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The Efficiency of Adiabatic Expansion

- by -

A. W. Morley, Ph. D.
of the Department of Aircraft Propulsion

SUMMARY

The efficiency of a process of compression or expansion of a gas is commonly defined in terms of the change in energy which occurs as compared with the change required in isentropic flow. In another method the efficiency may be defined in terms of the fraction of the mechanical work lost in friction and converted into heat. Alternatively, if the process is adiabatic, the efficiency may be defined in terms of the fraction of the enthalpy increment which is re-converted into heat by the frictional effects. This latter method is applied here to adiabatic subsonic expansion of a gas in steady flow and some simple relationships of a general nature are established. The application to simplified flow through a turbine nozzle is then considered with particular reference to the choking mass flow.

MCL

NOTATION

H	-	Quantity of Heat	
J	-	Joule's equivalent	
ρ	-	Density	
g	-	Acceleration due to gravity	
P	-	Pressure (absolute)	P_1 upstream
T	-	Temperature (absolute)	$\overline{T_1}$ upstream
C_p, C_v	-	Specific heats at constant pressure and constant volume respectively (per unit mass)	
R	-	Gas constant (per unit mass)	
v	-	Stream velocity	
I	-	Enthalpy (for unit mass)	
η	-	Efficiency factor ($0 \leqslant \eta \leqslant 1$)	
M	-	Mach number of stream velocity	
γ	-	C_p/C	
x	-	$(2 - \eta)\gamma/(\gamma - 1)$	

/ Introduction. ....

1. Introduction

When considering the adiabatic flow of a gas it is common practice to allow for the frictional losses by relating the change in energy of the gas during the process to the change that would take place were it possible to carry it out isentropically. The isentropic efficiency, as it is then called, is the ratio of the ideal change of energy to the actual change for a compression, and of the actual change to the ideal change for a turbine, so that in both cases efficiency values are less than unity.

This method of reckoning the efficiency has the merit of simplicity and practical utility. In the case of expansion through a nozzle however it leads us to the erroneous conclusion that the maximum flow will occur at a Mach number less than unity depending on the value taken for the efficiency thus defined. Only when this efficiency is made 100% will the calculated maximum flow correspond with the true choking condition. With efficiencies less than unity the flow through the nozzle and the momentum discharge will be over estimated.

These discrepancies were considered some time ago by Moyes [1], and Hudson [2], and though explained physically by Frossel's experiments [3] the theoretical difficulties are not entirely resolved.*

It is of interest to examine other definitions of efficiency suitable for one-dimensional adiabatic flow. We seek a simple means of allowing for frictional losses over the range of subsonic flow. It is not possible to consider in a simple manner the losses incurred once speeds are reached at which shock waves are formed in the flow.

2. Basic equations and definition of efficiency of a compressive adiabatic process

According to the First Law of Thermodynamics heat and work are mutually convertible, and one unit of heat is worth J units of work.

Consider the effect of an increment of heat added to a chemically inert gas flowing steadily between insulating walls. If the temperature of the gas increases by δT its heat content is increased by an amount $C_v \delta T$ per lb. The remainder of the heat will appear as mechanical work performed by the gas, each unit mass of which will expand against the stream pressure, thus doing an increment of mechanical work equal to the product of the pressure (P) and increase in specific volume $\delta(1/\rho)$.

Now part of any mechanical work developed within the expanding gas will inevitably be lost in eddies and frictional work and will be re-converted into heat.

This frictional heat, which the gas supplies continuously to itself, is thus added to any increment of heat supplied to the gas from an external source.

Writing δH as the increment of heat supplied externally, δF as the work reconverted into heat, the total heat increment (in work units) received by the gas is:

$$J\,\delta H + \delta F$$

Of this quantity $JC_v\,\delta T$ is accounted for by the temperature change. The remainder:

$$J\,\delta H + \delta F - JC_v\,\delta T$$

disappears as heat, having been converted into the mechanical work done by the gas $P\,\delta(1/\rho)$.

/As a direct ....

* If the losses are considered in terms of a wall-friction coefficient, theory finds the choking condition correctly. See Ref. 6.

