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The Potential due to a Source moving through a Compressible Fluid and Applications to some Rotary Derivatives of an Aerofoil

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SUMMARY

The first part of this note concerns the evaluation of the potential at a fixed point in space due to an arbitrarily moving source. The method is then applied to the calculation of the disturbance at a point fixed relative to a source moving in a helical path where conditions are invariant with time. An explicit relation for the potential is obtained if the rate of rotation is assumed small, and the results are applied to the calculation of the pressure distribution on a wing in a uniform rotary motion in yaw at supersonic speeds. The quasi-static yawing derivative of the rolling moment is then calculated for an infinite aspect ratio wing. It is found that the curvature of the path of the wing must be taken into account, except in the particular case of zero sweepback of the wing leading edge. Below a certain supersonic Mach Number the rolling moment is unstable, and this effect is most pronounced for high sweepback.

The results are based on a consideration of the classical wave equation for the potential in a compressible, but inviscid, gas. The construction of the required potential follows the method of Lienard and Wiechert in the electro-magnetic theory of the moving point charge.

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*Whilst engaged on this investigation the author held the Busk Studentship.

C O N T E N T S

NOTATION

M	$= \sqrt{M_1^2 + M_2^2}$.
M_o	$= \omega R_o/a$.
M_1	$= \omega \rho/a$.
M_2	$= V/a$.
R	= distance of field point from axis of rotation.
R_o	= distance of centre of rotation from reference point on wing centre line.
R'	= distance between effective source and field point.
U	= velocity of movement of wing leading edge due to rotation.
V	= forward speed along axis in helical motion.
V_o	$= \omega R_o$.
$\underline{V}$	= vector velocity of source at its effective position.
$\bar{X}$	= distance behind wing apex of reference point on wing centre line.
(X,Y)	system of rectangular coordinates taken parallel and perpendicular to wing centre line respectively (see Figure 6).
a	speed of sound.
b	wing span.
c	x-coordinate of leading edge of wing at $z = 0$.
d	$= R - R_o$.
m	source strength.
n	distance from wing leading edge.
p	pressure.
p_o	free-stream pressure.
Δp	pressure difference between top and bottom surfaces of wing.
r	distance between source and field point.
s	(see Figure 2).
t	time.
x,y,z	(paras.1-3) system of cartesian coordinates.
	(para.4 et seq.) system of helical coordinates (see Figures 3 and 4).

NOTATION (contd.)

α wing incidence.

β $= \sqrt{M^2 - 1}$.

γ angle between leading edge and radius vector to axis of rotation.

γ_o angle of sweep of wing leading edge.

ξ, η, ζ position of effective source element.

ρ radius of curvature of helical path.

σ source volume distribution.

τ retarded time $= t - \frac{R'}{a}$.

ϕ velocity potential.

ψ (see Figure 2).

ω rate of angular rotation.

1. Introduction

In any discussion of the stability of an aircraft in one plane there are three conditions of flight to be studied. For instance, in yawing motion, we may consider firstly the conditions resulting from a fixed angle of sideslip ($\beta \neq 0$ but $r = 0$); secondly, those resulting from a uniform rotary motion in yaw (with $\beta = 0$ but $r \neq 0$); and finally, the conditions in forward motion with oscillation in yaw ($\beta \neq 0$ but $r = 0$). In general, motion might consist of sideslip, rotation and yaw combined, and similar conclusions apply to the motion in other planes. The first and second of the conditions may be deemed 'steady motion' because conditions on the wing surface are invariant with time, whereas the last mentioned motion involves changing conditions on the wing surface during the oscillations. The first condition is amenable to discussion by the usual methods involving uniform motion of the elements of the aircraft through the air in straight lines. The second is typical of the type of motion which we shall consider here, where elements of the aircraft wing surface are all moving in a circular path about some common fixed centre of rotation. In the case of a uniform rolling motion the elements of the wing are moving in a helical path through space with a common axis of rotation, and because a circular movement in the plane of yaw or pitch is no more than a particular case of a helical movement, we shall consider the latter as being the general rotary movement in which conditions at points fixed relative to the aircraft are steady.

In a steady rotary motion the rolling, pitching or yawing derivatives of a wing may be defined in terms of the difference between the forces (or moments) acting on the wing in such a rotary motion and those engendered while moving in a straight path at the same forward speed (measured at some fixed reference point on the wing centre line). The derivatives are, in fact, written as the limit

$$\lim_{\Omega \to 0} \frac{\text{Difference in force (or moment)}}{\Omega} \div \text{dimensional constant}$$

where Ω is the appropriate rate of rotation, and in this limiting form they take what is called their 'quasi-static' value.

It is often found that some investigators of these 'quasi-static' rotary derivatives have used the argument that, as $\Omega \to 0$, the free-stream over the aerofoil becomes parallel, and hence the linearised equation for the potential:

$$(M^2 - 1)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \ldots\ldots(1.1)$$

for steady straight motion (at the representative local Mach Number M) is valid at any point on the wing.

Clearly, though indeed it is a convenient simplification, there is a need to investigate how far such an assumption may be justified.

In this investigation - founded upon a previous thesis[1] - an attempt is made to bring the 'curvature effect' into account. In the first instance we shall describe the method of construction of the velocity potential, due to an arbitrarily moving source, which satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{a^2}\frac{\partial^2 \phi}{\partial t^2} \quad \ldots\ldots(1.2)$$

A source is chosen,for, under certain conditions of the supersonic flow, the surface element of an aerofoil may be replaced by a source distribution whose density is adjusted to satisfy the local boundary conditions on that surface. We have seen that the most general steady motion of a surface element of the aircraft is a helix, so that - as an illustration of this method of constructing the potential - we shall therefore consider the potential due to a source moving on a helical path.

An explicit statement of this potential is not generally possible to obtain, as such a statement requires the solution first of a transcendental equation. However, we shall find an approximate explicit solution, bearing in mind that we are only interested in conditions of the flow at distances from the moving source which are small compared with the radius of curvature; for, in the study of quasi-static derivatives, the rate of angular rotation may be considered as infinitesimal, so that in a uniform rotary motion at a finite speed of flight, V_1, it follows that the radius of curvature is of the order of V/Ω and is (in the limit) large compared to, say, the wing dimensions.

Finally, we shall apply the results we obtain in this way to the study of the pressure field due to a wing describing a circular path with uniform supersonic velocity about some fixed centre of rotation. Such a condition corresponds to that of a steady rotary motion in yaw and the results we obtain are valid if the angular rate of rotation becomes vanishingly small. It is our intention to show that the curvature of the path can be an important first-order effect, even in this limiting condition.

For simplicity we shall confine our discussion to a study of those parts of the wing (with a supersonic but swept-back leading edge) where there are no tip effects, and no interference from the wing centre section. Nor shall we consider the effects of wing thickness, twist or dihedral. We shall indicate finally the lines of possible extension of this theory to more general problems. It is mainly the intention here to state the fundamental results and indicate the method of solution, by way of an example, rather than to attempt an exhaustive survey of possible applications.

2. The Potential of an Arbitrarily Moving Source of Varying Strength*

We are here concerned in the construction of the potential of an arbitrarily moving source of varying density which satisfies the Laplace Wave Equation (1.2). A solution of this problem has been obtained by Lienard[2] and Wiechert[3], who found it in analogous form when considering the electro-magnetic field due to a moving point charge. Although in a sense 'classical', their solution will be summarised here in a form relevant to our discussion. The notation used in this and other paragraphs is listed in the front of this report.

We consider a source, then, of arbitrary strength and motion which at the instant t occupies an element of volume $\delta x\ \delta y\ \delta z$. At this instant we assume that matter is locally introduced into the fluid at the rate $4\pi\rho\sigma\ \delta x\ \delta y\ \delta z$ per unit time.

At the element occupied by the source

$$\nabla^2\phi - \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\sigma \qquad \ldots\ldots (2.1)$$

Far away from the source $\sigma = 0$, or in spherical coordinates

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}$$

$$= \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} \qquad \ldots\ldots (2.2)$$

of which a solution is

$$\phi = \frac{1}{r} f(r - at) \qquad \ldots\ldots (2.3)$$

where r is the distance between the source and the field point.

As the source is approached

$$\phi \rightarrow \frac{1}{r} f(-at) \quad \text{and} \quad \frac{\phi_{tt}}{a^2\phi_{rr}} \rightarrow r^2\,\frac{f''(-at)}{f(at)} \qquad \ldots\ldots (2.4)$$

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*After this work was completed the attention of the author was drawn, by Professor G.N. Ward, to the fact that the analogous problem of an electron moving in a curved path at a speed greater than the speed of light was dealt with by G.A. Schott in his book 'Electromagnetic Radiation' (Camb. Univ. Press, 1912). It will be appreciated that it was only prior to the wide acceptance of the Theory of Relativity that such problems were regarded as fruitful topics for research.

and hence (2.1) reduces to Poisson's equation; so that as $r \to 0$,

$$\phi \to \iiint \frac{\sigma\, dx\, dy\, dz}{r} . \qquad \ldots\ldots(2.5)$$

Comparing (2.4) and (2.5), we therefore find that

$$\sigma\, dx\, dy\, dz = f(-\, at).$$

Similarly, for field points away from the source we may introduce the retarded time τ, where

$$\tau \;=\; t - \frac{r}{a}$$

so that

$$f(-\, a\tau) \;=\; f(r - at)$$

and consequently we may write

$$\phi \;=\; \iiint \frac{[\sigma]\, dx\, dy\, dz}{r} \qquad \ldots\ldots(2.6)$$

where $[\sigma]$ is the value of σ at the retarded time τ.

