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The Theory of General Instability of Cylindrical Shells

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SUMMARY

Using a new approach to the theoretical study of thin-walled cylinders with discrete reinforcing members developed in this paper, the problem of general instability of such structures is solved with more than usual generality. The principal stages are indicated which lead to the characteristic equation of the general problem in the form of a determinant of order three times the number of reinforcing members, i.e. stringers or rings. The less general problem of distributed stringers and discrete rings is solved completely and it is shown that for the case of one ring at the middle of the cylinder, buckling with axial symmetry, the characteristic equation can be reduced to a very simple closed form.

The method of solution, developed below, must not only be judged in its relation to the problem under consideration. It will be found to be fundamental to the theory of plates and shells in the sense that most problems having an exact solution for the case of the homogeneous structure, can now likewise be solved in the presence of reinforcing members.

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NOTATION

a	Radius of cylinder
a_{mn}, b_{mn}, c_{mn}	Arbitrary coefficients
d_n, f_n, g_n	Arbitrary coefficients in basic solution
h	Skin thickness
$k_x h$	Effective skin thickness for the case of distributed stringers.
k_m, k_n	See (2.1.4)
ℓ	Length of cylinder
m, n	Summation variables
u, v, w	Longitudinal, circumferential, radial displacements.
x, θ	Cylindrical coordinates
x_i	Position of ring
θ_i	Position of stringer
$A'_R = \dfrac{A_R}{a^2}$	
A_R	Area of cross section of ring
E	Young's modulus
I_R	Second moments of area of ring section
I_x	Second moments of area of distributed stringers per unit length of circumference
$I'_R = \dfrac{I_R}{a^4}$	
$\alpha = \dfrac{h^2}{12a^2}$	
$\beta = \dfrac{I_x(1-\nu^2)}{a^2 h}$	
$\alpha_f, \alpha_d, \beta_f, \beta_d, \gamma_f, \gamma_d$	(See 2.2.7)

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$\lambda = \frac{m\pi a}{\ell}$

κ, ψ See (3.2.4)

$\mu = \frac{n}{2}$

ν Poisson's ratio

$\phi = \frac{N_x(1-\nu^2)}{Eh}$

$\Delta()_{x=x_i}$ Magnitude of discontinuous change of function at $x=x_i$

$\Delta_{\lambda n}$ See (2.2.7)

$K = k_x + \nu^2(1-k_x)$

N_x Direct stress resultant in x direction

$O(\frac{1}{\lambda^2})$ Of order $1/\lambda^2$

Q_x Radial shear stress resultant

δ_{ij} Kronecker's delta

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1. INTRODUCTION

The use of structures with thin skins and reinforcing stringers and ribs is due to two factors, the desire to save weight and the desire to save material. While the latter aim has only arisen during the last few years, causing their use in all types of construction, the former has been the guiding principle of aeronautical engineering ever since man made his first flight in a machine heavier than air.

Great difficulties have been experienced in subjecting such structures to an exact theoretical analysis. A considerable amount of work has been done on the stability problems of discretely reinforced rectangular plates, using energy methods. This work is described in some detail in Ref.1, and one of the principal exponents of this approach in England is H.L. Cox who has published a number of papers on the subject (e.g. Ref.6). But this method leads always to an infinite system of simultaneous linear equations, and very often a great deal of ingenuity is required to reduce these to a closed expression.

To the author's knowledge, all attempts to solve stability and other problems in reinforced shell theory have been based on initial, very approximate, assumptions, and even then have led to such involved calculations that it is often very difficult, even for the specialist, to follow the reasoning and to grasp the physical meaning of the final results (e.g. Ref. 2-4).

The method of this report seems to avoid many of these disadvantages, and in some sense will be seen to be equivalent to the energy method, since results, obtained by it, probably could also be deduced in a round about manner by that method. Its strength lies with the fact that it has a simple physical interpretation. Its application is by no means restricted to the problem under consideration, nor, as a matter of fact, to problems of stability. The present paper deals only with the application of the method to the problem of general instability of shells. It is hoped that it will justify the claims made here, and it is the intention of the author to support these statements in the near future by a more comprehensive report, giving the solution of other important problems in the theory of plates and shells.

1.1 Statement of the Problem

The problem of elastic instability is one of the most important of the theory of elasticity, and its nature is explained in most relevant text books. For this reason, no attempt will be made here to describe it in detail. As in the later work, emphasis will be on the part played by the reinforcing members.