As a direct result of the First Law, we have therefore in symbols:

$$J\,\delta H + \delta F - JC_v\,\delta T = P\,\delta(1/\rho)$$

or, in differential form:

$$JdH + dF = JC_v dT + Pd(1/\rho) \qquad \ldots\ldots\ldots \quad (1)$$

In an "adiabatic" process dH is zero by definition. Equation (1) then reduces to

$$dF = JC\ dT + Pd(1/\rho) \qquad \ldots\ldots\ldots \quad (2)$$

It is convenient here to introduce the enthalpy change dI (heat units) which is given by the equation:

$$dI \approx C_p\,dT$$

$$= JC_v\,dT + d(P/\rho) \qquad \ldots\ldots\ldots \quad (3)$$

Equation (2) may now be re-written as:

$$dF = JdI - dP/\rho \qquad \ldots\ldots\ldots \quad (4)$$

If the adiabatic process were perfectly efficient dF would be zero, and the enthalpy change would be exactly equivalent to the mechanical work done.

It is important to note that friction is always positive so that care must be taken to ensure that dF and dI are of the same sign for a compression in which the enthalpy content is increased, and of opposite sign in an expansion where the enthalpy content is decreased. In any case some of the available enthalpy change is wasted.

In connection with some calculations by Pabst [4] on the efficiency of multi-stage compressors, where the conditions of adiabatic flow apply approximately, the fraction of the enthalpy lost in friction is used to define the efficiency of compression η . The equation used is:

$$dF \approx (1 - \eta)\,JdI \qquad \ldots\ldots\ldots \quad (5)$$

Since dI is positive this equation can be satisfied with values of η between zero and unity.

If we substitute this value for dF in equation (4) and also write:

$$dI = C_p\,dT$$

$$\rho = P/RT$$

$$C_p - C_v = R/J$$

we obtain the differential equation:

$$\eta\,\frac{\gamma}{\gamma - 1} \cdot \frac{dT}{T} = \frac{dP}{P}$$

If we follow Pabst and regard η and γ as constants, integration from state (1) to state (2) gives:

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\eta\frac{\gamma}{\gamma - 1}} \qquad \ldots\ldots\ldots \quad (6)$$

/ The temperature ....

The temperature exponent $\eta\gamma/(\gamma - 1)$ is directly comparable with that for a "polytropic" process which follows the law:

$$P \propto \rho^{n}$$

in which case:

$$\frac{P_2}{P_1} = \frac{T_2}{T_1}^{\frac{n}{n-1}} \qquad \text{........ (7)}$$

Equating the two exponents given by equations (6) and (7) gives:

$$\eta = \frac{\gamma - 1}{\gamma} \cdot \frac{n}{n - 1} \qquad \text{........ (8)}$$

or

$$n = \frac{\eta\gamma}{1 - \gamma(1 - \eta)} \qquad \text{........ (9)}$$

A curve for n in terms of η is given in the reference (4) and the connection between η and the isentropic efficiency also illustrated.

Since η is defined in terms of the infinitesimal change of state, it is called the "small stage efficiency" and because it appears in the "polytropic" exponent as a factor multiplying the "ideal" exponent $(\gamma/\gamma - 1)$ (see equation 8) it is also called the "polytropic efficiency".

In practical cases of adiabatic or approximately adiabatic compression, the efficiency η is usually in the range 0.85 to 0.98 say. Equation (8) shows that the index n is then greater than γ, and, therefore, the pressure - volume curve for adiabatic compression is steeper than the ideal isentropic curve, as is well known.

3. An adiabatic expansion

Adiabatic expansion is characterised by a decreasing enthalpy content, i.e. dI is negative. The heat generated internally as the result of friction is positive. Following equation (5) therefore, and defining the efficiency η' in terms of the lost enthalpy, we write:

$$dF = (\eta' - 1)\, J\, dI \qquad \text{........ (10)}$$

Here if η' were unity there would be no friction, i.e. the process becomes isentropic; and decreasing efficiency corresponds increasing (positive) frictional reheat.

Equation (4), from the First Law, still holds good. Substituting in this equation the above formula for dF, and again treating η' and γ as constants gives:

$$(2 - \eta)\,\frac{\gamma}{\gamma - 1}\int\frac{dT}{T} = \int\frac{dP}{P} + \text{constant} \qquad \text{........ (11)}$$

or

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{(2 - \eta')\frac{\gamma}{\gamma - 1}} \qquad \text{........ (12)}$$

/ By comparing ....