When determining the potential of a moving source, it is necessary to note that the retarded time varies over the volume occupied by the source. This is even important if the total volume it occupies is infinitesimal. Suppose we wish to evaluate the potential at (x',y',z') at time t'. Let (ξ,η,ζ) be the position of any element of the source at the retarded time $\tau = t' - R'/a$ where R' is the distance from the point (ξ,η,ζ) to the general field point (x',y',z').

We may speak of (ξ,η,ζ) as the 'effective position' of the element of the source, since this element contributes only to the potential we are in search of when it is at (ξ,η,ζ).

The retarded time will be different at any given instant at different parts of the source. Let us consider the conditions at some standard retarded time τ_o. Let the position of the element under consideration at $\tau = \tau_o$ be (ξ_o,η_o,ζ_o), and its velocity V_o at this instant. Let (ξ,η,ζ) and (ξ_o,η_o,ζ_o) be the components of two vectors $\underline{l}$ and $\underline{l}_o$, respectively; then since the element which is at (ξ_o,η_o,ζ_o) at time $\tau = \tau_o$ has moved to (ξ,η,ζ) by time τ

$$\underline{l} = \underline{l}_o + \underline{V}_o(\tau - \tau_o) + \tfrac{1}{2}\,\frac{\partial \underline{V}_o}{\partial t}\,(\tau - \tau_o)^2 + \ldots\ldots$$

Remembering that τ is a function of ξ, η and ζ, we have

$$\frac{\partial \underline{l}_o}{\partial \xi} - \frac{\partial \underline{l}}{\partial \xi} = -\frac{\partial \tau}{\partial \xi}\left\{\underline{V}_o + \frac{\partial \underline{V}_o}{\partial t}(\tau - \tau_o) + \frac{1}{2}\frac{\partial^2 \underline{V}_o}{\partial t^2}(\tau - \tau_o)^2 + \ldots\right\} \text{ etc.}$$

with similar expressions for the other derivatives.

Those elements of the source which have their effective positions inside a small element of volume $d\xi\, d\eta\, d\zeta$ occupy, at the fixed time τ_o, an element of volume $d\xi_o\, d\eta_o\, d\zeta_o$, the ratio of these elements being given by the usual Jacobian

$$\left|\frac{\partial(\xi_o, \eta_o, \zeta_o)}{\partial(\xi, \eta, \zeta)}\right| = \pm \begin{vmatrix} \frac{\partial \xi_o}{\partial \xi} & \frac{\partial \xi_o}{\partial \eta} & \frac{\partial \xi_o}{\partial \zeta} \\ \frac{\partial \eta_o}{\partial \xi} & \frac{\partial \eta_o}{\partial \eta} & \frac{\partial \eta_o}{\partial \zeta} \\ \frac{\partial \zeta_o}{\partial \xi} & \frac{\partial \zeta_o}{\partial \eta} & \frac{\partial \zeta_o}{\partial \zeta} \end{vmatrix}.$$

Evaluating the determinant we find

$$\frac{d\xi_o\, d\eta_o\, d\zeta_o}{d\xi\, d\eta\, d\zeta} = \left| 1 - \text{grad}\,\tau\left[\underline{V}_o + \dot{\underline{V}}_o(\tau - \tau_o) + \frac{1}{2}\ddot{\underline{V}}_o(\tau - \tau_o)^2 + \ldots\right]\right|.$$

Thus, if $\underline{V}$ is the velocity of the source when the element considered reaches its effective position,

$$\text{grad}\,\tau = \frac{1}{a}\,\text{grad}\,R'$$

and

$$\underline{V} = \underline{V}_o + \dot{\underline{V}}_o(\tau - \tau_o) + \frac{1}{2}\ddot{\underline{V}}_o(\tau - \tau_o)^2 + \ldots$$

We find that

$$\frac{d\xi_o\, d\eta_o\, d\zeta_o}{d\xi\, d\eta\, d\zeta} = \left|1 - \frac{\underline{V}}{a}.\,\text{grad}\,R'\right| = |K|,\ \text{say} \ldots\ldots (2.7)$$

where K is called the Doppler factor, and equals $\partial t/\partial \tau$.

Suppose now that we let (ξ_o, η_o, ζ_o) be the effective position of the centre of the source, so that $\tau \triangleq \tau_o$, $V \triangleq V_o$, etc., then the potential at (x', y', z') at time t' is given from (2.6) and (2.7) by

$$\phi = \iiint \frac{\sigma\, d\xi_o\, d\eta_o\, d\zeta_o}{|K|\, r}$$

where all quantities are evaluated at the retarded time τ_o;

i.e. neglecting variations in r as measured from various parts of the source

$$\phi = \frac{m_o}{|K| R'}$$

where m_o is the strength of the source at time τ_o.

Since there are in general a number of effective positions of the source - depending on its speed and path - we may write generally

$$\phi = \sum_n \frac{m_n}{|K_n| R'_n} \quad \ldots\ldots (2.8)$$

where m_n and K_n are respectively the source strength and the Doppler factor of the source at time $\tau = \tau_n$, when the source is at the effective position (ξ_n, η_n, ζ_n) distance R'_n from (x', y', z').

3. Pulse Waves

A physical interpretation of this equation (2.8) may be obtained by assuming that at each instant of time the moving source emits pulse waves which travel away from the point of emission in the form of spherical waves, expanding with velocity a.

Let the source at time t be at the point (x, y, z). The configuration of the pulse waves will be considered at the present time t_*. At this time the source occupies the position (x^*, y^*, z^*). The path of the source is denoted by Γ' in Figure 1.

In this figure, it will be seen that at a point (X, Y, Z, t^*) there are two pulse waves which intersect simultaneously as the source reaches (x^*, y^*, z^*). Hence the potential at (X, Y, Z, t^*) depends on the strength and kinematics of the source as it passes through the 'effective points' (x_1, y_1, z_1), (x_2, y_2, z_2), ...(x_n, y_n, z_n).

If the source never reaches the speed of sound it may quite simply be shown that n = 1 or 0, i.e. there is just one effective point and no other. For a source which, however, during its motion has exceeded the speed of sound, these may be regions for which n = 0 or n > 1.

To illustrate these deductions, we shall now consider the specific example of a source moving in a helix.

4. The Potential Near a Source of Constant Strength moving along a Helical Path at Constant Speed

The geometry of the path and the system of coordinates is shown in Figure 2, and the distance between the effective position of the source and the field point is R' where

$$R' = \sqrt{(y+V\tau)^2 + r^2 + \rho^2 - 2r\rho\cos(\omega\tau - \psi)} \quad \ldots\ldots (4.1)$$

where τ is the time of emission of an effective source previous to the present time t.

By definition of retarded time,

$$at = a\tau + R' \quad \ldots\ldots (4.2)$$

whereas

$$K = \frac{\partial t}{\partial \tau} = 1 + \frac{1}{a}\frac{\partial R'}{\partial t}.$$

Hence, the Doppler factor

$$K = 1 + \frac{V(y+V\tau) + r\rho\omega\sin(\omega\tau - \psi)}{aR'} \quad \ldots\ldots (4.3)$$

The problem is now to solve equations (4.1) and (4.2) to find τ, i.e. to solve

$$a(t-\tau) = \sqrt{(y+V\tau)^2 + r^2 + \rho^2 - 2r\rho\cos(\omega\tau - \psi)} \quad \ldots\ldots (4.4)$$

by which process we find the times of emission of <u>all</u> the effective sources.

Let us suppose that the source has been moving along the helix since time t_o. Then any solution of (4.4), $\tau = \tau_m$, must evidently satisfy the inequality

$$t_o \leqslant \tau_m < t. \quad \ldots\ldots (4.5)$$

If there are n solutions

$$\tau_1 > \tau_2 > \cdots\cdots > \tau_n \quad \text{and} \quad \tau_n \geqslant t_o$$

then evidently after the source has travelled a finite length of time $(t - \tau_n)$ the potential $\emptyset$ becomes independent of the starting time t_o.

We now define the new variables

$$\omega\tau - \psi = s/\rho \qquad \omega\rho/a = M_1$$

$$\omega(t-\tau) = (x-s)/\rho \qquad V/a = M_2$$

$$r = \rho + z$$

and without loss in generality we may consider the potential at (x,y,z) at time t = 0. Squaring both sides of (4.4) and substituting the new variables we find

$$\frac{(x-s)^2}{M_1^2} = \left\{- y + (x-s)\frac{M_2}{M_1}\right\}^2 + z^2 + 4\rho(\rho+z)\sin^2\frac{s}{2\rho}$$

i.e.

$$-\frac{s^2}{\rho^2}(M_2^2 - 1) + 2\frac{s}{\rho}\left\{\frac{x}{\rho}(M_2^2 - 1) - \frac{y}{\rho}M_2 M_1\right\}$$

$$-\left\{\frac{x^2}{\rho^2}(M_2^2 - 1) + \frac{y^2+z^2}{\rho^2}M_1^2 - \frac{2xy}{\rho^2}M_2 M_1\right\}$$

$$= 4M_1^2\left(1 + \frac{z}{\rho}\right)\sin^2\frac{s}{2\rho}\,. \qquad \ldots\ldots(4.6)$$

The assumption will now be made that we are interested only in the potential at points whose distance from the source is small compared with ρ. For the moment we may regard this merely as a convenient method by which an explicit solution of (4.6) may be obtained. The assumption means that

$$\left(\frac{x^2 + y^2 + z^2}{\rho^2}\right)$$

may be treated as small compared with unity: and consequently so also may (s^2/ρ^2). Correct to first order terms in the small quantities (4.6) becomes

$$\frac{s^2}{\rho^2}(M_2^2 + M_1^2 - 1) - \frac{2s}{\rho}\left\{\frac{x}{\rho}(M_2^2 - 1) - \frac{y}{\rho}M_1 M_2\right\}$$

$$+\left\{\frac{x^2}{\rho^2}(M_2^2 - 1) + \frac{y^2+z^2}{\rho^2}M_1^2 - \frac{2xy}{\rho^2}M_2 M_1\right\} = 0.$$

Here $M_1^2 + M_2^2 = M^2$, say, where M^2 is the Mach Number of the source relative to the gas. We must now consider the nature of the roots of this equation for both $M > 1$ and $M < 1$ (i.e. the supersonic and subsonic cases).

