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T H E   C O L L E G E   O F   A E R O N A U T I C S

C R A N F I E L D

The Aerodynamic Derivatives with respect
to Rate of Yaw for a Delta Wing with Small Dihedral
at Supersonic Speeds

-by-

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SUMMARY

Expressions are derived for the yawing deri-
vatives on the assumptions of the linearised theory of
flow for a delta wing with small dihedral flying at
supersonic speeds at small incidence.

The non-dimensional derivatives are numerically
decreasing functions of Mach number.   The non-
dimensional rolling and yawing derivatives are also
numerically decreasing functions of aspect ratio.

When the wing lies entirely within the apex
Mach cone there is a leading edge suction force
proportional to incidence which makes a destabilising
contribution to the yawing moment and side force which may
be of the same magnitude as that from the induced excess
pressure distribution.

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1.   Introduction

A number of papers have been published during
the past two years in which expressions are derived on
the basis of the linearised theory for the various force
coefficients acting on a delta wing flying at supersonic
speeds.   (C_L, C_D, C_M in Refs. 1 and 2; L_v,N_v,Y_v in Ref.3;
L_p,M_q in Ref.4)   The present paper, in which the
aerodynamic derivatives with respect to yawing are calculated,
completes the list of derivatives with respect to linear
and angular velocities relevant to the stability calculations
for the delta wing.

It is clear that a steady rate of yaw is not
possible if small deviations from a neutral position are
only to be considered.   In consequence the interpretation
of derivative with respect to yawing is a matter of
convention.   In this paper the forces are taken to
be those that would result from a hypothetical steady
motion with the same instantaneous velocity distribution
at the boundaries: the pitching derivatives are calculated
on the same basis in Ref.4.   The hypothetical pressure
distribution at the aerofoil differs from the true by
an amount of the same order in the frequency parameter, $r\,c/V$,
(for notation see next section): the errors in the
resultant force derivatives are reduced to second order
in $r\,c/V$ by the addition of appropriate sideslip acceleration
derivatives.   The latter may be of the same magnitude as
the yawing derivatives in the form assumed here,
and a short account of them will be given in a subsequent
report.

The present investigation is confined to a
wing of small dihedral at small incidence, of which the
two halves are flat.   The deviations from the neutral
position are assumed small and in particular it is
assumed that, if both leading edges lie within the apex
Mach cone when in neutral position, they will remain so
when disturbed, and vice versa.

When the leading edges lie within the apex Mach
cone, a solution to the potential equation is obtained by
extending the method of cone fields introduced by Stewart
(Ref.1) to cover velocity distributions which are of the
first degree in the space co-ordinates.   When the leading
edges protrude through the Mach cone the problem reduces
to the integration of a simple source distribution.

2.   Notation

V = Free stream velocity

δ = Angle of dihedral

γ = Semi vertex angle

α = Angle of incidence

c = Max. chord

S = Wing area

s = semi span

A = Aspect ratio. $(4s^2/S)$

L = Rolling moment

N = Yawing moment

y = Side force

r = Angular rate of yaw

ρ = Air density

M = Free stream Mach No.

$\beta = \sqrt{M^2-1}$

$\lambda = \beta \tan \delta$

$k^2 = 1-k'^2 = 1-\lambda^2, \lambda \leqslant 1$

K, E = Complete elliptic integrals of 1st and 2nd
kind of modulus k

$\ell_r = \dfrac{\partial L}{\partial r}/\rho VSs^2$ = non-dimensional rolling moment
derivative

$n_r = \dfrac{\partial N}{\partial r}/\rho VSs^2$ = non-dimensional yawing moment
derivative

$y_r = \dfrac{\partial Y}{\partial r}/\rho VSc$ = non-dimensional side force
derivative

3. Results

The non-dimensional aerodynamic derivatives quoted are referred to body axes and not to wind axes. The rolling axis is taken to be the axis of the aerofoil, that is the line common to the two halves. It will be noted that the signs of n_r and y_r are reversed if the x and z axes are taken in directions opposite to the arrangement of Fig. 4a.

For the wing entirely within the apex Mach cone $(\lambda < 1)$:-

(i) $$\ell_r = \frac{\xi}{6} \cdot \frac{(8 - 7\lambda^2)E - \lambda^2(4 - 3\lambda^2)K}{(2 - \lambda^2)E - \lambda^2 K}$$

(ii) $$n_r = -\frac{1}{\pi} \delta^2 \cot\gamma \frac{(6 - 5\lambda^2)E - \lambda^2(3 - 2\lambda^2)K}{(2 - \lambda^2)E - \lambda^2 K}$$

$$+ \frac{1}{E} \alpha\delta \cot\gamma \cosec\gamma \frac{(2E - \lambda^2 K)(1 - \lambda^2)^{3/2}}{(2 - \lambda^2)E - \lambda^2 K}$$

(iii) $$y_r = -\frac{4}{3\pi} \delta^2 \tan\gamma \frac{(6 - 5\lambda^2)E - \lambda^2(3 - 2\lambda^2)K}{(2 - \lambda^2)E - \lambda^2 K}$$

$$+ \frac{4}{3E} \alpha\delta \cos\gamma \frac{(2E - \lambda^2 K)(1 - \lambda^2)^{3/2}}{(2 - \lambda^2)E - \lambda^2 K}$$

For the leading edges protruding through the apex Mach cone $(\lambda > 1)$:-

(i) $$\ell_r = \delta/2\lambda$$

(ii) $$n_r = -\frac{\delta^2 \cot\gamma}{\pi(\lambda^2 - 1)} \left\{ 1 + \frac{2\lambda^2 - 3}{\sqrt{\lambda^2 - 1}} \sec^{-1}\lambda \right\}$$

(iii) $$y_r = -\frac{4\delta^2 \tan\gamma}{3\pi(\lambda^2 - 1)} \left\{ 1 + \frac{2\lambda^2 - 3}{\sqrt{\lambda^2 - 1}} \sec^{-1}\lambda \right\}$$

/It can ...

It can be shown that the derivatives are continuous as the parameter λ passes through unity.