Consider a stressed skin structure with ribs and stringers firmly attached to it in such a way that the elastic axes of the reinforcements lie in the middle plane of the skin. While this assumption is not essential, it will be made here for the sake of simplicity, the modifications arising from its removal being obvious. It is known from the theory of structures that, if such a structure deforms, certain of the stress-resultants undergo discontinuous changes at the reinforcing members. From the point of view of the skin, these discontinuous changes manifest themselves in local pressures and shears. On the other hand, from the point of view of the stringer, they can be conceived as loading on the member causing its deformation. While the reinforcing members are of finite width in practice, it is most convenient in theoretical work to idealize them into lines. As a result of this step, the pressures and shears, mentioned above, become infinite, since the line of contact is devoid of area.

After these remarks, the problem under consideration may be formulated as follows: "To find the characteristic equation for the determination of critical loads of a cylinder with thin skin, subject to localized infinite pressures and shears, which depend on the deformation of the shell and the stiffness of the reinforcing members.

1.2 General Discussion of the Method of Solution

A certain amount of hesitation may be shown at first at the idea of introducing infinite pressures. On the other hand, in many other branches of mathematical physics such localized infinities have been used for many years in the form of line sources. The method of solution, to be used here, thus will be seen to be based on a concept which, to the author's knowledge, has been little used in the theory of structures although it occurs in a disguised manner, for example, in Ref.5 in the solution of the problem of a plate loaded over a portion of its surface. But it has been arrived at in an arbitrary manner and the solution used there does not by itself satisfy the differential equation.

What may be termed the source solution of the theory of thin plates, will here be shown to fully satisfy the differential equation under loads, peculiar to the presence of discrete reinforcing members. Due to the indeterminateness of the pressures, as in the case of sources, for example, in aerodynamic theory, the source or basic solution will contain one arbitrary constant, or sets of arbitrary constants, which therefore can be used to satisfy the "internal" boundary conditions at the stringers or rings.

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In the case of stability problems these conditions lead to a homogeneous system of linear equations in the above mentioned constants, and the condition for the existence of non-zero values of these constants presents the required characteristic equation.

Finally, one short remark will be made with regard to the mathematical character of these source solutions. It will be shown later on, that these solutions are in actual fact combinations of the complementary functions and the particular integrals of the non-homogeneous differential equations, which follow from the ordinary stability equations after introduction of the earlier stated type of loading. The solutions, which are in the form of Fourier series, contain a "singular" part in that in the present problem their first or third derivatives with respect to the coordinate at right angles to the stringer or rib are discontinuous. The "singular" part, which is of the form used by S. Timoshenko in Ref.5 is easily separated from the basic solution, and the remaining part can then be shown to be "regular". It is for this reason that the term basic solution has been preferred to that of source solution, because the latter term is normally used for solutions which are purely singular.

1.3 Assumptions

Some of the assumptions to be stated here have been referred to earlier, others will be introduced during the actual analysis. Nevertheless it will be worthwhile to state them here in full. They are:-

A.1 Thin shell theory is applicable.

A.2 The elastic axes of stringers and ribs lie in the middle plane of the skin.

A.3 Stringers and ribs have no torsional stiffness and may be idealized into lines.

A.4 The free edges are simply supported.

When dealing with the simplified shell theory, based on distributed stringers, the following additional assumptions will be required:

A.5 The stringers are so closely spaced that each circumferential wave in the buckled state contains several stringers.

A.6 The stringers add effective area to the skin only as far as the longitudinal direct stress flexural stiffness and radial shear stress are concerned.

A.7 The skin lacks torsional, flexural and shear stiffness and the circumferential displacements are small as compared with those in radial direction.

A.8 The rings have no stiffness for deformation out of their plane.

(The last four conditions are discussed in detail in Ref.3)

As far as A.4 is concerned, this condition will be seen to be automatically satisfied. However, it will be suggested here that the edge conditions may be varied in a manner which is very close to practical conditions, and which it would be very difficult to achieve without the help of the basic solutions. In fact, by choosing the length of the theoretical cylinder somewhat larger and by introducing very stiff rings in the vicinity of one or both ends of the theoretical cylinder in such a way that either the distance between the stiff rings or between one simply supported end and the stiff ring is equal to the length of the actual cylinder, one could solve the problem for the case of clamped ends. The choice of the stiffness of the artificially introduced rings would thus give the means of varying the degree of clamping. However, it is not proposed to investigate this point further in this report, and A.4 will hold throughout the subsequent work.