The solutions are given by

$$\frac{(1-M^2)(s-x)}{M_1} - (x\,M_1 + y\,M_2) = \pm\sqrt{(M_1 x + M_2 y)^2 + (1-M^2)(x^2+y^2+z^2)}. \qquad \ldots\ldots(4.7)$$

The required solutions are those which yield a value of s such that $\tau < 0$, since we are considering only the conditions at the instant $t = 0$, (vide the inequality (4.5)). We must therefore have $s < x$; now it follows that in (4.7)

$$|x\,M_1 + y\,M_2| \lesseqgtr \sqrt{(xM_1 + yM_2)^2 + (1-M^2)(x^2+y^2+z^2)}$$

according to whether $M \lesseqgtr 1$.

Having regard to the signs of the expressions in (4.7) it follows that there is just one root $s = s_1 < x$ if $M < 1$. But if $M > 1$, neither or both roots satisfy the required condition according to whether $(M_1 x + M_2 y)^2 \gtrless (M^2 - 1)(x^2 + y^2 + z^2)$.

In other words, to the accuracy of the approximations already made, the expression of the potential from (2.8), (4.2) and (4.3) is

$$\emptyset = \sum_n \frac{m_n}{|K_n| R'_n} = \sum_n \frac{m_n}{\left| a(t-\tau_n) + \dfrac{V(y+V\tau_n) + r\rho\omega\sin(\omega\tau - \psi)}{a} \right|} \qquad \ldots\ldots(4.8)$$

i.e.

$$\emptyset = \sum_n \frac{m_n}{\left| \dfrac{(1-M^2)}{M_1}(x-s_n) + M_2 y + M_1 x \right|}$$

and using (4.7) we therefore find:

(i) if $M < 1$, there is one root $s_1 < x$ and

$$\emptyset = \frac{m}{\sqrt{(1-M^2)(x^2+y^2+z^2) + (M_1 x + M_2 y)^2}} \qquad \ldots\ldots(4.9(i))$$

or (ii) if $M > 1$ and $(M_1 x + M_2 y) < \sqrt{(M^2-1)(x^2+y^2+z^2)}$

$$\emptyset = 0 \qquad \ldots\ldots(4.9(ii))$$

or (iii) if $M \quad 1$ and $(M_1 x + M_2 y) \qquad (M^2-1)(x^2+y^2+z^2)$

$$\emptyset = \frac{2m}{\sqrt{(M_1 x + M_2 y)^2 - (M^2-1)(x^2+y^2+z^2)}} \qquad \ldots\ldots(4.9(iii))$$

assuming that the source strength is invariant with time.

The condition expressed by the inequalities of (ii) and (iii) is that in supersonic flow there is no disturbance outside the 'Mach Cone' from the position of the source at $t = 0$ (i.e. the origin $x = y = z = 0$). It is a 'cone' with its vertex at the origin, it semi-vertical angle $\delta = \arcsin 1/M$, and its axis along the line $M_2x + M_1y = 0 = z$. This line is in fact the path of the source, but since the expression above is valid only for small x/ρ, y/ρ and z/ρ, the axis in this region is indistinguishable from the tangent to the path.

This solution was obtained by retaining only the terms os lowest order in (4.6). It corresponds with the conditions we normally associate with uniform linear motion. A correction term of next lowest order will now be sought. It follows from (4.6) that the term of next order contained in the expansion of the r.h.s. is

$$(M_1^2 \frac{z}{\rho}) \frac{s^2}{\rho^2}$$

which involves the third power of the small quantities s/ρ, z/ρ, etc. whereas the other terms contain second powers only.

The solution of (4.6) then becomes - in place of (4.7) -

$$\frac{(1-M^2)}{M_1}(s-x)-(xM_1+yM_2) = \pm\sqrt{\left\{(xM_1+yM_2)^2+(1-M^2)(x^2+y^2+z^2) - \frac{z}{\rho}\left[(M_2x-M_1y)^2 - x^2 + M_1^2 z^2\right]\right\}}. \qquad \ldots\ldots(4.10)$$

The root in the r.h.s. is now the term which appears in the denominator of the expression of ϕ, in place of the first order expressions in (4.9). The conditions change for $M \gtrless 1$ as before, provided we do not consider the condition $M \to 1$, since z/ρ must be considered as an infinitesimal whereas $(M-1)$ is finite; the 'Mach Cone' is now modified to a slightly different shape, viz. if $M > 1$ and

$$(xM_1 + yM_2) > \sqrt{(1-M^2)(x^2+y^2+z^2) - \frac{z}{\rho}\left[(M_2x - M_1y)^2 - x^2 + M_1^2 z^2\right]},$$

$$\phi = \frac{2m}{\sqrt{(xM_1+yM_2)^2+(1-M^2)(x^2+y^2+z^2) - \frac{z}{\rho}\left[(M_2x-M_1y)^2-x^2+M_1^2 z^2\right]}}$$

Since we are generally interested in the potential due to a distribution of sources, and not just a single one, it is more convenient to refer the axes to a field point as origin. Accordingly we take a new set of axes as shown in Figure 3 with $x' = y' = z' = 0$ as origin corresponding to a field point moving with the same forward and angular velocities as the source. Then if its axial and circumferential Mach Numbers are M_1' and M_2', respectively

$$x = x'(1 - \frac{z'}{\rho}) \qquad M_1 = M_1'(1 - \frac{z'}{\rho})$$

$$y = y' \qquad M_2 = M_2'$$

$$z = z' \qquad \rho = R' - z'.$$

Hence, from (4.11) the potential at the origin due to a source at (x', y', z') is, correct to the same order of approximation,

$$\phi = \frac{2m}{(x'M_1'+y'M_2')^2+(1-M'^2)(x'^2+y'^2+z'^2)+\frac{z'}{R'}\left[(M_2'x'-M_1'y')^2-x'^2+M_1'^2z'^2\right]} \qquad \ldots\ldots(4.12)$$

5. The Pressure on a Flat Plate Aerofoil in a Steady Rotary Motion in Yaw

As an application of the previous solution let us consider a wing rotating in the plane of yaw in such a way that the angle of sideslip is always zero. This is in fact the condition of an aircraft when it is turning in an unbanked circle, pointing always in the direction of motion. In such a manoeuvre, the angle between the direction of travel and the plane of symmetry of the aircraft is zero, so that there is no sideslip and no yaw: yet there is plainly a rotation; in the usual stability notation, r is non-zero although $\beta = 0$, (in particular we shall take $\dot{\beta} = 0$). There is an analogy here in the behaviour of a model aircraft attached to the end of a whirling-arm where there may be no change in the angle of incidence with time, but where there is a pitching effect induced by the curved path of the model through the air. In this case there is non-zero q, although $\dot{\alpha} = 0$.

In the application proposed, each element of surface of the aircraft wing is moving in a circle about a centre of rotation fixed in space and time, so that (because a circle is a degenerate form of helix) the analysis of the above paragraph is relevant. Moreover, if this rate of rotation (which we shall call ω) is sufficiently small, then the radius of the circle will be large compared with the dimensions of the aircraft and the approximations previously introduced will be valid: we shall consider then the condition $\omega \to 0$, although $\omega R/a$ is finite.

As the velocity along the helix (V) is zero, the total velocity of the aircraft is simply ωR_o, where R_o is measured to some fixed datum on the aircraft. We shall consider $\omega R_o > a$, so that the motion is supersonic: more particularly, we assume that the Mach Cone from any point on the wing surface lies behind the wing leading edge. Moreover, we shall consider only the flow over those parts of the wings where there is no effect from either the wing tips or centre section.

