. A version of the program for the kinetic equations corresponding to option 1 was also investigated in parallel at the same time.

The kinematic equations are linearly dependent, and so only a number of equations corresponding to one set of algebraic equations, from the two sets resulting from the discretization of Eqs. (4.25) and (4.26), should be used in the calculation of the distribution of vorticity on the boundary. The way in which these equations are combined to form this system of algebraic equations affects the results considerably. It was found, observing the behaviour of the results, that the best way of assembling this system of equations is to chose the algebraic equations of the two sets which deliver the biggest absolute value for the leading diagonal of the matrix of coefficients. Skerget and his group had a similar problem. In order to overcome this problem they suggested a special measure to generate this system of algebraic equations. In their case, the problem seems more serious because the matrix of coefficients corresponding to each kinematic equation is singular, see for example Skerget *et al.* (1990) and Alujevic *et al.* (1991). One of the reasons for adopting in this work, a different form for the kinematic equations from those adopted by Skerget was so as to avoid similar problems. However it seems this measure did not give much improvement.

The divergence of the results was caused by many factors. One of these was the inaccuracy in the evaluation of the normal derivative of vorticity on the boundary. The main reasons for this are the corner problem and the inaccuracy in the calculation of the domain integral of the kinetic equation. The corner problem and the treatment given to it in the program are discussed in Chapter 6. Some of the techniques attempted in this work to solve these problems were assessed using the results of a simple test case. The accuracy of the integrals was also assessed indirectly using this simple test case.

In conclusion, the program based on the kinetic equation, option 1, converged for the case studies tested, however only for low values of Reynolds Number. The program based on option 2 also converged but with a little more difficulty. It seems that further experimentation will be necessary in order to extend the convergence of results to higher values of Reynolds number for the program based on both options.

After the initial tests were carried out for the purpose of validating the program, three test cases normally adopted in the literature were solved: Couette, Poiseuille and driven-cavity flows. The first two cases are very simple but important since all results can be obtained analytically. Therefore, they were extensively used in the development stage of the program to allow the assessment of all the results in every part of the domain. The driven-cavity flow is considered a bench mark test case that is often used to validate programs for viscous flows. This case has no analytical solution and the numerical results available in the literature do not allow results for all variables involved at every position to be obtained. The driven-cavity flow is strongly nonlinear for high Reynolds number flow and all domain integrals are important. Here, however, the Reynolds number adopted in the test are very low, almost zero in most cases, in order to afford to achieve the convergence of results. The results given by the program for these cases are presented next.

7.3.2- Pre-Validation Analysis

These preliminary tests include all those carried out during the development of the program with the two main objectives of helping in the debugging process and to assess the accuracy of the results delivered by the program. The principal test carried out was based on the analysis of each module of the program elaborated for a specific purpose. Thus, the modules that calculate the distribution of vorticity on the boundary, evaluate the normal derivative of vorticity etc, were assessed individually. In order to do so, the correct input obtained from any solution available in the literature was introduced in the module analysed and the output compared with the correct answer. Note, however, that no iteration existed in this case. This process was very important in order to help find mistakes made in the early stages of the development of the program. More recently, it was very useful in the investigation of corner problems and helpful in assessing indirectly the accuracy of the integrations. Some conclusions are now discussed. It is worth mentioning that these tests were carried out in parallel with attempts to improve the convergence of results, where any new idea was afterwards tested in the program to see the effects on convergence.

Poiseuille flow was adopted as the test case for this analysis because of its simplicity, which allows any result to be obtained analytically, and because for some cases, the domain integrals do not vanish. This is especially interesting when the effect of domain integrals need to be investigated. Figure 7.7 presents the geometry of the problem analysed along with the boundary conditions, in terms of velocity, and the correct distribution of vorticity and its normal derivative on the boundary. Note, that in this case, in the derivatives of vorticity in x- and y-directions are respectively 0.0 and 8.0 everywhere in the domain.

The domain was discretized using four 8-node quadrangular cells. The boundary was discretized using eight quadratic boundary elements. Thus, there were 20 nodes on the boundary and 5 internal nodes. An additional node was included at each corner. The picture of the mesh adopted is shown in Figure 7.8.

The first objective met by these tests was the identification and correction of any mistake in the programs. Subsequently, the main objective was to analyse the effects of the techniques to treat corners, mainly, before using them in the main program. It was a relatively easy task to discover the mistakes in the program. However, the identification of the cause of inaccuracy of some results and the investigation of the measures to correct it was a very time consuming task indeed. The ultimate test was the verification to see how these measures affect the convergence of the results, which is the main problem.

It has already been mentioned in the introduction of this Section, that the way the system of equations used to calculate the distribution of vorticity on the boundary is assembled, affects the results. The problem is that the kinematic equations, given by Eqs. (4.25) and (4.26), deliver a system of equations with ($2xm_\Gamma$) equations and m_Γ unknowns. A system of m_Γ equations, however, should result from the fact they are linearly dependent. Note that a system of equations fully based on one of the kinematic BIEs cannot be adopted since, depending on the geometry of the problem, the direction cosines may vanish on certain parts of the boundary along with the term containing the boundary integral with vorticity in the integrand. The solution, therefore, is to combine Eqs. (4.25) and (4.26) to generate a system of algebraic equations. Some combinations were attempted in this work. Initially, the system of equations was generated by adding the corresponding coefficients of these two equations. However, it was discovered later that the behaviour of the distribution of vorticity obtained initially hardly changed through the iterations. In other words, the results changed quantitatively but the behaviour of the distribution of vorticity was maintained almost the same. The problem is that bad results at some nodes persisted and eventually divergence of the results occurred. The same problem appeared when the equations were subtracted, instead. However, bad results appeared then on opposite positions around nodes located on the middle of the boundary, compared with the previous results. A mean average of the results obtained by adding and subtracting the kinematic equations was also tried. The behaviour of the distribution of vorticity was improved substantially. However, even though convergence of results was achieved for some simple cases, the algorithm was still very unstable. The simple test was not able to identify the best way to do it, since the results given in the test, without iterations, for different possible ways of assembling the system of equations were roughly of the same level of accuracy. Only tests using the main program, in an iterative calculation, could show the best way of assembling the system of equations. The results of each iteration had to be followed step-by-step to see, mainly, the behaviour of the vorticity distribution on the boundary. Eventually, it was discovered that the system of equations assembled by choosing the equation that gives the biggest absolute value for the leading diagonal of the matrix of coefficients gives the best results.

At the initial stage of development of the programs, with problems in the calculation of the vorticity and normal derivative of vorticity distribution, attempts were made also to solve the problem using Eqs. (4.25), (4.26) and (4.24) of option 2 coupled. That is, a system of algebraic equations involving these equations was assembled in order to obtain the distributions of vorticity and normal derivative of vorticity on the boundary nodes. This measure did not bring much improvement to the result and was later abandoned since, in this case, it is much more difficult to analyse the particular problem for each equation.

The most important difficulty regarding the corner problem appeared in the calculation of the normal derivative of vorticity on the boundary, for the reasons already explained in Chapter 6. In the kinematics, the calculation of the vorticity distribution on the boundary presented a slight problem of inaccuracy. This was due to the presence of gaps on the boundary, introduced with the double-node technique, in order to avoid the problem mentioned previously related to the calculation of the distribution of normal derivative of vorticity on the boundary. In the beginning, when the system of equations to calculate the distribution of this parameter was assembled using the equations given by the sum of the two kinematic equations, an additional problem caused by matrix ill-conditioning appeared when the corner nodes were getting closer. In this case, at the corners, two equations with nearly the same coefficients were generated. This problem disappeared with the use of a system of equations based on the equation which gives the biggest leading diagonal coefficient. Thus, different equations are used to represent each corner node and in this case the gap can be made as small as zero without causing a problem. However, the gap had to be left at the corners because of the calculation of the normal derivative of vorticity. In order to reduce the deterioration of the accuracy of results for the vorticity distribution on the boundary caused by the presence of the gaps, the jump function technique was adopted to correct the coefficient C_i. Since the value of vorticity at a corner is unique, the technique of removing one of the two nodes of the corner and its corresponding equation from the system of equations, already attempted in the system of equations that calculates the normal derivative of vorticity, could be used. However, priority was not given to this task at that stage of the development of the program.

Preliminary tests showed that the jump function approach to treat corner gaps affecting the calculation of the distribution of vorticity on the boundary had improved the results. Altering the value of the gap without any additional measure tried before the introduction of jump function technique did not work. The corrections introduced by the jump function were also used to obtain the velocity components distribution at internal nodes using the kinematic equations. In order to observe the effect of the gap, the results for three different values for the gaps: $1.1\text{x}10^{-2}$, 10^{-3} and 10^{-8} were analysed. It was observed that the accuracy of the results for all variables improved with the reduction of the gap, except for the results for the normal derivative of vorticity. This conclusion was already expected since the problem with matrix ill-conditioning normally appears when the corner nodes get closer. The results for the value of the gap $1.1\text{x}10^{-3}$ are summarized in Tables 7.1 and 7.2, corresponding to options 1 and 2 for the kinetics, respectively. Note, for example, that the results for the normal derivatives of vorticity are not satisfactory, in both cases, and that there is a problem in the internal results of the derivative of vorticity in x-direction appearing only in the equations of the option 2.

On the other hand, the implicit calculation of the distribution of the normal derivative of vorticity using the kinetic equation was also affected to a great extent by the corners. The technique of calculating the coefficient C_i, based on the so-called rigid translation technique, was also adopted in this part of the program, but did not solve the problem. These approaches were used in most tests carried out. More recently, the technique of locating the source points corresponding to the corner nodes outside the domain was adopted, with advantages over the previous technique, to improve the accuracy of the distribution of the normal derivative of vorticity, as shown in Tables 7.3 and 7.4, for option 1 and 2 for the kinetics, respectively, obtained using the same conditions used to obtain the results given in Tables 7.1 and 7.2, except that the corner gap was $1.1\text{x}10^{-8}$. It can be seen that the results for the normal derivative of vorticity have been improved notably. Another advantage, in this case, is that now the corner gap can be eliminated since the problem of matrix ill-conditioning disappears, increasing the accuracy of all the results of the program because, among other reasons, more accurate results can be obtained for the vorticity on the boundary. The problem with this technique is that the position of the source point outside the domain (regular corner technique) may have to be optimised. In simple tests, using the mesh shown in Fig. 7.8, three distances from the boundary relative to the length of the boundary element were analysed: 0.25, 0.5 and 1.0. It was observed that the accuracy of results for the normal derivative of vorticity was improved with the increase of the distance, but not very much. Therefore, it was decided to adopt a distance from the boundary of 0.25 throughout this study. It is worth mentioning that the regular BEM was attempted, however, the results were not acceptable.

Another measure attempted in order to try to improve the results was the refinement of the mesh. Most of the tests using 4 internal cells in the domain were repeated using a mesh with 16 internal cells and 16 boundary elements of the same type as before. Thus, 36 boundary and 33 internal nodes were used. This measure improved the results slightly, however all the problems mentioned before continued. For example, the distribution of the normal derivative of vorticity on the fluid boundary preserved the same tendency of the results to deteriorate in the direction of the corners. Hence, it was decided to continue using the coarse mesh for the purpose of the simple analysis. In order to try to tackle the corner problem also, a localized mesh refinement in corner regions was introduced. Again this measure did not solve the problem. With regard to the convergence, the refinement of the mesh did not bring any noticeable improvement to the iterative process. In addition, triangular cells on the domain combined with linear boundary elements, were also tried in the discretization. It was observed that, in general, the accuracy of the results obtained with this kind of mesh was not as good as the results obtained using a quadrangular mesh similar to the one given in Fig. 7.8.

Looking at the kinetic equations, it is easy to conclude that the convective term has its effect magnified since it is divided by the kinematic viscosity which is, in general, very small. For example, for air at the temperature of 15 °C it assumes a value of about 1.45×10^{-5} m^2/s. Thus, considering that for a certain geometry and specified velocity, when the value of Reynolds number increases (viscosity decreases), the effect of the convective term becomes very important and so requires to be evaluated with good accuracy. The reason is that any error is also magnified by the effect of the viscosity as well. The problem with representation and evaluation of the convective term created serious problems for people working with FDM and FEM in the past and is nowadays a problem in the use of the BEM also. The consequence of any inaccuracy is that the algorithm becomes unstable and divergence of results may occur. In the simple tests carried out here, without involving iterations, the value of viscosity was assumed unity in order to remove any possibility of masking the results of the convective term, since the accuracy of the integrals involved themselves was also being investigated. However, the early divergence in the results of the programs developed in this work was caused by many factors including the corner problem. More recently, the program based on the equations using option 1 for the kinematic equation converged adequately for some test cases analysed, but only at low values of Reynolds number. The remaining problem is now thought to be caused by the problem associated with the evaluation of the convective term of the kinetic equations. The simple test carried out with this set of equations using the regular treatment for the corners, without iteration, showed that the accuracy of the results is excellent, see Table 7.3. The same test showed, however, that the accuracy of the results using the set of equations of option 2 are not so good, as shown in Table 7.4. Note also that the results for the vorticity on the boundary can be slightly improved using the regular corner treatment in the kinematic equations, as shown in Table 7.5 for option 1. One could explain that the good results of option 1 are due to the fact that, in that case, the domain integrals for the kinetic equations vanish and conclude that the problem is caused by the convective terms. In other words, it is a problem due the evaluation of the domain integrals of the kinetic equations of option 2. This could in fact be true. The conclusions so far are: the algorithm based on option 1 is more stable than the program of option 2 and it is likely that the accuracy of the technique adopted to evaluate some domain integrals are not adequate for viscous flow analysis. This could explain the fact that the convergence so far was achieved only for low values of Reynolds number.

The simple test was also adopted to assess the accuracy of the integration schemes and to optimize the number of quadrature points to be used. In this case, the accuracy of the results of the program were assessed, instead of trying to check each integral individually. It was decided to adopt an even number of points to integrate on the boundary in order to avoid any point coinciding with the node at the middle of the quadratic element. This measure was intended to prevent problems when the source point was located at that node. It was decided to use 8 quadrature points to integrate on the boundary for accuracy's sake, although results with 6 points are not bad at all. Special attention was paid to the integration on the domain. In this case, the test showed that 4 points to integrate in quadrangular element mesh is adequate. When a fine mesh was used, less than 4 points were tried in order to avoid any influence due the proximity of the integration points to the source point.

Initially, it was thought that the algorithm based on the kinetic equations given by option 1 was not suitable for this analysis. Because, in this case, with the use of the correct information for the Poiseuille flow, the domain integrals corresponding to the convective term vanish in the Eqs.(4.22), (4.27) and (4.28), since the derivatives of vorticity in x-direction is zero and the velocity component in y-direction is also zero. However, these equations were useful to assess indirectly the accuracy of the boundary integrals that appear in the kinetic equations of both options, since the domain integrals are not present in this case. In the option 2, the domain integrals do not disappear, thus their accuracy can also be evaluated indirectly. From the results corresponding to option 1, given in Table 7.3, it was possible to conclude that the results delivered are in general accurate. This means that, unless any error is being cancelled out, the method adopted to perform the boundary integrals is adequate. The corner gap was considered practically zero in this case, where the jump function approach was also considered in the kinematic part and the regular corner approach was adopted in the evaluation of the normal derivative of vorticity. These same boundary integrals appear in the equations of option 2 where the results obtained are not so good as can be seen in Table 7.4. The other boundary integral appearing in these equations has the same kernel of one of the two boundary integrals previously investigated. Therefore, since the same approach is adopted to perform these integrals, it is possible to conclude that the accuracy of the results of the boundary integral is satisfactory. After this observation, it may be concluded also that the inaccuracy of the results may be caused by the domain integrals, as was suggested previously. It seems that every analysis so far led to the conclusion that there is a problem in the evaluation of domain integrals of the kinetic part. However, there are some facts that are not well understood. For example, in the results for option 2, given in Table 7.4, the results for the derivative of vorticity in x-direction at internal nodes at the position (0.5,0.25) and (0.5,0.75), which are more close to the solid boundary, are not so accurate as are the results for the derivative of vorticity in y-direction, in spite of the fact that the same subroutine is used to evaluate the domain integral in both cases. The equations that calculate that two parameters are very similar. These nodes are not so close to the boundary to blame any problem due to the effect of quasi-singular boundary integrals for this. The final conclusion is that this problem needs further investigation, meanwhile in this work more attention will be paid to the program based on equations of option 1, since it seems more promising. The fact that the kinetic equations of options 1 and 2 are only slightly different, but mathematically equivalent, led to the expectation initially, that their solutions would present similar behaviour and problems. The different problems presented by each one can give an indication of the diversity of numerical problems that formulations so different as that of Skerget and the one based on approach 1 could present.

After the partial success of the programs, some measures were attempted with the objective of extending the convergence of results to higher value of Reynolds number, at least to 100 which is the value adopted in most test cases presented in the literature. Changing the value of the relaxation factor did not solve the problem at all. Apart from this, the most important measure attempted was to increase the Reynolds number step by step, as suggested in many works using FEM and BEM. In other words, the converged results for Re=0, for example, was used as an initial guess for calculations with Re=10 for example. These results were in turn used as an initial guess in the calculation for Re=20 and so on. This process is continued until the target value for the Reynolds number be achieved. This technique was tried many times in different situations using the driven-cavity flow as a test case. Divergence of results, however, always occurs for Reynolds number greater than 50.

Changing the value of the relaxation factor, it was felt that the algorithm is very unstable and requires a very small relaxation factor to converge. In other words, unless the parameters updated at every iteration, mainly the derivatives of vorticity, are severely under-relaxed the convergence becomes difficult to be achieve. One problem is that, in this case, the converged results are obtained after an exaggerated number of iterations, along with the fact that the calculations becomes very computer time consuming. In the case of the program using the equations of option 1, for example, the derivatives of vorticity are under-relaxed using a typical relaxation factor of around 0.03. The objective of this is to reduce drastically the effect of the domain integrals. This under-relaxation affects indistinctly the kinematic and kinetic domain integrals, but the major problem seems to be the domain integrals of the kinematic part since they are divided by the kinematic viscosity. One measure that is currently under investigation is to multiply the domain integrals by a sort of scale factor, instead of using the under relaxation of the parameter in the integral. Thus, the domain integrals of the kinematics can be damped down with different strengths of the kinetics domain integrals, by using different factors. This could help to accelerate convergence. A final conclusion will be only emerge after a series of tests have been carried out.

7.3.3- Test Cases

7.3.3.1 Couette Flow

The Couette flow is represented by the flow between two flat parallel plates where the upper plate is moving at constant velocity. The solution of this flow is known to be given, in the case where gradients of pressure do not exist, by a linear variation of the axial velocity with the vertical distance, where on the lower plate the velocity is zero and at the position of the upper plate the flow assume the velocity of this plate. The transversal velocity is zero since the problem is one-dimensional. This simple model is adopted, in general, in tribology to describe the flow of lubricant in journal bearings. The case analysed here is described in Figure 7.9, where the upper plate is moving at unit velocity. The dimensions of the sides of the domain are assumed to have a unit value as well. In this situation the Reynolds number is given by the inverse of the kinematic viscosity.

The domain of the problem was discretized using a mesh with 16 domain cells and 16 boundary elements, as shown in Figure 7.10. Considering in this case the 8-node quadrangular type of cells on the domain and quadratic boundary elements, this gives 36 boundary and 33 internal nodes. The solution for this case given by the program, in the middle of the channel in the x-direction, is presented in Figure 7.11. It is possible to see that the results given by the program, based on a two-dimensional analysis, agree with the analytical solution. These results were achieved for Re = 10 after 108 iterations and a residual, calculated according Eq. (6.24), of 3×10^{-5}. In this case, both vorticity and derivative of vorticity were under-relaxed using the factors 0.3 and 0.03, respectively, although the program can converge with no need of relaxating the vorticity parameter. The initial distributions of the derivatives of vorticity were assumed to have a unit constant value which may be responsible for the large number of iterations required to achieve these results with this residual level.

7.3.3.2- Poiseuille Flow

This was another simple but important case considered. It represents the developed flow between two parallel plates, where the axial velocity profile is described by a parabolic equation in terms of the vertical distance of the channel. This is given in Figure 7.7 for a case where the velocity in middle of channel is unity. The dimension of the sides of the domain is also unity and again the Reynolds number for this case is given by the inverse of the kinematic viscosity.

The same mesh as given in Figure 7.10 was adopted to obtain the results presented in Figure 7.12, for axial velocity profile, and in Figure 7.13 for vorticity distribution, at the nodes located at the vertical line in the middle of the channel in the x-direction. It can be seen that the results agree with the analytical solution with good accuracy. The most noticeable discrepancy is observed in the results for the axial velocity in the middle of the channel, where there is an error of 2.5 %. These results were obtained for Re = 10 after 131 iterations with a residual of 10^{-5}. In this case only the derivatives of vorticity parameters were under-relaxed by a factor of 0.03.

7.3.3.3- Driven-Cavity Flow

This is one of the most adopted case studies in the literature. Different from the other cases, where a two-dimensional analysis was adopted to deal with one-dimensional problems, this is in fact a two-dimensional problem. It is described by a flow inside a rectangular cavity, as shown in Figure 7.14, where the upper lid is moving at constant velocity. The flow movement is, in this case therefore, induced by the upper surface of the cavity. Since the dimensions of the sides of the cavity were assumed to be unity, the Reynolds number is defined as before. Here, the results for Re = 0 and 10 were obtained and the comparisons with the literature presented. Explanations for the difference observed between these results are also discussed.

The works of Kelmanson (1983b), Karageorghis *et al.* (1989) and Rodriguez-Prada *et al.* (1990) for Re=0 were adopted as a reference to assess the results of this work. This problem, normally called Stoke's flow, is described in these works by a biharmonic equation in terms of the vorticity and stream function. The important point is that in their formulation the domain integrals does not appear. In this case, only the boundary needs to be discretized. The indirect BEM was adopted to obtain iteratively the results in the second work, where as many as 136 nodes were used to discretized the solution domain. In this case, due to the symmetry of the problem in question only half of the cavity was adopted as the solution domain. On the other hand, Kelmanson (1983b) and Rodriguez-Prada *et al.* (1990) adopted the regular BEM to solve the same problem. In the first work the problem was solved iteratively using different approaches. The best result was obtained with the approach that adopts an analytical solution of the results at the upper corners, where a boundary mesh with 200 constant elements was employed. In Rodriguez-Prada *et al.* (1990) this problem was solved directly without iterations. They typically adopted a boundary mesh with linear elements and 64 nodes in the analysis. It should be pointed out that these works, although use the BEM, they applied different formulations from the one adopted in this work.

In this work, the results were obtained using the same mesh given in Figure 7.10, which has, therefore, less nodes on the boundary than those works mentioned above. The results for Re=0 were obtained for many values of residual where a relaxation factors of 0.3 and 0.033 were adopted for vorticity and the derivatives of vorticity, respectively. A high value for the kinematic viscosity (10^5) was imposed to simulate the situation of zero Reynolds number flow. For this value of viscosity the convective terms in the kinetic equations have their effect reduced. However, instead of avoiding these terms it was decided to calculate them normally. The results for Re=10 were also obtained using the same parameters for several values of residual.

The reason behind the use of different values of residual is to discuss a problem observed in the results, which indicates that they begin to deteriorate after a certain number of iterations, compared with the results of the literature.

In Figure 7.15 the distribution of the x- component velocity along the vertical line in the middle of the cavity is presented for Re =0. The numerical solution obtained by Karageorghis *et al.* (1989) is used to assess the results of this work obtained for different numbers of iterations (values of residual), namely 65 (3.74 10^{-5}), 70 (2.99 10^{-5}), 80 (2.0 10^{-5}), 90 (1.41 10^{-5}), 102 (9.9 10^{-6}), 110 (8.077 10^{-6}), 120 (6.42 10^{-6}), 140 (4.332 10^{-6}) and 151(3.575 10^{-6}). It is possible to see that discrepancies exist between the results of the program compared to those from the literature, as shown in Figure 7.15, where some select cases are presented. The results of the program corresponding to 80 iterations seem to be very close to those of the literature. The difference increases with the number of iterations. In other words, for the lowest values of the residual the results deteriorate and fail to give a better solution. The most noticeable discrepancies are observed in the region corresponding to the inferior part of the cavity relative to the stagnation point. This means that the way of assessing the results based on the position of the stagnation point may be misleading. Karageorghis *et al.* (1989) and Rodriguez-Prada *et al.* (1990) found that the stagnation point was located on the middle line, at the distance of 0.76 and 0.75 from the bottom of the cavity, respectively. Kelmanson obtained also the value of 0.76. These are very near to the value 0.73 found in this work with 80 iterations and almost agree with the value 0.766 obtained with 120 iterations. Kelmanson (1983b) found that the vorticity at the stagnation point is -3.2021 which is a little bit lower than the value -3.4 obtained in this work for 80 iterations. With 120 iterations the program gave the value -4.06. Rodriguez-Prada *et al.* (1990) obtained the value -4.8 which they claimed that it agrees with the result given by another author.

The results corresponding to the distribution of vorticity on the upper lid, corresponding to 80 iterations, is compared in Figure 7.16 with the results of Rodriguez-Prada *et al.* (1990). The results of our work presented a smooth distribution, compared to theirs. This is mainly due to the fact that quadratic boundary elements were adopted here. At least in the middle of the lid a reasonable agreement between the results can be observed. The difference increases considerably to the extremities of the lid. The main reason for that, is that, at the extremities of the lid, the corners formed by the moving lid and stationary boundaries are considered singular points of the flow at which the vorticity and its normal derivative become unbounded, see Kelmanson (1983b). Different approaches may lead to different results at the corners. The difficulty of obtaining a solution in the regions of these corners were also mentioned by Rodriguez-Prada (1990) as one reason for the difference observed in their results at corners. As was pointed out in Kelmanson (1983b), since the formulations based on the FDM in general do not include the normal derivative of vorticity this major source of error is not inherent in the solution based on that method. Therefore, the works based on the FDM are normally adopted as reference to assess the accuracy of the results. It was observed here that the results on the lid did not vary significantly with the number of iterations.

The way the boundary conditions are imposed at the corners is another problem. In the FDM, for example, the velocity at these corners can assume either the value of the velocity of the lid or zero value at the stationary boundaries. It is recognized however that the inaccuracy thus introduced remains within a small region near to the corners. The situation seems to be different with regard the BEM. Kelmanson (1983b) solved the driven-cavity flow using the conventional BEM and came to a conclusion that this gives very poor convergence and the results are inaccurate in comparison with the literature. He blames the singular corner problem for that, which he says that errors are propagated to other variables. He improved the accuracy of the results and convergence introducing an analytical solution in the region of the corner in question. Rodriguez-Prada (1990) did not mention how they dealt with these corners. Here, since a double node technique was adopted to represent the corners, it was decided to assume that both corner nodes in this case belong to the lid, that is, they were imposed the conditions relative to the moving lid. The results considering one of the nodes of these corners as belonging to the stationary boundaries gave bad results. The conclusion is that the way the boundary conditions is imposed at these corners has great influence on the results. It may even be necessary to consider a special treatment to solve this problem as suggested in Kelmanson (1983b).

The problem related to these special corners and the fact that a coarse mesh relative to those works mentioned above has been used in this analysis can be considered as the main reasons for the behaviour of the results given by the program. Also, the accumulation of round off errors could be another possible cause for such performance.

The results shown in Figure 7.17 for Re=10 are compared with the result obtained by Graham & Oden (1986). Their results were obtained using the FEM and the penalty function approach. They adopted a fine mesh that allowed to obtain the values of the x-direction velocity in 16 nodes along the vertical line in the middle of the element. It can be seen that the behaviour observed in Figure 7.15 for Re=0 is also repeated in this case, where a solution closer to the results given by Graham & Oden (1986) can be found if a suitable number of iterations is selected. Again, a discrepancy between the results occurs mainly at the region of the flow corresponding to the inferior part of the cavity.

A plot of the iso-vorticity lines gave a better view of the results in the whole cavity. Figures 7.18 to 7.22 show the lines of constant vorticity inside of the cavity for Re=0 for several values of the number of iterations. These Figures allow a qualitative assessment of the results and enables one to see how the deformation of iso-vorticity lines progresses, due to the problem already mentioned above. In Figure 7.19, corresponding to 80 iterations for example, it can be seen that the lines are almost symmetric in relation to the vertical line in the middle of the cavity, as they should be. In fact, there is a slight asymmetry that may have been caused by the way the corner gap was located in this case. Travelling in the anti-clockwise direction on the boundary, the gap was located at the first corner node met. The other corner node was located right at the corner. Although the gap introduced was so small this may explain this behaviour of the lines of constant vorticity. The contour plot of Figure 7.19 shows many points of similarity to those presented in the literature, see for example, Karageorghis *et al.* (1989) and Rodriguez-Prada *et al.* (1990), mainly at the upper region of the cavity. However, it is possible to notice that the iso-vorticity lines are dislocated a little bit in direction to the bottom of the cavity compared with results given in the literature. The most important difference is that the iso-vorticity lines corresponding to values of vorticity -5, -4, -3, and so on do not start at the corners in question. This is another indication that the treatment to those corners given here, in terms of boundary conditions, may not be adequate. In Figure 7.16, for example, it is possible to see that Rodriguez-Prada *et al.* (1990) seems to have imposed a condition to enforce zero vorticity at these corners. Here, the structure of the program would have to be modified in order to impose such conditions. However, in Skerget *et al.* (1984) the driven-cavity flow is solved for Re=100 where the results in terms of velocity only were presented. These show good agreement with the other references, but they did not mention any special measure at these corners.

The variation of the residual with the number of iterations is illustrated in Figure 7.23 for the case with Re=0.

7.3.3.4- Additional Comments

The program was mainly validated using the first two test cases since some problems appeared with driven-cavity flow case, which seems to have been provoked by the singular corners of the moving lid. In order to solve this case properly more investigation is required, mainly in terms of special measures to treat these corners. Other simple cases need to be solved in the future to carry on with the improvement of the program. Also, it is very much desirable that a finer mesh is adopted as an attempt to improve the accuracy of the results, mainly with the use of refinement of the mesh in the regions of the corners. However, considering the excessive amount of CPU time required to achieve good convergence using the mesh adopted currently (typically, about 9 hours to undertake 140 iterations), which has prohibited the use of finer meshes properly, it will be necessary first to change the structure of the program to store the coefficients of influence. Thus, the integrations would be carried out just once. This measure will also reduce the problem related with the accumulation of round off errors since much less calculations will be performed.

Convergence for high values of Reynolds number seems to be now the next challenge in order to extend the applicability of the program. This certainly will require special measures in order to accelerate the convergence and to deal with high Reynolds number flow.

The convergence of the results was assessed in terms only of the vorticity parameter. In the two first tests, Couette and Poiseuille flows, it was possible to observe that when convergence is achieved the accuracy of the results in terms of velocity components is also adequate. However, the results for the derivatives of vorticity are not converged yet. This shows that more iterations is necessary in order to achieve the convergence for these parameters also. The results in the literature are, in general, presented in terms of velocity and vorticity parameters only and no information is given concerning the accuracy achieved with other parameters, as for example, the normal derivative of vorticity.

Information about number of the iterations, CPU time, relaxation factor adopted, etc. hardly appear in the literature. This would be very useful to assess the performance of the program in terms of these aspects. The program developed here encompasses the great majority of the steps necessary to develop a program based on the Skerget's formulations. The only important modification necessary to do that is in the way the distribution of vorticity on the boundary is evaluated. This would allow comparisons between the two formulations to be carried out. Most importantly, this would allow to investigate if any instability problem also appears in a program based on Skerget's approach using the techniques adopted here to evaluate integrals, treat corners and so on.

7.4- PENALTY FUNCTION FORMULATION

7.4.1- Introduction

The program based on the penalty function approach was developed based on a program for elasto-plastic stress analysis. This work began to be carried out only recently and so there was little time left to deal with iteration problems. Here the attempts to validate this program are discussed but no results for practical cases are presented since the program failed to converge conveniently for such cases.

7.4.2- Pre-validation analysis

In spite of the fact that many papers have pointed out that a program as such could be easily generated in that way, this was not confirmed here. The program itself was in fact modified to deal with viscous flow analysis without much difficulty. The problem, however, is that some parameters and constants change in value, compared with stress parameters, drastically, which may require some adjustment in the program to take into consideration these new values. For example, the kinematic viscosity replaces the Lame's constant μ', whose value is very high, around 10^{10} Pa, while the viscosity has normally a very low value. The other Lame's constant, whose value is also high, is replaced by the penalty parameter which is in general much higher. All these changes in the order of magnitude of those parameters are expected to change the pattern of the convergence behaviour and even may require some corrections in the program. For example, the parameter given by Eq. (5.10), which is equivalent to Poisson's ratio, assume here a value very near to 0.5 since the penalty parameter is always very high. This makes the constants d_{11}, d_{12}, d_{21} and d_{22} appearing as coefficients of the equations used to evaluate the derivatives of velocities on the boundary, discussed in Section K.2 of Appendix K, tend to have very high values. This seems to have not affected the results obtained so far, but that may create problems in some other cases.

The more serious modification introduced in the original program was related with the convective term. That term is defined by the domain integrals given by Eqs. (5.6) and (5.7), whose nonlinearity requires an iterative solution to Equations (5.4) and (5.5). The divergence of the results seems to be caused by a problem in the part of the program involved with the evaluation of this term. Unfortunately, the simple test cases, like Couette and Poiseuille flows, lead the domain integral to vanish and, therefore, the accuracy of this term could not be assessed using simple tests in the same fashion as was done in the vorticity-velocity formulation program. The program "converged" for these simple cases but what happened in fact was that the results came from a pseudo iterative process. Since in these cases the domain integrals vanish, in the first iteration, therefore, when the program naturally assumes that these domain integrals are zero, the results obtained should already be correct. Thus, after 2 or 3 iterations the program gives the results for these cases.These are illustrated in Figures 7.24 and 7.25 for Couette and Poiseuille flow, respectively, where the coarse mesh given in Figure 7.8 was used. These tests were at least useful to make sure that the part of the calculations involving the boundary integrals was working satisfactorily. The accuracy obtained are in general good, but varies slightly depending on the type of boundary conditions imposed, mainly due to corner problems. In this case, the corners were treated using the jump function technique discussed previously.

The Hagen-Poiseuille flow problem has been adopted as a test case where the domain integrals do not disappear. In this problem a constant velocity distribution is prescribed at the inlet of a channel formed by two parallel plates, whose length is enough to make the flow become fully developed. However, the results diverged so far. The problem has probably been caused by the treatment of the convective term, but it has been observed that relaxation of the results plays an important role. This problem in fact needs more investigation.

Chapter 8

Review and Recommendations for Further Work

The main objective of this research was the investigation of the BEM applied to solution of fluid flow problems. This was achieved satisfactorily in some cases and corresponding programs were developed and validations were carried out.