In Fig. 1 the quantity ℓ_r/δ is plotted against Mach number for different aspect ratios.
In Fig. 2 the quantity n_r/δ^2 for zero incidence and the contribution to $n_r/\alpha\delta$ due to leading edge suction are plotted similarly.   In a like manner the variation of y_r with Mach number and aspect ratio is shown in Fig. 3.

4.   Delta Wing Enclosed within the Apex Mach Cone

Linearising the equation of continuity for steady supersonic flow gives the Prandtl-Glauert equation:-

$$- \beta^2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where u,v,w are the induced velocity components in the x,y,z directions in the cartesian co-ordinate system indicated at Fig. 4a.

When the flow is irrotational there exists an induced velocity potential, φ, and it and u,v,w all satisfy the equation:-

$$- \beta^2 \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad (1)$$

Define $\omega = \eta + i\zeta = \beta \dfrac{y + iz}{x + \sqrt{x^2 - \beta^2 y^2 - \beta^2 z^2}}$

From the analogue of Donkin's general solution of degree zero of Laplace's equation in three dimensions (Ref. 5) it follows that the real part of any analytic function of ω is a solution of degree zero of equation (1) also satisfying Laplace's equation in η, ζ .

Suppose u^x, v^x, w^x are functions of degree zero in x,y,z, derived from a potential φ^+, and satisfying equation (1); we can take them to be the real parts of functions U,V,W of ω. It was shown in Ref.1 (compare

Ref.3)   that in these circumstances:-

$$- \frac{i\beta dU}{2\omega} = \frac{idV}{1 + \omega^2} = \frac{dW}{1 - \omega^2} \qquad (2)$$

It can be shown by dimensional arguments in
our present problem in which the aerofoil has an
angular velocity about the z-axis that the induced
velocity components are functions of degree one in
x, y, z in the region ahead of the trailing edge.
It is assumed that the motion is irrotational.   Therefore,
the first derivatives of the velocity components with
respect to x are of degree zero, are derivable from
a potential function and satisfy equation (1).   In
consequence these derivatives are the real parts of
functions U_x, V_x, W_x of ω , which are connected by
a relation of the form (2);   it follows likewise that
the derivatives with respect to y and z are the real
parts of functions U_y, V_y, W_y and U_z, V_z, W_z respectively,
which are similarly connected.

For the boundary conditions at the aerofoil
we make the usual assumptions of the linearised theory
of thin wings with small incidence and dihedral that the
kinematic boundary conditions are fulfilled at the normal
projection of the aerofoil on the x y - plane rather than
at the aerofoil itself.   In calculating the aerodynamic
derivatives with respect to yawing, referred to body
axes, we can, except for the purpose of assessing the
suction forces at the leading edges, ignore the incidence
without loss of generality.   Therefore, the boundary
condition at the aerofoil reduces to $w = - r \, x \delta$, y $>$ 0
and $w = + r \, x \delta$,   y $<$ 0, at z = 0.

The other relevant boundary is the shock wave
emanating from the apex of the delta wing, which in
accordance with the principles of the linearised theory
is taken to be the Mach cone corresponding to undisturbed
flow.   It is further assumed in the present problem
of a wing with small dihedral lying entirely within the
Mach cone that the shock wave is infinitely weak.   The
boundary condition reduces to the requirement that the
induced velocity should vanish at the apex cone.   Clearly
a sufficient condition, which will also be shown to be

necessary, is that the velocity components shall vanish
at one point and their first derivatives at all points
on the Mach cone.

Since the induced velocity vanishes at the Mach
cone we can write a number of relations of the form:-

$$\frac{u(x,y,z) - u(x^1,y,z)}{x - x^1} = \frac{y^1 - y}{x - x^1} \cdot \frac{u(x^1,y^1,z) - u(x^1,y,z)}{y^1 - y} \,,$$

where (x,y,z) and (x^1,y^1,z) are points on the Mach cone, from
which it can be shown that in any region on the Mach cone
where one velocity derivative is finite the remainder are
finite.

If u is to be zero over the Mach cone then at
the Mach cone in a region where its derivatives are
defined the following conditions hold:-

$$\left. \begin{aligned} x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} &= 0 \\ z \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial z} &= 0 \end{aligned} \right\} \tag{3}$$

so that

$$xz \frac{\partial u}{\partial x} + (y^2 + z^2)\frac{\partial u}{\partial z} = 0$$

Since $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$ everywhere and $x^2 = \beta^2(y^2 + z^2)$ on the
Mach cone, the last equation can be rewritten as :-

$$\beta \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} = 0 \tag{4}$$

Put

$$\frac{dW_x}{d\omega} = \frac{1 - \omega^2}{\omega^2} F(\omega)$$

so that

$$\frac{dU_x}{d\omega} = \frac{2}{\beta \omega} F(\omega)$$

and equation (4) reduces to:-

$$R \left\{ \int \int_{\omega_1}^{\omega_2} \frac{2iFd\omega}{\omega} + \int_{\omega_1}^{\omega_2} \frac{1 - \omega^2}{\omega^2} F\, d\omega \right\} = 0$$

/where ...

where ω_1, ω_2 are points in the same region in which

$\frac{\partial u}{\partial x}$ and $\frac{\partial w}{\partial x}$ are defined

Since on the Mach cone $|\omega| = 1$, we may substitute $e^{i\theta}$ for ω in the last equation and obtain:-

$$R \left\{ \sin \theta_2 \int_{\theta_1}^{\theta_2} F \, d\theta - \int_{\theta_1}^{\theta_2} F \sin \theta \, d\theta \right\} = 0$$

whence

$$R \int_{\theta_1}^{\theta_2} \cos\theta \left\{ \int_{\theta_1}^{\theta} F \, d\psi \right\} d\theta = 0$$

from which it follows that F is pure imaginary on $|\omega| = 1$ so that

$$\left(\frac{\partial u}{\partial x}\right)_{\theta_1} - \left(\frac{\partial u}{\partial x}\right)_{\theta_2} = \frac{2}{\beta} R \int_{\theta_1}^{\theta_2} F \, d\theta = 0$$

Therefore $\frac{\partial u}{\partial x}$ and similarly $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ are constant and clearly from equations (3) zero, in the region on the Mach cone where they are all three defined. In a like manner it may be shown that the other derivatives vanish in the same regions. Since these functions are continuous at points just inside the Mach cone it follows that they cannot be infinite in some regions and zero in the remainder and therefore either zero or infinite everywhere on the cone.