5.1 The Potential Function and the Mach 'Cone'

To proceed with the analysis, we note that - by reason of the above arguments - we may use (4.12) to describe the potential due to a point source on the wing, provided that we put $M_2 = 0$ and $M_1 = M$, say: then, dropping the primes in (4.12) the potential in the plane of rotation is

$$\phi\Big|_{y=0} = \frac{2m}{\sqrt{\left\{x^2 + (1-M^2)z^2 - \frac{z}{R}(x^2 - M^2z^2)\right\}}} \; . \quad \ldots\ldots (5.1)$$

The rate of rotation in the plane $y = 0$ is ω; and if the wing is at a vanishingly small incidence α, conditions in the plane $y = 0$ are effective, also those in the plane of the wing. The equation (5.1) applies for $M > 1$ where

$$x > \sqrt{\left\{(M^2 - 1)z^2 - \frac{z}{R}(x^2 + M^2z^2)\right\}}$$

i.e.

$$x > |z|\sqrt{M^2 - 1 - \frac{z}{R}} \; . \quad \ldots\ldots (5.2)$$

5.2 The Wing Geometry

The geometry of a part of the wing is shown in Figure 4. Here we consider that we wish to find the potential at the point A due to the rotation of the wing about the point O. If the length of the arc AD = c, then from the geometry of the triangle DPO, where P is any point of the wing leading edge.

$$\frac{R}{\sin(\pi - \gamma - \frac{x-c}{R})} = \frac{R - z}{\sin \gamma}$$

or for small values of (z/R),

$$z = (x-c)\left[\cot \gamma - \left(\frac{x-c}{R}\right)\left(\tfrac{1}{2} + \cot^2\gamma\right)\right] \quad \ldots\ldots (5.3)$$

5.3 The Source Distribution on the Wing

Suppose that the wing leading edge is 'supersonic'. Then conditions on the top and bottom surfaces are independent and we may simulate the pressure field over the upper surface as a plane distribution of sources (or sinks) on the wing. Now it may be shown using (4.11) that the displacement of the stream caused by a source of strength m is equal to an infinite cylinder of cross-section

$$2\pi . 2m \Big/ \left(\sqrt{1 + \frac{z}{\rho}} \, . \, \omega \rho\right)$$

normal to the direction of motion, if $m \rightarrow 0$. This follows from differentiation of ϕ in (4.11) with respect to $r = \sqrt{y^2 + z^2}$ to find the induced normal velocity, and then appropriate integration downstream for the condition $r \rightarrow 0$. However, if $r \rightarrow 0$, the variation of z on the limiting (surface) stream tube is negligible so that the displaced area is simply $4\pi m/\omega\rho$.

Let us now consider conditions on a plane surface over which the surface inclination to the direction of motion is equal to $-\alpha$ (as on the wing upper surface). Then taking coordinates on the surface s and n in the direction of, and perpendicular to, the resultant stream velocity, since in moving from s to s + δs the increase in flow area is $+\alpha\,\delta s\,\delta n$, where δn is the width of surface element considered, it follows that at a surface element of area δS

$$\tfrac{1}{2}\left(\frac{4\pi m}{\omega\rho}\right) = -\alpha\,\delta S$$

only the half of the displacement above the wing surface being considered.

In terms of the coordinates originating from a fixed field point, if the element of surface is at (x, y)

$$m = -\frac{\alpha}{2\pi}\,\omega(R - z)\,\delta S\,. \qquad \ldots\ldots(5.4)$$

5.4 The Potential due to the Lift Distribution

The potential at the point A (Figure 4) may be found by integrating the total effect due to the source distribution on the aerofoil surface. Since no disturbance is propagated outside the Mach 'Cone' of a source element, it follows that the range of integration may be confined within the region enclosed by ABC, defined by the leading edge BC given by (5.3) and the inequality (5.2). Thus, using (5.1) together with (5.4), the potential at A due to the wing lift is

$$\phi\Big|_{y=0} = -\frac{\alpha\omega}{\pi}\iint_{ABC}\frac{(R-z)\,dz\,dx}{\sqrt{(1-\frac{z}{R})x^2 - (M^2-M^2\,\frac{z}{R} - 1)z^2}}\,\ldots.(5.5)$$

where

$$\iint_{ABC} dz\,dx = \int_{z_C}^{z_B} dz \int_{x_{MC}}^{x_{LE}} dx$$

if z_B and z_C are respectively the values of z at B and C, and x_{MC} is the value of x at the intersection of the line z = const. through x_{LE}, with the Mach Cone ABC (x_{LE} lying on the leading edge BC).

Performing the integration with respect to x, since from (5.2)

$$x_{MC} = |z|\sqrt{M^2 - 1 - \frac{z}{R}}$$

we have, if $(M^2 - 1) = \beta^2$,

$$\phi\Big|_{y=0} = -\frac{\alpha\omega R}{\pi}\int_{z_C}^{z_B}\sqrt{1-\frac{z}{R}}\ \operatorname{argcosh}\left[\frac{x_{LE}}{|z|\sqrt{\beta^2-\frac{z}{R}}}\right]dz \qquad \ldots\ldots(5.6)$$

where by (5.3)

$$x_{LE} = c + z\tan\gamma\left[1+\frac{z}{R}\tan^2\gamma\left(\tfrac{1}{2}+\cot^2\gamma\right)\right],$$

and at the upper and lower limits, $z = z_B$ and z_C, $x_{LE} = |z|\sqrt{\beta^2 - \frac{z}{R}}$ by (5.2). This integral is evaluated in Appendix I. Using the result obtained there we have that

$$\phi\Big|_{y=0} = -\frac{\alpha\,\omega R\,c}{\sqrt{M^2-\sec^2\gamma}}\left\{1+\frac{c\tan\gamma\sec^2\gamma(M^2+2\sec^2\gamma)}{4(M^2-\sec^2\gamma)^2 R}+O\left(\frac{c^2}{R^2}\right)\right\} \qquad \ldots\ldots(5.7)$$

The first term is the same as that resulting from a uniform motion without any rotation, and the second provides the 'curvature effect', which is negligible if $\gamma \to 0$ only.

5.5 The Pressure Distribution

The pressure difference $(p - p_o)$ between the local pressure and that of the air at rest is given, to the first order in α, by the expression

$$p - p_o = \rho\frac{D\phi}{Dt} = \rho R\omega\frac{\partial\phi}{\partial c} \qquad \ldots\ldots(5.8)$$

or from (5.7)

$$p - p_o = -\frac{(\omega R)^2\alpha\rho}{\sqrt{M^2-\sec^2\gamma}}\left\{1+\frac{c\tan\gamma\sec^2\gamma(M^2+2\sec^2\gamma)}{2(M^2-\sec^2\gamma)^2 R}+O\left(\frac{c^2}{R^2}\right)\right\}.$$

By symmetry, the pressure difference between the upper and lower surfaces is

$$\triangle p = \frac{2\alpha\rho(\omega R)^2}{\sqrt{M^2-\sec^2\gamma}}\left\{1+\frac{c\tan\gamma\sec^2\gamma(M^2+2\sec^2\gamma)}{2(M^2-\sec^2\gamma)^2 R}+O\left(\frac{c^2}{R^2}\right)\right\} \qquad \ldots\ldots(5.9)$$

5.6 Change of Coordinates to those referred to the Wing Centre Line

We shall now refer to a system of coordinates referred to the centre line (root section) of the wing, shown in Figure 5 as LK. If OG is perpendicular to LK, and OG = R_o in length, we write

$$R = R_o + d\,, \qquad M_o = \frac{\omega R_o}{a} = \frac{V_o}{a}$$

and so $M = M_o(1 + \frac{d}{R_o})$. Further, we let $\gamma_o = \angle$ LKD be the sweepback of the wing leading edge, so that if GK = $\overline{X}$, from the geometry of the triangle ODH in Figure 5,

$$\gamma - \gamma_o \simeq \frac{\overline{X} - d\tan\gamma_o}{R_o} \qquad \ldots\ldots (5.10)$$

is small compared with unity. Thus correct to terms of first order in $1/R$,

$$\frac{R^2}{\sqrt{M^2-\sec^2\gamma}} = \frac{R_o^{\,2}}{\sqrt{M_o^{\,2}-\sec^2\gamma_o}}\left[1 + \frac{2d}{R_o} + (\gamma-\gamma_o)\frac{\sec^2\gamma_o\tan\gamma_o}{(M_o^{\,2}-\sec^2\gamma_o)} - \frac{d}{R_o}\,\frac{M_o^{\,2}}{(M_o^{\,2}-\sec^2\gamma_o)}\right]$$

$$= \frac{R_o^{\,2}}{\sqrt{M_o^{\,2}-\sec^2\gamma_o}}\left\{1 + \frac{d}{R_o(M_o^{\,2}-\sec^2\gamma_o)}\left[(M_o^{\,2}-\sec^2\gamma_o-\sec^4\gamma_o) + \frac{\overline{X}}{d}\sec^2\gamma_o\tan\gamma_o\right]\right\}.$$

Whence in (5.9), to the appropriate order of approximation

$$\Delta p = \frac{2\alpha\rho V_o^{\,2}}{\sqrt{M_o^{\,2}-\sec^2\gamma_o}}\left\{1 + \frac{\omega d}{V_o}\left[1 - \frac{\sec^4\gamma_o}{M_o^{\,2}-\sec^2\gamma_o}\right] + \frac{\omega\overline{X}}{V_o}\left[\frac{\sec^2\gamma_o\tan\gamma_o}{(M_o^{\,2}-\sec^2\gamma_o)}\right] + \frac{\omega c}{V_o}\left[\frac{\sec^2\gamma_o\tan\gamma_o(M_o^{\,2}+2\sec^2\gamma_o)}{(M_o^{\,2}-\sec^2\gamma_o)^2}\right]\right\}.$$

$$\ldots\ldots (5.11)$$

Without loss in accuracy in this expression we may replace the coordinates (c, d) in the correction terms due to the curvature of the path by the rectangular coordinates (X, Y): X measured from the leading edge parallel to the centre line, and Y measured from and perpendicular to the wing centre line (see Figure 6), i.e. if b is the wing span

$$\frac{\partial}{\partial\omega}(\triangle p) = \frac{2\,\alpha\,\rho\,V_o\,b}{\sqrt{M_o^2 - \sec^2\gamma_o}}\left\{\frac{x}{b}\left[\frac{\sec^2\gamma_o\,\tan\gamma_o(M_o^2 + 2\sec^2\gamma_o)}{(M_o^2 - \sec^2\gamma_o)^2}\right]\right.$$

$$\left. + \frac{\bar{x}}{b}\left[\frac{\sec^2\gamma_o\,\tan\gamma_o}{M_o^2 - \sec^2\gamma_o}\right] + \frac{y}{b}\left[1 - \frac{\sec^4\gamma_o}{M_o^2 - \sec^2\gamma_o}\right]\right\} \quad \text{......(5.12)}$$