2. DEDUCTION OF THE BASIC SOLUTIONS

In order to save space, the relevant differential equations and boundary conditions will be stated here without deduction. In the case of the general problem they may be found in section 84 of Ref.1, while for the case of the simplified problem they are essentially contained in Ref.3. As far as possible, the notation used agrees with that of Ref.1.

2.1 The General Case of a Circular Cylinder

The most general stability equations for the case of a thin walled homogeneous cylinder in compression are:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\,\frac{\partial^2 v}{a\,\partial x\,\partial\theta} - \frac{\nu}{a}\,\frac{\partial w}{\partial x} + \frac{1-\nu}{2}\,\frac{\partial^2 u}{a^2\,\partial\theta^2} = 0$$

$$\frac{1+\nu}{2}\,\frac{\partial^2 u}{\partial x\,\partial\theta} + \frac{a(1-\nu)}{2}\,\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{a\,\partial\theta^2} - \frac{\partial w}{a\,\partial\theta} +$$

$$+ \alpha\left[\frac{\partial^2 v}{a\,\partial\theta^2} + \frac{\partial^3 w}{a\,\partial\theta^3} + a\,\frac{\partial^3 w}{\partial x^2\,\partial\theta} + a(1-\nu)\,\frac{\partial^2 v}{\partial x^2}\right] - a\phi\,\frac{\partial^2 v}{\partial x^2} = 0 \qquad (2.1.1)$$

$$-\,a\phi\,\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial u}{\partial x} + \frac{\partial v}{a\,\partial\theta} - \frac{w}{a} -$$

$$-\,\alpha\left[\frac{\partial^3 v}{a\,\partial\theta^3} + (2-\nu)a\,\frac{\partial^3 v}{\partial x^2\,\partial\theta} + a^3\,\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{a\,\partial\theta^4} + 2a\,\frac{\partial^4 w}{\partial x^2\,\partial\theta^2}\right] = 0$$

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These equations represent the conditions of equilibrium in the longitudinal, circumferential and radial directions respectively at each point of the shell. As mentioned in the Introduction, these equations will be solved for the case, when infinite pressures and shears act either along a circle at a station x_i (case of a ring) or along a generator, specified by θ_i (case of a stringer). This loading condition will be produced by introducing on the right hand sides of these equations the following sets of loading functions:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} g_{ni} \sin\frac{n\theta}{2} k_m \cos\frac{m\pi x}{\ell}, \quad \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} d_{ni} \cos\frac{n\theta}{2} \sin\frac{m\pi x_i}{\ell} \sin\frac{m\pi x}{\ell},$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} f_{ni} \sin\frac{n\theta}{2} \sin\frac{m\pi x_i}{\ell} \sin\frac{m\pi x}{\ell} \qquad (2.1.2)$$

for the case of a ring,

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} g_{mi} \cos\frac{m\pi x}{\ell} \sin\frac{n\theta_i}{2} \sin\frac{n\theta}{2}, \quad \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} d_{mi} \sin\frac{m\pi x}{\ell} k_n \cos\frac{n\theta}{2},$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} f_{mi} \sin\frac{m\pi x}{\ell} \sin\frac{n\theta_i}{2} \sin\frac{n\theta}{2} \qquad (2.1.3)$$

for the case of a stringer,

where

$$k_1 = \begin{cases} \cos\dfrac{\pi x_i}{\ell} + \dfrac{\pi}{4} & k_m = \cos\dfrac{m\pi x_i}{\ell} + \dfrac{\cos\frac{m\pi}{2}}{1-m^2} \quad m \geqslant 2 \\[2ex] \cos\dfrac{\theta_i}{2} + \dfrac{\pi}{4} & k_n = \cos\dfrac{n\theta_i}{2} + \dfrac{\cos\frac{n\pi}{2}}{1-n^2} \quad n \geqslant 2 \end{cases} \qquad (2.1.4)$$

have been chosen in such a way that the cosine and sin series used in (2.1 2) and (2 1 3) are completely equivalent (See also section 2.3)

It is easily verified that these loading functions have the required properties, since the relevant parts of the Fourier series converge to zero in a conventional(Abel) Sense for all values of x or θ in the range $0 \leq x \leq \ell$ or $0 \leqslant \theta \leqslant 2\pi$ except when $x = x_i$, $\theta = \theta_i$, the series becoming infinite there.