Now for thin aerofoils which are approximately in the xy - plane and which are symmetrical with respect to y the induced velocity potential is antisymmetrical with respect to z, so that $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial z}$ vanish at all points in the xy - plane not on the aerofoil. Therefore all the derivatives vanish everywhere on the Mach cone.

Consider the transformation:-

$$\mathrm{cn}(\gamma, k) \equiv \mathrm{cn}(\rho + i\sigma, k) = \frac{2i\omega}{1 - \omega^2}$$

where $\mathrm{cn}(\gamma, k)$ is the Jacobian elliptic function of modulus k in Glaisher's notation. The interior of the Mach cone is represented in the γ - plane by the interior of the rectangle with vertices $\pm 2iK'$, $K \pm 2iK'$, the imaginary axis between $\pm 2iK'$ corresponding to the Mach cone and the

/parallel

parallel line between $K \pm 2iK'$ to the aerofoil. (See Fig. 4c).

Since the flow is irrotational it is clear that $V_z - W_y = W_x - U_z = U_y - V_x = 0$. and it will be seen that in the τ - plane

$$\beta \frac{dU_x}{cn\,\tau} = \frac{idU_y}{sn\,\tau} = dU_z = \frac{idV_x}{sn\,\tau} \tag{5}$$

$$= -\frac{dV_y}{\beta sn\tau sc\tau} = \frac{idV_z}{\beta sc\,\tau} = dW_x = \frac{idW_y}{\beta sc\,\tau} = \frac{dW_z}{\beta nc\,\tau}$$

It now remains to choose one function, say $\frac{dW_x}{d\tau}$, satisfying the boundary conditions in such a way that the velocity potential is single valued and the aerodynamic forces are finite.

B y reasons of symmetry $\frac{\partial w}{\partial x} = 0$ on $y = 0$ and therefore, referring to Fig. 4c, $\frac{dW_x}{d\tau}$ must be pure imaginary on the lines OC, AB and A'B'. On integrating $\frac{dW_x}{d\tau}$ along OCB $\frac{\partial w}{\partial x}$ must jump in value from 0 to $+ r\delta$ at the point C and on integrating a long OCB' to $- r\delta$. in order that the boundary conditions at the aerofoil may be met: hence $\frac{dW_x}{d\tau}$ must have a simple pole with a residue of imaginary part $2r\delta/\pi$ at C($\tau = K$) and similarly poles with residues of opposite sign at B,B' ($\tau = K \pm 2iK'$). In addition we require $\frac{\partial w}{\partial x}$ to be constant and $\frac{\partial w}{\partial y}$ to vanish over the two halves of the aerofoil, which need is met by choosing $\frac{dW_x}{d\tau}$ to be real on BB'.

The boundary condition at the Mach cone that t he first derivatives of the velocity components should vanish requires $\frac{dW_x}{d\tau}$ to be real on the imaginary axis, AA', to have no singularities on AA' that contribute to the real part of its integral and to have at least a simple zero at P and P' ($\tau = \pm iK'$).

For the velocity components to be single

valued $\frac{dW_x}{d\tau}$ must have no branch points or poles with residues inside the rectangle AA'B'B.   The aerodynamic forces will be finite provided $\frac{dU_x}{d\tau}$ and $\frac{dU_y}{d\tau}$ do not have any singularities of too high an order on the aerofoil, BB'.

The necessary function is found to be:-

$$\frac{dW_x}{d\tau} = isc\tau\,nd^4\tau\,(A\,dn^2\tau + Bk'^2) \qquad (6)$$

where   A   and   B   are real constants such that $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ have the correct values on the two halves of the aerofoil.   Any other function of this form would lead to an inadmissible singularity of one or more of the functions such as $\frac{dW_x}{d\tau}$ at either $\tau = \pm iK'$ or $\tau = = K \pm iK'$

This function has a residue $-i(A + B)/k'^3$ at the point C, $\tau = K$, and therefore $A + B = -\frac{2}{\pi}r\delta k'^3$

From equations (5) and (6) we have:-

$$\frac{\partial w}{\partial y} = \beta R \int_0^\tau sc^2\tau\,nd^4\tau(Adn^2\tau + Bk'^2)d\tau$$

$$= \frac{1}{k'^4}\beta R\left\{k'^2(A + \tfrac{4B}{3})K(\tau) - (2A + \tfrac{1}{3}(7+k^2)B)E(\tau)\right.$$

$$+ k^2(A + \tfrac{1}{3}(4+k^2)B)sn\tau dc\,\tau + (A+B)sn\tau cd\,\tau$$

$$\left. + \tfrac{1}{3}Bk^2k'^2 sn\tau cn\tau nd^3\tau\right\}$$

$$= \frac{1}{k'^4}\beta\left\{k'^2(A + \tfrac{4B}{3})K - (2A + \tfrac{1}{3}(7+k^2)B)E\right\}$$

for $\tau = K + i\sigma$

Therefore in order that $\frac{\partial w}{\partial y}$ may vanish on the aerofoil:-

$$B = \frac{6k'^3 r\,\delta}{\pi}.\ \frac{2E - k'^2 K}{k^2(K+E) - (K-E)}$$

Also from equations (5) and (6) we have:-

$$\frac{\partial u}{\partial x} = \frac{1}{\beta} R \int_0^{\tau} i sn\tau\, nd^4\tau\, (Adn^2\tau + Bk'^2)d\tau$$

$$= -\frac{1}{k'^2\beta} R\left\{ (A + B)icd\tau - \frac{1}{3} Bk^2 icd^3\tau \right\}$$

Now $cn\tau = \dfrac{2i\omega}{1 - \omega^2} = \dfrac{i\beta y}{\sqrt{x^2 - \beta^2 y^2}}$ on $z = 0$, so

that $icd\tau = y\Big/\sqrt{x^2\tan^2\gamma - y^2}$ for $z = +0$ and is
of opposite sign for $z = -0$

Hence

$$\frac{\partial u}{\partial x} = -\frac{1}{k'^2\beta}\left\{ \frac{(A + B)y}{(x^2\tan^2\gamma - y^2)^{\frac{1}{2}}} + \frac{Bk^2 y^3}{3(x^2\tan^2\gamma - y^2)^{3/2}} \right\}$$

on the upper surface of the aerofoil with opposite sign
on the lower surface.