5.7 The Unswept Wing

Evidently for an unswept wing $\gamma_o = 0$, and

$$\frac{\partial}{\partial\omega}(\triangle p) = \frac{2\,\alpha\rho V_o\,y}{\beta}\left(\frac{M_o^2 - 2}{M_o^2 - 1}\right). \quad \text{......(5.13)}$$

Other investigators[4,5] have found that for this condition

$$\frac{\partial}{\partial\omega}(\triangle p) = \frac{2\,\alpha\rho\,V_o\,y}{\beta}\left(\frac{1}{M_o^2 - 1}\right). \quad \text{......(5.14)}$$

It appears that this result is in error due to neglect of the variation in the induced normal velocity across the aerofoil. In its derivation, the source strength over the span is assumed constant, which by (5.4) is not so. The expression (5.13) may in fact be obtained if we assume that at the point A the equation for straight uniform flow is valid, i.e.

$$(M^2 - 1)\,\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

where the Mach Number M is chosen as that of the local flow at A. Hence, with no sweepback, Harmon's use in reference 5 of the two-dimensional Ackeret Theory may be justified, at positions outside the wing tip Mach Cones.

5.8 The Rolling Moment on a Wing of Infinite Aspect Ratio

We now consider the forces on a wing of infinite aspect ratio, allowing the wing chord to tend to zero, but its span b to remain finite. Then the spanwise distribution of pressure yields a difference caused by the curved motion of amount, from (5.12)

$$\frac{\partial}{\partial\omega}(\triangle p) = \frac{2\alpha\rho V_o b}{\sqrt{M^2 - \sec^2\gamma}}\left\{\frac{Y}{b}\left[1 - \frac{\sec^4\gamma_o}{M_o^{\,2} - \sec^2\gamma_o}\right]\right\}. \quad \ldots\ldots(5.15)$$

Integrating over the span from -b/2 to +b/2, since the contribution from the regions inside the wing apex and tip Mach Cones is negligible, the rolling moment is given by

$$l_r = \lim_{\omega\to 0}\left\{\frac{\text{moment}}{\frac{1}{2}\rho V_o \omega b^2 S}\right\} = -\frac{\alpha}{6}\,\frac{M_o^{\,2} - \sec^2\gamma_o - \sec^4\gamma_o}{(M_o^{\,2} - \sec^2\gamma_o)^{3/2}}$$

$$\ldots\ldots(5.16)$$

where S is the wing area.

This derivative changes sign at $M_o = \sec\gamma_o\sqrt{1+\sec^2\gamma_o}$: and the values of this critical Mach Number are plotted against γ_o, the angle of sweep, in Figure 7.

The effect is due to the fact that at low supersonic speeds the lift force at a given angle of attack decreases with increasing speed: thus those parts of the wing on the inside of the turn may, if the aircraft speed is low enough, be developing more lift since there the speed is lower than on the other half-wing (because this is moving faster). The result is that the wing tends to bank into a turn in the opposite sense to that which is actually taking place; the derivative l_r is then positive, whereas the stable rolling action - into the turn - is characterised by a negative value of l_r.

It will be seen from equation (5.15) that even in the example of an infinite aspect ratio swept wing, the curvature of the path is an important effect, since by an extension of the strip theory one would expect, by analogy with the case of the unswept wing, that one would obtain an expression like (5.13), viz.

$$\frac{\partial}{\partial\omega}(\triangle p) = \frac{2(\alpha\cos\gamma_o)\rho V_o Y}{\sqrt{M^2\cos^2\gamma_o - 1}}\left(\frac{M_o^{\,2}\cos^2\gamma_o - 2}{M_o^{\,2}\cos^2\gamma_o - 1}\right)$$

$$= \frac{2\alpha\rho V_o b}{\sqrt{M^2 - \cos^2\gamma_o}}\left\{\frac{Y}{b}\left[1 - \frac{\sec^2\gamma_o}{M_o^{\,2} - \sec^2\gamma_o}\right]\right\}$$

which differs from (5.15). This difference may be accounted as due to the effects of the curved flow, producing a change in the effective angle of sweep over the span.

6. Conclusions

We may conclude, therefore, that it is possible to construct the general form which the potential must take (equation (2.8)) in an arbitrary motion. The explicit form of the potential function appropriate to any prescribed motion may then be written down if we know the effective positions of the source. In general, the potential at a point fixed relative to the moving source will be a function of time, but in some types of motion which we may deem 'steady', the potential at such a point is invariant with time. As an example of such a motion the movement in a helical path is considered, but then it appears that the effective positions of the source may only be found if we solve a transcendental equation (4.4). Further analysis is only possible if we obtain an explicit solution of this equation, which is possible if we consider the curvature of the path to be infinitesimal: such an assumption means that the radius of curvature of the path becomes very large compared with the other dimensions, or in other words, we consider the potential at points relatively near the source compared with the radius of curvature. This choice of small curvature is compatible with the requirement that we shall ultimately calculate the quasi-static stability derivatives due to the curvature of the path, since these are valid only if the rate of angular rotation is infinitesimal.

The appropriate form of the potential can then be obtained for such a motion (equation (4.11)), and as an example we consider the potential due to a distribution of sources on a swept wing of an aircraft at incidence performing a uniform supersonic rotary motion in yaw, with vanishingly small rate of rotation, ω. The first-order solution in terms of ω for the pressure distribution is then given by (5.12) outside the region affected by the wing apex and taps. This differs from that obtained by other investigators, and is compatible with the assumption that the conditions may be found using a linearly moving source of appropriate velocity, only if there is no sweepback on the wing. Finally, the rolling moment due to a rotary motion in yaw (l_r) is calculated for an infinite aspect ratio wing - see equation (5.16): the moment changes sign below a certain critical supersonic Mach Number dependent on the angle of sweep (see Figure 5) and becomes 'unstable' - producing a roll out of the turn. This is due to the fact that at low supersonic speeds lift decreases with an increase in speed bringing about a higher lift on the half-wing on the inside of the turn where the local velocity is smaller. The effect is particularly important for high sweepback angles.

7. Extensions of the Method

There is no reason why the analysis cannot be applied to tip effects and centre-section effects on a wing of finite aspect ratio - although the calculation involved will be laborious. Also it is possible to calculate the rotary derivatives in a number of other steady conditions by precisely similar methods. Such investigations show, for example, that the quasi-static rolling derivative due to roll may be calculated using sources moving along straight lines, and this applies too for an aerofoil moving with a uniform rotary motion in pitch provided that there is no dihedral on the wing. It seems that the work may have some relevance to propeller theory, in those regions of the blade where the speed is supersonic.

Acknowledgements

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APPENDIX I

Evaluation of an Integral

It is required in para. 5.4 to evaluate the integral on the right-hand side of (5.6): this is

$$I = \int_{z_C}^{z_B} \sqrt{1 - \frac{z}{R}}\ \text{argcosh}\left[\frac{x_{LE}}{|z|\sqrt{\beta^2 - \frac{z}{R}}}\right] dz.$$

We need only consider the value correct to first order terms in (z/R) - assumed small; substituting, now,

$$t = \frac{x_{LE}}{|z|\sqrt{\beta^2 - \frac{z}{R}}} = \frac{c + z\tan\gamma\left[1 + \frac{z}{R}\tan^2\gamma\left(\frac{1}{2} + \cot^2\gamma\right)\right]}{|z|\ \beta\left(1 - \frac{z}{2R\beta^2}\right)}$$

we find that, from (5.2),

$$I = \int_1^{\infty}\left\{\left[\left(\frac{dz}{dt}\right)\sqrt{1 - \frac{z}{R}}\right]_{z<0} - \left[\left(\frac{dz}{dt}\right)\sqrt{1 - \frac{z}{R}}\right]_{z>0}\right\}\text{argcosh } t\ dt.$$

Integrating by parts:

$$I = -\int_1^{\infty}\left\{\left(z - \frac{z^2}{4R}\right)_{z<0} - \left(z - \frac{z^2}{4R}\right)_{z>0}\right\}\frac{dt}{\sqrt{t^2 - 1}}\ .$$