In particular, the program developed to deal with potential flows in cascades completely fulfilled the requirements. It is expected, therefore, that it can be applied directly to the solution of the blade-to-blade flow in a cascade of blades of axial turbomachinery with an additional boundary condition related to the physics of the problem.

The programs for viscous flow solution based on the vorticity-velocity formulation were developed using new sets of equations, which produced a different algorithm from those previously adopted in the literature. Validations of these programs were carried out using simple test cases. Convergence was achieved for most cases at low values of Reynolds number. However, the programs failed to converge at high values of Reynolds number.

Another program to solve viscous flows, based on the penalty function approach, was also developed. However, only a pre-validation analysis was carried out, which showed that this program does not converge when the convective term is included.

The original contributions of this work included the following:

(i) The development of a program with isoparametric boundary elements to deal with potential flow. This includes the periodicity boundary conditions normally considered in cascade flow analysis; (ii) The development of a program originally employed for elasto-plastic analysis to solve viscous flows using the penalty function approach, with some modifications in the solution algorithm when compared with existing publications; (iii) The main contribution is the proposal and derivation of a new set of integral equations using the vorticity-velocity formulation and the development of a corresponding computer program.

Suggestions for future work are as follows:

(i) The extension of the solution of potential flow in cascades for axial turbomachinery blades. This solution could be matched with the boundary layer solution, obtained using conventional methods such as FDM allowing also a simulation of viscous flow in cascades.

(ii) Continuation of the validation of the program based on the penalty function approach to obtain successful convergence. Schemes based on the FDM, for example, could be adopted to test the effects on the convergence and comparisons made against the results obtained using the BIE.

Note:

It is unlikely that the BEM alone, using the vorticity-velocity formulation could be applied to high Reynolds number flows. For example, the application of the method to real flows in cascades of turbomachinery at Reynolds numbers, typically, between 10^5 and 10^6 is not recommended. In this case, a combination of BEM with FEM and FDM to the solution of the entire flow field will be necessary.

References

ALUJEVIC, A.
KUHN, G.
SKERGET, P.
Boundary Element fo the Solution of Navier-Stokes Equations, Computer Methods in Applied Mechanics and Engineering, vol. 91, pp. 1187-1201,1991.

BAKER, A. J.
Finite Element Computational Fluid Mechnics, MacGraw-Hill & Hemisphere Publishing Corporation, Series in Comput. Meth. in Mech. and Thermal Science, 1983.

BANERJEE, P. K.
BUTTERFIELKD,
R.Boundary Element Method in Engineering Science, McGraw Hill, London, 1981.

BASKHROME, E.
HAMED, A.
A New Approach in Cascade Flow Analysis Using the Finite Element Method, AIAA Journal, vol. 19, nº 1, pp. 65-71, 1981.

BESKOS, D. E.
Introduction to Boundary Element Methods, Chapter 1, In: Boundary Element Methods in Mechanics, Vol. 3, Ed. by D. E. Beskos, Elsevier Science Publishers, Netherlands, 1987.

BHARADVAJ, B. K.
MORINO, L.
DEL MARCO, S.
A boundary Element Method for Unsteady Viscous Flows, BETECH 87, Proc. of the 2nd Boundary Element Technology Conference, MIT, USA, June, pp. 421-430, 1987.

BORCHERS, W.
HEBEKER, F. K.
The Boundary Element Spectral Method and Applications in 3-D Viscous Hydrodynamics, Boundary Elements VIII, Proc. of the 8th Int. Conf. on BEM, Tokyo, Japan, Sept., CMP & Springer-Verlag, Vol. 2, pp. 823-828, 1986.

BREBBIA, C. A.
FERRANTE, A.
Computational Hydraulics, Butterworths, UK, 1983.

BREBBIA, C. A.
TELLES, J. C. F.
WROBEL, L. C.
Boundary Element Techniques: Theory and Applications in Engineering, Springer-Verlag, Berlin, 1984.

BREBBIA, C. A.
WROBEL, L. C.
Viscous Flow Problems by the Boundary Element Method, Computational Techniques for Fluid Flow, Vol. 5, In: Recent Advances in Numerical Methods in Fluids, Eds. C. Taylor, J. A. Johnson and W. R. Smith, pp. 1-21, 1986.

BUSH, M. B.
TANNER, R. I.
Numerical Solution of Viscous Flow Using Integral Equation Methods, Int. J. for Numerical Methods in Fluids, Vol. 3, pp. 72-92, 1983.

CAMP, C. V.
GIPSON, G. S.
A Boundary Element Method for Viscous Flow at Low Reynolds Number, Boundary Elements IX, Proc. of the 9th Int. Conf. on BEM, Stuttgart, Germany, Sept., CMP&Springer-Verlag, vol. 3, pp. 419-430, 1987.

CAMP, C. V.
GIPSON, G. S.
A Boundary Element Method for Viscous Flows at Low Reynolds Numbers, Engineering Analysis, vol. 6, nº 3, pp. 144-151, 1989.

CARTER, I. N.
An Analysis of the Kutta-Joukovski-Carafoli Condition at Axial Profile Cascade Using the Boundary Element Method, Proc. of the 6th Int. Boundary Element Technology Conf., Southampton, pp. 51-61, 1991.

CHORIN, A. J.
A Numerical Method for Solving Incompressible Viscous Flow Problems, Journal of Comput. Physics, vol. 2, pp. 12-26, 1967.

DARGUSH, G. F.
BANERJEE, P. K.
Advanced Boundary Element Methods for Steady Incompressible Termoviscous Flow, In: Boundary Element Methods in Nonlinear Fluid Dynamics, Development in Boundary Element Methods - 6, Elsevier Applied Science, London, Chapter 2, pp. 55-84, 1990.

EL-REFAEE, M. M.
WU, J. C.
LEKOUDIS, S. G.
Solutions of the Compressible Navier-Stokes Equations Using the Integral Method, AIAA Journal, vol. 20, nº 3, pp. 356-362, 1982.

EL-ZAFRANY, A. — Boundary Element Method, Lecture Notes ME359, Cranfield Institute of Technology, SME, Oct., Cranfield, UK, 1989a.

EL-ZAFRANY, A. — Corner Jump Function for Two-Dimensional Elasticity Problems, Internal Report, SME, Cranfield Institute of Technology, 1989b.

EL-ZAFRANY, A. — Boundary Elements Methods, Ellis Horwood Publishers, UK, To appear in 1993.

FAROUK, B.
FUSEGI, T. — A Coupled Solution of the Vorticity-Velocity Formulation of the Incompressible Navier-Stokes Equations, Int. J. Numer. Meth. in Fluids, vol. 5, pp. 1017-1034, 1985.

FLETCHER, C. A. J. — Computational Galerkin Methods, Springer-Verlag, New York, 1984.

GUJ, G.
STELLA, F. — Numerical Solutions of High-Re Recirculating Flows in Vorticity-Velocity Form, Int. J. Numer. Meth. in Fluids, vol. 8, pp. 405-416, 1988.

GUNZBURGER, M.
MUNDT, M.
PETERSON, J. — Experiences with Finite Element Methods for the Velocity-Vorticity Formulation of Three-Dimensional, Viscous, Incompressible Flows, In: Computational Methods in Viscous Aerodynamics, Elsevier&CMP, Chapter 8, pp. 231-271, 1990.

GRAHAM, F. C.
ODEN, J. T. — Finite Elements: Fluid Mechanics, vol. VI, Prentice-Hall, 1986.

GRESHO, P. M. — Incompressible Fluid Dynamics: Some Fundamental Formulation Issues, Annu. Rev. Fluid Mech., 23, pp. 413-453, 1991.

HABASHI, W. G.
GUEVREMONT, G.
PEETERS, M. F.
PRZYBYTKOWSKI, S.M.
HAFEZ, M. M. — Finite Element Vorticity Based Methods for the Solution of the Compressible Navier-Stokes, In: Computational Methods in Viscous Aerodynamics, Elsevier&CMP, Chapter 7, pp. 197-230, 1990.

HARLOW, F. H.
WELCH, J. E.
Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with Free Surface, Physics of Fluids, vol. 8, pp. 2182-2189, 1965.

HEBEKER, F. K.
Efficient Boundary Element Methods for Three-Dimensional Viscous Flows, Boundary Elements VII, Proc. of the 7th Int. Conf. on BEM, Como, Italy, Sept., CMP&Springer-Verlag, vol. 2, pp. 9-37 to 9-44, 1985.

HEBEKER, F. K.
Characteristic and Boundary Elements for Navier-Stokes Flow, Boundary Elements IX, Proc. of the 9th Int. Conf. on BEM, Stuttgart, Germany, Sept., CMP&Springer-Verlag, vol. 3, pp. 433-442, 1987.

HEBEKER, F. K.
On Lagrangean and Boundary Element Methods for some Unsteady Isothermal Navier-Stokes Flow, Boundary Elements X, Proc. of the 10th Int. Conf. on BEM, Southampton, Sept., CMP&Springer-Verlag, vol. 2, pp. 373-380,1988.

HEBEKER, F. K.
On Lagrangean and Boundary Element Methods for a Model Problem of Viscous Compressible Motions, Notes on Numerical Fluid Mechanics, vol. 5, Finite Approximations in Fluid Mechanics II, Ed. by Ernst Heinrich Hirschel, Vieweg&Sohn, pp. 191-196, 1989.

HUGHES, T. J. R.
LIU, W. K.
BROOKS, A.
Finite Element Analysis of Incompressible Viscous Flows by the Penalty Function Formulation, Journal of Comput. Physics, vol. 30, pp. 1-60, 1979.

HUNG, S. C.
KINNEY, R. B.
Unsteady Viscous Flow over a Grooved Wall: A Comparison of Two Numerical Methods, Int. J. Numer. Meth. in Fluids, vol. 8, pp. 1403-1437, 1988.

INGHAM, D. B.
KELMANSON, M. A.
A Boundary Integral Equation Method for the Study of Slow Flow within Bearings Geometries, Proc. of the 5th Int. Conf. on BEM, Springer-Verlag, 1983.

JIALIN, Z. — On the Mathematical Foundations of the Boundary Element Method, Proc. on the Int. Conf. on Boundary Elements, Beijing, China, Oct., Pergamon Press, pp. 135-142, 1986.

KARAGEORGHIS, A.
FAIRWEATHER, G. — The Simple Layer Potential Method of Fundamental Solutions for Certain Biharmonic Problems, Int. J. Numer. Methods in Fluids, vol. 9, pp. 1221-1234, 1989.

KAKUDA, K.
TOASKA, N. — Boundary Element Approach to Viscous Flow Problems Based on the Time Splitting Technique, Boundary Elements VII, Proc. of the 12th Int. Conf. on BEM, Sapporo, Japan, Sept., CMP&Springer-Verlag, Vol. 2, pp. 63-72, 1990.

KELMANSON, M. A. — Boundary Integral Equation Solution of Viscous Flows with Free Surfaces, Journal of Engineering Mathematics, vol. 17, pp. 329-343, 1983a.

KELMANSON, M. A. — Modified Integral Equation Solution of Viscous Flow Near Sharp Corners, Computer and Fluids, vol.11, n° 4, pp. 307-324, 1983b.

KHADER, M. S. — A Surface Integral Numerical Solution for Laminar Developed Duct Flow, Journal of Applied Mechanics, ASME, vol. 48, pp.695-700, 1981.

KITAGAWA, K.
BREBBIA, C. A.
WROBEL, L. C.
TANAKA, M. — Boundary Element Analysis of Viscous Flow by Penalty Function Formulation, engineering Analysis, vol. 3, n° 4, pp. 194-200, 1986.

KITAGAWA, K.
WROBEL, L. C.
BREBBIA, C. A.
TANAKA, M. — Modelling Thermal Transport Problems Using the Boundary Element Method, Proc. of Int. Conf. on Development and Application of the Computer Techniques to Enviromental Studies, Southampton, pp. 715-731, 1986.

KITAGAWA, K. BREBBIA, C. A. WROBEL, L. C. TANAKA, M. — Viscous Flow Analysis Including Thermal Convection, Boundary Elements IX, Proc. of the 9th Int. Conf. on BEM, Stuttgart, Germany, Sept., CMP&Springer-Verlag, vol. 3, pp. 459-476,1987.

KITAGAWA, K. BREBBIA, C. A. TANAKA, M. WROBEL, L. C. — A Boundary Element Analysis of Natural Convection Problem by Penalty Function Formulation, Boundary Elements X, Proc. of 10th Int. Conf. on BEM, Southampton, Sept., CMP&Springer-Verlag, vol. 2, pp. 323-341,1988.

KITAGWA, K. — Boundary Element Analysis of Viscous Flow, Lecture Notes in Engineering, 55, Springer-Verlag, 1990.

KUROKI, T. ONISHI, K. TOSAKA, N. — Thermal Fluid Flow with Velocity Evaluation Using Boundary Elements and Penalty Function Method, Boundary Elements VII, Proc. of the 7° Int. Conf. on BEM, Como, Italy, Sept. CMP&Springer-Verlag, vol. 1, pp. 2-107 to 2-114, 1985.

KUHN, G. SKERGET, P. ALUJEVIC, A. ZAGAR, I. — Three-Dimensional Bouyancy Driven Flows in Inclined Cylinders, Boundary Element XI, Proc. of the 11th Int. Conf. on BEM, Cambridge, USA, Aug., CMP&Springer-Verlag, vol. 2, pp. 71-80, 1989.

LEONARD, B. P. — A Stable and Accurate Convective Modelling Procedure Based on Quadratic Upstream Interpolation, Comput. Meth. Appl. Mech. Eng., vol.19, pp. 59-98, 1979.

LIGHTHILL, M. J. — Boundary Layer Theory, Chapter II, In: Laminar Boundary Layer, Ed. L. Rosenhead, Oxford University Press, UK, 1963.

MERCHAND, W. COLLAR, A. R. — Flow of an Ideal Fluid past a cascade of Blades, A.R.C. R. & M. n° 1893, 1941.

MILLER, M. J. SEROVY, G. K. — Deviation Angle Estimation for Axial-Flow Compressors Using Inviscid Flow Solutions, J.

Engng. for Power, ASME, pp. 163-172, 1975.

MITRA, A. K.
INGBER, M. S.
Resolving Difficulties in the BIEM Caused by Geometric Corners and Discontinuous Boundary Conditions, Boundary Elements IX, Proc. of 9th Int. Conf. on BEM, Stuttgart, Germany, vol. 1, pp. 519-532, 1997.

ODEN, J. T.
WELLFORD, L.C. Jr.
Analysis of Flow of Viscous Fluids by the Finite Element Method, AIAA Journal, vol. 10, nº 12, pp. 1590-1599, 1972.

ODEN, J. T.
ZIENKIEWICZ, O. C.
GALLAGHER, R. H.
TAYLOR, C.
Finite Elements Methods in Flow Problems, Proc. of the 1st Int. Symposium. on FEM in Flow Problems, UAH Press, 1974.

OGANA, W.
Boundary Element Solution of the Transonic Integro-Differential Equation, Engineering Analysis, vol. 6, nº 4, pp. 170-179, 1989.

OGANA, W.
Boundary Element Methods in Three-Dimensional Transonic Flows, Boundary Elements XI, Proc. of the 11th Int. Conf. on BEM, Cambridge, USA, Aug., CMP&Springer-Verlag, vol.2, pp. 473-484, 1989.

ONISHI, K.
KUROKI, T.
TANAKA, M.
An Application of Boundary Element Method to Incompressible Laminar Viscous Flows, Engineering Analysis, vol. 1, nº 3, pp. 122-127, 1984.

ONISHI, K.
TOSAKA, N.
TANAKA, M.
Conduction, Convection, and Radiation in Heat Transport by BEM, Boundary Element VII, Proc. of the 7th Int. Conf. on BEM, Como, Italy, Sept., CMP&Springer-Verlag, vol. 1, pp. 2-3 to 2-12, 1985.

ONISHI, K.
Boundary Element Analysis of Thermal Fluid Flow Using ψ,ω, Proc. of 4th Int. Symp. on Innovative Numerical Methods in Engineering, Atlanta, USA, Mar., pp. 269-274, 1986.

ONISHI, K.
KUROKI, T.
TOSAKA, N.
Further Development of BEM in Thermal Fluid Dynamics, In: Boundary Element Methods in Nonlinear Fluid Dynamics, Development in BEM - 6, Elsevier Applied Science, London, Chapter 9, pp. 319-345, 1990.

OSEEN, C. W.
Neuere Methoden und Ergebnisse in der Hydrodynamik, Akad. Verlagsgesellschaft, Leipzig, DDR, 1927.

PANTON, R. L.
Incompressible Flow, John Wiley&Sons, NY, USA, 1984.

PATANKAR, S. V.
SPALDING, D. B.
A Calculation Procedure for Heat, Mass and Momentum Transfer in Three-Dimensional Parabolic Flows, Int. Journal of Heat and Mass Transfer, vol. 15, pp. 1787-1806, 1972.

PATANKAR, S. V.
Numerical Heat Transfer and Fluid Flow, Mcgraw-Hill, New York, 1980.

PATTERSON, C.
SHEIKH, M. A.
A Regular Boundary Element Method for Fluid Flow, Int. Journal for Numer. Meth. in Fluids, vol. 2, pp. 239-251, 1982.

PATTERSON, C.
SHEIKH, M. A.
On the Use of Higher Order Weighting Functions in the Boundary Integral Method for Fluid Flow, Proc. of the 3rd Int. Conf. on Numer. Meth. in Laminar and Turbulent Flow, Seattle, Aug., Pineridge Press, pp. 1163-1172, 1983.

PIVA, R.
GRAZIANY, G.
MORINO, L.
Boundary Integral Equation Method for Unsteady Viscous and Inviscid Flows, Advanced BEM, Proc. of the IUTAM Symp., San Antonio, Texas, Apr., pp. 297-304, 1987.

PRAGER, W.
Die Druckverteilung an Korpern in Ebener Potenttialstromung, Physikalische Zeitschrift, vol. 29, pp. 865-869, 1928.

PUN, W. M.
SPALDING, D.B.
A General Computer Program for Two-Dimensional Elliptic Flows, Report n° HTS/76/2, Imperial College, Mech. Eng. Dept., London, Amended, 1977.

RICCARDELLA, P.
An Implementation of the Boundary Integral Techniques for Plane Problems in Elasticity and Elasto-Plasticity, Ph.D. Thesis, Carnegie-Mellon University, Pittsburgh, 1973.

REK, Z.
SKERGET, P.
HRIBERSEK, M.
BEM for Laminar and Turbulent Flow, Z. angew. Math. Mech., vol. 71, n° 6, T635-T638, 1991.

RODRIGUEZ - PRADA, H A.
PIRONI, F. F.
SÁES, A. E.
Fundamental Solutions of the Stream Function - Vorticity Formulation of Navier-Stokes Equations, Int. Journal for Numerical Meth. in Fluids, vol. 10, pp. 1-12, 1990.

ROACHE, P. J.
Computational Fluid Dynamics, Hermosa Publishers, Albuquerque, New Mexico, USA, 1982.

SKERGET, P.
BREBBIA, C. A.
The Solution of Convective Problems in Laminar Flow, Proc. of 5th Int. Conf. on BEM, Hiroshima, Japan, Springer-Verlag, pp. 251-274, 1983.

SKERGET, P.
ALUJEVIC, A.
BREBBIA, C. A.
The Solution of Navier-Stokes Equations in Terms of Vorticity-Velocity Variables by Boundary Eelements, Boundary Elements VI, Proc. of the 6th Int. Conf. on BEM, Southampton, July, CMP&Springer-Verlag, pp. 4-41 to 4-56, 1984.

SKERGET, P.
ALUJEVIC, A.
BREBBIA, C. A.
Analysis of Laminar Flows with Separation Using BEM, Proc. of 7th Int. Conf. on BEM, Como, Italy, Sept., CMP&Springer-Verlag, pp. 9-23 to 9-36, 1985a.

SKERGET, P.
ALUJEVIC, A.
BREBBIA, C. A.
Analysis of Laminar Fluid Flows by Boundary Elements, Proc. of 4th Int. Conf. on Numer. Meth. in Laminar and Turbulent Flow, Swansea, pp. 22-38, 1985b.

SKERGET, P.
ALUJEVIC, A.
BREBBIA, C. A.

Nonlinear Fluid Problems by Boundary Elements, Proc. of 3rd Int. Conf. on Numerical Methods in Nonlinear Problems, Dubrovnik, pp. 834-846, 1986a.

SKERGET, P.
ALUJEVIC, A.
BREBBIA, C. A.

Vorticity-Velocity-Pressure Boundary Integral Formulation, Boundary Element VIII, Proc. of 8th Int. Conf. on BEM, Tokio, Japan, Sept., CMP&Springer-Verlag, pp. 829-834, 1986b.

SKERGET, P.
ALUJEVIC, A.
BREBBIA, C. A.

BEM for Laminar Motion of Isochoric Viscous Fluid, BETECH 87, Proc. of 2nd Int. Conf. on Boundary Element Technology, MIT, USA, June, pp. 397-419, 1987.

SKERGET, P.
BREBBIA, C. A.
KUHN, G.

Boundary Elements for Mixed-Convection Flow Problems, Proc. of 3rd Int. Conf. on Boundary Element Technology, Rio de Janeiro, Brazil, Springer-Verlag, pp. 17-25, 1987.

SKERGET, P.
ALUJEVIC, A.
KUHN, G.
BREBBIA, C. A.

Natural Convection Flow Problems by BEM, Boundary Elements IX, Proc. of the 9th Int. Conf. on BEM, Stuttgart, Germany, Sept., CMP& Springer-Verlag, vol.3, pp. 401-417, 1987.

SKERGET, P.
BREBBIA, C. A.
KUHN, G.

Time Dependent Three-Dimensional Laminar Isochoric Viscous Fluid Flow by Boundary Element Method, Proc. of 10th Int. Conf. on BEM, Shouthampton, UK, Sept., CMP&Springer- Verlag, pp. 57-74, 1988.

SKERGET, P.
KUHN, G.
ALUJEVIC, A.

Analysis of Mixed-Convection Flow Problems by BEM, Z. angew. Math. Mech., vol. 68, nº 5, pp. T413-T416, 1988.

SKERGET, P.
ALUJEVIC, A.
ZAGAR, I.
KUHN, G.

Time Dependent Recirculation Flow by the Boundary Element Method, Z. angew. Math. Mech., vol. 69, nº 4, T270-T272, 1989.

SKERGET, P.
KUHN, G.
ALUJEVIC, A.
BREBBIA, C. A.
Time Dependent Transport Problems by BEM, Advances in Water Resources, vol. 12, nº 1, Mar., pp. 9-20, 1989.

SKERGET, P.
ALUJEVIC, A.
REK, Z.
KUHN, G.
Boundary Element Method for General Viscous Flow Problems, Z. angew. Math. Mech., vol. 70, nº 6, pp. T713-T716, 1990.

SPALDING, D. B.
A Novel Finite-Difference Formulation for Differential Expressions Involving both First and Second Derivatives, Int. Journal of Num. Meth. in Eng., vol. 4, pp. 551-559, 1972.

SPEZIALE, C. G.
On the Advantages of the Vorticity-Velocity Formulation of the Equations of Fluid Dynamics, Journal of Computational Physics, 73, pp. 476-480, 1987.

SUGAVANAM, A.
WU, J. C.
Numerical Study of Separte Turbulent Flow over Airfoils, AIAA Journal, vol. 20, nº 4, pp. 464-470, 1987.

TAYLOR, C.
HOOD, P.
A Numerical Solution of the Navier-Stokes Equations Using the Finite Element Technique, Comput. Fluid, vol. 1, nº 1, pp. 73-100, 1973.

TANAKA, M.
KITAGAWA, K.
BREBBIA, C. A.
WROBEL, L. C.
A Boundary Element Investigation of Natural Convection Problems, Advances in Water Resources, vol.11, nº 3, pp. 139-143, 1988.

THOMPSON, D. S.
Finite Element Analysis of Flow Through a Cascade of Aerofoils, CUED/A Turbo/TR 45, Dept. of Eng., University of Cambridge, 1973.

THOMPSON, J. F.
WARSI, Z. U. A.
MASTIN, C. W.
Boundary-Fitted Coordinate Systems for Numerical Solution of Partial Differential Equations - A Review, Journal of Comput. Physics, vol. 47, pp. 1-108, 1982.

TOSAKA, N.
ONISHI, K.

Boundary Integral Equations Formulations for Steady Navier-Stokes Equations Using the Stokes Fundamental Solutions, Engineering Analysis, vol. 2, nº 3, pp. 128-132, 1985.

TOSAKA, N.
KAKUDA, K.
ONISHI, K.

Boundary Element Analysis of Steady Viscous Flows Based on P-U-V, Proc. of 7th Int. Conf. on BEM, Como, Italy, Sept., CMP&Spring-Verlag, vol. 2, pp. 9-71 to 9-80, 1985.

TOSAKA, N.
KAKUDA, K.

Numerical Solutions of Steady Incompressible Viscous Flow Problems by the Integral Equation Method, Proc. of 4th Int. Symp. on Innovative Numerical Methods in Enegineering, Atlanta, USA, Mar., pp. 211-222, 1986a.

TOSAKA, N.
KAKUDA, K.

Numerical Simulations of the Unsteady-State Incompressible Viscous Flows Using an Integral Equation, Proc. of the Int. Conf. on BEM, Beijing, China, Oct., pp. 163-172, 1986b.

TOSAKA, N.
KAKUDA, K.

Numerical Simulations for Incompressible Viscous Flow Problems Using the Integral Equation Methods, Boundary Elements VIII, Proc. of 8th Int. Conf. on BEM, Tokyo, Japan, Sept., CMP&Springer-Verlag, vol. 2, pp. 813-822, 1986c.

TOSAKA, N.
FUKUSHIMA, N.

Integral Equation Analysis of Laminar Natural Convection Problems, Boundary Elements VIII, Proc. of 8th Int. Conf. on BEM, Tokyo, Japan, Sept., CMP&Springer-Verlag, vol. 2, pp. 803-812, 1986.

TOSAKA, N.

Numerical Methods for Viscous Flow Problems Using an Integral Equation, 3rd Int. Symp. on River Sedimentation, University of Mississipi, USA, pp. 1514-1525, 1986.

TOSAKA, N.
ONISHI, K.

Boundary Integral Equation Formulation for Unsteady Incompressible Viscous Fluid Flow by Time-Differencing, Engineering Analysis, vol. 3, n° 2, pp. 101-104, 1986a.

TOSAKA, N.
ONISHI, K.
Integral Equation Method for Thermal Fluid Flow Problems, Computational Mechanics 86: Theory and Applications, Tokyo, May, Springer-Verlag, pp. XI-103 to XI-108, 1986b.

TOSAKA, N.
FUKUSHIMA, N.
Integral Equation Analysis of Unsteady Natural Convection Problems, Proc. of 1st Japan-China Symp. on BEM, Karuizawa, Japan, June, pp. 123-132, 1987.

TOSAKA, N.
KAKUDA, K.
Numerical Simulation of Laminar and Turbulent Flows by Using an Integral Equation, Boundary Elements IX, Proc. of the 9th Int. Conf. on BEM, Stuttgart, Germany, Sept., CMP&Springer-Verlag, vol. 3, pp. 489-502, 1987.

TOSAKA, N.
FUKUSHIMA, N.
Numerical Simulation of Laminar Natural Convection Problems by the Integral Equation Method, Proc. of 5th Int. Conf. on Numerical Methods in Thermal Problems, Providence, RI, Pineridge Press, vol. 1, pp. 500-511, 1988.

TOSAKA, N.
Integral Equation Formulations with the Primitive Variables for Incompressible Viscous Fluid Flow Problems, Computational Mechanics, n° 4, pp. 89-103, 1989.

TOSAKA, N.
FUKUSHIMA, N.
Integral Equation Analyses of Natural Convection Problems in Fluid Flow, Boundary Element Methods in Heat Transfer, Eds. Wrobel, L. C. and Brebbia, C. A., CMP&Elsevier Applied Science, UK, Chapter 8, pp. 235-267, 1992.

TOSAKA, N.
KAKUDA, K.
SATO, H.
Incompressible Viscous Flow by the Integral Equation Method, Boundary Elements XII, Proc. of 12th Int. Conf. on BEM, Sapporo, Japan, CMP& Verlag, vol. 2, pp. 93-104, 1990.

TOSAKA, N. KAKUDA, K.	Newtonian and Non-Newtonian Unsteady Flow Problems, In: Boundary Element Methods in Nonlinear Fluid Dynamics, Developments in BEM - 6, Elsevier Applied Science, London, Chapter 5, pp. 151-181, 1990.
TREFEZT, E.	Uber die Kontraktion Kreisformiger Flussigkeitsstrahlen, Zeitschrift fur Mathematik und Physik, vol. 64, pp. 34-61, 1917.
WALKER, S. P. FENNER, R. T.	Treatment of Corners in BIE Analysis of Potential Problems, Int. Journal for Num. Methods in Engineering, vol. 28, n° 11, pp. 2569-2581, 1989.
WU, J. C. THOMPSON, J. F.	Numerical Solutions of Time-Dependent Incompressible Navier-Stokes Equations Using an Integro-Differential Formulation, Computer& Fluids, vol.1, pp. 197-215, 1973.
WU, J. C. SPRING, A. H. SANKAR, N. L.	A Flowfield Segmentation Method for the Numerical Solution of Viscous Flow Problems, In: Lecture Notes in Physics, n° 35, Springer-Verlag, Berlin, pp. 452-457,1974.
WU, J. C. WAHBAH, M. M.	Numerical Solution of Viscous Flow Equations using Integral Representations, Lecture Notes in Physics, n° 59, Springer-Verlag, pp. 448-453, 1976.
WU, J. C.	Finite Element Solution of Flow Problems Using Integral Representations, Proc. of the 2nd Int. Symp. on Finite Element Methods in Flow Problems, Italy, June, pp. 205-216, 1976.
WU, J. C.	Prospects for the Numerical Solution of General Viscous Flow Problems, Proc. of the Lockheed-Georgia Company Viscous Flow Symp., USA, June, pp. 39-104, 1977.
WU, J. C. WAHBAH, M. M. SUGAVANAM, A.	Some Numerical Solutions of Turbulent Flow Problems by the use of Integral Representation, Proc. of the Symp. on Application of Computer

Methods in Engineering, University of Southern California, vol. II, pp. 983-992, 1977.

WU, J. C.
SUGAVANAM, A.
Method for the Numerical Solution of Turbulent Flow Problems, AIAA Journal, vol. 16, nº 9, pp. 948-955, 1978.

WU, J. C.
GULCAT, U.
Separate Treatment of Attached and Detached Flow Regions in General Viscous Flows, AIAA Journal, vol. 19, nº 1, 1981.

WU, J. C.
Problems of General Viscous Flow, Development in BEM - 2, Ed. by P. K. Banerjee and R. P. Shaw, Elsevier Applied Science Publishers Ltd, London, Chapter 4, pp. 69-109, 1982.

WU, J. C.
RIZK, Y. M.
SANKAR, N. L.
Problems of Time-Dependent Navier-Stokes Flow, Development in BEM - 3, Ed. by P. K. Banerjee and S. Mukherjee, Elsevier Applied Science Publishers, London, Chapter 6, pp. 137-169, 1984.

WU, J. C.
Fundamental Solutions and Numerical Methods for Flow Problems, International Journal for Numerical Methods in Fluids, vol. 4, pp. 185-201, 1984.

WU, J. C.
Boundary Element Methods and Inhomogeneous Elliptic Differential Equations, Boundary Elements VII, Proc. of 7th Int. Conf. on BEM, Como, Italy, Sept., CMP&Springer-Verlag, vol. 2, pp. 9-25 to 9-104, 1985.

WU, J. C.
Fundamental Solutions and Boundary Element Methods, Engineering Analysis, vol. 4, nº 1, pp. 2-6, 1987.

WU, J. C.
WANG, C. M.
Recent Advances in Solution Methods for Unsteady Viscous Flows, Boundary Element Methods in Nonlinear Fluid Dynamics, Developments in BEM - 6, Chapter 8, Ed. by P. K. Banerjee and L. Morino, Elsevier Applied Science Publishers, London, 1990.

YOUNGREEN, G. K.
ACRIVOS, A.
Stokes Flow Past a Particle of Arbitrary Shape: a Numerical Method of Solution, J. Fluid Mechanics, vol. 69, part 2, pp. 377-403,1975.

ZAGAR, I.
SKERGET, P.
ALUJEVIC, A.
HUDINA, M.
Combined Forced and Free Laminar Convection in the Entrance Region of a Tube, Boundary Elements XII, Proc. of 12th Int. Conf. on BEM, Sapporo, Japan, Sept., CMP&Springer-Verlag, vol. 2,pp. 85-92, 1990.

ZAGAR, I.
SKERGET, P.
ALUJEVIC, A.
Boundary Domain Integral Method for Space Time Dependent Viscous Incompressible Flow. In: Boundary Integral Methods: Theory and Applications - (IABEM-90), Eds. Morino, L. and Piva, R., Roma, Italy, Springer-Verlag, pp.510-519, 1991.

ZAGAR, I.
SKERGET, P.
ALUJEVIC, A.
HRIBERSEK, M.
Pressure Drop Calculations for Tube Bundles in Crossflow by the Boundary-Domain Integral Method, Z. angew. Math. Mech., vol.72, n° 5, T395-T398, 1992.

ZIENKIEWICZ, O. C.
CHEUNG, Y. K.
Finite Elements in the Solution of Field Problems, The Engineer, pp. 507-510, 1965.

ZUOSHENG, Y.
On the Boundary Element Method for Compressible Flow about Bodies, Boundary Elements VII, Proc. of 7th Int. Conf. on BEM, Como, Italy, Sept., CMP&Springer-Verlag, vol. 2, pp. 9-45 to 9-48, 1985.

ZUOSHENG, Y.
Boundary Element Method for Transonic Flow about Three-Dimensional Wings, Proc. of the Int. Conf. on Boundary Elements, Beijing, China, Oct., Pergamon Press, pp. 173-179,1986.

ZUOSHENG, Y. Integral Equation Method for Compressible Potential Flows, Proc. of 1st Japan-China Symposium on Boundary Element Methods, June, Karuizawa, Japan, pp. 107-114, 1987.

Appendix A

Navier-Stokes Equations

The main points concerning the derivation of Navier-Stokes equations are presented in order to introduce the concepts discussed in the main text. The rigorous and complete derivation of all equations can be found in many textbooks on fluid dynamics.

The Navier-Stokes equations are the result of the application of Newton's second law of motion to fluids. This states that the rate of change of momentum of a fluid particle is equal to the net force acting on it.

The rate of change of momentum per unit of volume is obtained by applying the operator D( )/Dt, known as the substantive derivative, given by

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \vec{v}.\nabla(\ )$$

on $\rho\ \vec{v}$, which is the momentum per unit of volume. Thus:

$$\rho\frac{D\vec{v}}{Dt} = \rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\ .\ \nabla\vec{v} \tag{A.1}$$

this represents the two contributions for the rate of change in $\rho\ \vec{v}$: the rate of change of momentum at a fixed point and the change of momentum that occurs when the fluid particle travels to a region where the velocity is different.