Again from equations (5) and (6)

$$\frac{\partial u}{\partial y} = R \int_0^{\tau} sn^2\tau\, nc\,\tau nd^4\gamma(Adn^2\tau + Bk'^2)d\tau$$

$$= R\left\{ \frac{A + B}{k'^3}\left[ch^{-1}dc\tau - k'sd\tau \right] - \frac{1}{3}Bk^2 sd^3\tau \right\}$$

$$= -\frac{A + B}{k'^3}\left\{ ch^{-1}\frac{x\tan\gamma}{|y|} - \frac{x\tan\gamma}{\sqrt{x^2\tan^2\gamma - y^2}} \right\}$$

$$+ \frac{Bk^2 x^3}{3\beta^3(x^2\tan^2\gamma - y^2)^{3/2}}$$

on the upper surface and with opposite sign on the lower
surface.

Hence $u = -\dfrac{A + B}{k'^3}y\; ch^{-1}\dfrac{x\tan\gamma}{|y|} + \dfrac{Bk^2}{3k'^3}\dfrac{xy\tan\gamma}{\sqrt{x^2\tan^2\gamma - y^2}}$

$$(7)$$

on the upper surface of the aerofoil.

/The excess ...

The excess pressure is approximately $-\rho uV$ and therefore the force derivatives at zero incidence may be readily calculated from expression (7). However, when the aerofoil is at incidence the interaction between the two fields give rise to a leading edge suction which contributes to the yawing moment and sideforce.

It is shown in Appendix IV of Ref.2. that, if the total induced velocity perpendicular to the leading edge is of the form $\left\{c\xi^{-\frac{1}{2}} + \text{bounded terms}\right\}$, where ξ is the distance in from the leading edge, then the suction force is $\pi\rho c^2 k \cos\gamma$ per unit length.

The velocity along the leading edge $y = x\tan\gamma$ induced by the yawing is $(u\cos\gamma + v\sin\gamma)$ and

$$\frac{\partial}{\partial x}(u\cos\gamma + v\sin\gamma) = \frac{1}{\rho}\cos\gamma\, R\int_0^{\tau}\left\{\frac{k'\,\mathrm{sn}\,\tau + i\mathrm{cn}\,\tau}{\mathrm{cn}\,\tau\,\mathrm{dn}^4\,\tau}\right.$$

$$\left. - \mathrm{sn}\,\tau(A\mathrm{dn}^2\,\tau + B^2 k')\right\}d\tau$$

which vanishes at the leading edge ($\tau = K - iK'$), and $\frac{\partial}{\partial y}(u\cos\gamma + v\sin\gamma)$ vanishes in like manner.

Hence the velocity induced by the yawing perpendicular to the leading edge is $(u\,\mathrm{cosec}\,\gamma + \text{bounded terms})$. The velocity potential induced by incidence alone is by Ref. 2 $V\alpha\sqrt{x^2\tan^2\gamma - y^2}/E$

Therefore the total induced velocity perpendicular to the leading edge is:-

$$\left\{\frac{V\alpha}{E} + \frac{Bk^2 y}{3k'^3}\right\}\frac{x\sec\gamma}{\sqrt{x^2\tan^2\gamma - y^2}} \quad + \quad \text{bounded terms}$$

$$= \left\{\frac{V\alpha}{E} + \frac{Bk^2}{3k'^3}x_0\right\}\sqrt{\frac{x_0\tan\gamma\sec\gamma}{2\xi}} + \text{bounded terms}$$

where $x = x_0 + \xi\sin\gamma$ and $y = x_0\tan\gamma - \xi\cos\gamma$.

Neglecting second order quantities the suction force resulting from yawing when at incidence is therefore:-

$$\pi\rho\,\alpha VBk^3 x^2 \tan\gamma/3Ek'^3 \qquad\qquad (8)$$

Integration of the pressure distribution obtained from expression (7) and of the suction forces at expression (8) yields the values of the non-dimensional derivatives ℓ_r, n_r, y_r given in Section 3. for $\lambda < 1$.

5.   Delta Wing with Leading Edges outside the Apex Mach cone

Since the region of influence of a disturbance at a point is contained within the Mach cone emanating from that point , the flow at the upper surface of the aerofoil is independent of the conditions at the lower surface, and vice versa, when the leading edges protrude through the apex Mach cone. The flow at the upper surface is represented, therefore, by a potential function which satisfies the boundary conditions at the top surfaces without regard to the lower surface, and such a function is obtained by integrating a distribution of elementary solutions or sources that give the correct values of the normal velocity locally. The required source distribution is of strength $-\dfrac{r\delta x}{\pi}$ for $y > 0$ and $+\dfrac{r\delta x}{\pi}$ for $y < 0$ for the upper surface.

The induced potential at the upper surface is therefore:-

$$\varphi(x,y) = -\frac{r\delta}{\pi}\int_{0}^{c}\int_{0}^{x_o\tan\gamma}\frac{x_o\,dx_o\,dy_o}{\sqrt{(x-x_o)^2-\beta^2(y-y_o)^2}}$$

$$+\ \frac{r\delta}{\pi}\int\int_{-x_o\tan\gamma}^{c}{}^{0}\frac{x_o\,dx_o\,dy_o}{\sqrt{(x-x_o)^2-\beta^2(y-y_o)^2}}$$

In Fig. 4d P is the point (x,y), OL_1 and OL_2 the leading edges, and PL_1 and PL_2 are the boundaries where $(x-x_o)^2-\beta^2(y-y_o)^2 = 0$

Put $x_o = x - \beta q\,\dfrac{1+t^2}{1-t^2}$, $y_o = y-q\dfrac{2t}{1-t^2}$

/The value ...