But, by definition

$$\frac{z}{c} = \frac{1}{s\beta t - \tan\gamma}\left\{1 + \frac{z}{R}\ \frac{1}{s\beta t - \tan\gamma}\ \left[\frac{st}{2\beta} + \tan\gamma + \frac{\tan^3\gamma}{2}\right]\right\},$$

where $s = \text{sgn } z$. Thus, correct to first order terms in z/R,

$$\frac{1}{c}\left\{\left(z - \frac{z^2}{4R}\right)_{z>0} - \left(z - \frac{z^2}{4R}\right)_{z<0}\right\}$$

$$= \frac{2\beta t}{(\beta^2 t^2 - \tan^2\gamma)}\left\{1 + \frac{c}{R}\left[\left(\tan\gamma + \frac{\tan^3\gamma}{2}\right)\frac{\beta^2 t^2 + 3\tan^2\gamma}{(\beta^2 t^2 - \tan^2\gamma)^2}\right.\right.$$

$$\left.\left. + \frac{\tan\gamma}{2\beta^2}\ \frac{3\beta^2 t^2 + \tan^2\gamma}{(\beta^2 t^2 - \tan^2\gamma)^2} - \frac{\tan\gamma}{2}\ \frac{1}{\beta^2 t^2 - \tan^2\gamma}\right]\right\}$$

$$= \frac{2\beta t}{\beta^2 t^2 - \tan^2\gamma}\left\{1 + \frac{c\tan\gamma}{2R}\left[\frac{(\sec^2\gamma + 3/\beta^2)}{\beta^2 t^2 - \tan^2\gamma} + \frac{4(1 + \sec^2\gamma + 1/\beta^2)\tan^2\gamma}{(\beta^2 t^2 - \tan^2\gamma)^2}\right]\right\}$$

The required integral is then made up of integrals of the type

$$\int_1^\infty \frac{2\beta t}{(\beta^2t^2-\tan^2\gamma)^n}\,\frac{dt}{\sqrt{t^2-1}} = \beta^{1-2n}\int_1^\infty \frac{ds/\sqrt{s-1}}{\left(s-\frac{\tan^2\gamma}{\beta^2}\right)^n} \qquad (s=t^2)$$

$$= 2\beta^{1-2n}\int_0^\infty \frac{du}{\left[u^2+\left(1-\frac{\tan^2\gamma}{\beta^2}\right)\right]^n}$$

$$(u=\sqrt{s-1})$$

$$= \frac{\pi}{\sqrt{\beta^2-\tan^2\gamma}} \qquad (n=1)$$

$$= \frac{\pi(2n-3)!}{(n-1)!\,(n-2)!\,2^{2n-3}}\cdot\frac{1}{(\beta^2-\tan^2\gamma)^{n-\frac{1}{2}}}$$

$$(n \geqslant 2).$$

Hence, we find that

$$I = \frac{\pi c}{\sqrt{M^2-\sec^2\gamma}}\left\{1+\frac{c\tan\gamma}{4R}\left[\frac{\sec^2\gamma+3/\beta^2}{M^2-\sec^2\gamma}+\frac{3(1+\sec^2\gamma+1/\beta^2)\tan^2\gamma}{(M^2-\sec^2\gamma)^2}\right]\right\}$$

$$= \frac{\pi c}{\sqrt{M^2-\sec^2\gamma}}\left\{1+\frac{c\tan\gamma}{4R(M^2-\sec^2\gamma)^2}\left[\sec^2\gamma(M^2+2\sec^2\gamma)\right]\right\}.$$

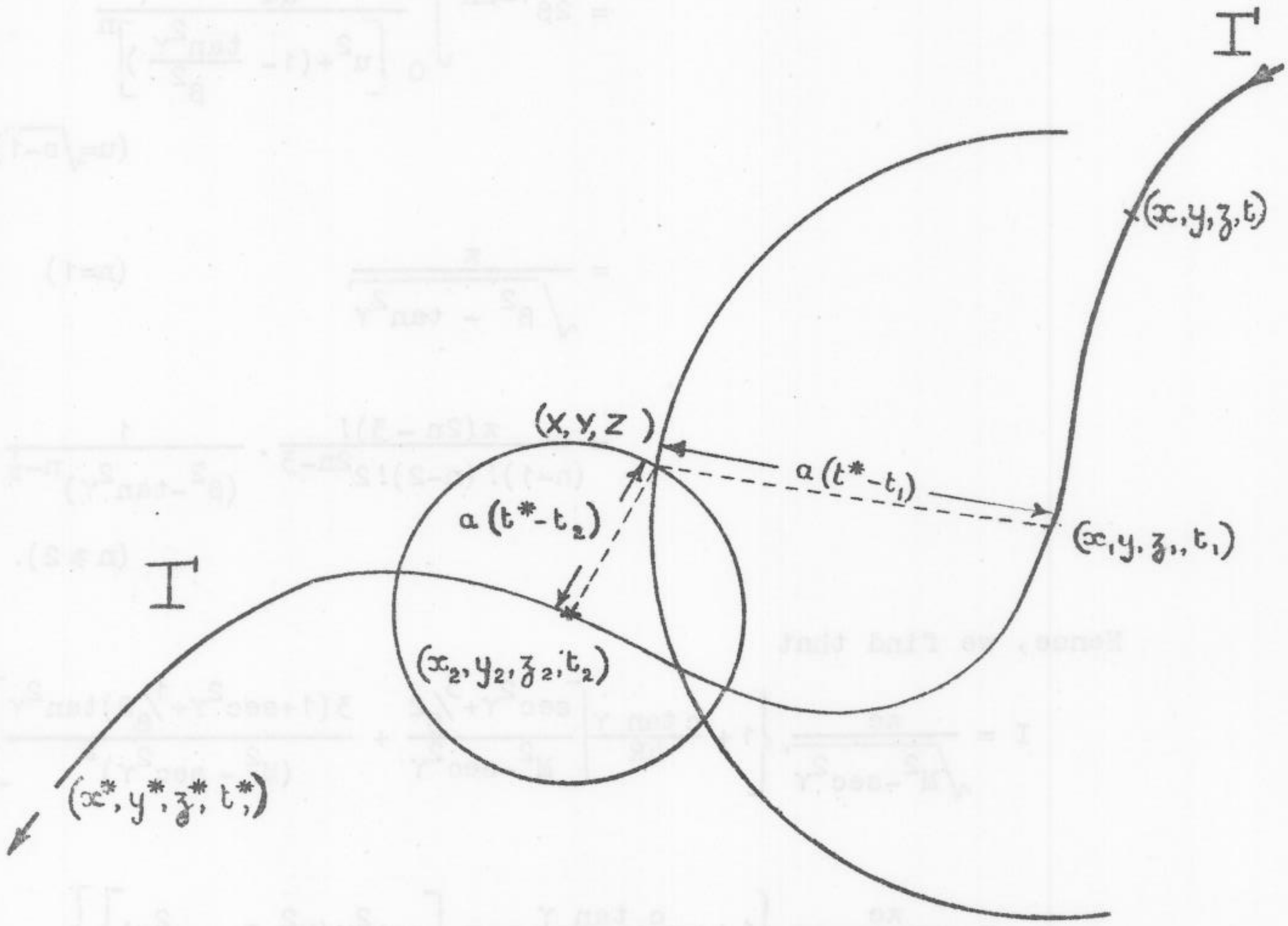

FIG. 1. PULSE WAVES (SEE PARA. 3)

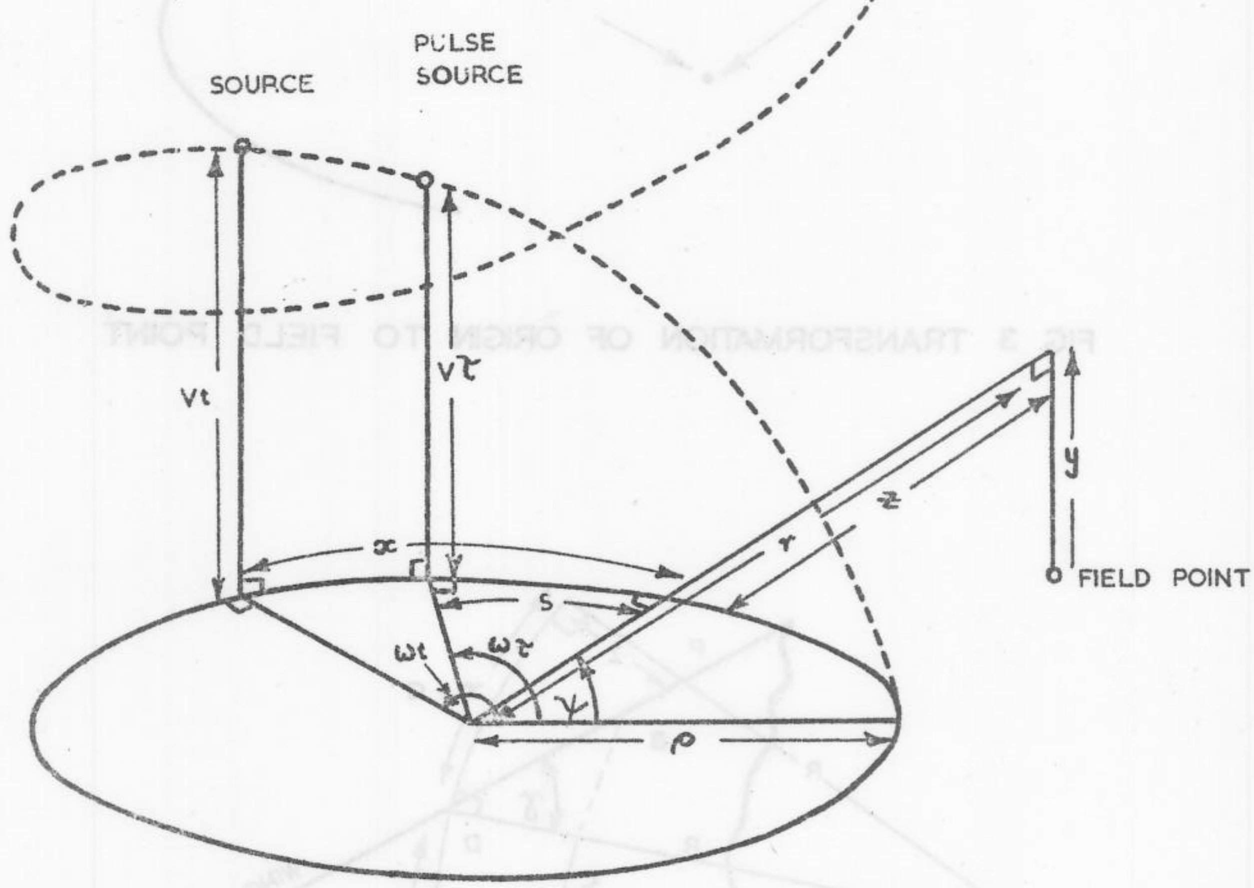

FIG. 2. CO-ORDINATES USED IN PARA. 4. FOR A SOURCE MOVING ALONG A HELIX.