From the above, the left-hand side of the dynamic equation, representing Newton's second law of motion, is $\rho\ D\vec{V}/Dt$. The right-hand side is the sum of the forces per unit volume acting on the fluid particle.

There are basically two kinds of forces acting on the fluid particle : the external and the internal forces. The external force, represented by $\vec{f}$, acts on the bulk of the fluid domain. The internal forces are due to both pressure and viscous action acting on the surface of the element of fluid. Both the pressure and viscous action generate stresses across any arbitrary surface within the fluid which are related to the velocity field. In consequence, they are intrinsic parts of the equations of motion. Therefore, the force on a particle of fluid is the net result of the effect of the stresses over its surface plus the external force.

The stress tensor, σ, and rate of strain tensor, ϵ, are briefly considered, and then the consequences of a linear relationship between them is determined.

The total stress acting in a fluid has contributions from both pressure and viscous effects. These can be represented through the following expression:

$$\sigma = -pI + \tau$$

where p is the pressure and I and τ are the identity tensor and the viscous stress tensor, respectively. Note that τ contains both normal and tangential viscous components which add or subtract from the pressure.

However, the net force on a fluid particle is given by the difference in the stresses acting across opposite faces. Consequently, the total force per unit volume is given by

$$\nabla . \sigma = -\nabla . (pI) + \nabla . \tau$$

but from the property of tensors the first term on the right-hand side of the above equation can transformed into

$$\nabla.(pI) = \nabla p$$

and so

$$\nabla.\sigma = -\nabla p + \nabla.\tau \qquad \text{(A.2)}$$

Conversely, rates of strain are related to the velocity gradient tensor, ζ, given by

$$\zeta = \nabla\vec{v}$$

There are, however, some velocity gradient fields that involve no changes in the length of any material line. No distortion or viscous effects appear therefore. It is, in fact, the symmetrical combination of velocity gradients that give rise to rates of strain which produce deformation, a motion consisting of extension and shearing strains. Hence, the rate of strain tensor is the symmetric part of ζ,

$$\epsilon = \frac{1}{2}\left(\nabla\vec{v} + (\nabla\vec{v})_c\right) \qquad \text{(A.3)}$$

where $(\nabla\vec{v})_c$ represents the conjugated of the velocity gradient tensor. Additionally, it can be shown that the anti-symmetric part corresponding to a tensor causing a solid-body-like rotation motion, is given by

$$\eta = \frac{1}{2}\left(\nabla\vec{v} - (\nabla\vec{v})_c\right)$$

and its three independent entries may be expressed in the form of the vorticity vector $\vec{\omega}$, defined by:

$$\vec{\omega} = \nabla\wedge\vec{v}$$

Thus, vorticity is another local property of the flow field in the same way as momentum, kinetic energy, etc, i. e. parameters that are somehow related to the velocity field through any kinematic relation.

For a Newtonian fluid, τ is linearly related to ϵ, through a constitutive relation which can be written:

$$\tau = \lambda(\nabla . \vec{v})I + (\xi + \chi)\epsilon$$

where λ is the so-called second coefficient of viscosity and the two arbitrary constants, ξ and χ, are related to the first coefficient of viscosity, μ, applying the following relation :

$$\xi + \chi = 2\mu$$

The presence of λ in the previous equation introduces some difficulties. This parameter in general should assume the value (-3/2 μ) according to the Stoke's assumption. The latter considers that there is no difference between the thermodynamics and mechanical pressures if the fluid is undergoing an expansion or compression. The following expression for the viscous stress tensor results:

$$\tau = -\frac{3}{2}\mu(\nabla . \vec{v})I + 2\mu\epsilon$$

However, since the second term of this equation is eliminated in incompressible flow, the second viscosity parameter can play no role in such flows. Hence, for incompressible flow:

$$\tau = 2\mu\epsilon \qquad \text{(A.4)}$$

Replacing Eqs. (A.4) and (A.3) into Eq.(A.2):

$$\nabla . \sigma = -\nabla p + \mu \nabla . \ (\nabla \vec{v} + (\nabla \vec{v})_c) \qquad \text{(A.5)}$$

if μ is taken to be constant. This equation can further be manipulated to give:

$$\nabla . \sigma = -\nabla p + \mu(\nabla^2 \vec{v} + \nabla(\nabla . \vec{v})) \qquad \text{(A.5)}$$

since

$$\nabla . \nabla \vec{v} = \nabla^2 \vec{v}$$

and

$$\nabla . (\nabla \vec{v})_c = \nabla(\nabla . \vec{v})$$

Note that the last term of above equation disappears for incompressible flow, and so

$$\nabla . \sigma = -\nabla p + \mu \nabla^2 \vec{v} \qquad \text{(A.6)}$$

Hence, Newton's second law for fluids can be represented by

$$\rho \frac{D\vec{v}}{Dt} = \vec{f} + \nabla . \sigma \qquad \text{(A.7)}$$

and finally, substituting (A.1) and (A.6) into (A.7), Navier-Stokes equations for unsteady flow analysis of an incompressible Newtonian fluid with constant properties, is given by:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} . \nabla \vec{v} = \frac{\vec{f}}{\rho} - \frac{1}{\rho} \nabla p + \mu \nabla^2 \vec{v} \qquad \text{(A.8)}$$

The second term on the right-hand side of Eq. (A.8) represents the force due to pressure and the third term represents the diffusion process of velocity. On the left-hand side, the first term represents the transport convective of velocity and the second term represents the variation of velocity with time.

The first term on the right-hand side of this equation represent the contribution of those forces, such as gravity, that have to be included in the specification of the problem. However, in most situations the action of these forces can be considered negligible. In this case, the cause of motion is either imposed by pressure differences or relative movement, that is, no body forces are applied. Although, almost every flow takes place in a gravity field, the gravitational body force acts significantly only on density differences. The natural convection flow is an example where the effects of this term must be considered, otherwise it can be eliminated. Besides, when the body forces is due to the gravity field, which is conservative, this can be introduced into the pressure term. The body force term will be maintained in the derivation of the equations for the sake of generality only.

Appendix B

Formulae

Some useful vectorial properties, theorems and Dirac delta function properties are presented to support the derivations discussed in the main text.

B.1- VECTOR OPERATOR IDENTITIES

The variables $\vec{A}$, $\vec{B}$ and $\vec{C}$ denote vector fields and f denotes a scalar field. Thus,

$$\nabla\wedge(\nabla f) = 0 \qquad \text{(B.1)}$$

$$\nabla.(\nabla\wedge\vec{A}) = 0 \qquad \text{(B.2)}$$

$$\nabla.(f\vec{A}) = f(\nabla.\vec{A}) + \vec{A}.\nabla f \qquad \text{(B.3)}$$

$$\nabla\wedge(f\vec{A}) = f(\nabla\wedge\vec{A}) + \nabla f\wedge\vec{A} \qquad \text{(B.4)}$$

$$\vec{A}\wedge(\vec{B}\wedge\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - (\vec{A}.\vec{B})\vec{C} \qquad \text{(B.5)}$$

$$\nabla.(\vec{A}\wedge\vec{B}) = \vec{B}.(\nabla\wedge\vec{A}) - \vec{A}.(\nabla\wedge\vec{B}) \qquad \text{(B.6)}$$

$$\nabla\wedge(\vec{A}\wedge\vec{B}) = (\vec{B}.\nabla)\vec{A} - \vec{B}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{B} + \vec{A}(\nabla.\vec{B}) \tag{B.7}$$

$$\nabla(\vec{A}.\vec{B}) = (\vec{B}.\nabla)\vec{A} + (\vec{A}.\nabla)\vec{B} + \vec{B}\wedge(\nabla\wedge\vec{A}) + \vec{A}\wedge(\nabla\wedge\vec{B}) \tag{B.8}$$

$$\nabla\wedge(\nabla\wedge\vec{A}) = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A} \tag{B.9}$$

$$\vec{A}\wedge(\nabla\wedge\vec{A}) = \frac{1}{2}\nabla(\vec{A}.\vec{A}) - (\vec{A}.\nabla)\vec{A} \tag{B.10}$$

$$\nabla\wedge[\vec{A}\wedge(\nabla\wedge\vec{A})] = -\nabla\wedge[(\vec{A}.\nabla)\vec{A}] \tag{B.11}$$

$$\begin{aligned}\nabla\wedge[\vec{A}\wedge(\nabla\wedge\vec{A})] &= [(\nabla\wedge\vec{A}).\nabla]\vec{A} \\ &- (\nabla\wedge\vec{A})(\nabla.\vec{A}) - (\vec{A}.\nabla)(\nabla\wedge\vec{A})\end{aligned} \tag{B.12}$$

B.2- THEOREMS AND IDENTITIES

B.2.1-Integration by Parts Theorems

Let h(x,y) and g(x,y) be continuous functions with defined continuous partial derivatives, up to the requested order, within a close domain Ω bounded by Γ. Then,

$$\iint_\Omega h\frac{\partial g}{\partial x}dx\,dy = \oint_\Gamma hg\ell\, d\Gamma - \iint_\Omega \frac{\partial h}{\partial x}g\,dx\,dy \tag{B.13}$$

$$\iint_\Omega h\frac{\partial g}{\partial y}\,dx\,dy = \oint_\Gamma hgm\, d\Gamma - \iint_\Omega \frac{\partial h}{\partial y}g\,dx\,dy \tag{B.14}$$

and

$$\iint_\Omega h(\nabla^2 g)\,dx\,dy = \oint_\Gamma h\frac{\partial g}{\partial n}d\Gamma - \oint_\Gamma g\frac{\partial h}{\partial n}d\Gamma + \iint_\Omega g(\nabla^2 h)\,dx\,dy \tag{B.15}$$

where ℓ and m are the direction cosines.

B.2.2- Green's Theorem for Vectors

Consider that $\vec{D}$ and $\vec{E}$ are two continuous vector functions single-valued, finite and have continuous second derivatives within a domain Ω bounded by a close boundary Γ, or a close outer boundary and one or more closed inner boundaries, the latter enclosed by the former. Then, Green's theorem for vectors can be conveniently represented in following form convenient for the derivations presented in this work :

$$\begin{aligned}\iint_\Omega [\vec{D}.(\nabla\wedge\nabla\wedge\vec{E}) - \vec{E}.(\nabla\wedge\nabla\wedge\vec{D})]\,dx\,dy \\ = \oint_\Gamma [\vec{E}\wedge(\nabla\wedge\vec{D}) - \vec{D}\wedge(\nabla\wedge\vec{E})].\vec{n}\,d\Gamma\end{aligned} \tag{B.16}$$

where $\vec{n}$ is the unit vector normal to the boundary.

B.2.3-Gauss' Divergence Theorem

Suppose that a region Ω in a vector field $\vec{G}(x,y)$ is bounded by a closed boundary Γ and that the vector function $\vec{G}(x,y)$ is uniform continuous on Γ and within the region Ω. It can then be shown that

$$\iint_{\Omega} \nabla . \vec{G}\, dx dy = \oint_{\Gamma} \vec{G} . \vec{n}\, d\Gamma \tag{B.17}$$

where $\vec{n}$ is the unit vector normal to the boundary.

B.3- PROPERTIES OF DIRAC DELTA FUNCTION

Some useful properties of Dirac Delta function used throughout this work, compiled from El-Zafrany (1993), are presented. These properties are :

$$\delta\,(x-x_i, y-y_i) = 0, \quad \text{for} \quad (x,y) \neq (x_i,y_i) \tag{B.18}$$

$$\delta\,(x-x_i, y-y_i) = \infty, \quad \text{for} \quad (x,y) = (x_i,y_i) \tag{B.19}$$

$$\iint_{\Omega} f(x,y)\, \delta\,(x-x_i, y-y_i)\, dx dy = C_i f(x_i,y_i) \tag{B.20}$$

$$\iint_{\Omega} f(x,y) \frac{\partial \delta}{\partial x}(x-x_i, y-y_i)\, dx dy = -C_i \frac{\partial f(x,y)}{\partial x}\Big|_{(x_i,y_i)} \tag{B.21}$$

$$\iint_{\Omega} f(x,y)\,\frac{\partial \delta}{\partial y}(x-x_i,y-y_i)\,dx\,dy = -C_i\,\frac{\partial f(x,y)}{\partial y}\Big|_{(x_i,y_i)} \tag{B.22}$$

$$\iint_{\Omega} f(x,y)\,\frac{\partial^2 \delta}{\partial x\,\partial y}(x-x_i,y-y_i)\,dx\,dy = C_i\,\frac{\partial^2 f(x,y)}{\partial x\,\partial y}\Big|_{(x_i,y_i)} \tag{B.23}$$

where f(x,y) is any scalar continuous field function defined inside the domain Ω and C_i is a constant whose value depends on the position of the source point (x_i,y_i), as follows:

$C_i = 1$	if (x_i,y_i) is inside the domain Ω;
$C_i = 0$	if (x_i,y_i) is outside the domain Ω;
$C_i = \alpha_i / 2\pi$	if (x_i,y_i) is on the boundary Ω, where α_i is the corner angle at this point, as shown in Figure B.1. Note that if the boundary is smooth $\alpha_i = \pi$ and so $C_i = 0.5$.

Appendix C

Coefficients of Influence

The coefficients of the discretized equations, Eqs. (6.9) to (6.25), are given below:

Boundary Integrals:

$$h_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi) J(\xi) \frac{\partial u^*}{\partial n} d\xi$$

$$hx_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi) J(\xi) \frac{\partial u^*}{\partial x} d\xi$$

$$hy_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi) J(\xi) \frac{\partial u^*}{\partial y} d\xi$$

$$hxn_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi) J(\xi) \frac{\partial^2 u^*}{\partial x \partial n} d\xi$$

$$hvy_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi)\,J(\xi)\,v_n \frac{\partial u^*}{\partial y}\, d\xi$$

$$g_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi)\,J(\xi)\,u^*\, d\xi$$

$$gm_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi)\,J(\xi)\,m u^*\, d\xi$$

$$gl_j^{(e)}(x_i,y_i) = \int_0^1 N_j(\xi)\,J(\xi)\,l u^*\, d\xi$$

$$gxx_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,G_{11}\, d\xi$$

$$gyx_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,G_{21}\, d\xi$$

$$gxy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) G_{12}\, d\xi$$

$$gyy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) G_{22}\, d\xi$$

$$fxx_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) F_{11}\, d\xi$$

$$fyx_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) F_{21}\, d\xi$$

$$fxy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) F_{12}\, d\xi$$

$$fyy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) F_{22}\, d\xi$$

$$gxxx_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{11}}{\partial x}\, d\xi$$

$$gyxx_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{21}}{\partial x} d\xi$$

$$gxxy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{11}}{\partial y} d\xi$$

$$gyxy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{21}}{\partial y} d\xi$$

$$gxyx_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{12}}{\partial x} d\xi$$

$$gyyx_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{22}}{\partial x} d\xi$$

$$gxyy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{12}}{\partial y} d\xi$$

$$gyyy_j^{(e)} = \int_0^1 N_j(\xi) J(\xi) \frac{\partial G_{22}}{\partial y} d\xi$$

$$fxx x_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{11}}{\partial x}\,d\xi$$

$$fyx x_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{21}}{\partial x}\,d\xi$$

$$fxx y_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{11}}{\partial y}\,d\xi$$

$$fyx y_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{21}}{\partial y}\,d\xi$$

$$fxy x_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{12}}{\partial x}\,d\xi$$

$$fyy x_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{22}}{\partial x}\,d\xi$$

$$fxy y_j^{(e)} = \int_0^1 N_j(\xi)\,J(\xi)\,\frac{\partial F_{12}}{\partial y}\,d\xi$$

$$fyyy_j^{(e)} = \int_0^1 N_j(\xi)\, J(\xi) \frac{\partial F_{22}}{\partial y}\, d\xi$$

Domain Integrals:

$$hyc_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta) \frac{\partial u^*}{\partial y}\, d\xi d\eta$$

$$hxc_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta) \frac{\partial u^*}{\partial x}\, d\xi d\eta$$

$$hxx_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta) \frac{\partial^2 u^*}{\partial x^2}\, d\xi\, d\eta$$

$$hyy_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta) \frac{\partial^2 u^*}{\partial y^2}\, d\xi\, d\eta$$

$$hxy_j^{(c)}(x_i,y_i) = \iint_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta) \frac{\partial^2 u^*}{\partial x\, \partial y}\, d\xi\, d\eta$$

$$gc_j^{(c)}(x_i,y_i) = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, u^*\, d\xi\ \ d\eta$$

$$gxxc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, G_{11}\, d\xi\, d\eta$$

$$gyxc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, G_{21}\, d\xi\, d\eta$$

$$gxyc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, G_{12}\, d\xi\, d\eta$$

$$gyyc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, G_{22}\, d\xi\, d\eta$$

$$gxxxc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{11}}{\partial x}\, d\xi\, d\eta$$

$$gxxyc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{11}}{\partial y}\, d\xi\, d\eta$$

$$gyxxc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{21}}{\partial x}\, d\xi\, d\eta$$

$$gyxyc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{21}}{\partial y}\, d\xi\, d\eta$$

$$gxyxc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{12}}{\partial x}\, d\xi\, d\eta$$

$$gxyyc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{12}}{\partial y}\, d\xi\, d\eta$$

$$gyyxc_j^{(c)} = \iint\limits_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta)\, \frac{\partial G_{22}}{\partial x}\, d\xi\, d\eta$$

$$gyyyc_j^{(c)} = \iint_{\xi-\eta} N_j(\xi,\eta)\, J(\xi,\eta) \frac{\partial G_{22}}{\partial y}\, d\xi\, d\eta$$

where u^*, F_{11}, F_{12}, F_{21}, F_{22}, G_{11}, G_{12}, G_{21} and G_{22} are the fundamental solutions defined in the main text.

Appendix D

BIE for Vorticity-Velocity Formulation : Part I

For incompressible fluid flow the Continuity equation ($\nabla \cdot \vec{v} = 0$) states that the velocity field is solenoidal. Therefore, there is a vector potential function, say $\vec{\Psi}$, related to the velocity field, such that

$$\nabla \wedge \vec{\psi} = \vec{v} \tag{D.1}$$

which satisties that statement. However, since the curl of a gradient of any scalar function is zero, it can be seen that $\vec{\Psi}$ is indeterminade in the field of gradient of a function, and so Eq. (D.1) does not uniquely define $\vec{\Psi}$. Another assumption has now to be made. The assumption normally adopted is that $\vec{\Psi}$ is also solenoidal, and so

$$\nabla \cdot \vec{\psi} = 0 \tag{D.2}$$

Taking the curl of both sides of Eq. (D.1) the following differential equation results,

$$\nabla \wedge (\nabla \wedge \vec{\psi}) = \vec{\omega} \tag{D.3}$$

The corresponding weighted-residual expression for Eq. (D.3), with a weight residual function $\vec{\Psi}^*$ and including a solution satisfying the given boundary conditions, can be represented as follows:

$$\iint_{\Omega} \vec{\psi}^* \cdot [\nabla \wedge (\nabla \wedge \vec{\psi})] \, dxdy - \iint_{\Omega} \vec{\psi}^* \cdot \vec{\omega} \, dxdy = 0 \tag{D.4}$$

Let $\vec{D} = \vec{\Psi}^*$ and $\vec{E} = \vec{\Psi}$ in the Green's theorem for vectors in the form given by Eq. (B.16) of Appendix B. The results is:

$$\iint_\Omega \{\vec{\psi}^* . [\nabla \wedge (\nabla \wedge \vec{\psi})] - \vec{\psi} . [\nabla \wedge (\nabla \wedge \vec{\psi}^*)]\} dx dy = \oint_\Gamma [\vec{\psi} \wedge (\nabla \wedge \vec{\psi}^*) - \vec{\psi}^* \wedge (\nabla \wedge \vec{\psi})] . \vec{n} \, d\Gamma \tag{D.5}$$

and substituting Eq. (D.5) into (D.4) it gives

$$\iint_\Omega \vec{\psi} . [\nabla \wedge (\nabla \wedge \vec{\psi}^*)] dx dy - \iint_\Omega \vec{\psi}^* . \vec{\omega} \, dx dy + \oint_\Gamma [\vec{\psi} \wedge (\nabla \wedge \vec{\psi}^*) - \vec{\psi}^* \wedge (\nabla \wedge \vec{\psi})] . \vec{n} \, d\Gamma = 0 \tag{D.6}$$

this is the so-called inverted weighted-residual expression of Eq. (D.4).

Let now $\vec{\Psi}^*$ be represented in terms of one scalar function u^* and an arbitrary constant vector $\vec{e}$ as follows

$$\vec{\psi}^* = \nabla u^* \wedge \vec{e} \tag{D.7}$$

or, using the relations (B.4) and (B.2) of the Appendix B,

$$\vec{\psi}^* = \nabla \wedge (u^* \vec{e}) \tag{D.8}$$

It can be shown that this satisfies the assumption given by Eq. (D.2), using the relation (B.2).

In order to work with the integrand of the first domain integral in the Eq. (D.6), it is convenient to take first the curl of the Equation (D.7),

$$\nabla \wedge \vec{\psi}^* = \nabla(\vec{e}\,.\,\nabla u^*) - (\nabla^2 u^*)\,\vec{e} \tag{D.9}$$

from the relations (B.7) and (B.8) of Appendix B. Also, it can be shown from the previous equation, Eq. (D.9), that

$$\nabla \wedge (\nabla \wedge \vec{\psi}^*) = (\vec{e} \wedge \nabla)\,\nabla^2 u^* \tag{D.10}$$

obtained using the relations (B.1) and (B.4).

Suppose now that u^* is the solution corresponding to the following equation

$$\nabla^2 u^* = -\,\delta(x-x_i, y-y_i) \tag{D.11}$$

where $\delta(x-x_i\ ,\ y-y_i)$ is Dirac delta function and (x,y) and (x_i,y_i) are the coordinates of field and source points, respectively. This equation is well known from the literature on the BEM whose solution is given in Chapter 3. It is normally called of fundamental solution and is given as follows :

$$u^* = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$$

for two-dimensional problems, where r stands as defined in Chapter 3.

Hence, substituting Eq. (D.11) into (D.10), the following expression results

$$\nabla \wedge (\nabla \wedge \vec{\psi}^*) = -(\vec{e} \wedge \nabla)\,\delta(x-x_i, y-y_i) \tag{D.12}$$

Introducing the above equation into Eq. (D.6) gives

$$\begin{aligned}\iint_{\Omega} \vec{\psi}\,.\,(\vec{e} \wedge \nabla)\,\delta(x-x_i, y-y_i)\,dx\,dy + \iint_{\Omega} \vec{\psi}^*\,.\,\vec{\omega}\,dx\,dy \\ - \oint_{\Gamma} [\vec{\psi} \wedge (\nabla \wedge \vec{\psi}^*) - \vec{\psi}^* \wedge (\nabla \wedge \vec{\psi})]\,.\,\vec{n}\,d\Gamma = 0\end{aligned} \tag{D.13}$$

Finally, substituting Eq. (D.7) or (D.8) into the above equation and working with each term, an integral expression in terms of velocity, vorticity and the fundamental solution, only results. This derivation is presented in Appendix E and the final boundary integral equation is

$$C_i \vec{v}(x_i,y_i) + \oint_{\Gamma} (\vec{v}\,.\,\vec{n})\, \nabla u^*\, d\Gamma = \oint_{\Gamma} (\vec{v} \wedge \vec{n}) \wedge \nabla u^*\, d\Gamma + \iint_{\Omega} \vec{\omega} \wedge \nabla u^*\, dx\,dy \tag{D.14}$$

Appendix E

BIE for Vorticity-Velocity Formulation : Part II

In this Appendix each term of the boundary integral Equation (D.13) of Appendix D will be manipulated in order to isolate the arbitrary vector $\vec{e}$. Thus, from the first integral on the left-hand side of Eq. (D.13) :

$$(i) \quad I_1 = \iint_{\Omega} \vec{\psi} \cdot [(\vec{e} \wedge \nabla) \; \delta(x-x_i, y-y_i)] \, dx\,dy$$

Note that to manipulate this integral attention must be paid to the derivatives of δ $(x-x_i, y-y_i)$ appearing in the integrand. It can be shown that the integral above results in

$$I_1 = - C_i [\vec{\psi} \cdot (\vec{e} \wedge \nabla)] \big|_{(x_i, y_i)}$$

using the properties of the Dirac delta function given in the Appendix B, where C_i also stands according to the definition presented in that Appendix. Now manipulating this expression in order to isolate $\vec{e}$, gives:

$$\begin{aligned} I_1 &= - C_i [(\nabla \wedge \vec{\psi}) \cdot \vec{e}] \big|_{(x_i, y_i)} = - C_i [\vec{v} \cdot \vec{e}] \big|_{(x_i, y_i)} \\ &= - C_i \vec{v}(x_i, y_i) \cdot \vec{e} \end{aligned}$$

using Eq. (D.1), the property of the triple scalar product, and also the fact that $\vec{e}$ is considered constant everywhere.

(*ii*) $$I_2 = \iint_\Omega \vec{\psi}^* . \vec{\omega}\, dx dy$$

Introducing Eq. (D.7) into this integral, gives:

$$I_2 = \iint_\Omega (\nabla u^* \wedge \vec{e}) . \vec{\omega}\, dx dy = \iint_\Omega (\vec{\omega} \wedge \nabla u^*) . \vec{e}\, dx dy$$

from the property of the triple scalar product.

(*iii*) $$I_3 = \oint_\Gamma [\vec{\psi} \wedge (\nabla \wedge \vec{\psi}^*)] . \vec{n}\, d\Gamma$$

Introducing eq. (D.9) into the integrand of this integral, gives

$$\oint_\Gamma \{\vec{\psi} \wedge [\nabla(\vec{e} . \nabla u^*) - (\nabla^2 u^*) \vec{e}]\} . \vec{n}\, d\Gamma$$

This new integral can be split into two terms, I_{3a} and I_{3b}. Term I_{3a} is represented by

$$I_{3a} = \oint_\Gamma \{\vec{\psi} \wedge [\nabla(\vec{e} . \nabla u^*)]\} . \vec{n}\, d\Gamma$$

However,

$$\nabla(\vec{e} . \nabla u^*) \wedge \vec{\psi} = \nabla \wedge [(\vec{e} . \nabla u^*) \vec{\psi}] - (\vec{e} . \nabla u^*)(\nabla \wedge \vec{\psi})$$

from the relation (B.4). Hence, substituting this expression into integral I_{3a} gives

$$I_{3a} = \oint_\Gamma \{(\vec{e} . \nabla u^*)(\nabla \wedge \vec{\psi}) - \nabla \wedge [(\vec{e} . \nabla u^*) \vec{\psi}]\} . \vec{n}\, d\Gamma$$

or

$$I_{3a} = \oint_{\Gamma} [(\vec{e}\,.\,\nabla u^{*})\,\vec{v}]\,.\,\vec{n}\,d\Gamma \; - \; \oint_{\Gamma} \{\nabla\wedge[(\vec{e}\,.\,\nabla u^{*})\,\vec{\psi}]\}\,.\,\vec{n}\,d\Gamma$$

from eq. (D.1), and manipulating the integrand of the first integral, gives:

$$I_{3a} = \oint_{\Gamma} [(\vec{v}\,.\,\vec{n})\,\nabla u^{*}]\,.\,\vec{e}\,d\Gamma \; - \; \oint_{\Gamma} \{\nabla\wedge[(\vec{e}\,.\,\nabla u^{*})\,\vec{\psi}]\}\,.\,\vec{n}\,d\Gamma \qquad (E.1)$$

The last integral can be manipulated further using Gauss' Divergence theorem given in Appendix B, Eq. (B.17). Thus,

$$\oint_{\Gamma} \{\nabla\wedge[(\vec{e}\,.\,\nabla u^{*})\,\vec{\psi}]\}\,.\,\vec{n}\,d\Gamma = \iint_{\Omega} \nabla\,.\,[\nabla\wedge(\vec{e}\,.\,\nabla u^{*})\,\vec{\psi}]\,dx\,dy \qquad (E.2)$$

However, using the vector property (B.4) it can be shown that the integrand of this domain integral can be represented by

$$\nabla\,.\,[\nabla\wedge(\vec{e}\,.\,\nabla u^{*})\,\vec{\psi}] = \nabla\,.\,\{(\vec{e}\,.\,\nabla u^{*})(\nabla\wedge\vec{\psi}) + [\nabla(\vec{e}\,.\,\nabla u^{*})]\wedge\vec{\psi}\}$$

and employing the vector properties (B.3) and (B.6),

$$\begin{aligned} &= (\vec{e}\,.\,\nabla u^{*})\,\nabla\,.\,(\nabla\wedge\vec{\psi}) + (\nabla\wedge\vec{\psi})\,.\,\nabla(\vec{e}\,.\,\nabla u^{*}) \\ &+ \vec{\psi}\,.\,[\nabla\wedge\nabla(\vec{e}\,.\,\nabla u^{*})] - \nabla(\vec{e}\,.\,\nabla u^{*})\,.\,(\nabla\wedge\vec{\psi}) \end{aligned}$$

Finally, according to the vector properties (B.2) and (B.1) the first and third terms, respectively, of the previous expression vanish. Additionally, the second and last terms cancel each other. Therefore, the domain integral in question is zero as is the last term on the right-hand side of expression (E.1). This conclusion can readily be obtained through the use of the vector property (B.2) in the domain integral of eq. (E.2). Thus, term I_{3a} becomes

$$I_{3a} = \oint_{\Gamma} [(\vec{v}\,.\,\vec{n})\,\nabla u^{*}]\,.\,\vec{e}\,d\Gamma \qquad (E.3)$$

Term I_{3b} is given by

$$I_{3b} = \oint_{\Gamma} [\vec{\psi} \wedge (\nabla^2 u^* \vec{e})] \,.\, \vec{n} \, d\Gamma$$

and again using Gauss' divergence theorem, gives

$$I_{3b} = \oint_{\Gamma} [\vec{\psi} \wedge (\nabla^2 u^* \vec{e})] \,.\, \vec{n} \, d\Gamma = \iint_{\Omega} \nabla . [\vec{\psi} \wedge (\nabla^2 u^* \vec{e})] \, dx dy$$

and using the vector property (B.6),

$$= \iint_{\Omega} [(\nabla^2 u^* \vec{e}) \,.\, (\nabla \wedge \vec{\psi}) \; - \; \vec{\psi} \,.\, (\nabla \wedge (\nabla^2 u^* \vec{e}))] \, dx dy$$

Manipulating the first term and using the property of the triple scalar product in the second term, it can be shown that:

$$= \iint_{\Omega} \{\vec{e} \,.\, [(\nabla^2 u^* (\nabla \wedge \vec{\psi})] \; - \; [(\vec{\psi} \wedge \nabla) \nabla^2 u^*] \,.\, \vec{e}\} \, dx dy$$

However, since

$$\nabla^2 u^* = - \, \delta (x - x_i, y - y_i)$$

and the previous integral can be represented by

$$= - \iint_{\Omega} \{[(\nabla \wedge \vec{\psi}) \, \delta (x - x_i, y - y_i)] \,.\, \vec{e} \; - \; [(\vec{\psi} \wedge \nabla) \, \delta (x - x_i, y - y_i)] \,.\, \vec{e}\} \, dx dy$$

Using the Dirac delta function properties (B.20) and (B.21), given in Appendix B, in the first and second terms of the previous integral, respectively, gives

$$= - C_i [(\nabla \wedge \vec{\psi}) . \vec{e}] |_{(x_i,y_i)} - C_i [(\vec{\psi} \wedge \nabla) . \vec{e}] |_{(x_i,y_i)}$$

It is now simple to conclude that these two terms cancel out. Therefore, $I_{3b} = 0$.

In conclusion,

$$I_3 = I_{3a} = \oint_\Gamma [(\vec{v} . \vec{n}) \nabla u^*] . \vec{e} \, d\Gamma$$

(iv) $I_4 = \oint_\Gamma [\vec{\psi}^* \wedge (\nabla \wedge \vec{\psi})] . \vec{n} \, d\Gamma$

Introducing Eqs. (D.1) and (D.7) into I_4, it results

$$I_4 = \oint_\Gamma [(\nabla u^* \wedge \vec{e}) \wedge \vec{v}] . \vec{n} \, d\Gamma$$

and using the property of the triple scalar product,

$$I_4 = \oint_\Gamma (\vec{v} \wedge \vec{n}) . (\nabla u^* \wedge \vec{e}) \, d\Gamma$$

Applying again the same property, gives

$$I_4 = \oint_\Gamma \vec{e} . [(\vec{v} \wedge \vec{n}) \wedge \nabla u^*] \, d\Gamma$$

Finally, introducing the integrals I_1, I_2, I_3 and I_4 into the corresponding terms in Eq. (D.13), the following expression results:

$$-C_i\,\vec{v}(x_i,y_i)\,.\,\vec{e} + \iint_{\Omega} (\vec{\omega} \wedge \nabla u^*)\,.\,\vec{e}\,dx\,dy$$

$$- \oint_{\Gamma} [(\vec{v}\,.\,\vec{n})\,\nabla u^*]\,.\,\vec{e}\,d\Gamma + \oint_{\Gamma} [(\vec{v} \wedge \vec{n}) \wedge \nabla u^*]\,.\,\vec{e}\,d\Gamma = 0$$

and since $\vec{e}$ is arbitrary, the following integral expression results:

$$C_i\,\vec{v}(x_i,y_i) + \oint_{\Gamma} [(\vec{v}\,.\,\vec{n})\,\nabla u^*]\,d\Gamma = \oint_{\Gamma} [(\vec{v} \wedge \vec{n}) \wedge \nabla u^*]\,d\Gamma + \iint_{\Omega} (\vec{\omega} \wedge \nabla u^*)\,dx\,dy \tag{E.4}$$

Appendix F

Futher Manipulation of the BIE for Vorticity-Velocity Formulation

Considering a two-dimensional internal flow problem, the vorticity vector has only one component, perpendicular to the domain Ω (which can conveniently be treated as a scalar), the unit normal vector points outside and the unit tangential vector points in the anti-clockwise direction on the boundary Γ.

In this case, the unit normal vector is represented by:

$$\vec{n} = \ell\hat{i} + m\hat{j} \tag{F.1}$$

and the tangential vector is given by

$$\vec{t} = -m\hat{i} + \ell\hat{j} \tag{F.2}$$

where ℓ and m represent the direction cosines.