In order to solve the differential equations (2.1.1) with the loading systems (2.1.2) or (2.1.3), substitute for the displacements the following series:

/ u = ...

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin \frac{n\theta}{2} \cos \frac{m\pi x}{\ell}$$

$$v = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} b_{mn} \cos \frac{n\theta}{2} \sin \frac{m\pi x}{\ell} \tag{2.1.5}$$

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{mn} \sin \frac{n\theta}{2} \sin \frac{m\pi x}{\ell}$$

which after some intermediate calculations reduce the differential equations to the following system of equations for the determination of the arbitrary constants of (2.1.5)

$$\begin{bmatrix} \lambda^2 + \frac{1-\nu}{2}\mu^2 & \frac{1+\nu}{2}\mu\lambda & \nu\lambda \\ \frac{1+\nu}{2}\mu\lambda & \frac{1-\nu}{2}\lambda^2 + \mu^2 + \alpha(1-\nu)\lambda^2 + \alpha\mu^2 - \lambda^2\emptyset & \mu + \alpha\mu(\lambda^2+\mu^2) \\ \nu\lambda & \mu\left[1 + \alpha\{\mu^2 + (2-\nu)\lambda^2\}\right] & 1 - \lambda^2\emptyset + \alpha(\lambda^2+\mu^2)^2 \end{bmatrix} \begin{bmatrix} a_{mn} \\ b_{mn} \\ c_{mn} \end{bmatrix}$$

$$= \Delta(\lambda\mu) \begin{bmatrix} a_{mn} \\ b_{mn} \\ c_{mn} \end{bmatrix} = \begin{cases} \begin{bmatrix} g_{ni} & k_m \\ d_{ni} & \sin\frac{m\pi x_i}{\ell} \\ f_{ni} & \sin\frac{m\pi x_i}{\ell} \end{bmatrix} & \text{for the case of a ring} \\ \begin{bmatrix} g_{ni} & \sin\frac{n\theta_i}{2} \\ d_{ni} & k_n \\ f_{ni} & \sin\frac{n\theta_i}{2} \end{bmatrix} & \text{for the case of a stringer} \end{cases} \tag{2.1.6}$$

The determinant of this system of equations vanishes for critical loads, i.e. values of $\emptyset$, which correspond to the buckling of the homogeneous cylinder, since

$$\Delta(\lambda_{\mu}) = 0$$

is the characteristic equation of the homogeneous problem. Hence (2.1.6) will normally have finite solutions for the a_{mn}, b_{mn}, c_{mn}. The solutions (2.1.5) with the appropriate values of the constant coefficients are the basic solutions of this problem.

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2.2 The Simplified Theory of a Circular Cylinder with Distributed Stringers.

Using A.5 to A.7 of section 1.3, the differential equations take in this case the following form:

$$K\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x\partial\theta} + \frac{1-\nu}{2}\frac{\partial^2 u}{a^2\partial\theta^2} - \frac{\nu}{a}\frac{\partial w}{\partial x} = 0$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial\theta} + a\frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{a\partial\theta} + \frac{\partial^2 v}{a\partial\theta^2} = 0 \qquad (2.2.1)$$

$$-a\phi\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial u}{\partial x} + \frac{\partial v}{a\partial\theta} - \frac{w}{a} - a^3\beta\frac{\partial^4 w}{\partial x^4} = 0$$

The two stress resultants, affected by A.6 and used in deducing (2.2.1) are now given by

$$N_x = \frac{hE}{1-\nu^2}\left[K\frac{\partial u}{\partial x} + \frac{\nu}{a}\left(\frac{\partial v}{\partial\theta} - w\right)\right], \quad Q_x = -EI_x\frac{\partial^3 w}{\partial x^3} \qquad (2.2.2)$$

where

$$K = k_x + \nu^2(1-k_x), \quad \beta = \frac{I_x(1-\nu^2)}{a^2 h} \qquad (2.2.3)$$