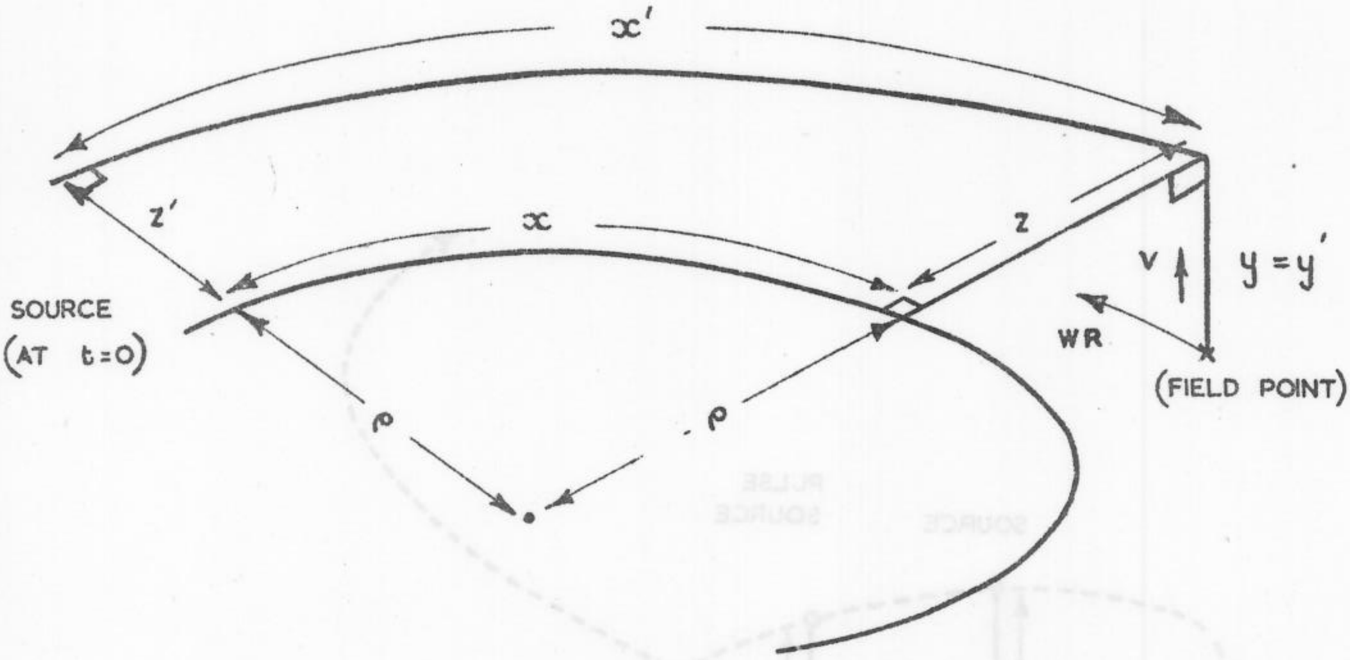

FIG 3 TRANSFORMATION OF ORIGIN TO FIELD POINT

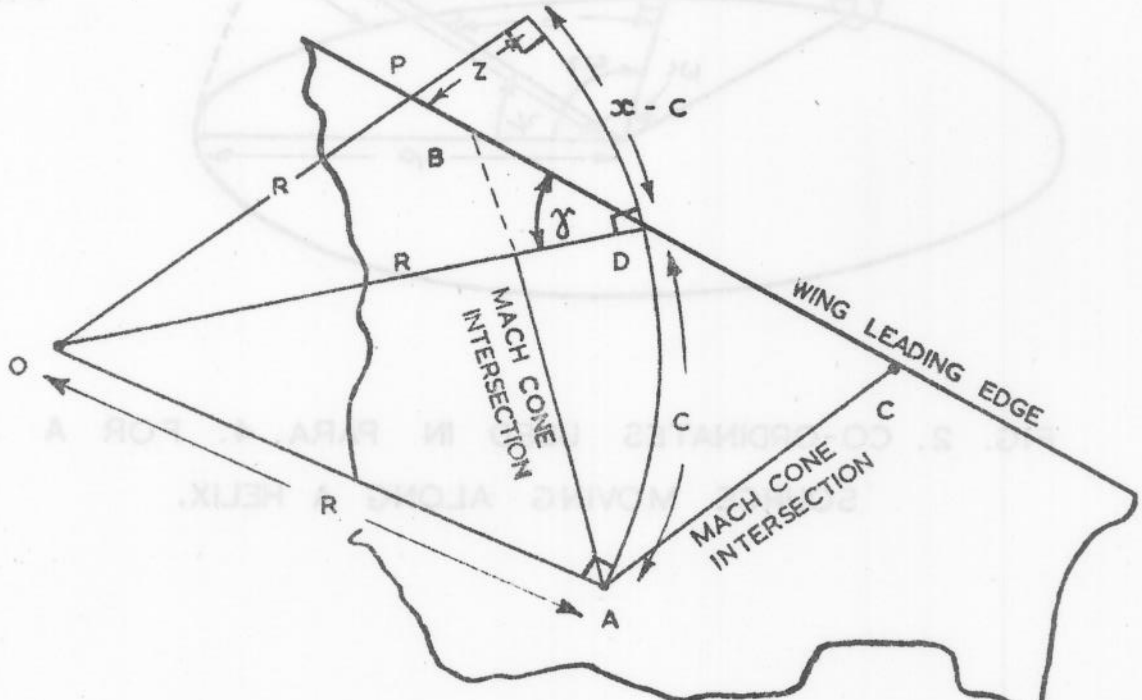

FIG. 4. COORDINATES ON WING SURFACE

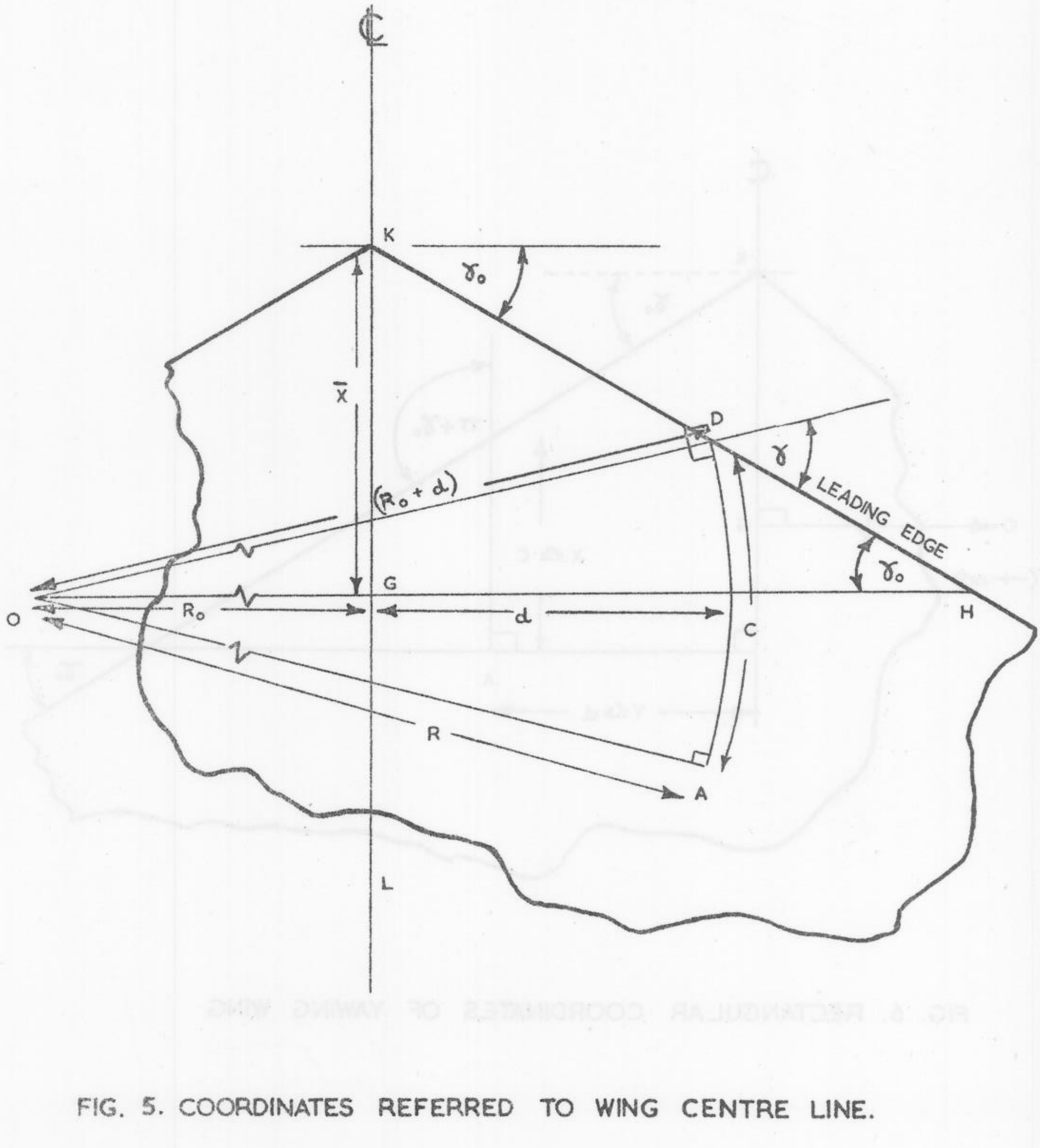

FIG. 5. COORDINATES REFERRED TO WING CENTRE LINE.