The value of q and t vary as follows:-

When (x_0, y_0) is on (i) PL_1, $t = -1$

(ii) PL_2, $t = +1$

(iii) OP, $t = t_0 = \dfrac{x - \sqrt{x^2 - \beta^2 y^2}}{\beta y}$

(iv) OX, $q = q_0 = \dfrac{1-t^2}{2t} \; y$

(v) OL_1, $q = q_1 = \dfrac{(x \tan \gamma - y)(1-t^2)}{\lambda(1+t^2) - 2t}$

(vi) OL_2, $q = q_2 = \dfrac{(x \tan \gamma + y)(1-t^2)}{\lambda(1+t^2) + 2t}$

When P is inside the Mach cone so that $x > \beta y > 0$, we have

$$\phi = \frac{2r\delta}{\pi} \left\{ \int_0^{q_1} \int_{-1}^{t_0} + \int_0^{q_0} \int_0^1 - \int_{t_0}^{q_2} \int_{t_0}^1 \frac{x(1-t^2) - \beta q(1+t^2)}{(1-t^2)^2} dq \; dt \right\}$$

Having integrated with respect to q we can differentiate under the integral signs with respect to x, since $q_0 = q_1 = q_2$ when $t = t_0$, and:-

$$u = \frac{2r\delta\lambda}{\pi} \int_{-1}^{t_0} \left\{ \frac{x \tan \gamma}{(\lambda t - 1)^2 + \lambda^2 - 1} - \frac{2\lambda t (x \tan \gamma - y)}{\left[(\lambda t - 1)^2 + \lambda^2 - 1 \right]^2} \right\} dt$$

$$- \frac{2r\delta\lambda}{\pi} \int_{t_0} \left\{ \frac{x \tan \gamma}{(\lambda t + 1)^2 + \lambda^2 - 1} + \frac{2\lambda t (x \tan \gamma + y)}{\left[(\lambda t + 1)^2 + \lambda^2 - 1 \right]^2} \right\} dt$$

$$+ \frac{2r\delta}{\pi} \int_{t_0}^1 \frac{y \, dt}{t} \tag{9}$$

$$= \frac{2r\delta}{\pi} \frac{y}{(\lambda^2 - 1)^{3/2}} \tan^{-1} \left\{ x \tan \gamma \sqrt{\frac{\lambda^2 - 1}{x^2 \tan^2 \gamma - \lambda^2 y^2}} \right\}$$

$$+ \frac{2r\delta}{\pi} \frac{(\lambda^2 - 2) x \tan \gamma}{(\lambda^2 - 1)^{3/2}} \tan^{-1} \left\{ y \sqrt{\frac{\lambda^2 - 1}{x^2 \tan^2 \gamma - \lambda^2 y^2}} \right\} + \frac{2r\delta y}{\pi} \mathrm{ch}^{-1} \frac{x}{\beta |y|}$$

the same result being obtained for $x > -\beta y > 0$

For a point outside the Mach cone, $\beta y > x$,

$$\varphi = \frac{2r\delta}{\pi} \int_0^{q_1} \int_{-1}^{+1} \frac{x(1-t^2)-\beta q(1+t^2)}{(1-t^2)^2} \, dq \, dt$$

By putting $t_o = 1$ in equation (9), we obtain

$$u = r\delta \, \frac{x \tan\gamma \, (\lambda^2-2) + y}{(\lambda^2 - 1)^{3/2}},$$
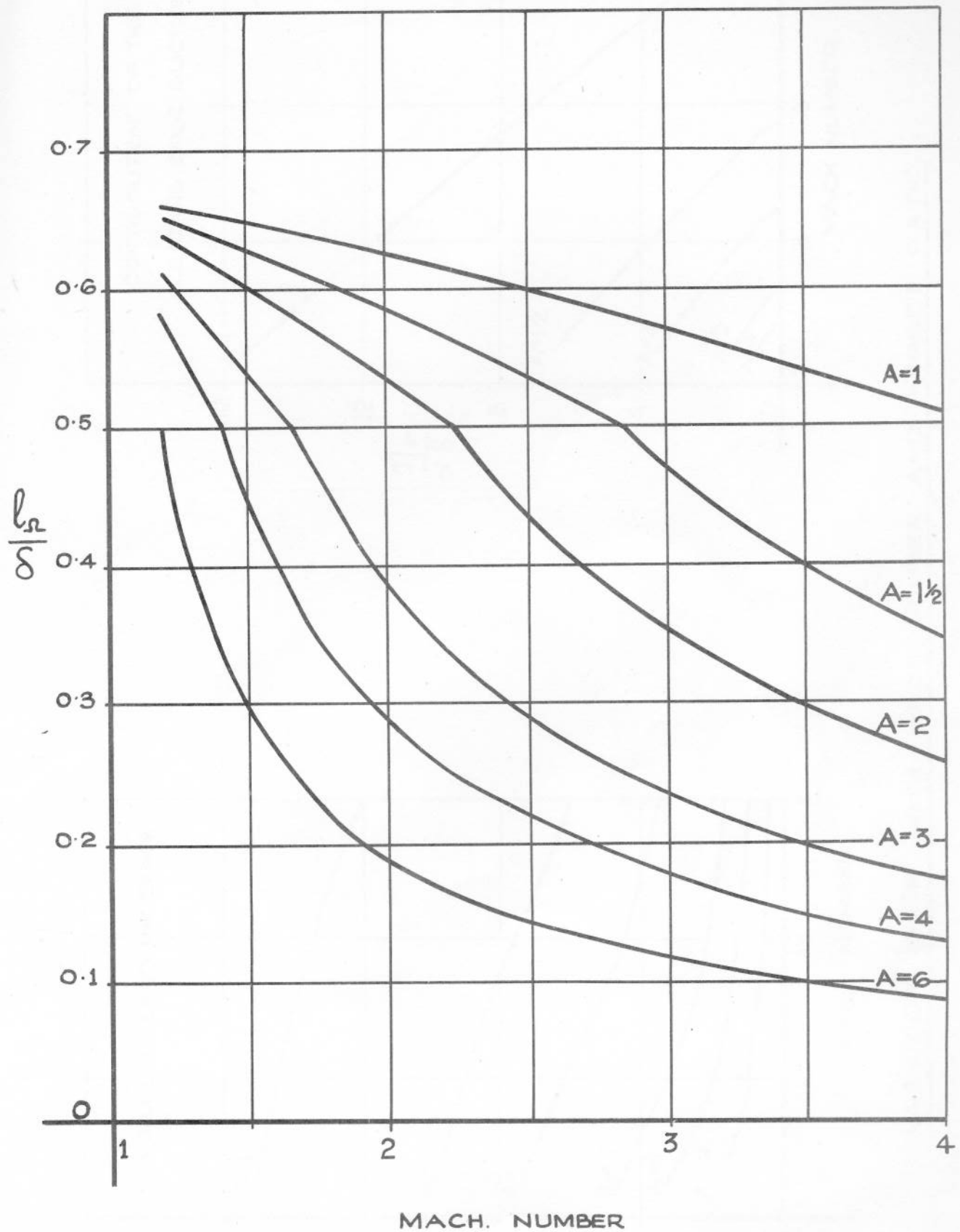
and similarly for $\beta \, y < -x$

$$u = - r\delta \, \frac{x \tan\gamma \, (\lambda^2 - 2) - y}{(\lambda^2 - 1)^{3/2}}$$