The equations (2.2.1) are the conditions of equilibrium for an orthotropic shell. Using A.8, the loading system for the case of one ring becomes

$$0\,, \quad \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} d_{ni}\sin\frac{m\pi x_i}{\ell}\cos n\theta\sin\frac{m\pi x}{\ell}\,, \quad \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} f_{ni}\sin\frac{m\pi x_i}{\ell}\sin n\theta\sin\frac{m\pi x}{\ell}$$

$$(2.2.4)$$

and proceeding in the same manner as in the general case one arrives at the system of equations

$$\begin{bmatrix} K\lambda^2 + \frac{1-\nu}{2}n^2 & \frac{1+\nu}{2}n\lambda & \nu\lambda \\ \frac{1+\nu}{2}n\lambda & \frac{1-\nu}{2}\lambda^2 + n^2 & n \\ \nu\lambda & n & 1+\beta\lambda^4-\phi\lambda^2 \end{bmatrix}\begin{bmatrix} a^i_{mn} \\ b^i_{mn} \\ c^i_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ d_{ni}\sin\frac{m\pi x_i}{\ell} \\ f_{ni}\sin\frac{m\pi x_i}{\ell} \end{bmatrix}$$

$$(2.2.5)$$

/ Again . .

Again, the vanishing of the determinant of this system gives the characteristic equation for the homogeneous orthotropic cylinder. Assuming the critical loads of the cylinder with a ring to be different from those of the homogeneous shell, one finds from (2.2.5)

$$a^i_{mn} = \left[d_{ni}\alpha_d + f_{ni}\alpha_f\right] \sin\frac{m\pi x_i}{\ell} = \frac{\sin\frac{m\pi x_i}{\ell}}{\Delta_{\lambda n}} \left[d_{ni} \left\{ \frac{1+\nu}{2}(\phi - \beta\lambda^2)\lambda^2 - \frac{1-\nu}{2} \right\} n\lambda \right.$$
$$\left. + f_{ni} \frac{1-\nu}{2} \left\{ n^2 - \nu\lambda^2 \right\} \lambda \right]$$

$$b^i_{mn} = \left[d_{ni}\beta_d + f_{ni}\beta_f\right] \sin\frac{m\pi x_i}{\ell} = \frac{\sin\frac{m\pi x_i}{\ell}}{\Delta_{\lambda n}} \left[d_{ni} \left\{ \left(K\lambda^2 + \frac{1-\nu}{2} n^2\right)(1+\beta\lambda^4 - \phi\lambda^2) - \nu^2\lambda^2 \right\} \right.$$
$$\left. + f_{ni} \left\{ \frac{1+\nu}{2}\nu\lambda^2 - \left(K\lambda^2 + \frac{1-\nu}{2} n^2\right) \right\} n \right]$$

$$c^i_{mn} = \left[d_{ni}\gamma_d + f_{ni}\beta_f\right] \sin\frac{m\pi x_i}{\ell} = \frac{\sin\frac{m\pi x_i}{\ell}}{\Delta_{\lambda n}} \left[d_{ni} \left\{ \lambda^2 \left(\frac{1+\nu}{2}\nu - K\right) - \frac{1-\nu}{2} n^2 \right\} \right.$$
$$\left. + f_{ni} \left\{ \frac{1-\nu}{2}(K\lambda^4 + n^4) + (K-\nu)\lambda^2 n^2 \right\} \right] \qquad (2.2.6)$$

where

$$\Delta_{\lambda n} = \left[\left\{ \beta\lambda^2 - \phi \right\} \left\{ \frac{1-\nu}{2}(n^4 + K\lambda^4) + n^2\lambda^2 (K-\nu) \right\} + \frac{1-\nu}{2}\lambda^2 (K - \nu^2) \right] \lambda^2 \qquad (2.2.7)$$

Thus, similarly to (2.1.5) one has in this case the basic solutions

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[d_{ni}\,\alpha_d + f_{ni}\,\alpha_f \right] \sin\frac{m\pi x_i}{\ell} \sin n\theta \cos\frac{m\pi x}{\ell}$$

$$v = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[d_{ni}\,\beta_d + f_{ni}\,\beta_f \right] \sin\frac{m\pi x_i}{\ell} \cos n\theta \sin\frac{m\pi x}{\ell} \qquad (2.2.8)$$

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[d_{ni}\,\gamma_d + f_{ni}\,\gamma_f \right] \sin\frac{m\pi x_i}{\ell} \sin\frac{m\pi x}{\ell} \sin n\theta$$

These solutions will be used in section 3. to obtain the characteristic equation for the critical loads. However, the understanding of the work of that section will be greatly helped by the short discussion of the properties of the basic solutions, which is given in the next section.