The velocity vector is represented in term of components as follows:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \tag{F.3}$$

where v_x and v_y are the velocity components in x and y directions, respectively. Thus, the normal component of the velocity is represented by

$$v_n = \ell v_x + m v_y \tag{F.4}$$

while the tangential component is

$$v_t = -m v_x + \ell v_y \tag{F.5}$$

In what follows, each term of Eq. (4.11) will be represented in terms of components. Therefore, referring to the integrand of the contour integral on the left-hand side of that integral, we have

$$\vec{v}.\vec{n} = v_n$$

and

$$\nabla u^* = \frac{\partial u^*}{\partial x}\hat{i} + \frac{\partial u^*}{\partial y}\hat{j}$$

and so,

$$(\vec{v}.\vec{n})\nabla u^* = v_n\left(\frac{\partial u^*}{\partial x}\hat{i} + \frac{\partial u^*}{\partial y}\hat{j}\right) \tag{F.6}$$

Now, the integrand of the contour integral on the right-hand side of Eq. (4.11) will be represented in terms of components, but considering first that

$$\vec{v}\wedge\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 0 \\ l & m & 0 \end{vmatrix}$$

$$\vec{v}\wedge\vec{n} = (m v_x - \ell v_y)\hat{k}$$

and so

$$\vec{v}\wedge\vec{n} = -v_t\hat{k}$$

from Eq. (F.5). Hence,

$$(\vec{v} \wedge \vec{n}) \wedge \nabla u^* = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -v_t \\ \dfrac{\partial u^*}{\partial x} & \dfrac{\partial u^*}{\partial y} & 0 \end{vmatrix}$$

which produces

$$(\vec{v} \wedge \vec{n}) \wedge \nabla u^* = v_t \left(\frac{\partial u^*}{\partial y} \hat{i} - \frac{\partial u^*}{\partial x} \hat{j} \right) \tag{F.7}$$

The integrand of the domain integral of Eq. (4.11) is represented in terms of components by:

$$\vec{\omega} \wedge \nabla u^* = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ \dfrac{\partial u^*}{\partial x} & \dfrac{\partial u^*}{\partial y} & 0 \end{vmatrix}$$

or

$$\vec{\omega} \wedge \nabla u^* = -\,\omega \left(\frac{\partial u^*}{\partial y} \hat{i} + \frac{\partial u^*}{\partial x} \hat{j} \right) \tag{F.8}$$

Finally, substituting Eqs. (F.6), (F.7) and (F.8) into Eq. (4.11), gives the following equation:

$$C_i (v_x \hat{i} + v_y \hat{j}) + \oint_\Gamma v_n \left(\frac{\partial u^*}{\partial x} \hat{i} + \frac{\partial u^*}{\partial y} \hat{j} \right) d\Gamma$$
$$= \oint_\Gamma v_t \left(\frac{\partial u^*}{\partial y} \hat{i} - \frac{\partial u^*}{\partial x} \hat{j} \right) d\Gamma + \iint_\Omega \omega \left(-\frac{\partial u^*}{\partial y} \hat{i} + \frac{\partial u^*}{\partial x} \hat{j} \right) dx\,dy$$

from which it can be deduced that the two components of Eq. (4.11), in x and y directions, are respectively:

$$C_i\, v_x(x_i,y_i) + \oint_\Gamma v_n \frac{\partial u^*}{\partial x}\, d\Gamma = \oint_\Gamma v_t \frac{\partial u^*}{\partial y}\, d\Gamma - \iint_\Omega \omega \frac{\partial u^*}{\partial y}\, dx\,dy \tag{F.9}$$

and

$$C_i\, v_y(x_i,y_i) + \oint_\Gamma v_n \frac{\partial u^*}{\partial y}\, d\Gamma = -\oint_\Gamma v_t \frac{\partial u^*}{\partial x}\, d\Gamma + \iint_\Omega \omega \frac{\partial u^*}{\partial x}\, dx\,dy \tag{F.10}$$

Considering now Eq. (4.18), each term of this equation will be represented in terms of components. The integrand of the boundary integral on the left-hand side of Eq. (4.18) can be represented in terms of components, but as a first step the vectorial product of the term in parentheses is manipulated first. Then,

$$\vec{\omega} \wedge \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & w \\ \ell & m & 0 \end{vmatrix}$$

this results in

$$\vec{\omega} \wedge \vec{n} = \omega(-m\hat{i} + \ell\hat{j}) = \omega\,\hat{t}$$

Eq. (F.2) is now used to show that

$$(\omega \wedge \vec{n}) \wedge \nabla u^* = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega m & \omega \ell & 0 \\ \frac{\partial u^*}{\partial x} & \frac{\partial u^*}{\partial y} & 0 \end{vmatrix}$$

or

$$(\vec{\omega} \wedge \vec{n}) \wedge \nabla u^* = -\omega(\ell \frac{\partial u^*}{\partial x} + m \frac{\partial u^*}{\partial y})\vec{k} = -\omega \frac{\partial u^*}{\partial n}\vec{k} \qquad \text{(F.11)}$$

The integrand of the second contour integral on the right-hand side of Eq. (4.18) represented in terms of components is

$$\nabla u^* \wedge \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial u^*}{\partial x} & \frac{\partial u^*}{\partial y} & 0 \\ l & m & 0 \end{vmatrix}$$

or

$$\nabla u^* \wedge \vec{n} = (\frac{\partial u^*}{\partial x} m - \frac{\partial u^*}{\partial y} l)\hat{k} = -\frac{\partial u^*}{\partial t}\hat{k} \qquad \text{(F.12)}$$

Regarding the integrand of the domain integral of Eq. (4.18), as a first step we have

$$\vec{v} \wedge \omega = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 0 \\ 0 & 0 & \omega \end{vmatrix}$$

$$\vec{v} \wedge \omega = \omega\,(v_x\hat{i} - v_y\hat{j})$$

and so

$$(\vec{v} \wedge \vec{\omega}) \wedge \nabla u^* = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega v_x & -\omega v_y & 0 \\ 0 & 0 & \omega \end{vmatrix}$$

$$(\vec{v} \wedge \vec{\omega}) \wedge \nabla u^* = \omega\,(v_x \frac{\partial u^*}{\partial y} + v_y \frac{\partial u^*}{\partial x})\,\hat{k} \tag{F.13}$$

Substituting Eqs. (F.11), (F.12) and (F.13) into Eq. (4.18), gives

$$\begin{aligned} C_i\,\omega(x_i, y_i) + \oint_\Gamma \omega \frac{\partial u^*}{\partial n} d\Gamma &= -\frac{1}{\nu} \oint_\Gamma h \frac{\partial u^*}{\partial t} d\Gamma \\ &+ \frac{1}{\nu} \iint_\Omega \omega\,(v_x \frac{\partial u^*}{\partial y} + v_y \frac{\partial u^*}{\partial x})\,dx\,dy \end{aligned} \tag{F.14}$$

Appendix G

BIE for Elasticity Analysis

The equations for plane-strain elasticity analysis in terms of displacement for an isotropic homogeneous material are briefly discussed. Both differential and integral formulations are presented which are based on the derivations given in reference El-Zafrany (1993).

G.1- DIFFERENTIAL FORMULATION

The basic governing differential equations are summarized below :

Strain-displacement relationships

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned} \tag{G.1}$$

Stress-strain relationships

$$\begin{aligned} \sigma_x &= d_{11}\,\varepsilon_x + d_{12}\,\varepsilon_y \\ \sigma_y &= d_{21}\,\varepsilon_x + d_{22}\,\varepsilon_y \\ \tau_{xy} &= d_{33}\,\gamma_{xy} \end{aligned} \tag{G.2}$$

where :

$$d_{11} = d_{22} = 2\mu'(1 - \nu')/(1 - 2\nu')$$
$$d_{12} = d_{21} = 2\mu'\nu'/(1 - 2\nu') \qquad (G.3)$$
$$d_{33} = \mu'$$

Equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \qquad (G.4)$$

with the following equations, at any point on the boundary :

$$T_x = \ell\,\sigma_x + m\,\tau_{xy}$$
$$T_y = \ell\,\tau_{xy} + m\,\sigma_y \qquad (G.5)$$

where ϵ_x,ϵ_y and γ_{xy} are the strain tensor components, u and v are the displacement components, σ_x, σ_y and τ_{xy} are the stress tensor components, f_x and f_y are the body force components (force per unit of volume), T_x and T_y are the traction components on the boundary, ℓ and m are the cosine directions and μ' and ν' are the shear modulus and Poisson's ratio, respectively, which are related by

$$\mu' = \frac{E}{2(1 + \nu')}$$

where E is Young's modulus.

Substituting Eqs. (G.1) into (G.2), the stress components may be expressed in terms of displacement components. Substituting the resulting equations into the equations of equilibrium, Eq. (G.4), the system of governing equations is recast in the following elliptic partial differential equations in terms of displacement components u and v, only:

$$\nabla^2 u + \frac{1}{(1-2\nu')}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x/\mu' = 0$$
$$\nabla^2 v + \frac{}{(1-2\nu')}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y/\mu' = 0 \qquad \text{(G.6)}$$

or putting them in vectorial form,

$$\nabla^2 \vec{q} + \frac{1}{(1-2\nu')}\nabla(\nabla.\vec{q}) + \vec{F}/\mu' = 0 \qquad \text{(G.7)}$$

where $\vec{q} = u\hat{i} + v\hat{j}$ and $\vec{F} = f_x\hat{i} + f_y\hat{j}$, which are the displacement vector and the vector due to the body forces, respectively. Finally, Eq. (G.7) can be written in a form containing only Lamé's constants λ' and μ' as coefficients, which are related to the Poisson's ratio, ν', through the following expression:

$$\nu' = \frac{\lambda'}{2(\lambda' + \mu')} \qquad \text{(G.8)}$$

and so Eq. (G.7) can be represented in the following form, which is more appropriate for the present purpose:

$$\mu'\nabla^2\vec{q} + (\lambda' + \mu')\nabla(\nabla.\vec{q}) + \vec{F} = 0 \qquad \text{(G.9)}$$

The boundary conditions for this elliptic partial differential equation are either the displacement or the traction components on the boundary. The most usual boundary conditions are when the traction component in one direction has a prescribed value and so the boundary point is free to move in such a direction. On the other hand, when the displacement component in one direction is specified as zero, the boundary point is restrained in that direction.

G.2- INTEGRAL FORMULATION

The boundary integral equation corresponding to the plane-strain problem in terms of displacement is derived. This equation is normally obtained using the inverse weighted-residual expression corresponding to the differential equations based on approximate solutions which satisfy exactly any given boundary condition. In order to obtain the initial weighted-residual expression, consider a two-dimensional problem defined on a domain Ω, which has a single or multiple-connected boundary Γ. Consider also that u and v are the displacement components that satisfy the given boundary conditions exactly. Hence, a weighted-residual statement may be expressed in terms of two arbitrary weighting functions, u^* and v^*, as follows :

$$\begin{aligned}&\iint_\Omega \left[u^* \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x \right) \right. \\ &\left. + v^* \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y \right) \right] dx\,dy = 0\end{aligned} \tag{G.10}$$

where the equations of equilibrium are adopted instead of Eqs. (G.6). In this way the derivations are much simpler. Using integration-by-parts theorems and the basic equations given by Eqs. (G.1) to (G.5), and manipulating the terms, the inverse weighted-residual expression corresponding to Eq. (G.10) can be reduced to the following integral equation :

$$\begin{aligned}&\oint_\Gamma (T_x^* u + T_y^* v)\, d\Gamma + \iint_\Omega \left(f_x^* u + f_y^* v \right) dx\,dy \\ &= \oint_\Gamma (T_x u^* + T_y v^*)\, d\Gamma + \iint_\Omega \left(f_x u^* + f_y v^* \right) dx\,dy\end{aligned} \tag{G.11}$$

Provided that, for convenience the weighting functions, u^* and v^*, are the solutions of a similar problem, with the same domain but under different loading conditions. Thus, the weighting functions must satisfy the following governing differential equations:

$$\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{xy}^*}{\partial y} + f_x^* = 0$$
$$\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} + f_y^* = 0 \qquad \text{(G.12)}$$

In order to obtain the weighting functions, consider a two-dimensional elasticity problem in an infinite domain with a state of loading defined by a concentrated force acting at a point (x_i , y_i) , with a uniform distribution in the z-direction. Then, from the definition of the two-dimensional Dirac delta function, a domain distribution of the load intensity equivalent to the applied force, may be expressed as follows:

$$f_x^* = e_x \delta(x-x_i, y-y_i)$$
$$f_y^* = e_y \delta(x-x_i, y-y_i) \qquad \text{(G.13)}$$

where e_x and e_y are the x- and y-components of the applied force per unit of thickness.

Eqs. (G.6) can be used to solve this special problem defined above, whose solution is known as the fundamental solution. Representing the displacement components for this problem, $u^*$and v^*, in terms of the components, G_x and G_y, of the so-called Galerkin's vector, $\vec{G}$, by definition:

$$u^* = \nabla^2 G_x^* - \frac{1}{2(1-\nu')}\frac{\partial}{\partial x}\left(\frac{\partial G_x^*}{\partial x} + \frac{\partial G_y^*}{\partial y}\right)$$
$$\qquad \text{(G.14)}$$
$$v^* = \nabla^2 G_y^* - \frac{1}{2(1-\nu')}\frac{\partial}{\partial y}\left(\frac{\partial G_x^*}{\partial x} + \frac{\partial G_y^*}{\partial y}\right)$$

and substituting these into Eqs. (G.6) applied to this problem, along with Eqs. (G.13), a biharmonic differential equation in terms of Galerkin's vector can be obtained:

$$\nabla^4 G_x^* + e_x\,\delta(x-x_i,y-y_i)/\mu' = 0$$
$$\nabla^4 G_y^* + e_y\,\delta(x-x_i,y-y_i)/\mu' = 0 \qquad (G.15)$$

Such equations can be solved more easily with the introduction of some adequate changes of a variable in order to have a Poisson-type equation, whose solution is already known from the literature. Thus, first define the components of Galerkin's vector in terms of a function g^* through the following equations:

$$G_x^* = g^* e_x$$
$$G_y^* = g^* e_y \qquad (G.16)$$

then Eqs. (G.15) can be reduced to the following single equation:

$$\nabla^4 g^* + \delta(x-x_i,y-y_i)/\mu' = 0 \qquad (G.17)$$

Additionally, defining u^* such that:

$$\nabla^2 g^* = u^*/\mu' \qquad (G.18)$$

then Eq. (G.17) can be rewritten in terms of the following Poisson's partial differential equation:

$$\nabla^2 u^* + \delta(x-x_i,y-y_i) = 0$$

which has the following solution:

$$u^* = \frac{1}{2\pi}\left[\ln(1/r) + C_1\right]$$

Substituting this result into the Eq. (G.18), and using Gauss' Divergence theorem, it can be shown that:

$$g^* = \frac{r^2}{8\pi\mu'}\left[\ln(1/r) + C_1 + 1\right] + C_2 \tag{G.19}$$

where C_1 and C_2 are arbitrary integration constants.

Alternatively, Eqs. (G.14) can be rewritten in terms of the function g^* using the Eqs. (G.16). The final expression can then be shown as:

$$u_k^*(x-x_i,y-y_i) = G_{k1}(x-x_i,y-y_i)\,e_x + G_{k2}(x-x_i,y-y_i)\,e_y$$

where the tensorial notation is being considered and so $(u_1^*,u_2^*) \equiv (u^*,v^*)$.

The fundamental solution parameter G_{kl} is expressed as follows :

$$G_{kl}(x-x_i,y-y_i) = \nabla^2 g^*\,\delta_{kl} - \frac{1}{2(1-\nu')}\left(\frac{\partial^2 g^*}{\partial x_k\,\partial x_l}\right) \tag{G.20}$$

Finally, Eq. (G.19) is substituted into the above equation and, after some manipulation and appropriated values being assigned for the constants C_1 and C_2, -1 and 0, respectively, it is possible to obtain the following expression for the displacement fundamental solution parameter:

$$G_{kl} = \frac{1}{8\pi\mu'(1-\nu')}\left[\{(3-4\nu')\left(\ln\left(\frac{1}{r}\right) -1\right) - 0.5\}\,\delta_{kl} + \frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\right] \tag{G.21}$$

where G_{kl} and r are functions of $(x-x_i\,,\,y-y_i)$ and the indices k and l can assume the values 1 or 2 to represent the x- or y-directions, respectively, so that (x_1,x_2) means (x,y). Hence, the fundamental solution for displacement can be shown to be explicitly:

$$\begin{aligned} u^* &= G_{11}\,e_x + G_{12}\,e_y \\ v^* &= G_{21}\,e_x + G_{22}\,e_y \end{aligned} \tag{G.22}$$

where u^* and v^* are also function of $(x\text{-}x_i\ ,\ y\text{-}y_i)$. Fundamental solutions for all other variables related to stress analysis, like strain, stress and traction, can be readily obtained by substituting Eqs. (G.22) into the basic equations relating all those parameters to the displacement, presented at the beginning of this Appendix. Other fundamental solutions are derived in El-Zafrany (1993). Only the final expression for the fundamental solution corresponding to traction is given here, since along with the displacement variable, they are the main variables of interest for the purpose of the analysis presented in the main text. Hence, the traction fundamental solution is given by :

$$\begin{aligned} T_x^* &= F_{11}\, e_x + F_{12}\, e_y \\ T_y^* &= F_{21}\, e_x + F_{22}\, e_y \end{aligned} \tag{G.23}$$

where F_{11}, F_{12}, F_{21} and F_{22} are obtained from the following expression :

$$\begin{aligned} F_{kl} = -\frac{1}{4\pi(1 - \nu')r}\Bigg[& 2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l} \\ & + (1 - 2\nu')\left(\ell_l\frac{\partial r}{\partial x_k} - \ell_k\frac{\partial r}{\partial x_l} + \frac{\partial r}{\partial n}\delta_{kl}\right)\Bigg] \end{aligned} \tag{G.24}$$

In this expression ℓ_1 and ℓ_2 refer to the directon cosines ℓ and m, respectively, and all other observations apply as before for Eq. (G.21).

Finally, the governing boundary integral equation for plane-strain problem is obtained considering that the weighting functions of inverse weighted-residual expression, Eq. (G.11), to be the fundamental solutions presented above. First, as a result of this assumption, the fundamental loading parameters defined by Eq. (G.13) are substituted into Eq. (G.11), where the Dirac delta function properties given in Appendix B can be used to produce the following expression:

$$C_i u(x_i,y_i)\, e_y + C_i v(x_i,y_i)\, e_y + \oint_\Gamma (T_x^* u + T_y^* v)\, d\Gamma = \oint_\Gamma (T_x u^* + T_y v^*)\, d\Gamma + \iint_\Omega (f_x u^* + f_y v^*)\, dx\, dy \tag{G.25}$$

The constant C_i can assume the values discussed in Appendix B. Now, substituting the fundamental displacements, Eqs. (G.22), and the fundamental traction, Eqs. (G.23), into the above equation, and also taking in consideration that the fact that the values e_x and e_y are arbitrary, the following boundary integral equations result, which are defined with respect to the source point (x_i,y_i):

$$C_i u(x_i,y_i) + \oint_\Gamma (F_{11} u + F_{21} v)\, d\Gamma = \oint_\Gamma (G_{11} T_x + G_{21} T_y)\, d\Gamma + U(x_i,y_i) \tag{G.26}$$

and

$$C_i v(x_i,y_i) + \oint_\Gamma (F_{12} u + F_{22} v)\, d\Gamma = \oint_\Gamma (G_{12} T_x + G_{22} T_y)\, d\Gamma + V(x_i,y_i) \tag{G.27}$$

where:

$$U(x_i,y_i) = \iint_\Omega (G_{11} f_x + G_{21} f_y)\, dx\, dy \tag{G.28}$$

and

$$V(x_i,y_i) = \iint_\Omega (G_{12} f_x + G_{22} f_y)\, dx\, dy \tag{G.29}$$

which represent domain load terms. All fundamental solution parameters, G_{kl} and F_{kl}, in these integrals are functions of $(x-x_i\ ,\ y-y_i)$. Eqs. (G.26) and (G.27) can be rewritten in a compact form using tensorial representation, as follows:

$$C_i\, q_l(x_i,y_i) + \oint_\Gamma q_k F_{kl}\, d\Gamma = \oint_\Gamma T_k\, G_{kl}\, d\Gamma + Q_l(x_i,y_i) \qquad \text{(G.30)}$$

where the subscript "i" refers to the source point while the other indices refer to the tensorial notation and so $k,l = 1,2$. Also, q_k refers to the displacement components and Q_l refers to the body force terms, so that $(q_1,q_2) \equiv (u,v)$ and $(Q_1,Q_2) \equiv (U,V)$.

The derivatives of displacement components are necessary, for example, for the calculation of strain and for stress analysis purposes. Boundary integral equations to calculate these parameters can easily be obtained by derivation of Eq. (G.30) with respect to the source point coordinates x_i and y_i, in the same way adopted in Chapter 3. These equations, however, are appropriate to obtain the derivatives of displacement only at internal points of the domain, since for points on the boundary they are expected to deliver inaccurate results due to the presence of boundary integrals with strong a singularity. On the boundary, a different approach has to be adopted normally using the interpolating functions of the discretization process. This will not be presented in this Appendix. The final boundary integral equations for derivatives of displacement, useful only for internal points calculations, are :

$$\frac{\partial q_k}{\partial x_m}(x_i,y_i) + \oint_\Gamma q_k \frac{\partial F_{kl}}{\partial x_m}\, d\Gamma = \oint_\Gamma T_k \frac{\partial G_{kl}}{\partial x_m}\, d\Gamma + \frac{\partial Q_l}{\partial x_m}(x_i,y_i) \qquad \text{(G.31)}$$

where:

$$\frac{\partial Q_l}{\partial x_m}(x_i,y_i) = \iint_\Omega f_k \frac{\partial G_{kl}}{\partial x_m} dx\,dy$$

$$\frac{\partial G_{kl}}{\partial x_m} = \frac{1}{8\pi\mu'(1-v')r}\left[2\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} + (3-4v')\frac{\partial r}{\partial x_m}\delta_{kl} - \frac{\partial r}{\partial x_k}\delta_{lm} - \frac{\partial r}{\partial x_l}\delta_{mk}\right] \tag{G.32}$$

and

$$\begin{aligned}\frac{\partial F_{kl}}{\partial x_m} = \frac{-1}{4\pi(1-v')r^2}\Bigg[& 2\frac{\partial r}{\partial n}\{4\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} + (1-2v')\frac{\partial r}{\partial x_m}\delta_{kl} \\ & - \frac{\partial r}{\partial x_k}\delta_{lm} - \frac{\partial r}{\partial x_l}\delta_{km}\} \\ & + 2\{(1-2v')(\ell_k\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} - \ell_l\frac{\partial r}{\partial x_m}\frac{\partial r}{\partial x_k}) - \ell_m\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_l}\} \\ & - (1-2v')(\ell_k\delta_{lm} - \ell_l\delta_{mk} + \ell_m\delta_{kl})\Bigg]\end{aligned} \tag{G.33}$$

Note that the coefficient C_i is not included in Eq. (G.31) since for internal nodes it assumes a value 1.0. See the information presented in Appendix B.

Eqs. (G.32) and (G.33) were obtained using the expression given by Eq. (4.63) of El-Zafrany (1993). Also, in these equations, the derivatives of any functions with respect to the source point coordinates, x_i and y_i, are replaced by the derivative with respect to the field point coordinate, using the expressions given by Eqs. (3.14) and (3.15) in Chapter 3.

Appendix H

Isoparametric Elements

H.1- BOUNDARY ELEMENTS

The boundary Γ of problem is divided in many pieces Γ_e according to the discretization process discussed in the main text. Each piece of boundary element is defined in terms of n_e nodes, which in the case of the so-called isoparametric element adopted in this work, are both the geometrical points and the field-function nodes of the boundary element. The use of the global coordinates (x,y) to describe the boundary element is not convenient for the sake of generality of the interpolation functions. Therefore, an intrinsic parameter ξ is defined, such that at the j^{th} local node, with the nodes at equal distanced, the coordinate gives:

$$\xi_j = \frac{j-1}{n_e-1}$$

where this expression is applied to a domain for the intrinsic coordinates defined within [0,1]. Note that although in the literature the domain [-1,1] is normally adopted, the domain [0,1] is adopted throughout this work following the suggestion of El-Zafrany (1993).

The use of this intrinsic coordinate makes the analysis much simple, since the same representation for the interpolation functions can be adopted in every boundary element, independent of the nodal cartesian coordinate.

The field point (x,y), which is any point moving on Γ_e, may be expressed in terms of ξ by means of the following Lagrangian interpolation equations:

$$x(\xi) = \sum_{j=1}^{n_e} x_j N_j(\xi) \tag{H.1}$$

and

$$y(\xi) = \sum_{j=1}^{n_e} y_j N_j(\xi) \tag{H.2}$$

where :

$$N_j(\xi) = \prod_{\substack{r=1 \\ r \neq j}}^{n_e} \frac{(n_e-1)\xi - (r-1)}{j-r}$$

and is the Lagrangian interpolation function adopted in this work.

From the definition of an isoparametric element, any field functions parameter, say θ, can be approximated in terms of its nodal values at the same nodes used to define geometrically the boundary element. Therefore,

$$\theta = \sum_{j=1}^{n_e} \theta_j N_j(\xi)$$

as given in the main text.

The infinitesimal length $d\Gamma$ of the boundary element may be expressed as follows:

$$d\Gamma = J(\xi)\, d\xi$$

where $J(\xi)$ is the Jacobian of this transformations given by:

$$J(\xi) = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

and from Eqs. (H.1) and (H.2) it can be shown that:

$$\frac{dx}{d\xi}(\xi) = \sum_{j=1}^{n_e} x_j \frac{dN_j}{d\xi}(\xi)$$

and

$$\frac{dy}{d\xi}(\xi) = \sum_{j=1}^{n_e} y_j \frac{dN_j}{d\xi}(\xi)$$

where:

$$\frac{dN_j}{d\xi}(\xi) = \sum_{\substack{r=1 \\ r\neq j}}^{n_e} \left(\frac{n_e-1}{j-r}\right) \prod_{\substack{s=1 \\ s\neq r \\ s\neq j}}^{n_e} \frac{(n_e-1)\,\xi-(s-1)}{j-s}$$

Finally, the direction cosines ℓ and m can be evaluated at any point over the boundary element by the following expressions:

$$\ell = \frac{dy}{d\Gamma} = \frac{dy}{d\xi}\frac{d\xi}{d\Gamma} = \frac{dy/d\xi}{J(\xi)}$$

and

$$m = -\frac{dx}{d\Gamma} = -\frac{dx}{d\xi}\frac{d\xi}{d\Gamma} = -\frac{dx/d\xi}{J(\xi)}$$

In this work, only two-node linear elements and three-node quadratic elements are used in the discretization of the boundary, as discussed in the main text. For linear elements the interpolation functions are given by:

$$N_1 = 1-\xi$$

and

$$N_2 = \xi$$

For the quradratic element, the interpolation functions are given by:

$$N_1 = (1-\xi)(1-2\xi)$$

$$N_2 = 4\xi(1-\xi)$$

and

$$N_3 = \xi(2\xi-1)$$

H.2- DOMAIN ELEMENTS

The domain is also divided into many small elements or cells Ω_c , each one defined in terms of n_c geometrical points. A similar technique is used to define the domain elements or cells in terms of a local system of coordinates, instead of using the global system of coordinates (x,y). Using this local system of reference, the interpolation function can be defined in terms of ξ and η coordinates, called the intrinsic coordinates of this system. In this work, such coordinates are defined in the domain [0,1]. Thus, a point given in terms of global coordinates (x,y) are defined as:

$$x = \sum_{j=1}^{n_c} x_j N_j(\xi,\eta)$$

and

$$y = \sum_{j=1}^{n_c} y_j N_j(\xi,\eta)$$

where $N_j(\xi,\eta)$ are the interpolation function. If an isoparametric cell is adopted, any field function of the problem can be represented by a similar expression given above using the same interpolation functions.

An infinitesimal area element (dx dy) for the cell may be expressed as follows:

$$dx\,dy = J(\xi,\eta)\, d\xi\, d\eta$$

where $J(\xi,\eta)$ is the Jacobian of this transformation, given by:

$$J(\xi,\eta) = \begin{vmatrix} \dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}$$

where:

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^{n_c} x_j \frac{\partial N_j}{\partial \xi}(\xi,\eta)$$

$$\frac{\partial y}{\partial \xi} = \sum_{j=1}^{n_c} y_j \frac{\partial N_j}{\partial \xi}(\xi,\eta)$$

$$\frac{\partial x}{\partial \eta} = \sum_{j=1}^{n_c} x_j \frac{\partial N_j}{\partial \eta}(\xi,\eta)$$

and

$$\frac{\partial y}{\partial \eta} = \sum_{j=1}^{n_c} y_j \frac{\partial N_j}{\partial \eta}(\xi,\eta)$$

In this work, three-node triangular and eight-node quadrangular cells were adopted in the discretization of the domain. The interpolation functions for the triangular cell are given by:

$$N_1 = 1-\xi-\eta$$

$$N_2 = \xi$$

and

$$N_3 = \eta$$

For the eight-node quadrangular cell, also called the Serendipty element, the interpolation functions are given by:

$$N_1 = (1-\xi)(1-\eta)(1-2\xi-2\eta)$$

$$N_2 = 4\xi(1-\xi)(1-\eta)$$

$$N_3 = \xi(1-\eta)(-1+2\xi-2\eta)$$

$$N_4 = 4\xi\eta(1-\eta)$$

$$N_5 = \xi\eta(-3+2\xi+2\eta)$$

$$N_6 = 4\xi\eta(1-\xi)$$

$$N_7 = (1-\xi)\eta(-1-2\xi+2\eta)$$

and

$$N_8 = 4(1-\xi)\eta(1-\eta)$$

Appendix I

Singular Integrations

The techniques used to overcome the singular kernels of the domain and boundary integrals appearing in the formulations discussed in the main text are presented here. These basically follow the suggestions given in El-Zafrany(1993).

I.1- BOUNDARY INTEGRALS

The only singular kernel appearing in the boundary integral that requires special treatment is the one that contains the logarithmic function, as given by the following general expression:

$$I = \int_0^1 F_1 \ln\left(\frac{1}{r}\right) d\xi \qquad (I.1)$$

where F_1 is a function that includes the other parameters appearing in the integrand, such as, Jacobian, interpolation function and the variable of the problem. The parameter r is the distance of the source point to the field point given as:

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2}$$

The integral above represents the integration over a boundary element, where ξ is the intrinsic variable with reference to that element. The singularity occurs when the boundary element contains the source point, and so, when the source point is approached, r tends to zero and the integral becomes singular.

Let us suppose that the source point, and thus the singularity, is placed at the j^{th} node (local numbering) of a boundary element, shown in Figure I.1. That node splits the boundary element into two parts, where the integral will be performed. Hence, considering that the limits of integration vary from the left to the right of that element, the integral given above is integrated from $\xi = 0$ to $\xi = \xi_j$ on part I and from $\xi = \xi_j$ to $\xi = 1$ on part II. Now, in order to remove the singularity, a suitable change of variable will be introduced in each of these parts. Thus, on part I, if

$$\frac{1}{r} = \left(\frac{\xi_j}{\xi_j - \xi}\right)\left(\frac{\xi_j - \xi}{\xi_j r}\right)$$

results in

$$\ln\left(\frac{1}{r}\right) = \ln\left(\frac{\xi_j}{\xi_j - \xi}\right) + \ln\left(\frac{\xi_j - \xi}{\xi_j r}\right)$$

Thus, the integration on part I contains two terms as follows:

$$\int_0^{\xi_j} F_1 \ln\left(\frac{1}{r}\right) d\xi = \int_0^{\xi_j} \left[F_1 \ln\left(\frac{\xi_j}{\xi_j - \xi}\right) + F_1 \ln\left(\frac{\xi_j - \xi}{\xi_j r}\right)\right] d\xi$$

Let a new variable be introduced in the first term on the right-hand side of the integral above, defined by:

$$\phi_1 = \frac{\xi_j - \xi}{\xi_j}$$

where the new limits of integration corresponding to this variable are introduced. The integration in terms of this new variable is performed from 0 to 1, in place of limits 0 to ξ_j, respectively, of the former variable of integration. Note also that

$$d\phi_1 = -\frac{d\xi}{\xi_j}$$

Hence,

$$\int_0^{\xi_j} F_1 \ln\left(\frac{1}{r}\right) d\xi = \int_0^{1} F_1 \ln\left(\frac{1}{\phi_1}\right) \xi_j \, d\phi_1 + \int_0^{\xi_j} F_1 \ln\left(\frac{1}{\rho_1}\right) d\xi \qquad \text{(I.2)}$$

where:

$$\rho_1 = \frac{\xi_j r}{\xi_j - \xi} = \frac{r}{\phi_1}$$

A similar procedure is introduced in the integral corresponding to the part II.

There,

$$\frac{1}{r} = \left(\frac{1 - \xi_j}{\xi - \xi_j}\right)\left(\frac{\xi - \xi_j}{(1 - \xi_j)\, r}\right)$$

and as a result

$$\int_{\xi_j}^{1} F_1 \ln\left(\frac{1}{r}\right) d\xi = \int_{\xi_j}^{1} \left[F_1 \ln\left(\frac{1 - \xi_j}{\xi - \xi_j}\right) + F_1 \ln\left(\frac{\xi - \xi_j}{(1 - \xi_j)\, r}\right)\right] d\xi$$

A new variable defined by

$$\phi_2 = \frac{\xi - \xi_j}{1 - \xi_j}$$

is introduced in the first term on the right-hand side of the previous integral, where new limits of integration have to be introduced, ie. from 0 to 1, to replace the previous ones, ξ_j to 1, respectively. Note also, that

$$d\phi_2 = \frac{d\xi}{(1 - \xi_j)}$$

Hence, the new integral is given as follows:

$$\int_{\xi_j}^{1} F_1 \ln\left(\frac{1}{r}\right) d\xi = \int_{0}^{1} F_1 \ln\left(\frac{1}{\phi_2}\right) (1 - \xi_j)\, d\phi_2 + \int_{\xi_j}^{1} F_1 \ln\left(\frac{1}{\rho_2}\right) d\xi \quad (I.3)$$

where:

$$\rho_2 = \frac{(1 - \xi_j)\, r}{\xi - \xi_j} = \frac{r}{\phi_2}$$

Substituting Eqs. (I.2) and (I.3) into Eq. (I.1) results:

$$I = I_0 + I_1 + I_2$$

where:

$$I_0 = \int_0^{\xi_j} F_1 \ln\left(\frac{1}{\rho_1}\right) d\xi + \int_{\xi_j}^{1} F_1 \ln\left(\frac{1}{\rho_2}\right) d\xi$$

$$I_1 = \int_0^{1} F_1 \ln\left(\frac{1}{\phi_1}\right) \xi_j \, d\phi_1$$

and

$$I_2 = \int_0^{1} F_1 \ln\left(\frac{1}{\phi_2}\right) (1 - \xi_j)\, d\phi_2$$

At this point, it is possible to see that the integrals I_1 and I_2 can be evaluated using the singular quadrature available in the literature, which applies to integrals of type:

$$\int_0^1 F(\phi) \ln\left(\frac{1}{\phi}\right) d\phi = \sum_{s=1}^{N_s} F(\phi_s) w_s$$

where N_s is the number of quadrature points and ϕ_s and w_s are the coordinates of the quadrature points and the corresponding weight, respectively. Note, that is this work, the modified quadrature for this case, given in El-Zafrany (1993), was adopted.