By integrating the pressure distribution given by these expressions for u, the values of the derivatives quoted in Section 3. for $\lambda > 1$ are obtained.
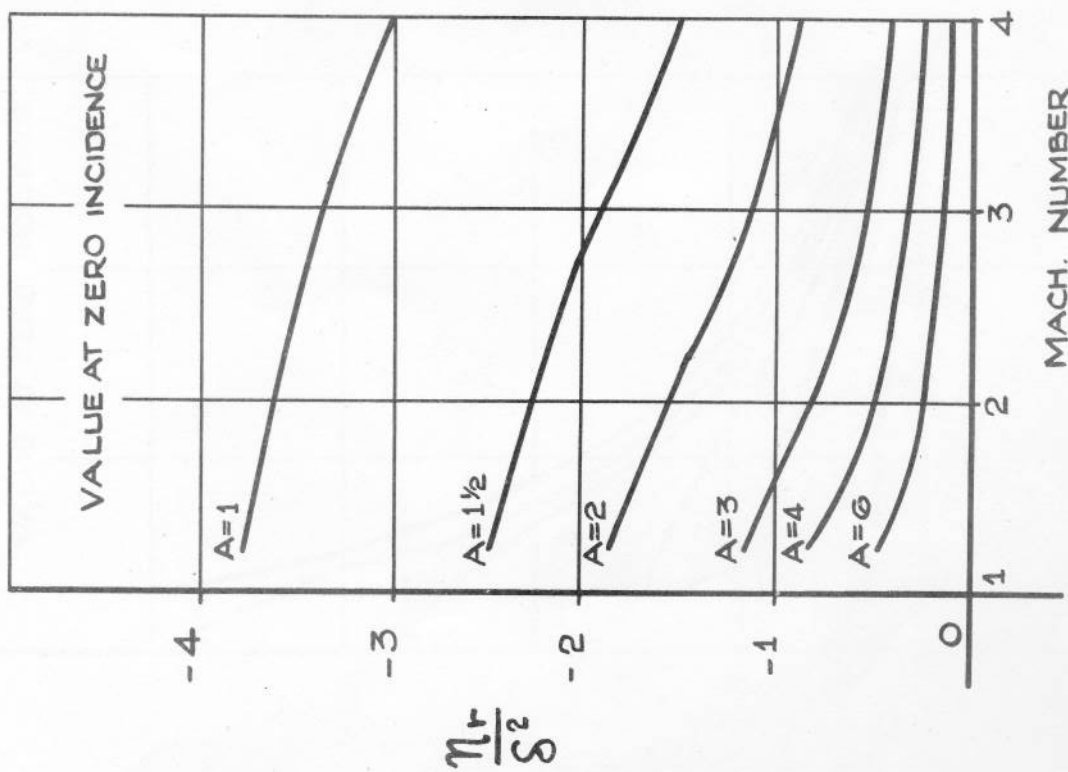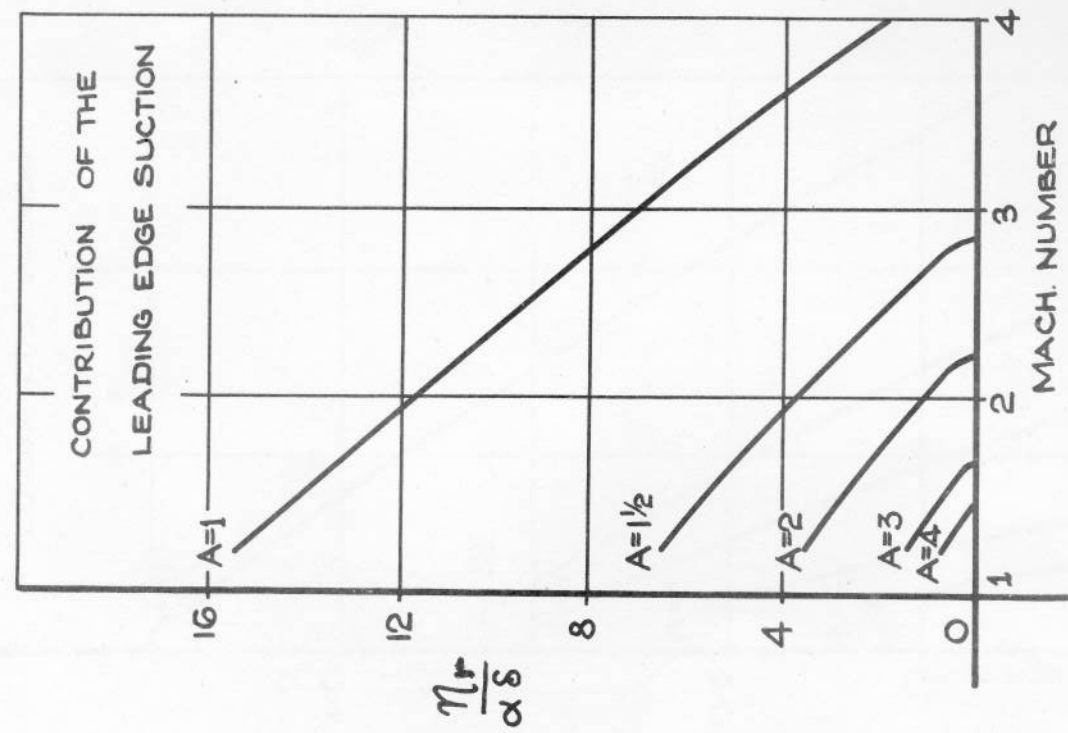
REFERENCES