2.3 Discussion of the Properties of the Basic Solutions

It is obvious from the form of the solutions, deduced above, that they satisfy the assumption of simply supported edges. So there remains only to show that these solutions are complete, that their relevant derivatives are discontinuous and that they are otherwise "regular"

It is seen from (2.2.6) and (2.2.7) that

$$b^i_{mn} = O(1/\lambda^2) \quad , \quad c^i_{mn} = O(1/\lambda^4) \tag{2.3.1}$$

while

$$b^i_{mn} - \frac{2}{1-\nu}\frac{d_{ni}}{2} = O(1/\lambda^4) \quad , \quad c^i_{mn} - \frac{f_{ni}}{\beta\lambda^4} = O(1/\lambda^6) \tag{2.3.2}$$

Thus the first and third derivatives of v and w with respect to x are discontinuous at x_i, since $\lambda = \frac{m\pi a}{\ell}$ and

$$\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{m\pi x_i}{\ell}\cos\frac{m\pi x}{\ell} = \begin{cases} -\frac{1}{2}\pi\frac{x}{\ell} & 0 \leq x \leq x_i \\ \frac{1}{2}\pi\left(1-\frac{x}{\ell}\right) & x_i \leq x \leq \ell \end{cases} \quad \text{for} \tag{2.3.3}$$

If Δ denotes the magnitude of the discontinuities, it is seen that

$$\Delta\left(\frac{\partial v}{\partial x}\right)_{x=x_i} = \frac{1}{1-\nu}d_{ni}\frac{\pi^2}{\ell} \quad , \quad \Delta\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=x_i} = -\frac{1}{\beta}f_{ni}\frac{\pi^4}{2\ell^3} \tag{2.3.4}$$

The expressions (2.3.2) show, that, as far as the differential equations (2.2.1) are concerned, the remainders of the basic solutions are regular in x. Similar arguments can be applied to the other derivatives of u, v, w which occur in the differential equations, and it can be shown in this manner that the discontinuities, indicated above, are the only ones of sufficiently low order to be of interest here. Further, since the basic solutions contain regular as well as singular parts they will be complete, and by a suitable uniqueness theorem they are the only solutions, as they satisfy the boundary conditions. This latter point will become yet more obvious in the next section dealing with the deduction of the characteristic equation and the "internal" boundary conditions.

3. THE CHARACTERISTIC EQUATION FOR THE SIMPLIFIED THEORY

It has been shown in section 2.3 that the basic solutions (2.2.8) possess analytical properties, which will be required in the process of satisfying the "internal" boundary conditions. As indicated in the Introduction, these conditions lead to a homogeneous system of equations for the determination of the d_{ni} and f_{ni}. Since there are only two sets of such constants for each ring, it will be expected that the characteristic determinant in the present case will be of order 2g, where g is the number of rings.

3.1 Circular Cylinder with Several Rings

The relevant boundary conditions at the rings are discussed in detail in Refs. 3 and 4. They are:

$$\left[I'_{Ri}\frac{\partial^3}{\partial\theta^3}\left\{\frac{\partial w}{\partial\theta}+v\right\} - A'_{Ri}\left\{\frac{\partial v}{\partial\theta}-w\right\}\right]_{x=x_i} = -I_{xi}\,\Delta\left(\frac{\partial^3 w}{\partial x^3}\right)_{x=x_i} \quad (i = 1,2,\dots,g) \qquad (3.1.1)$$

$$\left[I'_{Rl}\frac{\partial^2}{\partial\theta^2}\left\{\frac{\partial w}{\partial\theta}+v\right\} + A'_{Rl}\frac{\partial}{\partial\theta}\left\{\frac{\partial v}{\partial\theta}-w\right\}\right]_{x=x_l} = -\frac{h}{2(1+\nu)}\,\Delta\left(\frac{\partial u}{a\partial\theta}+\frac{\partial v}{\partial x}\right)_{x=x_l}$$

where the left hand sides are the elastic forces in the ring which have to be in equilibrium with the external loading, from the point of view of the ring, represented by the terms on the right hand sides.