When the source point is located at the extreme nodes of the boundary element either I_1 or I_2 is calculated. For $j=1$, $I_1 = 0$ and only the integration on part II is necessary. On the other hand, for $j=n$, (last node), $I_2 = 0$, only the integration on part I is necessary.

With regard to the remaining integral, I_0, it is more convenient to write its expression in a more compact form, as follows:

$$I_0 = \int_0^1 F_1 \ln\left(\frac{1}{\rho}\right) d\xi \qquad (I.4)$$

where ρ assumes the following values depending on the position of the points of integrations with relation to the source point:

$$\rho = \frac{\xi_j r}{\xi_j - \xi} \qquad \text{for } \xi \langle \xi_j$$

$$\rho = \frac{(1 - \rho_j) r}{\xi - \xi_j} \qquad \text{for } \xi \rangle \xi_j$$

The integral I_0 can be performed using the usual Guassian quadrature, since the singularity is avoided. However, it is valuable to note the fact that it is recommended to avoid any integration point lying at the source point. In the quadratic type of boundary element, for example, a number even of integration points is suggested to avoid such problem.

I.2- DOMAIN INTEGRAL

The tecnhiques adopted in this work to overcome the singularity of the domain integrals are based on an idea that can be applied to a variety of singular kernels. To explain how it works, let us consider the following general integral, whose kernel contains the derivative of the fundamental solution, say P^*, and represents most of the cases analysed in this work:

$$I = \iint_{\Omega} F \frac{\partial P^*}{\partial n_o} \, dx\, dy \tag{I.5}$$

where $F \equiv f(x,y)$ and represents the field variable. The kernel of this integral is supposed to be singular at the source point, whilst F is regular throughtout the domain Ω. The technique adopted to deal with this singularity is based on a very simple idea. It consists of the inclusion of two terms with opposite signs in the integral above with the aim of making the domain integral regular. This is represented by the following expression:

$$I = \iint_{\Omega} (F - F_i) \frac{\partial P^*}{\partial n_o} \, dx\, dy + F_i \iint_{\Omega} \frac{\partial P^*}{\partial n_o} \, dx\, dy \tag{I.6}$$

where F_i is a constant which is equal to the value of the function F at the source point. In this way, when the source point is approached, while the kernel goes to infinity the term ($F - F_i$) tends to zero and the integral becomes finite. This measure, roughly, allows the domain integral in question to be treated as regular, where the usual quadrature formulae can be used to evaluate it. This technique appears in many books on numerical analysis. It is worth mentioning that the singularity is not removed, in fact, it is weakened since the second derivative of the kernel in this case is still singular. It can however, be re-applied again and again in order to reduce the strengh of the singularity involved, although this is not undertaken here.

Alternatively, the remaining domain integral is singular and a special treatment is required. The way devised in this work to deal with this case is to transform the domain integral into a boundary integral, using some identities in which the singularity can be suitably treated. Thus, first expanding the kernel of the domain integral in question,

$$F_i \iint_\Omega \frac{\partial P^*}{\partial n_o}\, dx\, dy = F_i \iint_\Omega \left(\ell_o \frac{\partial P^*}{\partial x} + m_o \frac{\partial P^*}{\partial y} \right) dx\, dy$$

and using the basic theorem for the reduction of the double integral given in Appendix B, the integral above can be transformed into:

$$F_i \iint_\Omega \frac{\partial P^*}{\partial n_o}\, dx\, dy = F_i \ell_o \oint_\Gamma \ell P^* d\Gamma + F_i m_o \oint_\Gamma m P^* d\Gamma$$

or

$$F_i \iint_\Omega \frac{\partial P^*}{\partial n_o}\, dx\, dy = F_i \oint_\Gamma (\ell_o \ell + m_o m)\, P^*\, d\Gamma$$

and finally,

$$F_i \iint_\Omega \frac{\partial P^*}{\partial n_o}\, dx\, dy = F_i \oint_\Gamma (\vec{n}_o \cdot \vec{n})\, P^*\, d\Gamma \tag{I.7}$$

Note that the domain integrals resulting from the use of integration-by-parts theorem to obtain the Eq. (I.7) above, vanish since the derivatives of F_i are zero.

Hence, placing Eq. (I.7) into Eq. (I.6), the following expression results and is used in place of Eq. (I.5),

$$I = \iint_\Omega (F - F_i) \frac{\partial P^*}{\partial n_o}\, dx\, dy + F_i \oint_\Gamma (\vec{n}_o \cdot \vec{n})\, P^*\, d\Gamma \tag{I.8}$$

where the domain integral can be treated as regular and the boundary integral, if singular, can be evaluated, in most cases, using available techniques.

Some kernels appearing in the formulations investigated in this work which can be treated by the formula represented by Eq. (I.8), are given:

$$\iint_\Omega F \frac{\partial u^*}{\partial x}\, dx\, dy = \iint_\Omega (F - F_i) \frac{\partial u^*}{\partial x}\, dx\, dy + F_i \oint_\Gamma \ell u^*\, d\Gamma$$

and

$$\iint_\Omega F \frac{\partial u^*}{\partial y} \, dx\,dy = \iint_\Omega (F - F_i) \frac{\partial u^*}{\partial y} \, dx\,dy + F_i \oint_\Gamma m u^* \, d\Gamma$$

for kernels given by the derivatives of the fundamental solution in the x- and y-directions. Kernels containing the second derivatives of the fundamental solution produce the following expressions:

$$\iint_\Omega F \frac{\partial^2 u^*}{\partial x^2} \, dx\,dy = \iint_\Omega (F - F_i) \frac{\partial^2 u^*}{\partial x^2} \, dx\,dy + F_i \oint_\Gamma \ell \frac{\partial u^*}{\partial x} \, d\Gamma$$

$$\iint_\Omega F \frac{\partial^2 u^*}{\partial y^2} \, dx\,dy = \iint_\Omega (F - F_i) \frac{\partial^2 u^*}{\partial y^2} \, dx\,dy + F_i \oint_\Gamma m \frac{\partial u^*}{\partial y} \, d\Gamma$$

and

$$\iint_\Omega F \frac{\partial^2 u^*}{\partial x \partial y} \, dx\,dy = \iint_\Omega (F - F_i) \frac{\partial^2 u^*}{\partial x \partial y} \, dx\,dy + \frac{F_i}{2} \oint_\Gamma \left(\ell \frac{\partial u^*}{\partial y} + m \frac{\partial u^*}{\partial x} \right) d\Gamma$$

where u^* is the fundamental solution as defined by Eq. (3.9) in the main text. Note, that in the last expression, an average of the integrand of the boundary integral was adopted to reduce any inaccuracy due to the change of the order of derivation of the fundamental solution.

I.2.1- Special Cases

Kernels that do not include the derivative of the fundamental solution cannot be dealt with using Eq. (I.8). The problem is that the boundary integral cannot be obtained in the way given above, and so an alternative approach has to be used.

The integral given below is included in this category:

$$I = \iint_{\Omega} F u^* \, dx dy$$

and as before

$$\iint_{\Omega} F u^* \, dx dy = \iint_{\Omega} (F - F_i) u^* \, dx dy + F_i \iint_{\Omega} u^* \, dx dy$$

The technique used by El-Zafrany (1993) to reduce the domain integral, like the second one on the right-hand side of the above equation, to the boundary integral has been adopted in this work. This technique, in brief, consists of defining an auxiliary function $f^*(r)$ such that

$$\nabla^2 f^* = u^*$$

where u^* is the fundamental solution given by Eq. (3.9) presented in the main text. The function f^* can be found by solving the above equation and is given by:

$$f^*(r) = -\frac{r^2}{8\pi} (\ln r - 1) \tag{I.9}$$

Introducing Eq. (I.9) into the domain integral in question and, after integration by parts, gives:

$$\iint_{\Omega} F_i u^* \, dx dy = \iint_{\Omega} F_i \nabla^2 f^* \, dx dy = F_i \oint_{\Gamma} \frac{\partial f^*}{\partial n} \, d\Gamma$$

since F_i is a constant, which forces the other terms to vanish, and where

$$\frac{\partial f^*}{\partial n} = - \frac{r}{8\pi} (2\ln r - 1) \frac{\partial r}{\partial n}$$

The final expression for this case is, therefore:

$$I = \iint_{\Omega} (F - F_i) u^* dxdy - \frac{F_i}{8\pi} \oint_{\Gamma} r (2\ln r - 1) \frac{\partial r}{\partial n} d\Gamma$$

Note that the integrand of the boundary integral contains singularities. Fortunately, the only case that demands the use of a special technique is the one which contains the logarithmic function. This can however, be overcome using the technique discussed in Section I.1 presented previously.

Domain integrals similar to the previous case, where the kernel is given by the fundamental solution itself, appear in the formulation based on the penalty function as discussed in Chapter 5. In this case, however, due to the more complicated form of the fundamental solution G_{kl}, another technique to reduce the domain integral into a boundary integral has to be devised. This problem can be easy to handle using the integration by parts theorem if the fundamental solution is represented by Eq. (G.20), instead of Eq. (5.8). Thus,

$$\iint_{\Omega} F G_{kl} dxdy = \iint_{\Omega} (F - F_i) G_{kl} dxdy + F_i \iint_{\Omega} G_{kl} dxdy \tag{I.10}$$

where the fundamental solution G_{kl} (k, l =1,2) is represented by eq. (G.20), given by:

$$G_{kl} = \nabla^2 g^* \delta_{kl} - \frac{1}{2(1-\bar{v})} \frac{\partial^2 g^*}{\partial x_k \partial x_l}$$

and

$$g^* = - \frac{r^2}{8\pi v} \ln r$$

obtained from Eq. (G.19) after assuming that $C_1 = -1$ and $C_2 = 0$.

Now, introducing this into the second domain integral on the right-hand side of the expression given by Eq. (I.10) and using the integration by parts theorem, gives:

$$\iint_\Omega F G_{kl} \, dx dy = \iint_\Omega (F - F_i) G_{kl} \, dx dy$$
$$+ F_i \oint_\Gamma \left[\frac{\partial g^*}{\partial n} \delta_{kl} - \frac{1}{2(1-\bar{v})} \ell_k \frac{\partial g^*}{\partial x_l} \right] d\Gamma$$

where ℓ_k represent the direction cosines. Note that the other terms resulting from the transformation vanish because F_i is a constant. The terms in the boundary integral above can be expanded further as follows:

$$\frac{\partial g^*}{\partial n} = - \frac{r(2\ln r + 1)}{8\pi v} \frac{\partial r}{\partial n}$$

and

$$\frac{\partial g^*}{\partial x_l} = - \frac{r(2\ln r + 1)}{8\pi v} \frac{\partial r}{\partial x_l}$$

The final expression for each combination of the indeces k and l are given by:

$$\iint_{\Omega} F G_{11}\, dx\, dy = \iint_{\Omega} (F - F_i)\, G_{11}\, dx\, dy$$
$$+ F_i \oint_{\Gamma} \left[\frac{\partial g^*}{\partial n} - \frac{1}{2(1-\bar{\nu})}\, \ell \frac{\partial g^*}{\partial x} \right] d\Gamma$$

$$\iint_{\Omega} F G_{12}\, dx\, dy = \iint_{\Omega} (F - F_i)\, G_{12}\, dx\, dy + F_i \oint_{\Gamma} \left[- \frac{1}{2(1-\bar{\nu})}\, \ell \frac{\partial g^*}{\partial y} \right] d\Gamma$$

$$\iint_{\Omega} F G_{21}\, dx\, dy = \iint_{\Omega} (F - F_i)\, G_{21}\, dx\, dy + F_i \oint_{\Gamma} \left[- \frac{1}{2(1-\bar{\nu})}\, m \frac{\partial g^*}{\partial x} \right] d\Gamma$$

and

$$\iint_{\Omega} F G_{22}\, dx\, dy = \iint_{\Omega} (F - F_i)\, G_{22}\, dx\, dy$$
$$+ F_i \oint_{\Gamma} \left[\frac{\partial g^*}{\partial n} - \frac{1}{2(1-\bar{\nu})}\, m \frac{\partial g^*}{\partial y} \right] d\Gamma$$

Note that due to the presence of the logarithmic function in the integrand of the above boundary integrals, these are singular. However, the problem can be overcome using the technique discussed in Section I.1.

Appendix J

Calculation of C_i Coefficients

J.1- INTRODUCTION

When the double-node approach is adopted to treat corners, a geometrical discontinuity is introduced on the boundary, since a finite gap exists separating the two nodes. For the sake of accuracy, a correction should be introduced in the BIE of the problem to compensate for the error provoked by the gap.This correction is normally introduced in the evaluation of the coefficient C_i. When the variable in the BIE is considered a scalar, the approach given by Eq. (6.23) may be adopted. However, a special treatment is required when the variable of the problem contains two or more components.

One possibility is to obtain the coefficient using an approach that considers a smooth element connecting the two corner nodes in order to close the gap. Boundary integrals corresponding to the integration from one node to another on this boundary element is introduced in the BIE. The correction is obtained by calculating the limit of the integral when the gap tends to zero. All additional terms introduced in the original BIE as a result of the integration on the element at the corner are considered to be the new coefficient C_i. This should be used in place of the suggestion given in Appendix B. When the source point is away from the corner, these integrations vanish. They only participate when the source point is located at the corner.

J.2- ELASTICITY PROBLEM

In this work, it was decided to adopt an approach that can be considered as an extension to the concept of a rigid translation for problems whose dependent variables have two components. This approach is discussed by El-Zafrany (1989b) applied to the BIE of elasticity. Here, the main steps considering the application of the approach will be presented as applied also to the equations of elasticity discussed in Appendix G.

Consider the BIEs given by Eqs. (G.26) and (G.27) of Appendix G. If the structure analyzed is moved in rigid translations α and β, in the x-direction and in the y-direction, respectively, then there is no loading and no stress. In other words, the following conditions are applied in the equations:

$$\begin{aligned} T_x &= 0 \\ T_y &= 0 \\ U(x_i, y_i) &= 0 \\ V(x_i, y_i) &= 0 \\ u &= \alpha \\ v &= \beta \end{aligned}$$

Substituting these conditions into Eqs. (G.26) and (G.27), these can be rewritten as follows:

$$C_i\, \alpha + \alpha \oint_{\Gamma} F_{11}\, d\Gamma + \beta \oint_{\Gamma} F_{21}\, d\Gamma = 0$$

and

$$C_i \beta + \alpha \oint_{\Gamma} F_{12}\, d\Gamma + \beta \oint_{\Gamma} F_{22}\, d\Gamma = 0$$

For the special case of $\alpha = u_i$ and $\beta = v_i$ the above equations are rewritten as follows:

$$C_i u_i + \oint_{\Gamma} F_{11} u_i\, d\Gamma + \oint_{\Gamma} F_{21} v_i\, d\Gamma = 0 \qquad (J.1)$$

and

$$C_i v_i + \oint_{\Gamma} F_{12} u_i\, d\Gamma + \oint_{\Gamma} F_{22} v_i\, d\Gamma = 0 \qquad (J.2)$$

Subtracting Eq. (J.1) from Eq. (G.26) and Eq. (J.2) from Eq. (G.27), the boundary integral equations can be rewritten as follows:

$$\oint_{\Gamma} F_{11} (u - u_i)\, d\Gamma + \oint_{\Gamma} F_{21} (v - v_i)\, d\Gamma$$
$$= \oint_{\Gamma} G_{11} T_x\, d\Gamma + \oint_{\Gamma} G_{21} T_y\, d\Gamma + U(x_i, y_i)$$

and

$$\oint_{\Gamma} F_{12} (u - u_i)\, d\Gamma + \oint_{\Gamma} F_{22} (v - v_i)\, d\Gamma$$
$$= \oint_{\Gamma} G_{12} T_x\, d\Gamma + \oint_{\Gamma} G_{22} T_y\, d\Gamma + V(x_i, y_i)$$

The above equations should be applied to a problem with a continuous boundary Γ. However, when the double-node technique is used to treat corners, in order to correct the error introduced by the gap between the two corner nodes, the above equations should contain a correction, as follows, valid for one corner with two nodes "a" and "b":

$$\begin{aligned}&\oint_{\Gamma'} F_{11}(u-u_i)\,d\Gamma + \lim_{a\to b}\oint_a^b F_{11}(u-u_i)\,d\Gamma \\ &+ \oint_{\Gamma'} F_{21}(v-v_i)\,d\Gamma + \lim_{a\to b}\oint_a^b F_{21}(v-v_i)\,d\Gamma \\ &\quad = \oint_{\Gamma'} G_{11}T_x\,d\Gamma + \oint_{\Gamma'} G_{21}T_y\,d\Gamma + U(x_i,y_i)\end{aligned} \tag{J.3}$$

and

$$\begin{aligned}&\oint_{\Gamma'} F_{12}(u-u_i)\,d\Gamma + \lim_{a\to b}\oint_a^b F_{12}(u-u_i)\,d\Gamma \\ &+ \oint_{\Gamma'} F_{22}(v-v_i)\,d\Gamma + \lim_{a\to b}\oint_a^b F_{22}(v-v_i)\,d\Gamma \\ &\quad = \oint_{\Gamma'} G_{12}T_x\,d\Gamma + \oint_{\Gamma'} G_{22}T_y\,d\Gamma + V(x_i,y_i)\end{aligned} \tag{J.4}$$

where the corrections on the traction terms were not introduced because it can shown that:

$$\lim_{a\to b}\oint_a^b G_{kl}T_k\,d\Gamma = 0$$

and the boundary Γ is the sum of Γ' to the gap separating the corner nodes "a" and "b".

Considering a corner node (x_c,y_c), it can be also shown that:

$$\lim_{a \to b} \oint_a^b F_{kl} (u - u_i) \, d\Gamma = (u_c - u_i) \, \overline{F}_{kl} (x_i, y_i)$$

and

$$\lim_{a \to b} \oint_a^b F_{kl} (v - v_i) \, d\Gamma = (v_c - v_i) \, \overline{F}_{kl} (x_i, y_i)$$

where:

$$\overline{F}_{kl} = \lim_{a \to b} \oint_a^b F_{kl} \, d\Gamma$$

are the so-called "jump functions" due to the fact that they are non-zero only when the source point is at the corner, otherwise they are zero. In other words, they are effective only at corner nodes. Thus,

$$\overline{F}_{kl} (x_i, y_i) = 0 \quad \text{for} \quad (x_i, y_i) \neq (x_c, y_c)$$

and

$$\overline{F}_{kl} (x_i, y_i) \neq 0 \quad \text{for} \quad (x_i, y_i) \equiv (x_c, y_c)$$

However, when $(x_i,y_i) = (x_c,y_c)$, the additional conditions are introduced

$$(u_c - u_i) = 0;$$
$$(v_c - v_i) = 0$$

Hence, for any source point (x_i,y_i), it can be deduced that:

$$\lim_{a \to b} \oint_a^b F_{kl} (u - u_i) \, d\Gamma = 0$$

and

$$\lim_{a \to b} \oint_a^b F_{kl} (v - v_i) \, d\Gamma = 0$$

and so, Eqs. (J.3) and (J.4) can be rewritten as follows:

$$\oint_{\Gamma'} F_{11} (u - u_i) \, d\Gamma + \oint_{\Gamma'} F_{21} (v - v_i) \, d\Gamma$$
$$= \oint_{\Gamma'} G_{11} T_x \, d\Gamma + \oint_{\Gamma'} G_{21} T_y \, d\Gamma + U(x_i, y_i)$$

and

$$\oint_{\Gamma'} F_{12} (u - u_i) \, d\Gamma + \oint_{\Gamma'} F_{22} (v - v_i) \, d\Gamma$$
$$= \oint_{\Gamma'} G_{12} T_x \, d\Gamma + \oint_{\Gamma'} G_{22} T_y \, d\Gamma + V(x_i, y_i)$$

which are free from corner effects. In other words, the corner can be left as a finite gap in the boundary element mesh. Note, that this analysis is still valid for cases where the gap length tends to zero.

Finally, defining the following factors at the source point:

$$C_{kl}(x_i, y_i) = - \oint_{\Gamma'} F_{kl}(x - x_i, y - y_i) \, d\Gamma$$

then it can be deduced that:

$$C_{11}(x_i,y_i)\,u_i + C_{12}(x_i,y_i)\,v_i + \oint_{\Gamma'} F_{11}\,u\,d\Gamma + \oint_{\Gamma'} F_{21}\,v\,d\Gamma = \oint_{\Gamma'} G_{11}\,T_x\,d\Gamma + \oint_{\Gamma'} G_{21}\,T_y\,d\Gamma + U(x_i,y_i)$$

and

$$C_{21}(x_i,y_i)\,u_i + C_{22}(x_i,y_i)\,v_i + \oint_{\Gamma'} F_{12}\,u\,d\Gamma + \oint_{\Gamma'} F_{22}\,v\,d\Gamma = \oint_{\Gamma'} G_{12}\,T_x\,d\Gamma + \oint_{\Gamma'} G_{22}\,T_y\,d\Gamma + V(x_i,y_i)$$

where the coefficients C_{kl} are calculated using the coefficients of the discretized equations, as follows:

$$C_{kl}(x_i,y_i) = -\sum_{e=1}^{N_e}\sum_{j=1}^{n_e} f_{kl}^{(e)}(x_i,y_i)$$

and

$$f_{kl}^{(e)}(x_i,y_i) = \oint_{0}^{1} F_{kl}(x-x_i,y-y_i)\,J(\xi)\,d\xi$$

where the nomenclature used here is as defined in the discussion of the discretization in the main text.

In order to calculated the value of u_i and v_i at any generic source point, the following procedure has to be adopted. Let us first define the two expressions:

$$\Phi_i = \oint_{\Gamma'} F_{11} u \, d\Gamma + \oint_{\Gamma'} F_{21} v \, d\Gamma + \oint_{\Gamma'} G_{11} T_x \, d\Gamma + \oint_{\Gamma'} G_{21} T_y \, d\Gamma + U(x_i, y_i)$$

and

$$\Psi_i = \oint_{\Gamma'} F_{12} u \, d\Gamma + \oint_{\Gamma'} F_{22} v \, d\Gamma + \oint_{\Gamma'} G_{12} T_x \, d\Gamma + \oint_{\Gamma'} G_{22} T_y \, d\Gamma + V(x_i, y_i)$$

Hence,

$$C_{11}(x_i, y_i) \, u_i + C_{12}(x_i, y_i) \, v_i = \Phi_i$$

and

$$C_{21}(x_i, y_i) \, u_i + C_{22}(x_i, y_i) \, v_i = \Psi_i$$

These two equations form a system of equations that can be solved by means of Cramer's rule to produce:

$$u_i = \frac{C_{22} \Phi_i - C_{12} \Psi_i}{C_{11} C_{22} - C_{12} C_{21}}$$

and

$$v_i = \frac{C_{11}\Psi_i - C_{21}\Phi_i}{C_{11}C_{22} - C_{12}C_{21}}$$

Although this approach has been derived for elasticity equations, it can be applied also in the penalty function formulation to fluid flow analysis, according to the analogy between these two cases given in Chapter 5.

J.3- VORTICITY-VELOCITY FORMULATION

This approach can also be employed in the kinematic equations of the vorticity-velocity formulation. In this case, although the physical view of the conditions suggested is less evident, it can be assumed that the fluid flows with constant velocities α and β in the x- and y- directions, respectively. Thus, the flow is irrotational since in this case $\omega = 0$. Hence, the terms in the kinematic equations containing the vorticity parameter disappear and the final equations are given by:

$$C_i\alpha + (\alpha\ell + \beta m)\oint_\Gamma \frac{\partial u^*}{\partial x}\, d\Gamma - (-\alpha m + \beta\ell)\oint_\Gamma \frac{\partial u^*}{\partial y}\, d\Gamma = 0$$

and

$$C_i\beta + (\alpha\ell + \beta m)\oint_\Gamma \frac{\partial u^*}{\partial y}\, d\Gamma + (-\alpha m + \beta\ell)\oint_\Gamma \frac{\partial u^*}{\partial x}\, d\Gamma = 0$$

For the special case of $\alpha = v_{xi}$ and $\beta = v_{yi}$ the equations above can be transformed into the following equations:

$$C_i v_{x_i} + \oint_\Gamma \frac{\partial u^*}{\partial x} v_{n_i}\, d\Gamma - \oint_\Gamma \frac{\partial u^*}{\partial y} v_{t_i}\, d\Gamma = 0 \qquad (J.5)$$

and

$$C_i v_{y_i} + \oint_\Gamma \frac{\partial u^*}{\partial y} v_{n_i}\, d\Gamma + \oint_\Gamma \frac{\partial u^*}{\partial x} v_{t_i}\, d\Gamma = 0 \qquad (J.6)$$

where:

$$v_{n_i} = v_{x_i}\ell + v_{y_i} m$$

and

$$v_{t_i} = - v_{x_i} m + v_{y_i}\ell$$

Subtracting Eqs. (J.5) and (J.6) from Eqs. (4.25) and (4.26), respectively, gives the following boundary integral equations:

$$\oint_\Gamma \frac{\partial u^*}{\partial x} (v_n - v_{n_i})\, d\Gamma - \oint_\Gamma \frac{\partial u^*}{\partial y} (v_t - v_{t_i})\, d\Gamma + \oint_\Gamma \omega u^* m\, d\Gamma = \iint_\Omega u^* \frac{\partial \omega}{\partial y}\, dx\, dy$$

and

$$\oint_{\Gamma} \frac{\partial u^*}{\partial y} (v_n - v_{n_i})\, d\Gamma + \oint_{\Gamma} \frac{\partial u^*}{\partial x} (v_t - v_{t_i})\, d\Gamma$$
$$- \oint_{\Gamma} \omega u^* \ell\, d\Gamma = - \iint_{\Omega} u^* \frac{\partial \omega}{\partial y}\, dx\, dy$$

Following the same procedure described above for elasticity equations, the final representation for the BIEs for this problem can be obtained:

$$\oint_{\Gamma'} \frac{\partial u^*}{\partial x} (v_n - v_{n_i})\, d\Gamma - \oint_{\Gamma'} \frac{\partial u^*}{\partial y} (v_t - v_{t_i})\, d\Gamma$$
$$+ \oint_{\Gamma'} \omega u^* m\, d\Gamma = \iint_{\Omega} u^* \frac{\partial \omega}{\partial y}\, dx\, dy$$

and

$$\oint_{\Gamma'} \frac{\partial u^*}{\partial y} (v_n - v_{n_i})\, d\Gamma + \oint_{\Gamma'} \frac{\partial u^*}{\partial x} (v_t - v_{t_i})\, d\Gamma$$
$$- \oint_{\Gamma'} \omega u^* \ell\, d\Gamma = - \iint_{\Omega} u^* \frac{\partial \omega}{\partial x}\, dx\, dy$$

The final expression can be obtained directly from the equations above, in terms of the tangential and normal velocities.

$$C_x v_{n_i} - C_y v_{t_i} + \oint_{\Gamma'} \frac{\partial u^*}{\partial x} v_n\, d\Gamma - \oint_{\Gamma'} \frac{\partial u^*}{\partial y} v_t\, d\Gamma$$
$$+ \oint_{\Gamma'} \omega u^* m\, d\Gamma = \iint_{\Omega} u^* \frac{\partial \omega}{\partial y}\, dx\, dy \qquad (J.7)$$

and

$$C_y v_{n_i} + C_x v_{t_i} + \oint_{\Gamma'} \frac{\partial u^*}{\partial y} v_n \, d\Gamma + \oint_{\Gamma'} \frac{\partial u^*}{\partial x} v_t \, d\Gamma - \oint_{\Gamma'} \omega u^* \ell \, d\Gamma = - \iint_{\Omega} u^* \frac{\partial \omega}{\partial x} \, dx\, dy \qquad (J.8)$$

where:

$$C_x = \oint_{\Gamma'} \frac{\partial u^*}{\partial x} \, d\Gamma$$

and

$$C_y = \oint_{\Gamma'} \frac{\partial u^*}{\partial y} \, d\Gamma$$

However, it is convenient for the analysis, using these equations, that they give explicitly the components of the velocity in the x- and y- directions. Therefore, Eqs. (J.7) and (J.8) were manipulated further to produce the following:

$$C_n v_{x_i} - C_t v_{y_i} + \oint_{\Gamma'} \frac{\partial u^*}{\partial x} v_n \, d\Gamma - \oint_{\Gamma'} \frac{\partial u^*}{\partial y} v_t \, d\Gamma + \oint_{\Gamma'} \omega u^* m \, d\Gamma = \iint_{\Omega} u^* \frac{\partial \omega}{\partial y} \, dx\, dy$$

and

$$C_t v_{x_i} + C_n v_{y_i} + \oint_{\Gamma'} \frac{\partial u^*}{\partial y} v_n \, d\Gamma + \oint_{\Gamma'} \frac{\partial u^*}{\partial x} v_t \, d\Gamma - \oint_{\Gamma'} \omega u^* \ell \, d\Gamma = - \iint_{\Omega} u^* \frac{\partial \omega}{\partial x} \, dx\, dy$$

where:

$$C_n = \oint_{\Gamma'} \frac{\partial u^*}{\partial n} \, d\Gamma$$

and

$$C_t = \oint_{\Gamma'} \frac{\partial u^*}{\partial t} \, d\Gamma$$

corresponding to the derivatives of the fundamental solution in the normal and tangential directions, respectively. These can be evaluated using the coefficients resulting from the discretized equations, similar to the method used in the discussion of the elasticity equations.

Finally, in order to calculate explicitly the velocity components v_{xi} and v_{yi}, the following expressions are used:

$$v_{x_i} = \frac{\phi C_n + \Psi C_t}{C_n^2 + C_t^2}$$

and

$$v_{y_i} = \frac{\Psi C_n - \phi C_t}{C_n^2 + C_t^2}$$

where:

$$\phi = -\oint_{\Gamma'} \omega u^* m \, d\Gamma + \iint_{\Omega} u^* \frac{\partial \omega}{\partial y} \, dx \, dy$$

and

$$\Psi = \oint_{\Gamma'} \omega u^* \ell \, d\Gamma - \iint_{\Omega} u^* \frac{\partial \omega}{\partial x} \, dx \, dy$$

Appendix K

Derivatives on the Boundary

K.1- VORTICITY-VELOCITY FORMULATION

Suppose that the distribution of vorticity, ω, and its normal derivative are known on the boundary nodes. In this case, it is possible to obtain expressions to calculate the derivatives of vorticity with respect to the coordinates x and y.

Consider one n-node boundary element. It is possible to define on this element:

$$\omega(\xi) = \sum_{j=1}^{n} \omega_j N_j(\xi) \qquad (K.1)$$

$$\frac{\partial \omega}{\partial \xi}(\xi) = \sum_{j=1}^{n} \omega_j \frac{\partial N_j}{\partial \xi}(\xi) \qquad (K.2)$$

and

$$\frac{\partial \omega}{\partial \xi}(\xi) = \sum_{j=1}^{n} \left(\frac{\partial \omega}{\partial n}\right)_j N_j(\xi) \qquad (K.3)$$

where $N_j(\xi)$ is the interpolation function and ξ is the intrinsic coordinate over the boundary element. Thus, the parameters above can be represented in terms of the coordinate ξ.

The parameter in Eq. (K.2) can be also given in the following expression:

$$\frac{\partial \omega}{\partial \xi} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial \xi}$$

However, for this kind of elements the direction cosines ℓ and m are defined by:

$$\ell = \frac{dy}{d\xi}\frac{d\xi}{d\Gamma} = \frac{dy/d\xi}{J(\xi)}$$

and

$$m = -\frac{dx}{d\xi}\frac{d\xi}{d\Gamma} = -\frac{dx/d\xi}{J(\xi)}$$

and so

$$\frac{\partial \omega}{\partial \xi}\frac{1}{J(\xi)} = m\frac{\partial \omega}{\partial x} + \ell\frac{\partial \omega}{\partial y} \qquad \text{(K.4)}$$

where $J(\xi)$ is the Jacobian given by

$$J(\xi) = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

On the other hand, the normal derivative of vorticity, given by Eq. (3), in terms of nodal values and the interpolation function, is also defined by:

$$\frac{\partial \omega}{\partial n} = \ell \frac{\partial \omega}{\partial x} + m \frac{\partial \omega}{\partial y} \tag{K.5}$$

Solving the system of equations given by Eqs. (K.4) and (K.5), gives:

$$\frac{\partial \omega}{\partial x} = \ell \frac{\partial \omega}{\partial n} - m \frac{\partial \omega}{\partial \xi} \frac{1}{J(\xi)} \tag{K.6}$$

and

$$\frac{\partial \omega}{\partial y} = \ell \frac{\partial \omega}{\partial \xi} \frac{1}{J(\xi)} + m \frac{\partial \omega}{\partial n} \tag{K.7}$$

this allows the calculation of the derivatives of vorticity over the boundary explicitly.

K.2- PENALTY FUNCTION FORMULATION

The expression to be found in this case needs calculate the derivatives of the velocity components comprising four parameters and, therefore, four equations will be necessary. The first two equations come from the definition of traction given by Eqs. (G.5). Introducing the definition of stresses given by Eqs. (G.2) into Eqs. (G.5), it can be shown that:

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \ell d_{11} & \ell d_{12} & m d_{33} & m d_{33} \\ m d_{21} & m d_{22} & \ell d_{33} & \ell d_{33} \end{bmatrix} \begin{bmatrix} \dfrac{\partial v_x}{\partial x} \\ \dfrac{\partial v_y}{\partial y} \\ \dfrac{\partial v_y}{\partial x} \\ \dfrac{\partial v_x}{\partial y} \end{bmatrix}$$

where the parameters d_{11}, d_{12}, d_{21}, d_{22} and d_{33} stand as define in Appendix G.

The other two equations come from the application of Eq. (K.4) twice to the velocity components to obtain:

$$\frac{\partial v_x}{\partial \xi}\frac{1}{J(\xi)} = m\frac{\partial v_x}{\partial x} + \ell\frac{\partial v_x}{\partial y}$$

and

$$\frac{\partial v_y}{\partial \xi}\frac{1}{J(\xi)} = m\frac{\partial v_y}{\partial x} + \ell\frac{\partial v_y}{\partial y}$$

These two equations along with the system of equations for traction components form a set of four equations that allow the derivatives of the velocity components on the boundary to be obtained.

TABLES

Table 7.1: Results Using Option 1 with Corner Gap 1.1×10^{-3} and Jump Functions.