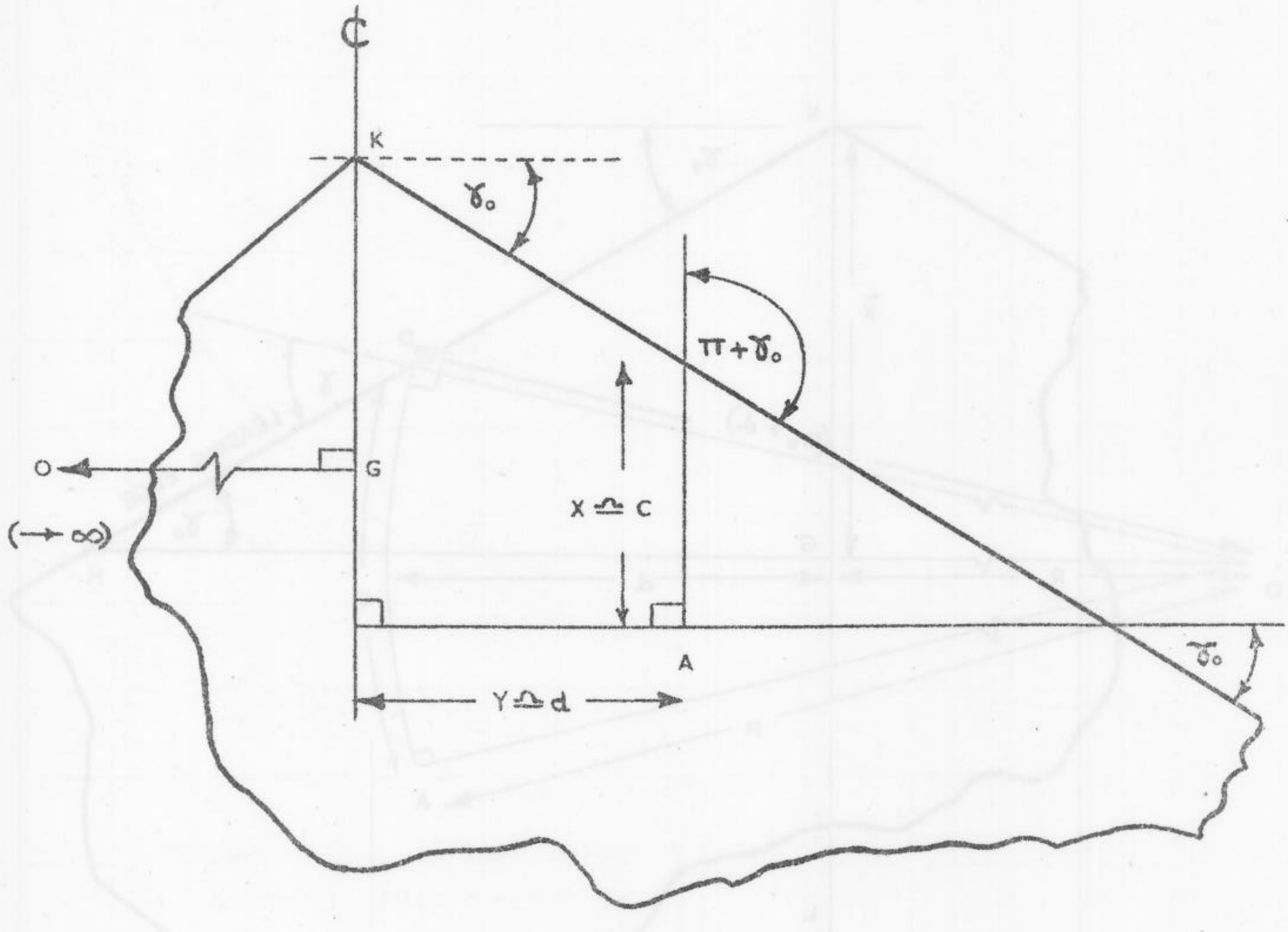

FIG. 6. RECTANGULAR COORDINATES OF YAWING WING

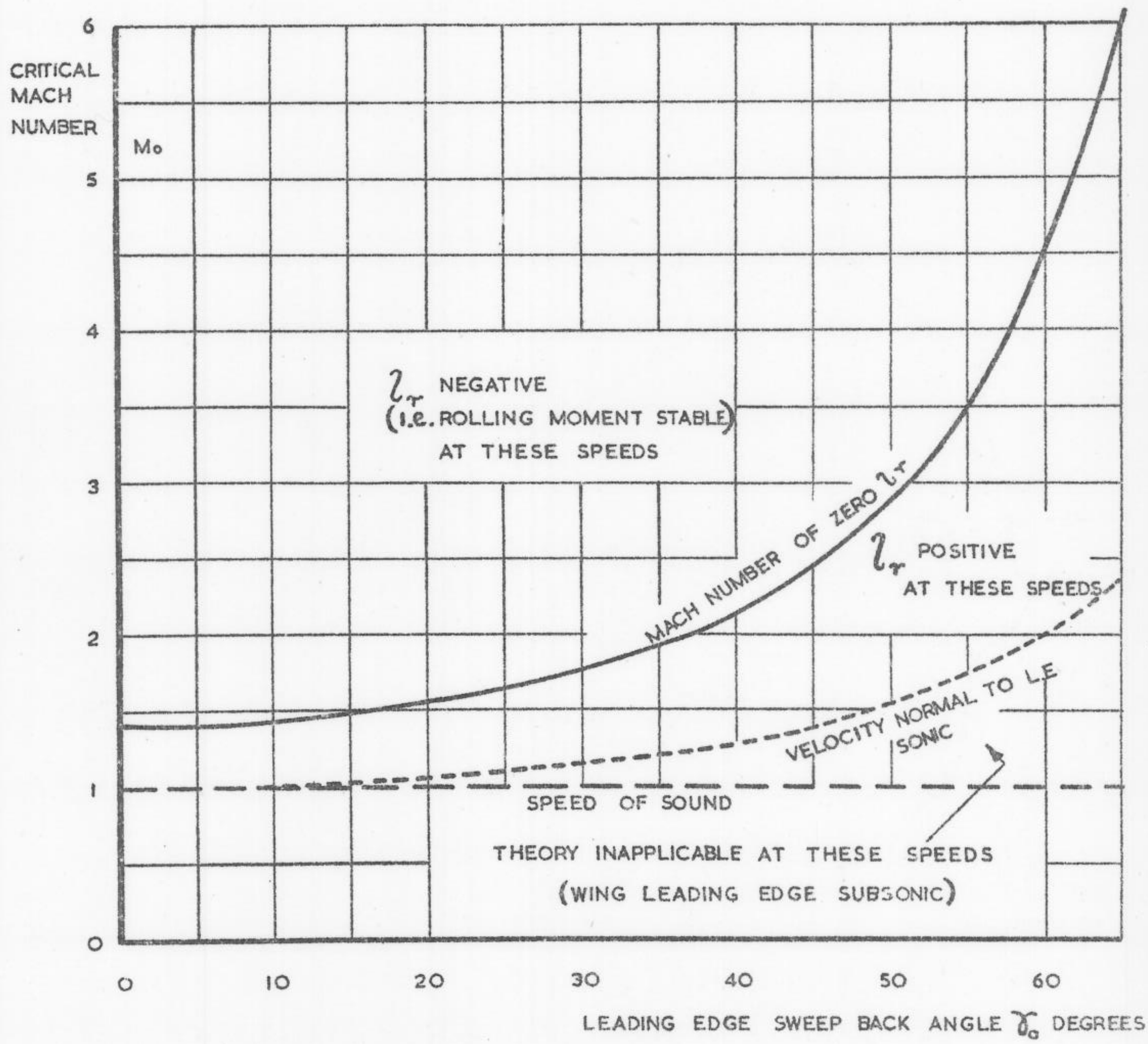

FIG. 7. THE SUPERSONIC MACH NUMBER AT WHICH A REVERSAL IN ROLLING MOMENT DUE TO YAW OCCURS ON A SWEPT WING OF INFINITE ASPECT RATIO.