If there is more than one ring, the appropriate expressions for the displacements may be obtained by adding several of the basic solutions (2.2.8). Substituting the composite solutions, thus obtained into (3.1.1) and using (2.3.4) one finds the following determinantal condition for the existence of non-zero values of the d_{ni} and f_{ni}:

$$\left|\begin{array}{c|c}
\frac{-I_x\pi^4}{\beta 2\ell^3}\delta_{ij} + \sum\limits_{m=1}^{\infty}\left[I_{Ri}(n^4\gamma_f+n^3\beta_f) + A'_{Ri}(n\beta_f+\gamma_f)\right]\sin\frac{m\pi x_i}{\ell}\sin\frac{m\pi x_j}{\ell} & \sum\limits_{m=1}^{\infty}\left[I'_{Ri}(n^4\gamma_d+n^3\beta_d) + A'_{Ri}(n\beta_d+\gamma_d)\right]\sin\frac{m\pi x_i}{\ell}\sin\frac{m\pi x_k}{\ell} \\
\hline
\sum\limits_{m=1}^{\infty}\left[I'_{Rl}(n^3\gamma_f+n^2\beta_f) + A'_{Rl}(n^2\beta_f+n\gamma_f)\right]\sin\frac{m\pi x_l}{\ell}\sin\frac{m\pi x_j}{\ell} & \frac{-h\pi^2}{2(1-\nu^2)}\delta_{lk} + \sum\limits_{m=1}^{\infty}\left[I'_{Rl}(n^3\gamma_d+n^2\beta_d) + A'_{Rl}(n^2\beta_d+n\gamma_d)\right]\sin\frac{m\pi x_l}{\ell}\sin\frac{m\pi x_k}{\ell}
\end{array}\right| = 0$$

$$= 0 \qquad (3.1.2)$$

where $i,\ j,\ l,\ k = 1,\ 2,\ \dots,\ g$.

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By (2.2.6) the coefficients under the sum signs above are

$$n^4\gamma_f + n^3\beta_f = \frac{n^4}{\Delta_{\lambda n}}\left[\frac{1-\nu}{2}\left\{K\lambda^4 + n^2(n^2-1)\right\} + \lambda^2\left\{K(n^2-1) + \nu\left(\frac{1+\nu}{2} - n^2\right)\right\}\right]$$

$$n\beta_f + \gamma_f = \frac{\lambda^2}{\Delta_{\lambda n}}\,\frac{1-\nu}{2}\left[K\lambda^2 - \nu n^2\right]$$

$$n^4\gamma_d + n^3\beta_d = \frac{n^3}{\Delta_{\lambda n}}\left[n^2\left\{\lambda^2\left(\nu\frac{1+\nu}{2} - K\right) + \frac{1-\nu}{2}(1-n^2)\right\} + \lambda^2\left\{\left(K\lambda^2 + \frac{1-\nu}{2}n^2\right)(\beta\lambda^2 - \emptyset\lambda) + (K-\nu^2)\right\}\right]$$

$$n\beta_d + \gamma_d = \frac{n\lambda^2}{\Delta_{\lambda n}}\left[\left(K\lambda^2 + \frac{1-\nu}{2}n^3\right)(\beta\lambda^2 - \emptyset) + \nu\frac{1-\nu}{2}\right] \tag{3.1.3}$$

where $\Delta_{\lambda n}$ is given by (2.2.7)

Equation (3.1.2) is the final expression for the determination of the critical loads. Inspection of the coefficients (3.1.3), remembering that $\Delta_{\lambda n} = O(\lambda^8)$ shows that the last two are of order λ^{-2} while the first two are of order λ^{-4}. Depending on the magnitude of n, it will normally be sufficient to retain only a few terms of these series. However, the numerical application of this equation should be the object of a separate investigation, and it will be satisfactory at present, that the series involved converge. In the following section, one special case will be studied in greater detail and it will be shown that for axially symmetrical buckling a closed expression can be obtained.