Boundary Results:

I	X	Y	WP	DWN
1	0.2500E+00	0.0000E+00	-0.3969E+01	-0.8054E+01
2	0.5000E+00	0.0000E+00	-0.4020E+01	-0.7793E+01
3	0.7500E+00	0.0000E+00	-0.3969E+01	-0.8054E+01
4	0.9989E+00	0.0000E+00	-0.4164E+01	-0.6517E+01
5	0.1000E+01	0.2500E+00	-0.1999E+01	0.1135E+00
6	0.1000E+01	0.5000E+00	-0.2555E-15	0.2778E-14
7	0.1000E+01	0.7500E+00	0.1999E+01	-0.1135E+00
8	0.1000E+01	0.9989E+00	0.4034E+01	0.1770E+01
9	0.7500E+00	0.1000E+01	0.3969E+01	0.8054E+01
10	0.5000E+00	0.1000E+01	0.4020E+01	0.7793E+01
11	0.2500E+00	0.1000E+01	0.3969E+01	0.8054E+01
12	0.1100E-02	0.1000E+01	0.4164E+01	0.6517E+01
13	0.0000E+00	0.7500E+00	0.1999E+01	-0.1135E+00
14	0.0000E+00	0.5000E+00	-0.2015E-15	0.8882E-15
15	0.0000E+00	0.2500E+00	-0.1999E+01	0.1135E+00
16	0.0000E+00	0.1100E-02	-0.4034E+01	-0.1770E+01
17	0.1100E-02	0.0000E+00	-0.4164E+01	-0.6517E+01
18	0.1000E+01	0.1100E-02	-0.4034E+01	-0.1770E+01
19	0.9989E+00	0.1000E+01	0.4164E+01	0.6517E+01
20	0.0000E+00	0.9989E+00	0.4034E+01	0.1770E+01

Domain Results:

I	X	Y	VX	VY	W	WX	WY
1	0.2500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
2	0.5000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
3	0.7500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
4	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	-0.3996E+01	0.0000E+00	0.8000E+01
5	0.1000E+01	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
6	0.1000E+01	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
7	0.1000E+01	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
8	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00	0.3996E+01	0.0000E+00	0.8000E+01
9	0.7500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
10	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
11	0.2500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
12	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.3996E+01	0.0000E+00	0.8000E+01
13	0.0000E+00	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
14	0.0000E+00	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
15	0.0000E+00	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.3996E+01	0.0000E+00	0.8000E+01
17	0.2500E+00	0.5000E+00	0.9999E+00	-0.4668E-17	-0.6635E-16	-0.7286E-16	0.7987E+01
18	0.5000E+00	0.2500E+00	0.7495E+00	-0.5233E-17	-0.1997E+01	-0.1284E-15	0.7992E+01
19	0.5000E+00	0.5000E+00	0.9997E+00	-0.3581E-17	-0.7633E-16	-0.2559E-16	0.7989E+01
20	0.5000E+00	0.7500E+00	0.7495E+00	0.5690E-17	0.1997E+01	0.4163E-16	0.7992E+01
21	0.7500E+00	0.5000E+00	0.9999E+00	0.5809E-17	-0.5898E-16	0.1084E-16	0.7987E+01

Table 7.2: Results Using Option 2 with Corner Gap $1.1\text{x}10^{-3}$ and Jump Functions.

Boundary Results:

I	X	Y	WP	DWN
1	0.2500E+00	0.0000E+00	-0.3969E+01	-0.8050E+01
2	0.5000E+00	0.0000E+00	-0.4020E+01	-0.7793E+01
3	0.7500E+00	0.0000E+00	-0.3969E+01	-0.8057E+01
4	0.9989E+00	0.0000E+00	-0.4164E+01	-0.6417E+01
5	0.1000E+01	0.2500E+00	-0.1999E+01	0.8030E-01
6	0.1000E+01	0.5000E+00	-0.2555E-15	0.2573E-14
7	0.1000E+01	0.7500E+00	0.1999E+01	-0.8030E-01
8	0.1000E+01	0.9989E+00	0.4034E+01	0.1586E+01
9	0.7500E+00	0.1000E+01	0.3969E+01	0.8057E+01
10	0.5000E+00	0.1000E+01	0.4020E+01	0.7793E+01
11	0.2500E+00	0.1000E+01	0.3969E+01	0.8050E+01
12	0.1100E-02	0.1000E+01	0.4164E+01	0.6617E+01
13	0.0000E+00	0.7500E+00	0.1999E+01	-0.1467E+00
14	0.0000E+00	0.5000E+00	-0.2015E-15	0.8882E-15
15	0.0000E+00	0.2500E+00	-0.1999E+01	0.1467E+00
16	0.0000E+00	0.1100E-02	-0.4034E+01	-0.1955E+01
17	0.1100E-02	0.0000E+00	-0.4164E+01	-0.6617E+01
18	0.1000E+01	0.1100E-02	-0.4034E+01	-0.1586E+01
19	0.9989E+00	0.1000E+01	0.4164E+01	0.6417E+01
20	0.0000E+00	0.9989E+00	0.4034E+01	0.1955E+01

Domain Results:

I	X	Y	VX	VY	W	WX	WY
1	0.2500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
2	0.5000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
3	0.7500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
4	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	-0.3996E+01	0.0000E+00	0.8000E+01
5	0.1000E+01	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
6	0.1000E+01	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
7	0.1000E+01	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
8	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00	0.3996E+01	0.0000E+00	0.8000E+01
9	0.7500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
10	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
11	0.2500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
12	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.3996E+01	0.0000E+00	0.8000E+01
13	0.0000E+00	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
14	0.0000E+00	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
15	0.0000E+00	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.3996E+01	0.0000E+00	0.8000E+01
17	0.2500E+00	0.5000E+00	0.9999E+00	-0.4668E-17	-0.7142E-16	-0.5791E-16	0.7985E+01
18	0.5000E+00	0.2500E+00	0.7495E+00	-0.5233E-17	-0.1997E+01	0.1432E-01	0.7992E+01
19	0.5000E+00	0.5000E+00	0.9997E+00	-0.3581E-17	-0.7699E-16	0.4786E-17	0.7989E+01
20	0.5000E+00	0.7500E+00	0.7495E+00	0.5690E-17	0.1997E+01	-0.1432E-01	0.7992E+01
21	0.7500E+00	0.5000E+00	0.9999E+00	0.5809E-17	-0.6265E-16	0.2591E-16	0.7988E+01

Table 7.3: Results Using Option 1 with Corner Gap 1.1×10^{-8} and Regular Corner Model.

Boundary Results:

I	X	Y	WP	DWN
1	0.2500E+00	0.0000E+00	-0.3998E+01	-0.8000E+01
2	0.5000E+00	0.0000E+00	-0.4002E+01	-0.8000E+01
3	0.7500E+00	0.0000E+00	-0.3998E+01	-0.8000E+01
4	0.1000E+01	0.0000E+00	-0.4010E+01	-0.8000E+01
5	0.1000E+01	0.2500E+00	-0.2000E+01	-0.1356E-05
6	0.1000E+01	0.5000E+00	-0.2782E-15	0.1082E-14
7	0.1000E+01	0.7500E+00	0.2000E+01	0.1356E-05
8	0.1000E+01	0.1000E+01	0.4000E+01	-0.1292E-04
9	0.7500E+00	0.1000E+01	0.3998E+01	0.8000E+01
10	0.5000E+00	0.1000E+01	0.4002E+01	0.8000E+01
11	0.2500E+00	0.1000E+01	0.3998E+01	0.8000E+01
12	0.1100E-07	0.1000E+01	0.4010E+01	0.8000E+01
13	0.0000E+00	0.7500E+00	0.2000E+01	0.1356E-05
14	0.0000E+00	0.5000E+00	0.6662E-15	0.4441E-15
15	0.0000E+00	0.2500E+00	-0.2000E+01	-0.1356E-05
16	0.0000E+00	0.1100E-07	-0.4000E+01	0.1292E-04
17	0.1100E-07	0.0000E+00	-0.4010E+01	-0.8000E+01
18	0.1000E+01	0.1100E-07	-0.4000E+01	0.1292E-04
19	0.1000E+01	0.1000E+01	0.4010E+01	0.8000E+01
20	0.0000E+00	0.1000E+01	0.4000E+01	-0.1292E-04

Domain Results:

I	X	Y	VX	VY	W	WX	WY
1	0.2500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
2	0.5000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
3	0.7500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
4	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
5	0.1000E+01	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
6	0.1000E+01	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
7	0.1000E+01	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
8	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
9	0.7500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
10	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
11	0.2500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
12	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
13	0.0000E+00	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
14	0.0000E+00	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
15	0.0000E+00	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
17	0.2500E+00	0.5000E+00	0.1000E+01	-0.1128E-16	-0.3253E-16	-0.1110E-15	0.8000E+01
18	0.5000E+00	0.2500E+00	0.7500E+00	-0.5258E-17	-0.2000E+01	-0.1284E-15	0.8000E+01
19	0.5000E+00	0.5000E+00	0.1000E+01	0.4337E-18	-0.3556E-16	-0.2819E-16	0.8000E+01
20	0.5000E+00	0.7500E+00	0.7500E+00	0.3835E-17	0.2000E+01	0.8847E-16	0.8000E+01
21	0.7500E+00	0.5000E+00	0.1000E+01	0.1117E-16	-0.4163E-16	0.6288E-16	0.8000E+01

Table 7.4: Results Using Option 2 with Corner Gap $1.1x10^{-8}$ and Regular Corner Model.

Boundary results:

I	X	Y	WP	DWN
1	0.2500E+00	0.0000E+00	-0.3998E+01	-0.8005E+01
2	0.5000E+00	0.0000E+00	-0.4002E+01	-0.8000E+01
3	0.7500E+00	0.0000E+00	-0.3998E+01	-0.7995E+01
4	0.1000E+01	0.0000E+00	-0.4010E+01	-0.8070E+01
5	0.1000E+01	0.2500E+00	-0.2000E+01	-0.3565E-01
6	0.1000E+01	0.5000E+00	-0.2782E-15	0.6661E-15
7	0.1000E+01	0.7500E+00	0.2000E+01	0.3565E-01
8	0.1000E+01	0.1000E+01	0.4000E+01	-0.3326E+00
9	0.7500E+00	0.1000E+01	0.3998E+01	0.7995E+01
10	0.5000E+00	0.1000E+01	0.4002E+01	0.8000E+01
11	0.2500E+00	0.1000E+01	0.3998E+01	0.8005E+01
12	0.1100E-07	0.1000E+01	0.4010E+01	0.7930E+01
13	0.0000E+00	0.7500E+00	0.2000E+01	-0.3565E-01
14	0.0000E+00	0.5000E+00	0.6662E-15	0.0000E+00
15	0.0000E+00	0.2500E+00	-0.2000E+01	0.3565E-01
16	0.0000E+00	0.1100E-07	-0.4000E+01	-0.3326E+00
17	0.1100E-07	0.0000E+00	-0.4010E+01	-0.7930E+01
18	0.1000E+01	0.1100E-07	-0.4000E+01	0.3326E+00
19	0.1000E+01	0.1000E+01	0.4010E+01	0.8070E+01
20	0.0000E+00	0.1000E+01	0.4000E+01	0.3326E+00

Domain Results:

I	X	Y	VX	VY	W	WX	WY
1	0.2500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
2	0.5000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
3	0.7500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
4	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
5	0.1000E+01	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
6	0.1000E+01	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
7	0.1000E+01	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
8	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
9	0.7500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
10	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
11	0.2500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
12	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
13	0.0000E+00	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
14	0.0000E+00	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
15	0.0000E+00	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
17	0.2500E+00	0.5000E+00	0.1000E+01	-0.1128E-16	-0.8676E-17	-0.2882E-17	0.8000E+01
18	0.5000E+00	0.2500E+00	0.7500E+00	-0.5258E-17	-0.2000E+01	0.4048E-03	0.8000E+01
19	0.5000E+00	0.5000E+00	0.1000E+01	0.4337E-18	-0.1061E-16	-0.8712E-18	0.8000E+01
20	0.5000E+00	0.7500E+00	0.7500E+00	0.3835E-17	0.2000E+01	-0.4048E-03	0.8000E+01
21	0.7500E+00	0.5000E+00	0.1000E+01	0.1117E-16	-0.1411E-16	0.1558E-17	0.8000E+01

Table 7.5: Results Using Option 1 with Corner Gap 1.1×10^{-8} and Regular Corner Model for Kinetics and Kinematics.

Boundary results:

I	X	Y	WP	DWN
1	0.2500E+00	0.0000E+00	-0.4000E+01	-0.8000E+01
2	0.5000E+00	0.0000E+00	-0.4000E+01	-0.8000E+01
3	0.7500E+00	0.0000E+00	-0.4000E+01	-0.8000E+01
4	0.1000E+01	0.0000E+00	-0.4000E+01	-0.8000E+01
5	0.1000E+01	0.2500E+00	-0.2000E+01	-0.1356E-05
6	0.1000E+01	0.5000E+00	-0.3282E-15	0.1082E-14
7	0.1000E+01	0.7500E+00	0.2000E+01	0.1356E-05
8	0.1000E+01	0.1000E+01	0.4000E+01	-0.1292E-04
9	0.7500E+00	0.1000E+01	0.4000E+01	0.8000E+01
10	0.5000E+00	0.1000E+01	0.4000E+01	0.8000E+01
11	0.2500E+00	0.1000E+01	0.4000E+01	0.8000E+01
12	0.1100E-07	0.1000E+01	0.4000E+01	0.8000E+01
13	0.0000E+00	0.7500E+00	0.2000E+01	0.1356E-05
14	0.0000E+00	0.5000E+00	0.7117E-15	0.4441E-15
15	0.0000E+00	0.2500E+00	-0.2000E+01	-0.1356E-05
16	0.0000E+00	0.1100E-07	-0.4000E+01	0.1292E-04
17	0.1100E-07	0.0000E+00	-0.4000E+01	-0.8000E+01
18	0.1000E+01	0.1100E-07	-0.4000E+01	0.1292E-04
19	0.1000E+01	0.1000E+01	0.4000E+01	0.8000E+01
20	0.0000E+00	0.1000E+01	0.4000E+01	-0.1292E-04

Domain results:

I	X	Y	VX	VY	W	WX	WY
1	0.2500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
2	0.5000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
3	0.7500E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
4	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
5	0.1000E+01	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
6	0.1000E+01	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
7	0.1000E+01	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
8	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
9	0.7500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
10	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
11	0.2500E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
12	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.4000E+01	0.0000E+00	0.8000E+01
13	0.0000E+00	0.7500E+00	0.7500E+00	0.0000E+00	0.2000E+01	0.0000E+00	0.8000E+01
14	0.0000E+00	0.5000E+00	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00	0.8000E+01
15	0.0000E+00	0.2500E+00	0.7500E+00	0.0000E+00	-0.2000E+01	0.0000E+00	0.8000E+01
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	-0.4000E+01	0.0000E+00	0.8000E+01
17	0.2500E+00	0.5000E+00	0.1000E+01	-0.1128E-16	-0.3253E-16	-0.1110E-15	0.8000E+01
18	0.5000E+00	0.2500E+00	0.7500E+00	-0.5258E-17	-0.2000E+01	-0.1284E-15	0.8000E+01
19	0.5000E+00	0.5000E+00	0.1000E+01	0.4337E-18	-0.3556E-16	-0.2819E-16	0.8000E+01
20	0.5000E+00	0.7500E+00	0.7500E+00	0.3835E-17	0.2000E+01	0.8847E-16	0.8000E+01
21	0.7500E+00	0.5000E+00	0.1000E+01	0.1117E-16	-0.4163E-16	0.6288E-16	0.8000E+01

FIGURES

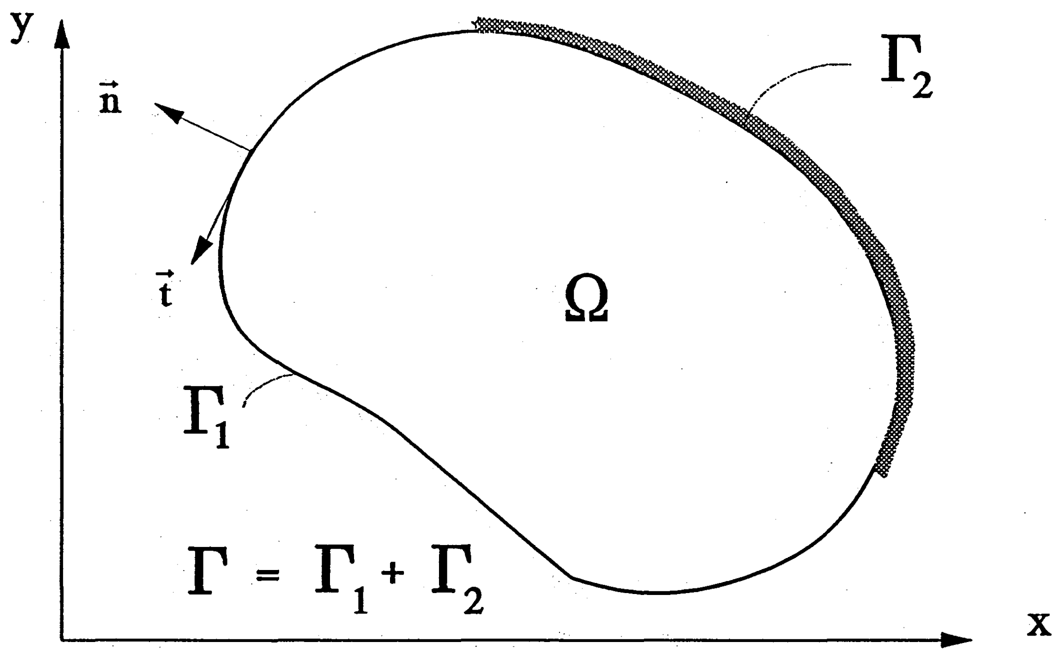

Figure 3.1: Nomenclature for a Two-Dimensional Poisson's Type Problem.

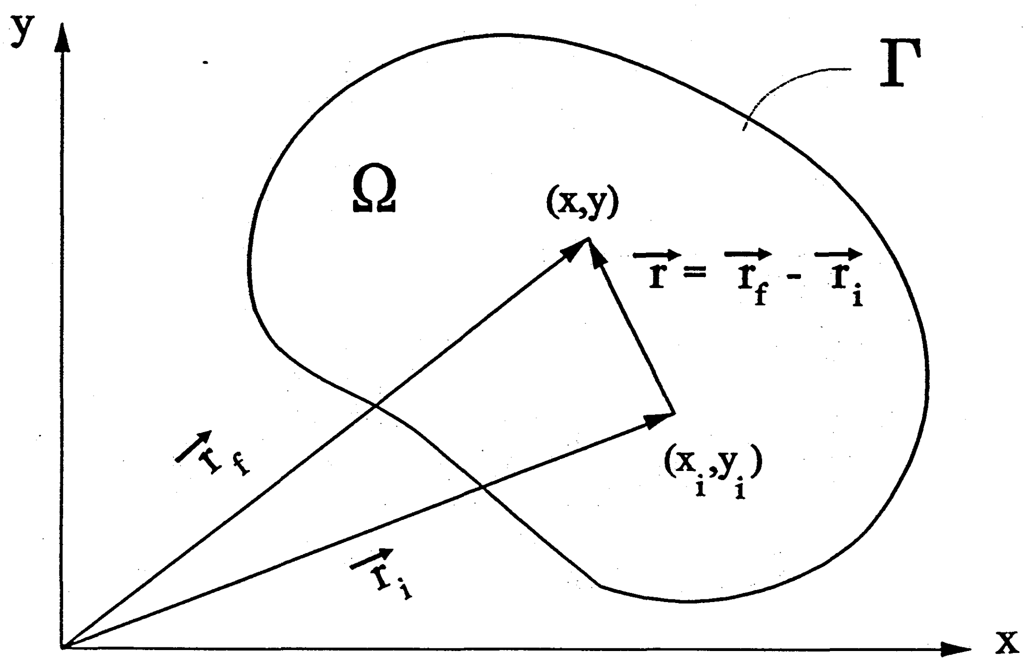

Figure 3.2: Definition of the Position Vector $\vec{r}$ Related to Source Point $(x_i\ ,\ y_i)$.

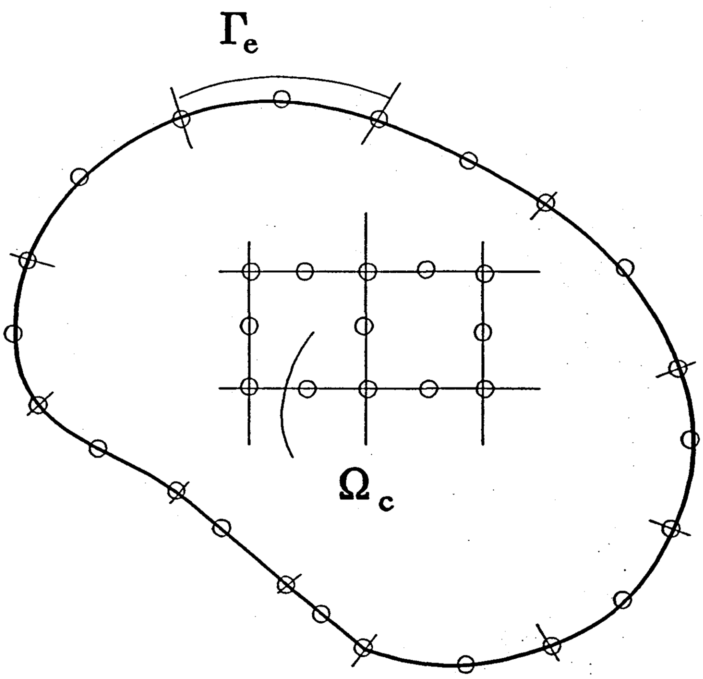

Figure 6.1: Discretization of the Boundary and the Domain.

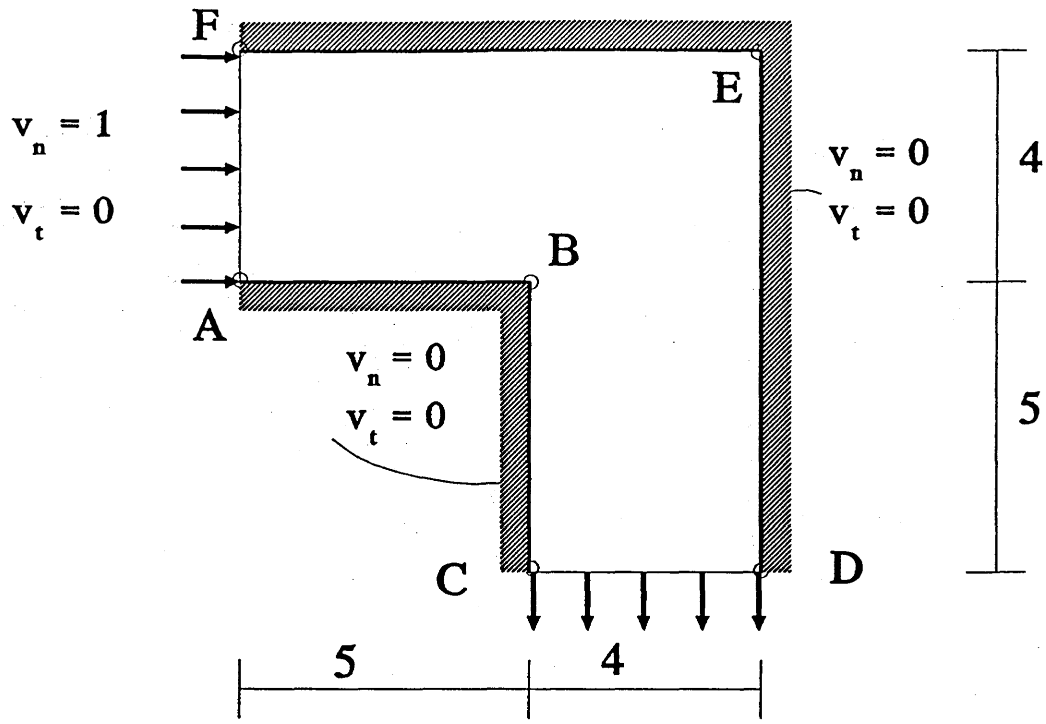

Figure 7.1: L-Shaped Channel Flow.

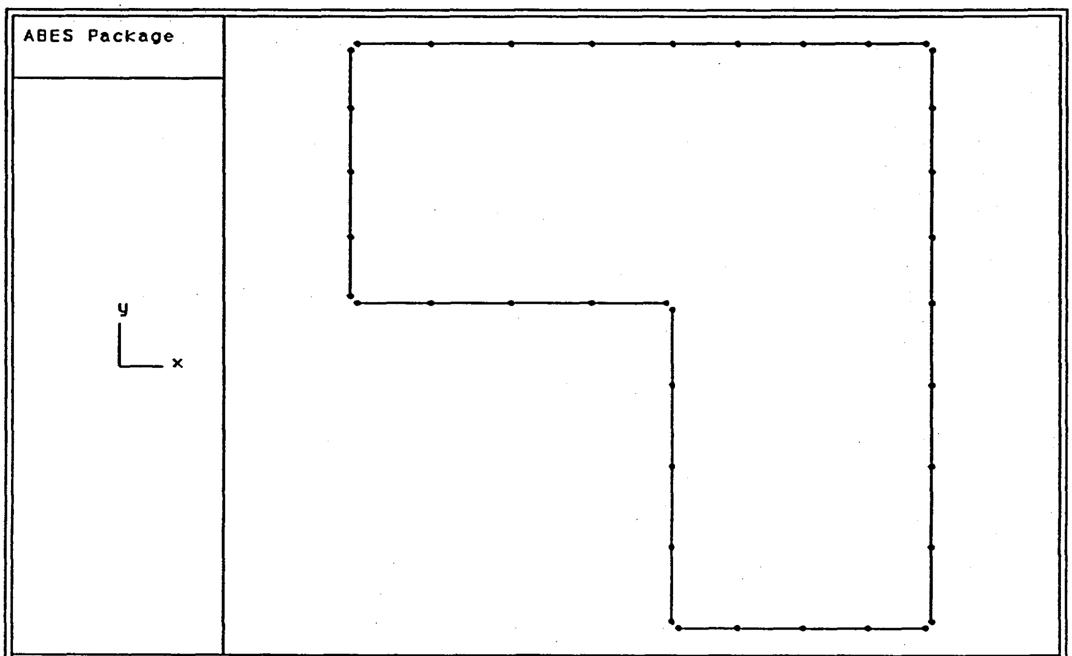

Figure 7.2 : Boundary Element Mesh for the L-Shaped Channel Test Case.

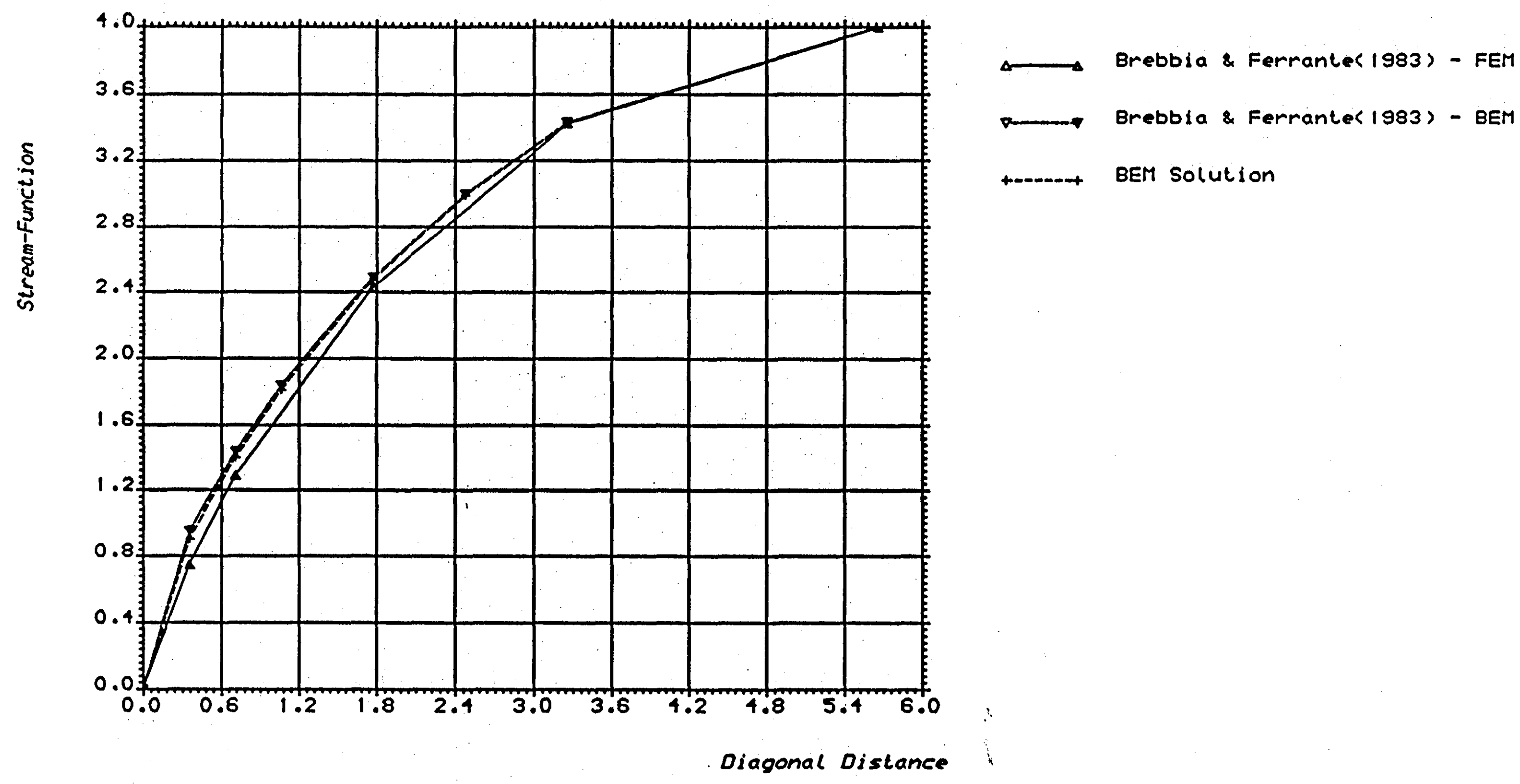

Figure 7.3 : Distribution of Stream-Function Parameter along the Diagonal BE.

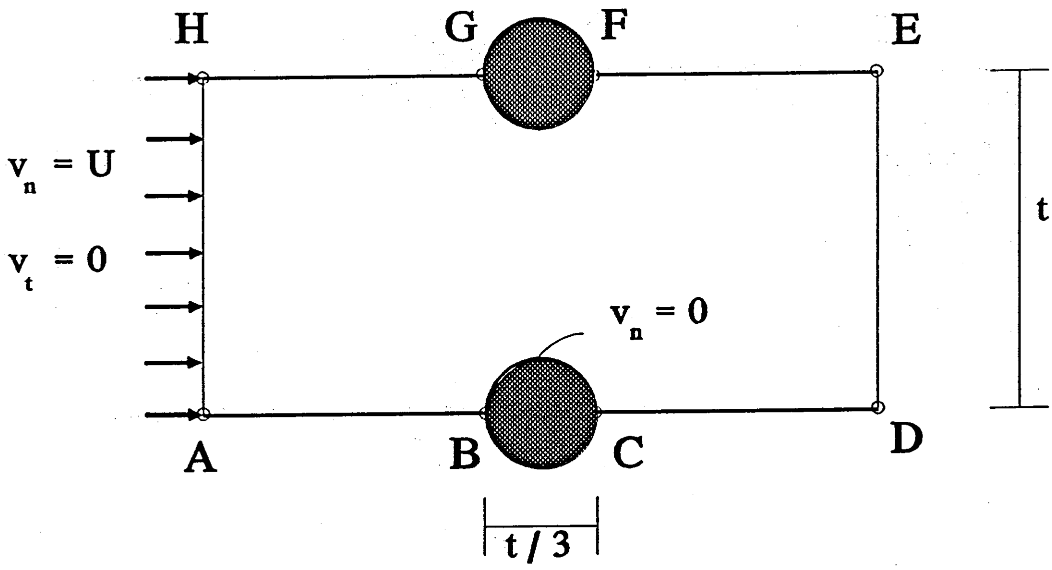

Figure 7.4 : Flow in a Cascade of Cylinders.

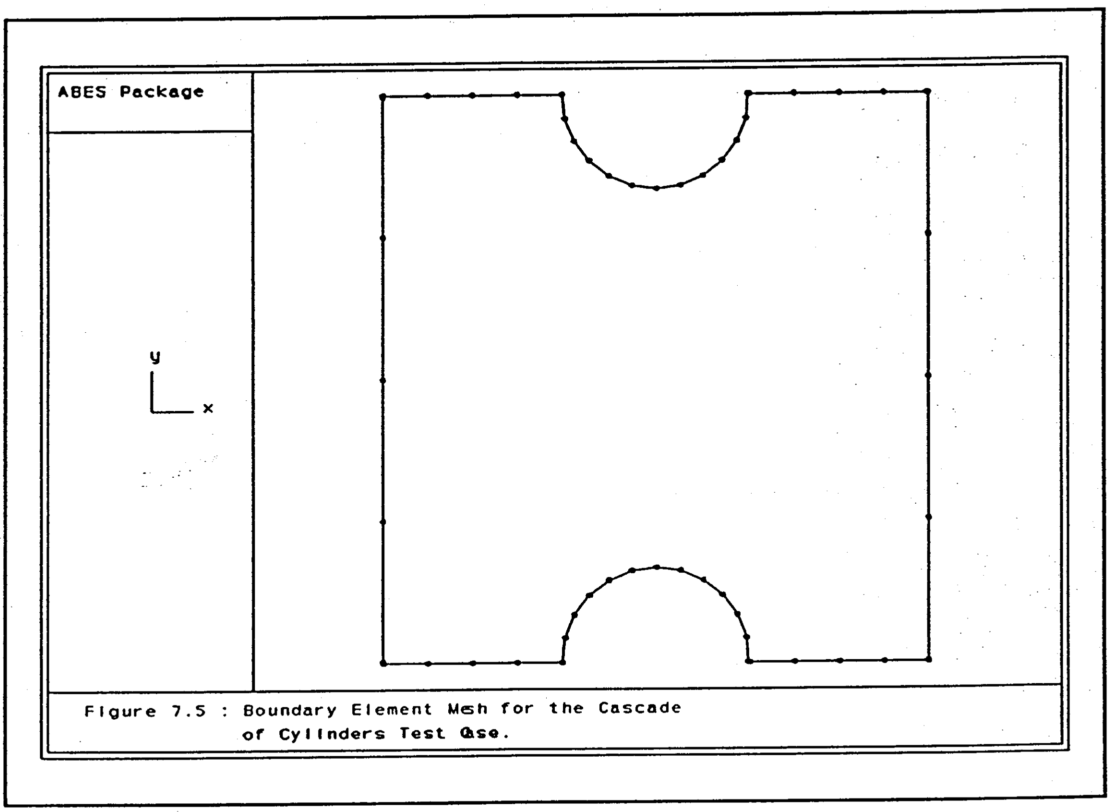

Figure 7.5 : Boundary Element Mesh for the Cascade of Cylinders Test Case.