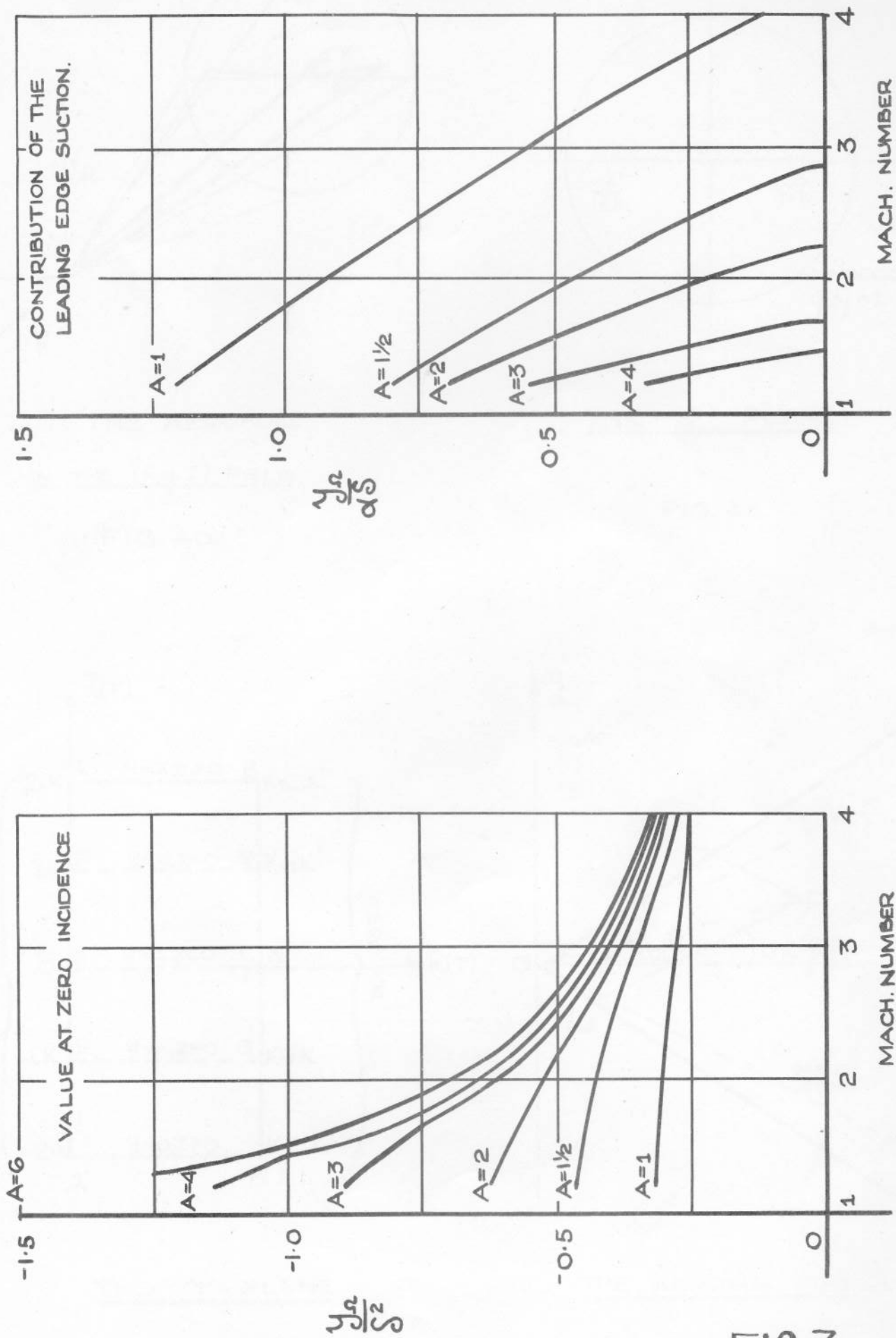|		
1	H.J. Stewart	The Lift of a Delta Wing at Supersonic Speeds. Quarterly of Applied Mathematics. vol.4. October, 1946.
2.	A. Robinson	Aerofoil Theory of a Flat Delta Wing at Supersonic Speeds. May,1946 to be published in A.R.C. R & M Series.
3.	A. Robinson J.H. Hunter-Tod	The Aerodynamic Derivatives with Respect to Sideslip for a Delta Wing with Small Dihedral at Supersonic Speeds. College of Aeronautics Report No. 12, 1947.
4.	A. Robinson	Rotary Derivatives of a Delta Wing at Supersonic Speeds. J. Roy. Ae. Soc. vol 52. November, 1948.
5.	E.W. Hobson	The Theory of Spherical and Ellipsoidal Harmonics (Section 106). Cambridge, 1931.

MACH. NUMBER

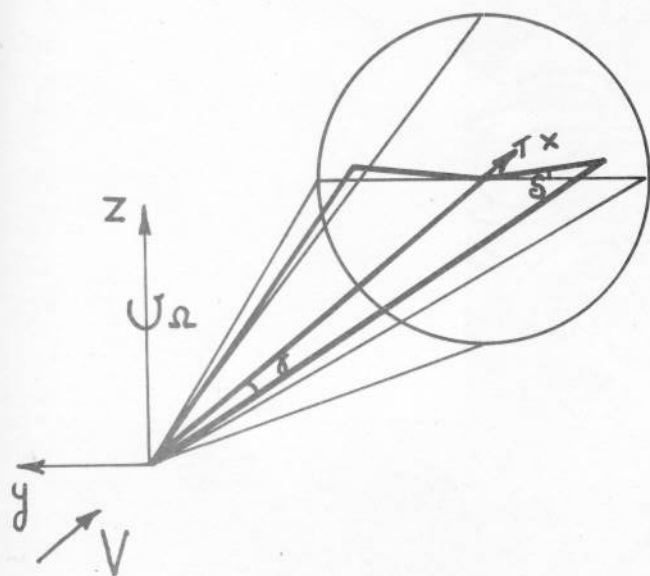VARIATION OF l_Ω WITH MACH NUMBER AND ASPECT RATIO

FIG. 1

CONTRIBUTION OF THE
LEADING EDGE SUCTION

$\dfrac{\eta_r}{\alpha \delta}$

16 — A=1

12

8 — A=1½

4 — A=2

A=3
A=4

0

1   2   3   4

MACH. NUMBER

VALUE AT ZERO INCIDENCE

$\dfrac{\eta_r}{\delta^2}$

A=1

A=1½

A=2

A=3

A=4

A=6

-4

-3

-2

-1

0

1   2   3   4

MACH. NUMBER

VARIATION OF η_r WITH MACH. NUMBER AND ASPECT RATIO.