3.2 Cylinder with one ring at the centre. Axially symmetric buckling

In this case (3.1.2) reduces to:

$$\begin{vmatrix} \dfrac{-I_x\pi^4}{2\ell^3\beta} + \sum\limits_{m=1}^{\infty}\left[I_R'(n^4\gamma_f+n^3\beta_f)+A_R'(n\beta_f+\gamma_f)\right]\sin^2\dfrac{m\pi}{2} & \sum\limits_{m=1}^{\infty}\left[I_R'(n^4\gamma_d+n^3\beta_d)+A_R'(n\beta_d+\gamma_d)\right]\cdot\sin^2\dfrac{m\pi}{2} \\ \sum\limits_{m=1}^{\infty}\left[I_R'(n^3\gamma_f+n^2\beta_f)+A_R'(n^2\beta_f+n\gamma_f)\right]\sin^2\dfrac{m\pi}{2} & -\dfrac{h\pi^2}{2(1-\nu^2)} + \sum\limits_{m=1}^{\infty}\left[I_R'(n^3\gamma_d+n^2\beta_d) + A_R'(n^2\beta_d+n\gamma_d)\right]\sin^2\dfrac{m\pi}{2} \end{vmatrix} = 0$$

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This equation is still rather complicated and because of the complexity of the coefficients (3.1.3) no attempt will be made to sum the series here. However, in the case of axially symmetrical buckling, i.e. when $n = 0$, a closed expression is easily obtained, as will now be shown; but unfortunately this case is of little practical importance.

For $n = 0$, (3.2.1) becomes, using (2.2.3), (2.2.6) and (2.2.7)

$$\frac{-I_x \pi^4}{2 \varrho^3 \beta} + \sum_{m=1}^{\infty} A'_R \gamma_f \sin^2 \frac{m\pi}{2} = - \frac{a^2 h \pi^4}{2 \varrho^3 (1-\nu^2)} + A'_R K \sum_{m=1}^{\infty} \frac{\sin^2 \frac{m\pi}{2}}{\left[(\beta \lambda^4 - \phi \lambda^2)K + K - \nu^2\right]} = 0$$

Summing the series, using the relation

$$\sum_{0}^{\infty} \frac{1}{(2m+1)^2 - \zeta^2} = \frac{\pi}{4\zeta} \tan \frac{\pi \zeta}{2} \tag{3.2.2}$$

which holds for all complex values of ζ, except for $\zeta = \pm 1, \pm 3$, etc. one finds finally

$$\frac{4 I_x}{\varrho^3} \pi^3 \kappa \sqrt{\psi^2 - \kappa^2} = \sqrt{\psi - \sqrt{\psi^2 - \kappa^2}} \tan \frac{\pi}{2} \sqrt{\psi + \sqrt{\psi^2 - \kappa^2}} - \sqrt{\psi + \sqrt{\psi^2 - \kappa^2}} \tan \frac{\pi}{2} \sqrt{\psi - \sqrt{\psi^2 - \kappa^2}} \tag{3.2.3}$$

where

$$\kappa = \sqrt{\frac{K - \nu^2}{K\beta}} = \sqrt{\frac{k_x a^2 h}{K I_x}}, \quad \psi = \frac{\phi}{2\beta} = \frac{N_x a^2}{2 E I_x}, \quad \sqrt{\psi \pm \sqrt{\psi^2 - \kappa^2}} \neq \pm 1, \pm 3, \ldots \tag{3.2.4}$$

Since N_x is positive for compression and all other quantities are positive, all the roots will be real provided $\psi^2 > \kappa^2$, i.e.

$$\frac{N_x^2 a^2}{4 E^2 I_x} > \frac{k_x h}{K} \sim h \tag{3.2.5}$$

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4. CONCLUSIONS

Using special solutions of the stability equations the problem of general instability of circular cylinders can be solved exactly and, in the general case, when there are only rings or stringers present, the characteristic equation for the critical loads will be a determinant of order three times the number of reinforcing members.

On the basis of a simplified shell theory, requiring assumptions additional to those of the theory of thin shells, the complete solution of the problem of a cylinder with distributed stringers and discrete rings has been obtained. In this case the order of the determinantal equation for the critical loads is twice the number of rings. Special consideration has been given to the case of a cylinder with one ring at the centre and a closed expression has been deduced for the case of axially symmetrical buckling.

The method of solution of problems of thin walled structures with stringers and rings, developed in this report, is equally applicable to problems, other than those of elastic stability. Its advantage lies with the fact that it does not lead to systems of equations with infinitely many unknowns. Since it is based on types of solutions of the equilibrium or stability equations which appear to be inherent to the case of plates or shells, reinforced by discrete members, it may well be said that these solutions may be conceived as a suggestion towards the use of such reinforcements.

5. ACKNOWLEDGEMENT

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/List ....

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