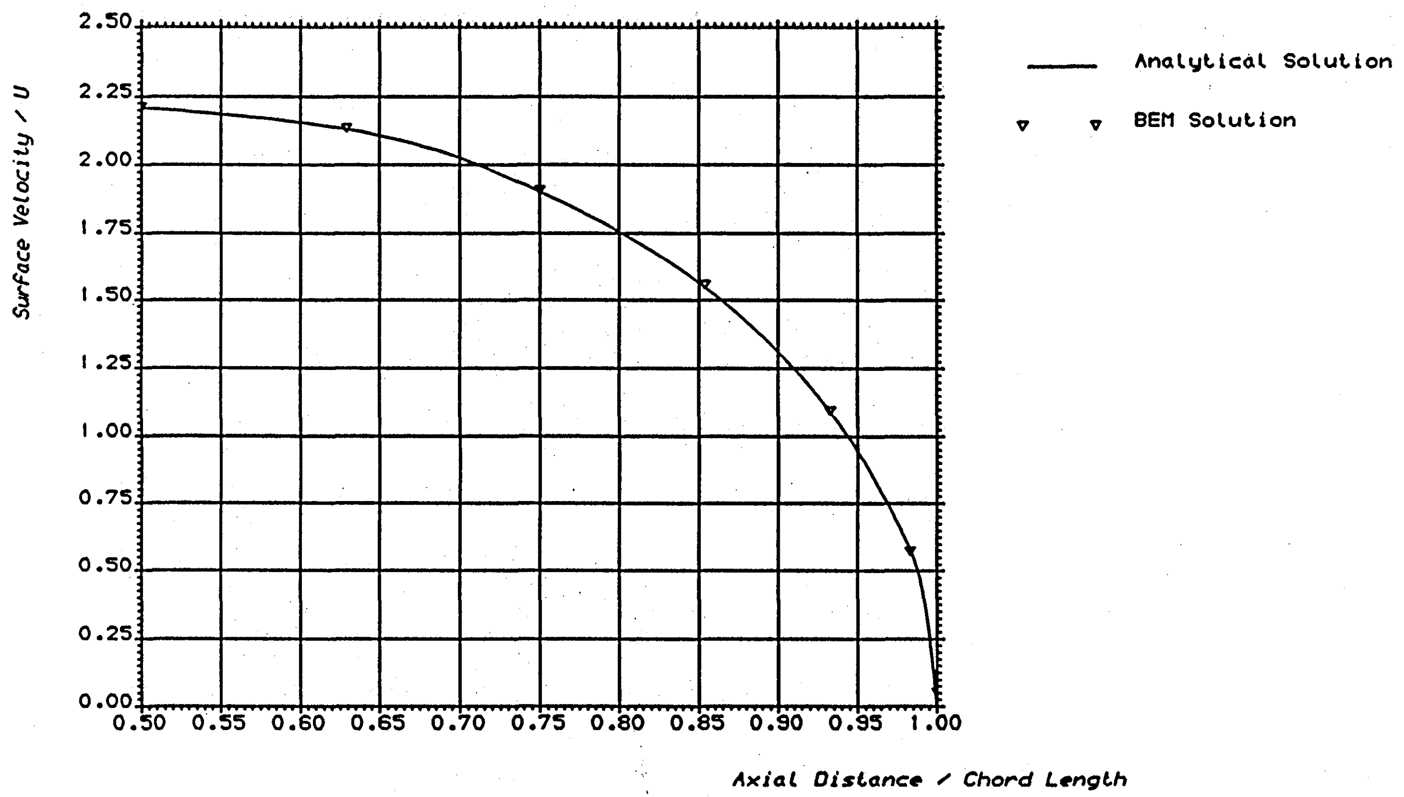

Figure 7.6: Velocity Distribution over one Quadrant.

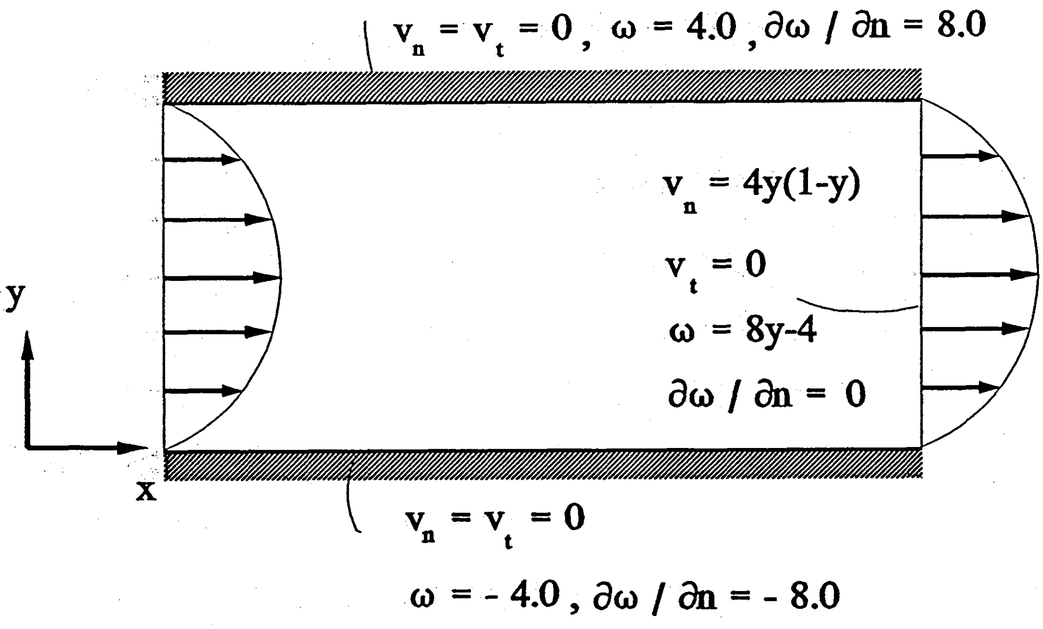

Figure 7.7: Poiseuille Flow.

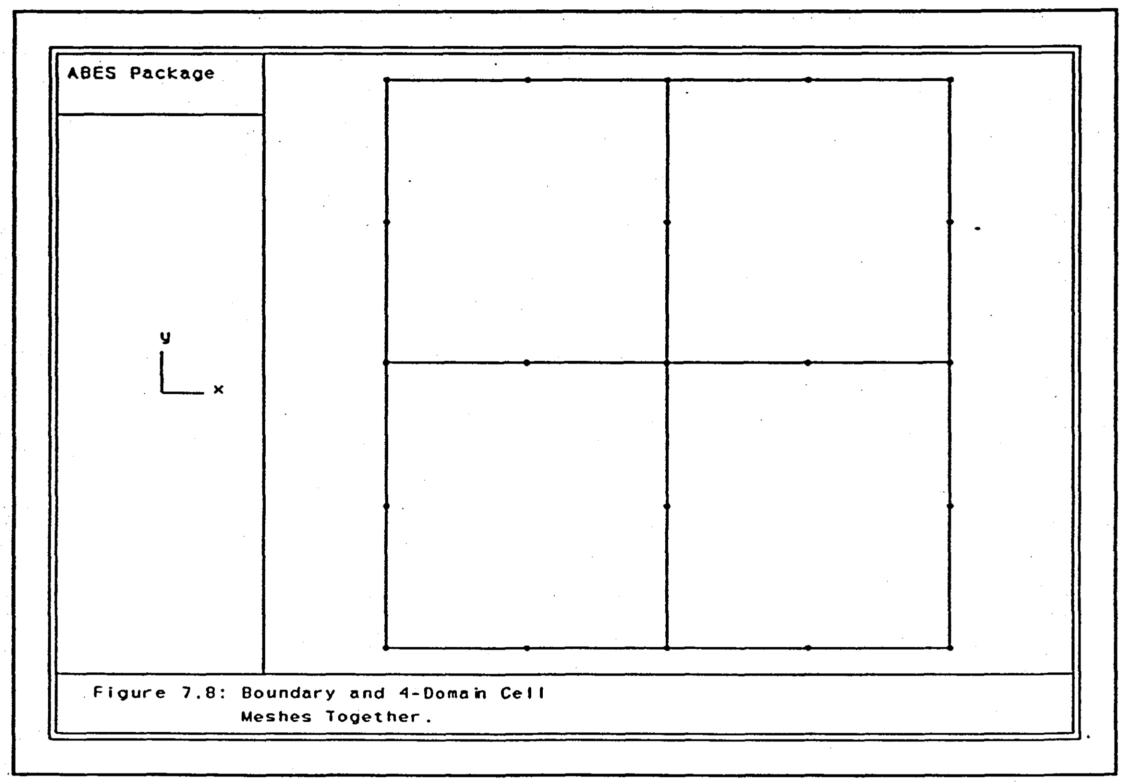

Figure 7.8: Boundary and 4-Domain Cell Meshes Together.

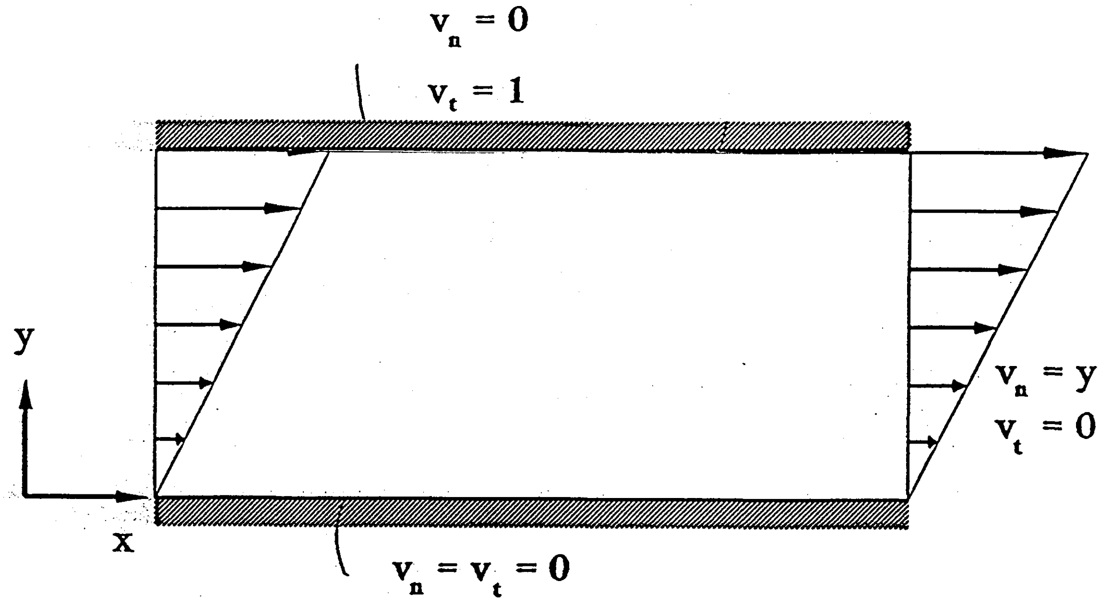

Figure 7.9 : Couette Flow.

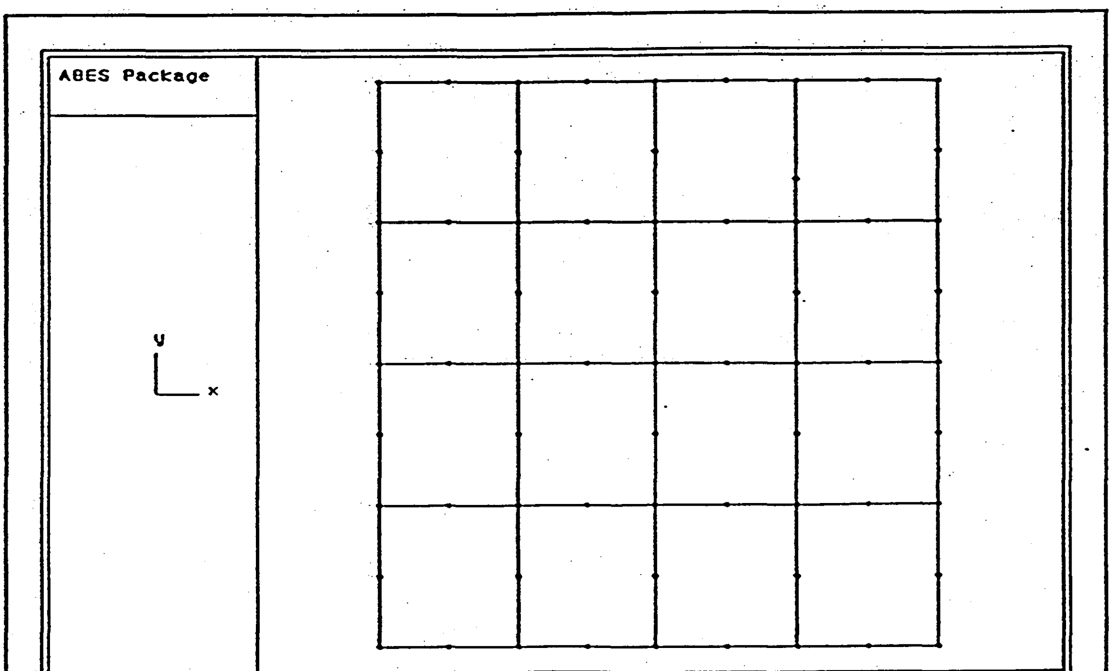

Figure 7.10: Boundary and 16-Domain Cell Meshes together.

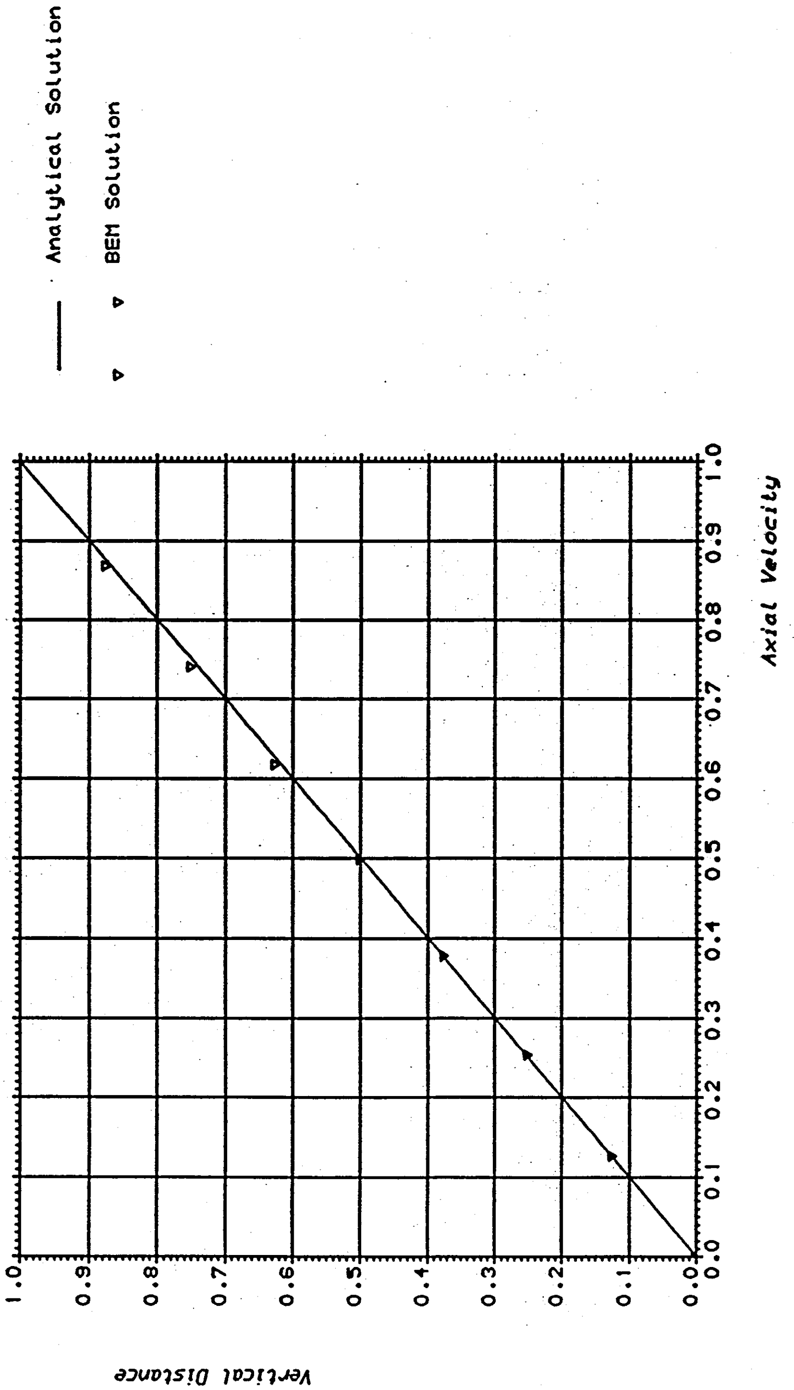

Figure 7.11: Axial Velocity Profile For Couette Flow with Re=10.

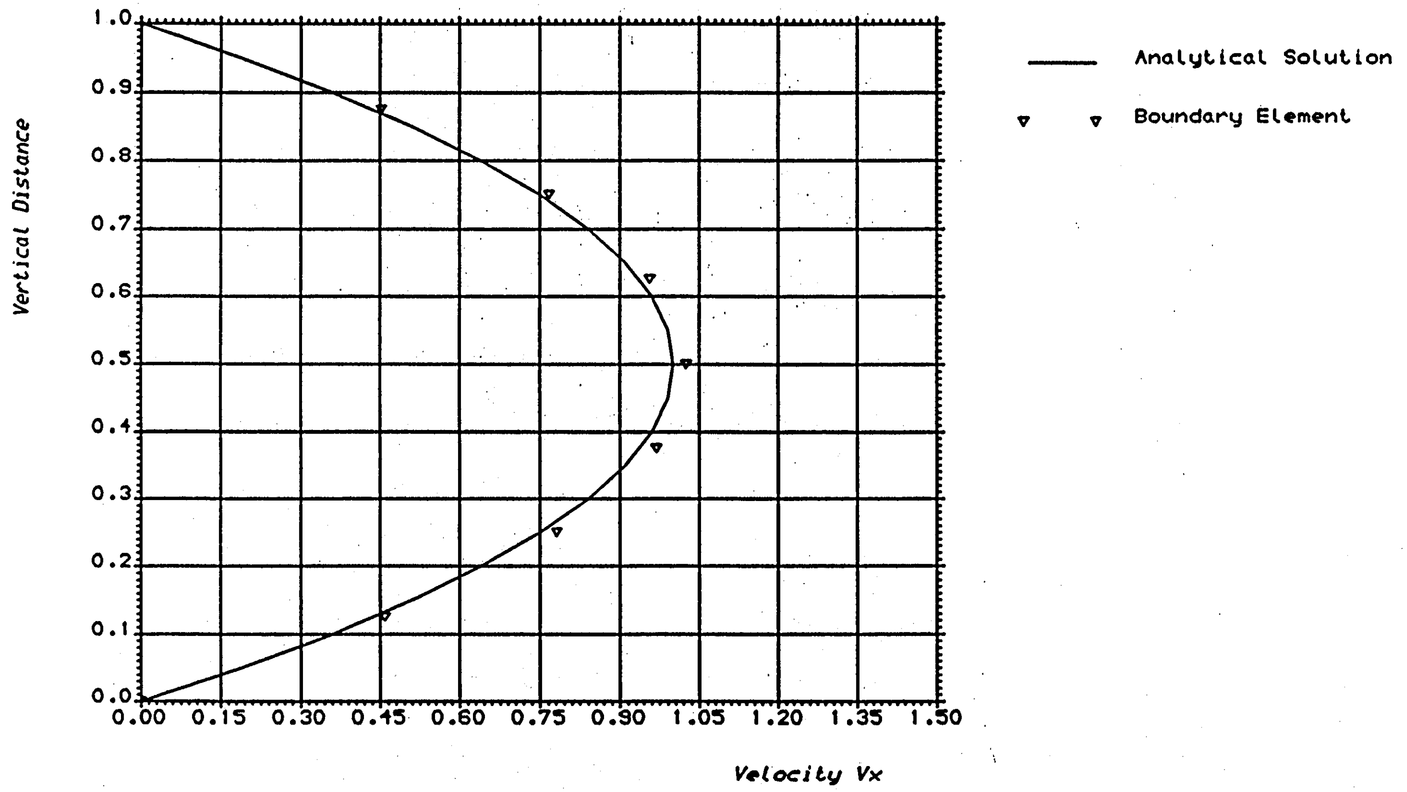

Figure 7.12: Axial Velocity Profile for Poiseuille Flow With Re=10.

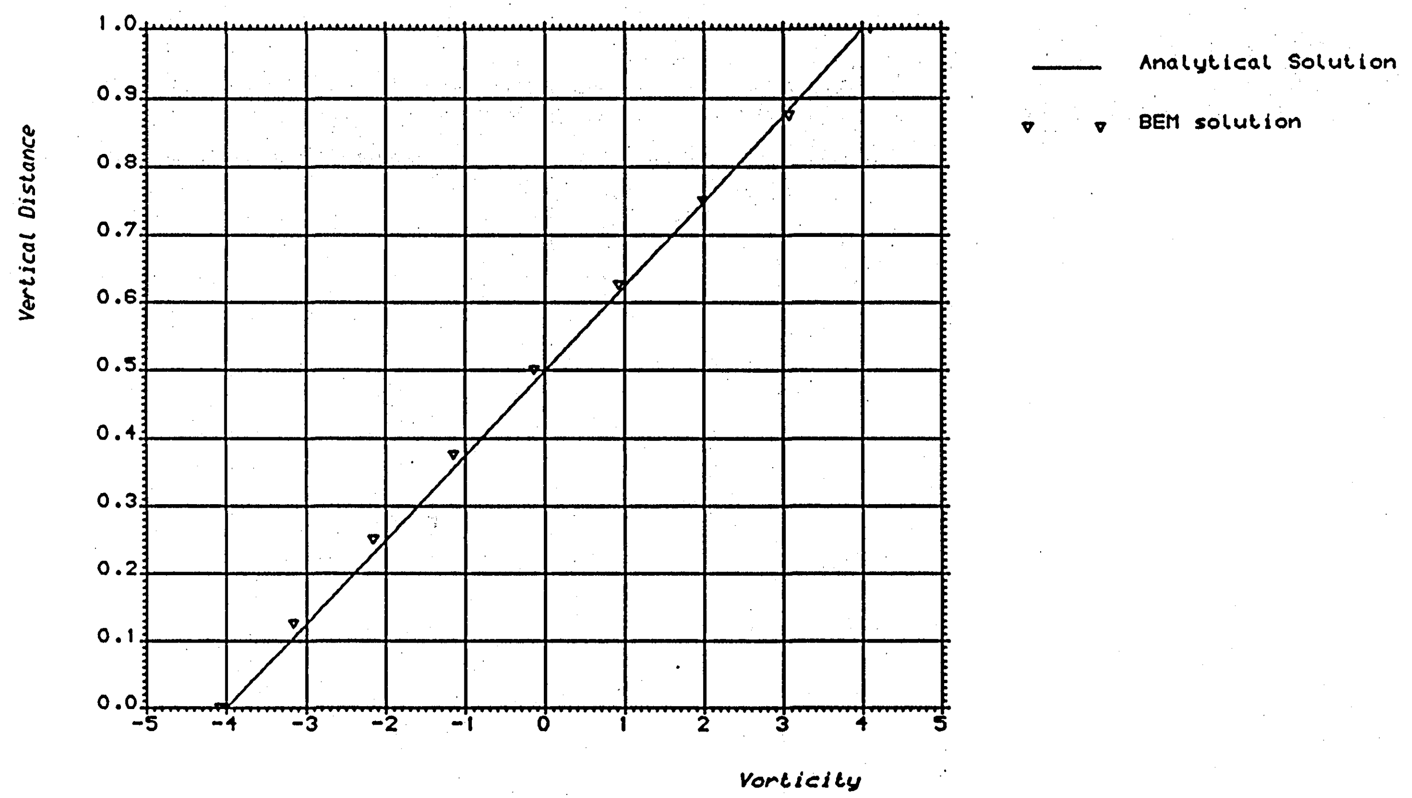

Figure 7.13 : Vorticity Distribution For Poiseuille Flow with Re. = 10.

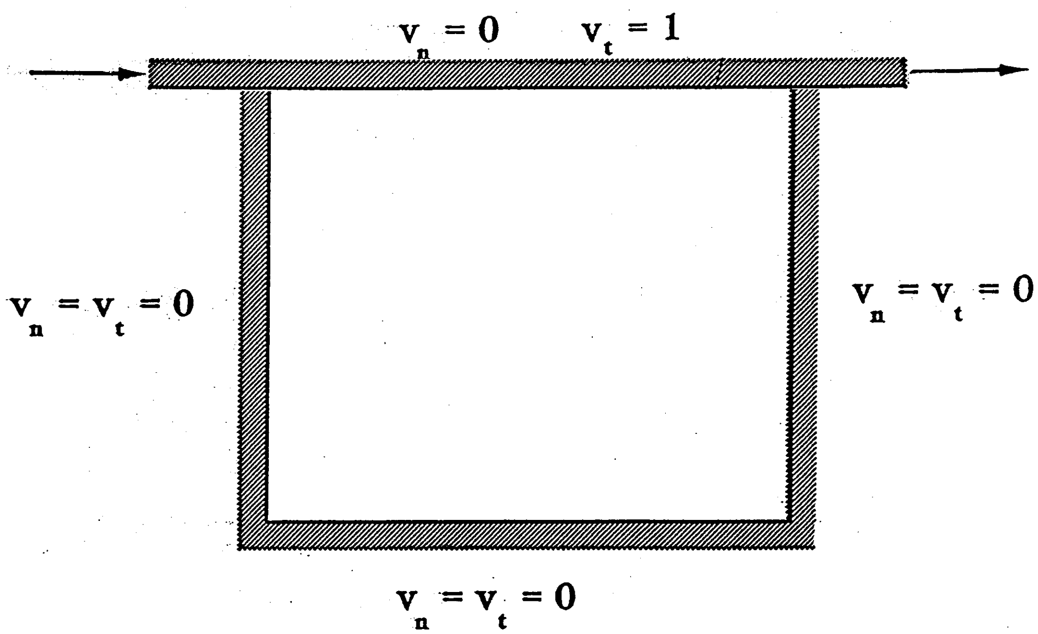

Figure 7.14: Driven-Cavity Flow.

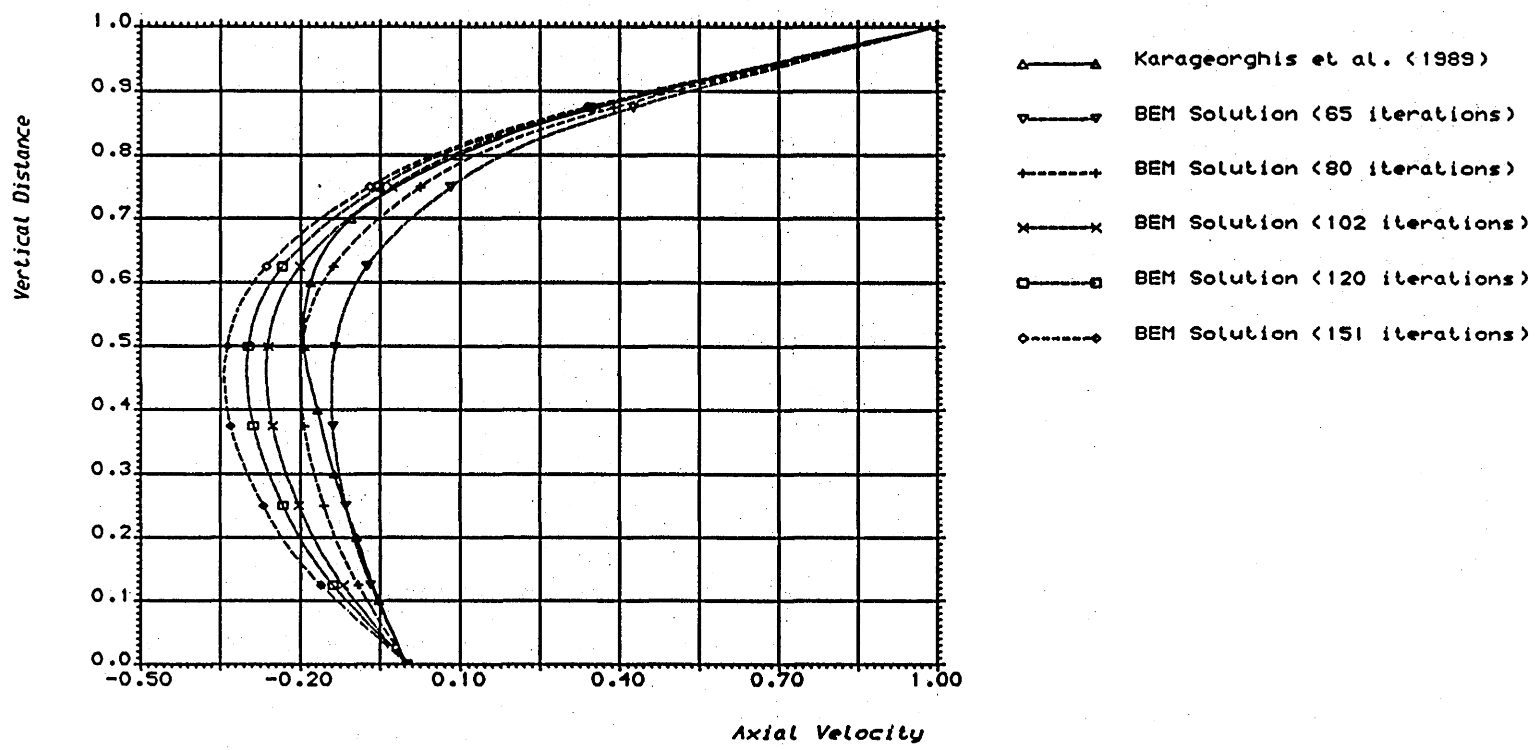

Figure 7.15: Axial Velocity Profiles along the Vertical Distance in the Middle of the Cavity (Re=0).

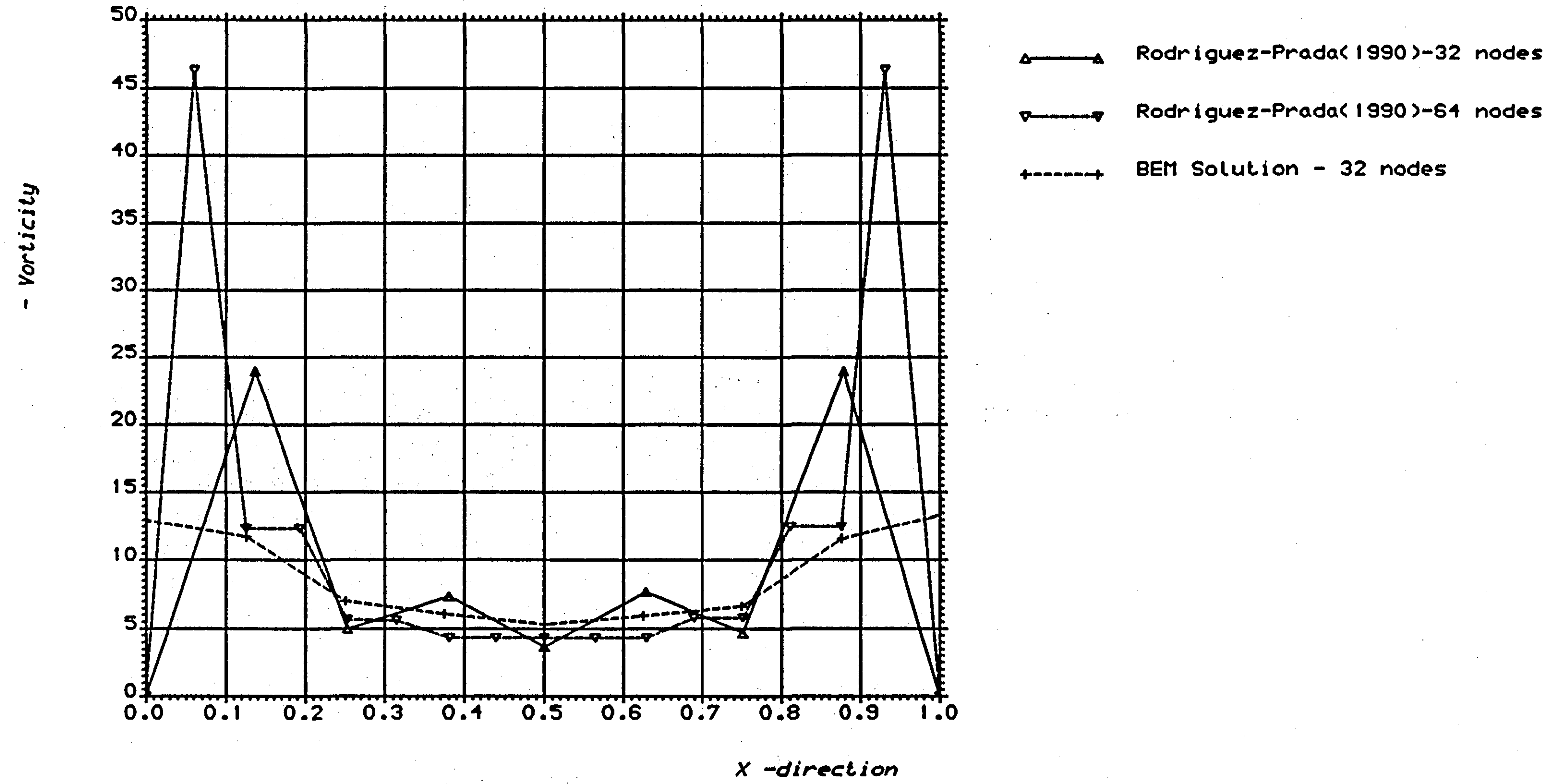

Figure 7.16 : Vorticity Distributions at the Upper Lid Nodes For Re =0.