FIG.2

CONTRIBUTION OF THE LEADING EDGE SUCTION.

$\dfrac{y_\Omega}{\alpha S}$

A=1

A=1½

A=2

A=3

A=4

MACH. NUMBER

VARIATION OF y_Ω WITH MACH. NUMBER AND ASPECT RATIO.

VALUE AT ZERO INCIDENCE

$\dfrac{y_\Omega}{S^2}$

A=6

A=4

A=3

A=2

A=1½
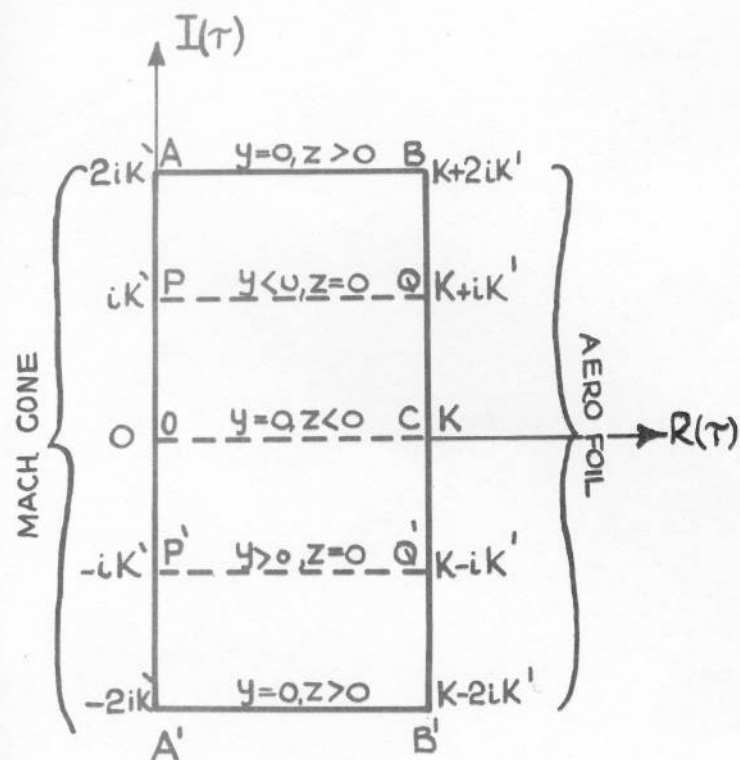
A=1

MACH. NUMBER

FIG.3

THE AEROFOIL

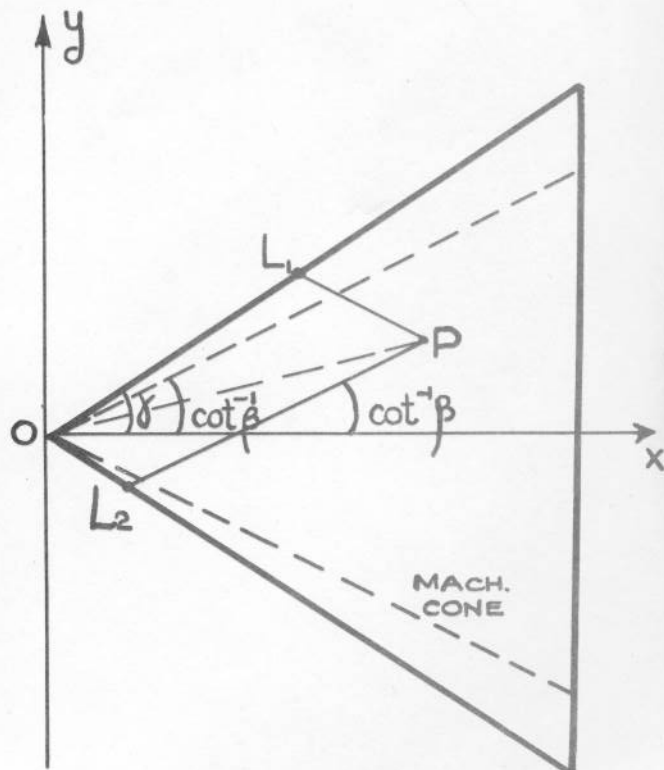IN THE (x, y, z) FIELD

FIG. 4a.

THE ω - PLANE
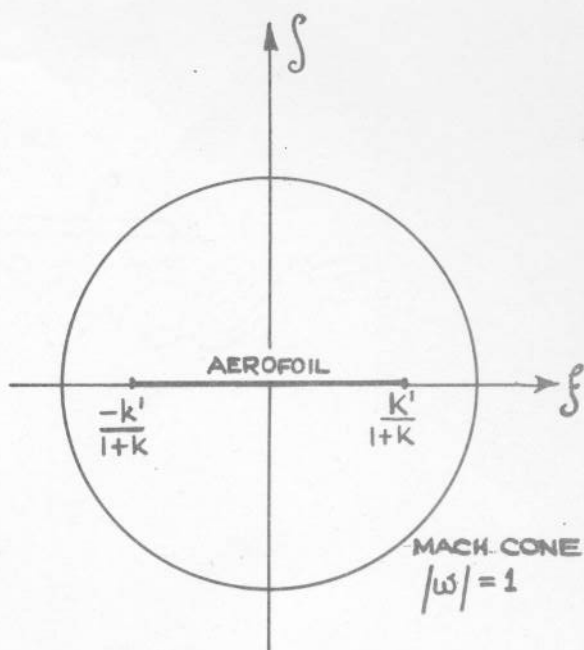
FIG. 4b.

THE τ - PLANE

FIG. 4C.

THE AEROFOIL FOR $\lambda > 1$

FIG. 4d.

FIG. 4