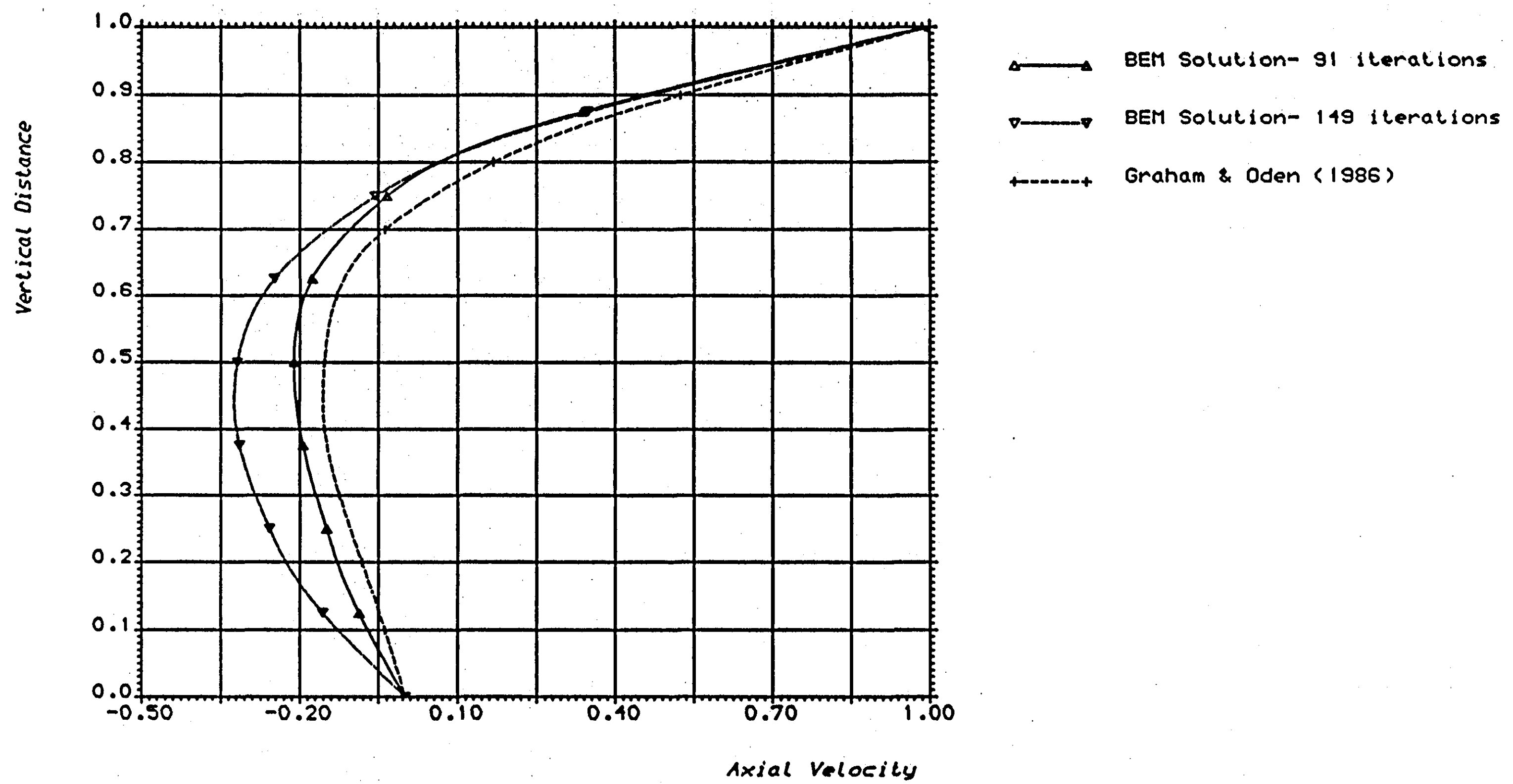

Figure 7.17: Axial Velocity Profiles along the Vertical Distance in the Middle of the Cavity for Re=10.

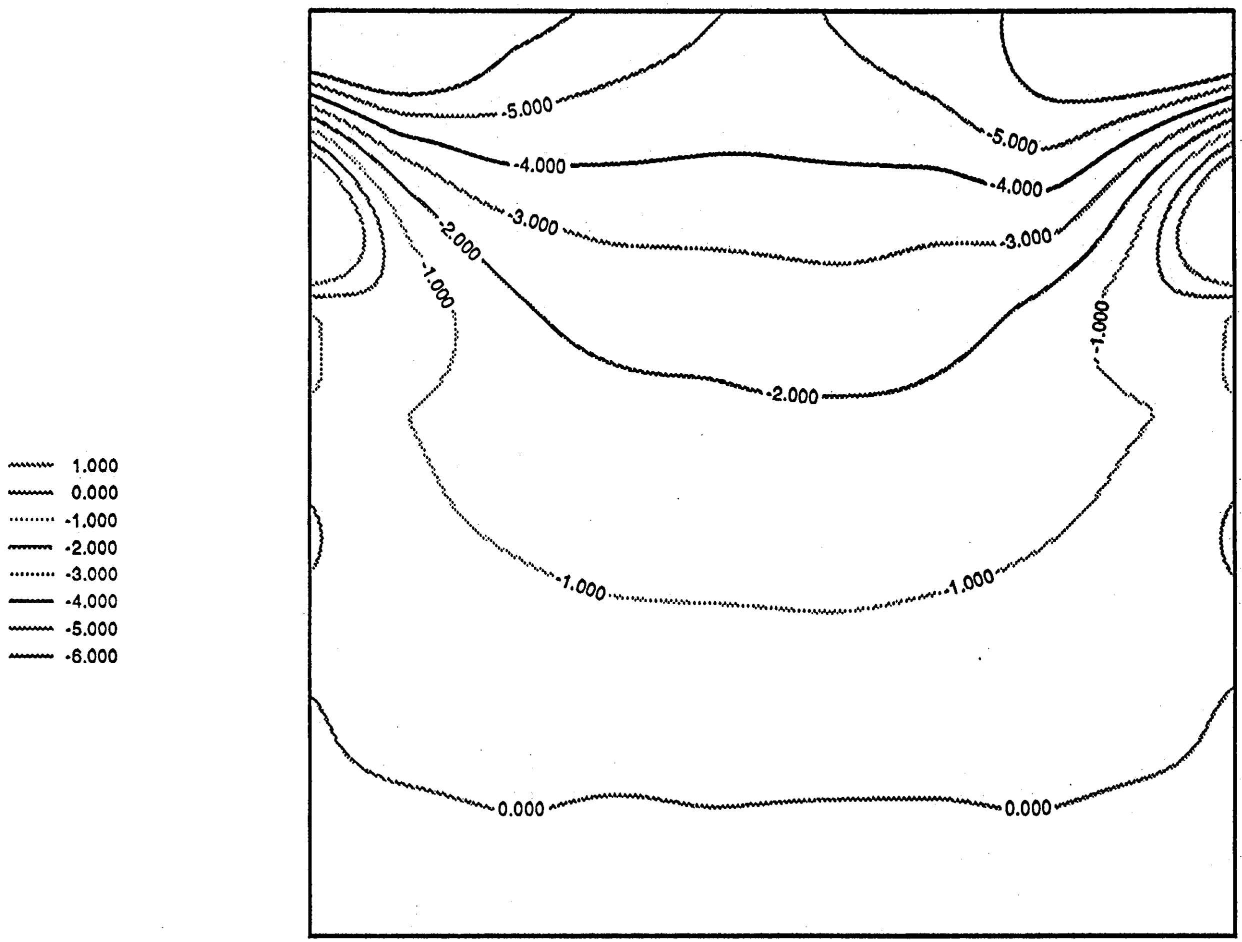

Figure 7.18 : Iso-vorticity Lines for Driven-Cavity Flow with Re = 0 (65 iterations).

VORTICITY

Run on 28-Apr-93, At 16:30:18

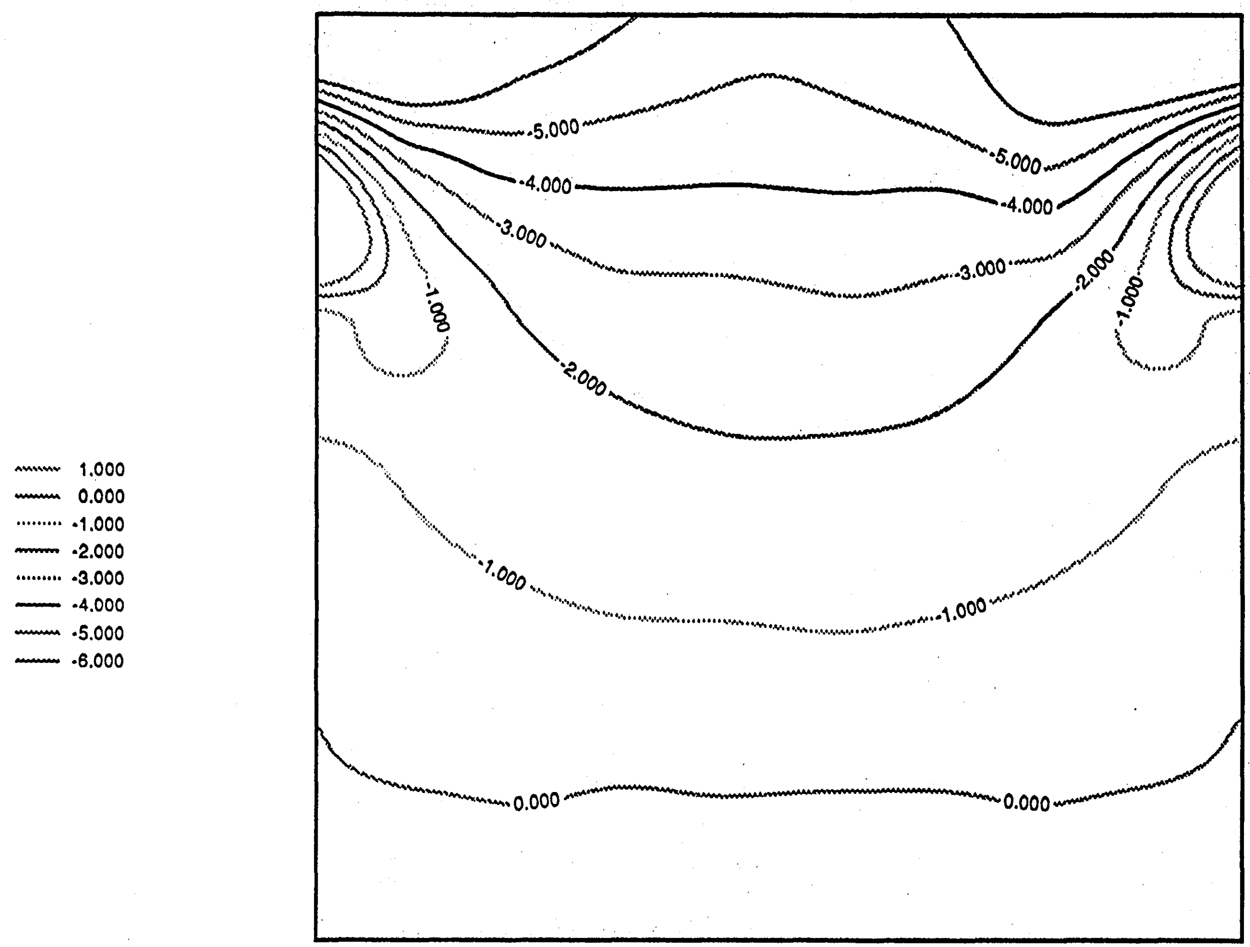

Figure 7.19 : Iso-vorticity Lines for Driven-Cavity Flow with Re = 0 (80 iterations).

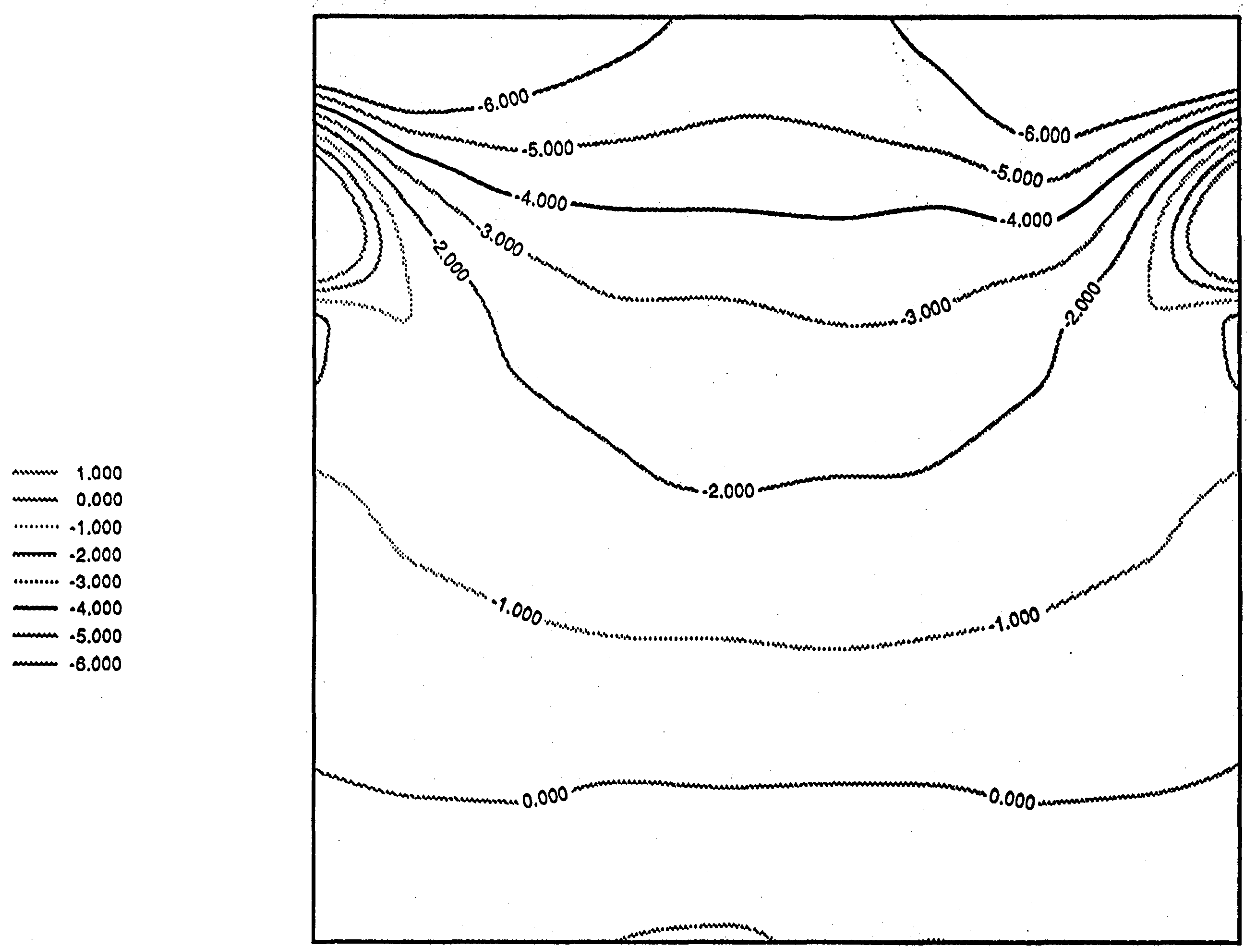

Figure 7.20 : Iso-vorticity Lines for Driven-Cavity Flow with Re = 0 (102 iterations).

VORTICITY

Run on 28-Apr-93, At 17:14:34

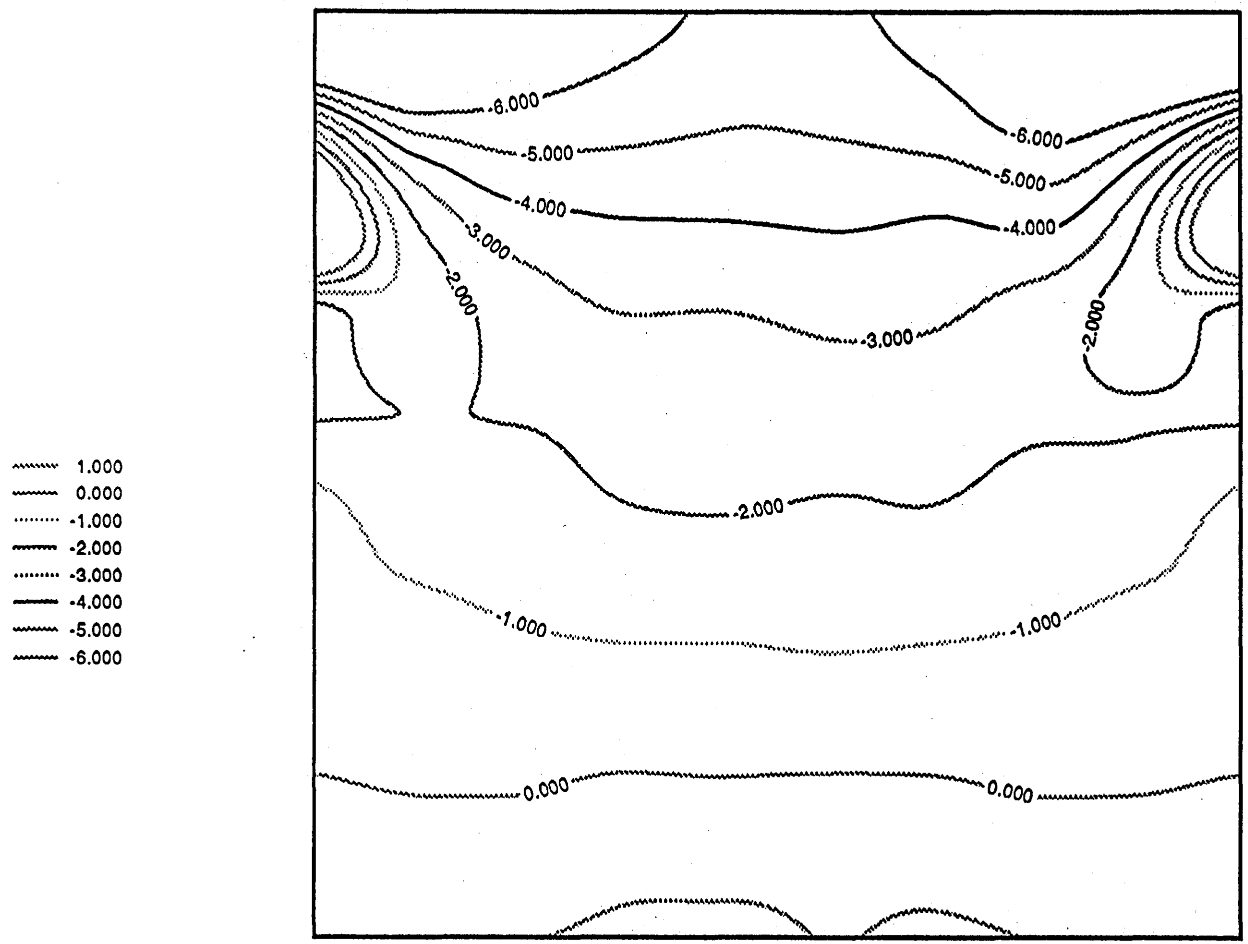

Figure 7.21 : Iso-vorticity Lines for Driven-Cavity Flow with Re = 0 (120 iterations).

VORTICITY

Run on 28-Apr-93, At 17:07:19

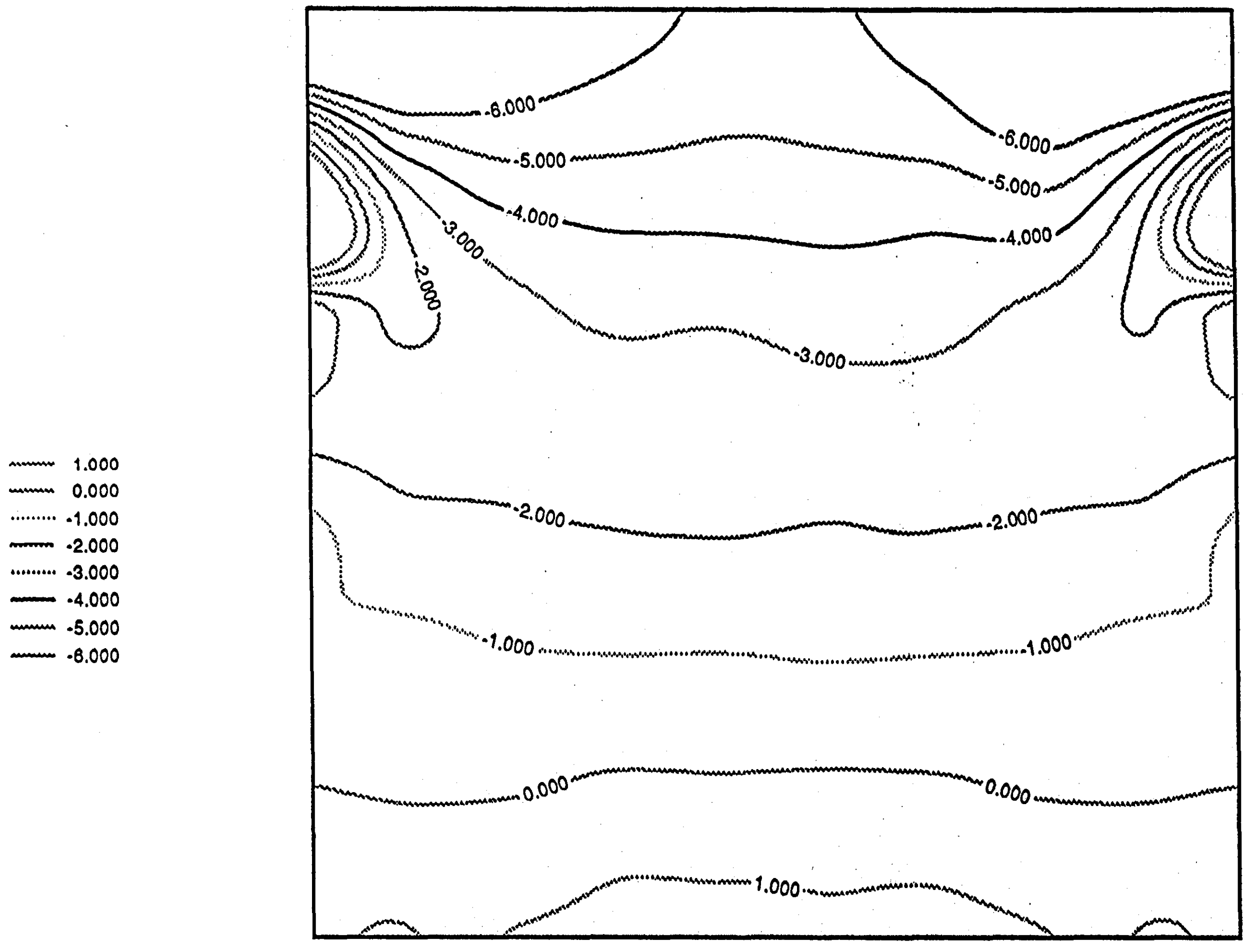

Figure 7.22 : Iso-vorticity Lines for Driven-Cavity Flow with Re = 0 (151 iterations).

VORTICITY

Run on 28-Apr-93, At 17:12:25

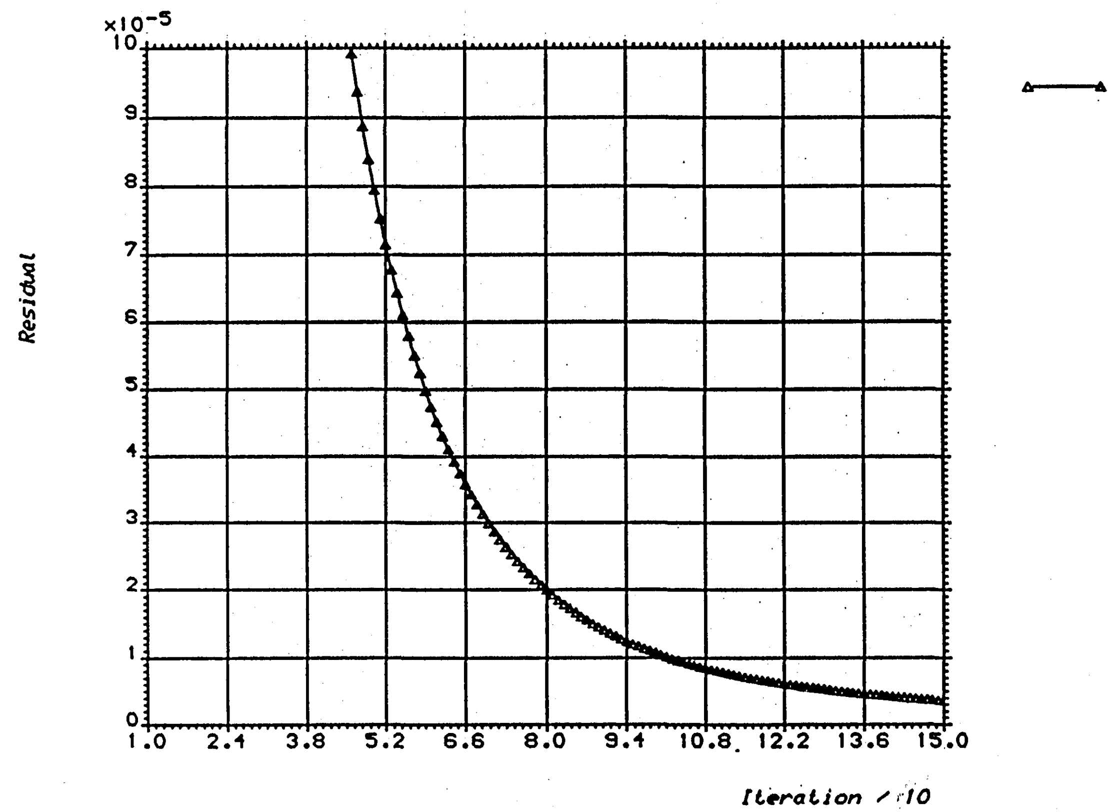

Figure 7.23 : Variation of the Residual with the Number of Iterations For Driven-Cavity Flow with Re=0.

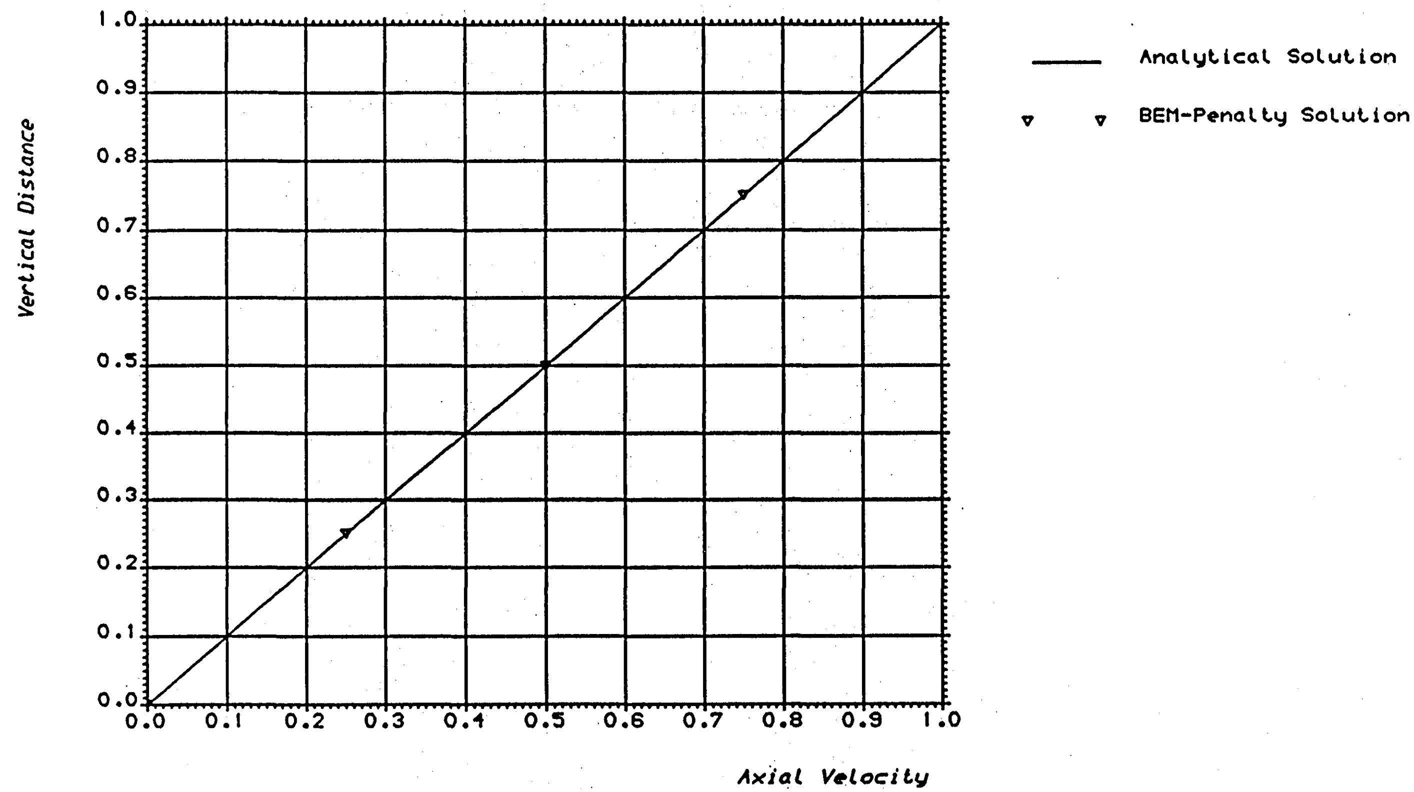

Figure 7.24: Axial Velocity Profile For Couette
Flow with Re=10.

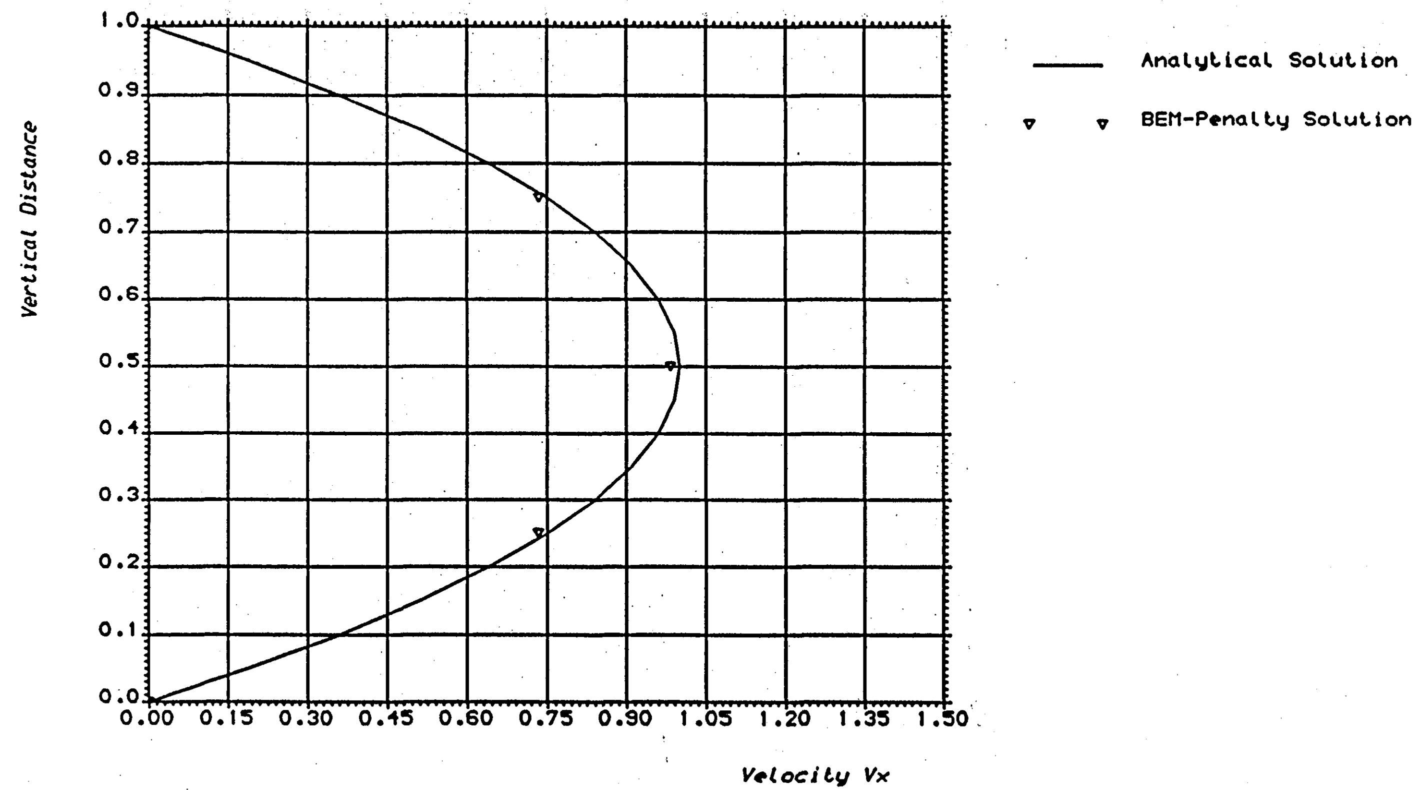

Figure 7.25: Axial Velocity Profile for Poiseuille Flow With Re=10.

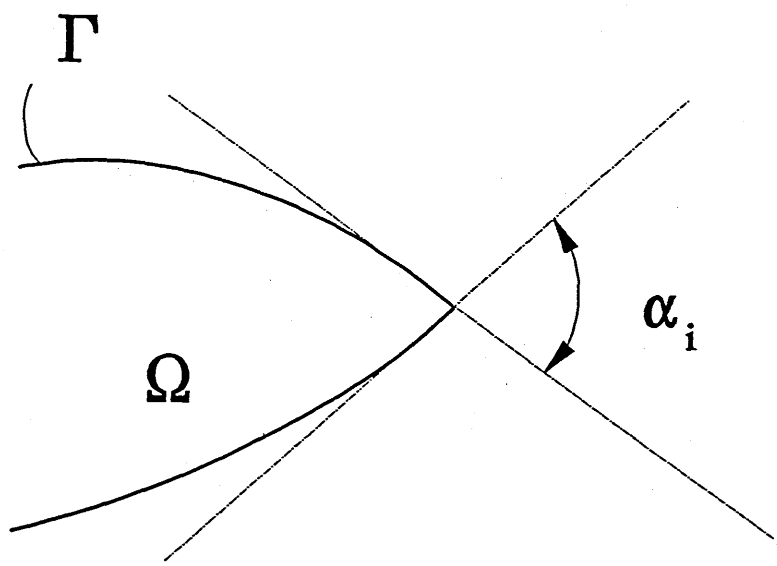

Figure B.1: Definition of Corner Angle α_i.

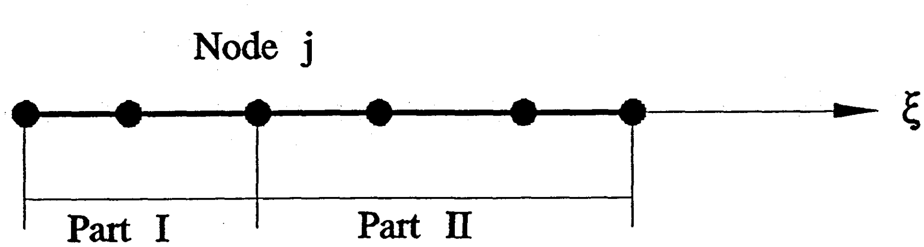

Figure I.1: A Boundary Element with its j^{th} Node as Source Point.