4. Critical conditions in a nozzle

Consider the flow through a turbine nozzle. Here the total energy content per lb. of gas is constant; so that with the assumption that the flow can be treated one-dimensionally and that changes in specific heat are not important, the total head temperature is constant along the nozzle. If the gas is at rest before it enters the nozzle with temperature T_1 then the total head temperature remains at T_1 throughout. Hence the temperature at a section where the Mach number is M will be given by:

$$T_1 = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

i.e.

$$M = \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}} \left(\frac{T_1}{T} - 1\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots \quad (19)$$

We now suppose that the flow is adiabatic with constant expansion efficiency (i.e. η' as defined in equation (10) is a constant). This can only apply approximately and then only up to the point where shock disturbances occur, beyond which the basic assumption of a continuous expansion can no longer hold.

Equation (12) gives the ratio of the pressure (P) at the point where the Mach number is M to the upstream pressure P_1 in the form:

$$\frac{P}{P_1} = \left(\frac{T}{T_1}\right)^{\frac{(2 - \eta)\gamma}{\gamma - 1}} \quad \ldots\ldots\ldots \quad (20)$$

A third relation between P, T and M is obtained from the continuity condition:

$$W = A \rho v = \text{a constant} \quad \ldots\ldots\ldots \quad (21)$$

where W is the mass flow, A the cross sectional area of the stream, ρ and v the mean sectional density and velocity respectively. This equation may be written:

$$W = A \cdot \frac{P}{RT} \cdot M \quad (\gamma g R T)^{*}$$

$$= APM \left(\frac{\gamma g}{RT}\right)^{\frac{1}{2}} = \text{a constant} \quad \ldots\ldots\ldots \quad (22)$$

Here g is the acceleration due to gravity.

Equations (19), (20) and (22) are sufficient for the calculation of P, T and M at all sections provided A and efficiency η are known.

For critical conditions M is unity, for then any further decrease of downstream pressure is no longer able to increase the velocity to the throat. From equation (19) we then have the well known condition for the critical temperature T_c at the throat:

$$\frac{T_c}{T_1} = \frac{2}{\gamma + 1} \quad \ldots\ldots\ldots \quad (23)$$

/ The critical ....

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* The appearance of g in this formula is due to the common practice amongst propulsion engineers of stating pressure in lb. per unit area and density in lb. per unit volume.

The critical pressure P_c follows from equation (16) :

$$\frac{P_c}{P_1} = \left(\frac{2}{\gamma + 1}\right)^{\frac{(2 - \eta')\gamma}{\gamma - 1}} \qquad \ldots\ldots\ldots \quad (24)$$

whence the critical mass flow from equation (22) is:

$$W_c = \frac{AP_1}{\sqrt{T_1}} \left\{ \left(\frac{\gamma g}{R}\right) \cdot \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma(3 - 2\eta') + 1}{\gamma - 1}} \right\}^{\frac{1}{2}} \qquad \ldots \quad (25)$$

Any further decrease in the ratio P/P_1, i.e. reduction in downstream pressure below P_c has no effect on the mass flow. Thus $W = W_c$ if:

$$\frac{P}{P_1} \leqslant \left(\frac{2}{\gamma + 1}\right)^{(2 - \eta')\frac{\gamma}{\gamma - 1}}$$

5. Condition for maximum mass flow

Experiment shows that the critical condition gives maximum mass flow[3]. If, however, equations (19) and (20) are used to eliminate M and T from equation (22), the nozzle mass flow is given by:

$$W = \frac{AP_1}{\sqrt{T_1}} \left(\frac{2g}{R} \frac{\gamma}{\gamma - 1}\right)^{\frac{1}{2}} \left(\frac{P}{P_1}\right)^{1 - \frac{x}{2}} \left\{ \left(\frac{P}{P_1}\right)^{-x} - 1 \right\}^{\frac{1}{2}} \qquad \ldots\ldots\ldots \quad (26)$$

where x is the polytropic pressure exponent $(\gamma - 1)/\{\gamma(2 - \eta')\}$. The maximum value of this expression does not correspond to the true critical flow.

We again treat γ and η' as constants, then with fixed initial conditions P_1 and T_1, W depends only on P/P_1. Then from equation (26) W is a maximum when:

$$\left(\frac{P}{P_1}\right)^{-x} = \frac{2 - x}{2 - 2x} = \frac{(3 - 2\eta')\gamma + 1}{(2 - 2\eta')\gamma + 2} \qquad \ldots\ldots\ldots \quad (27)$$

Calling the corresponding throat temperature T_m, we then have:

$$\frac{T_m}{T_1} = \frac{(2 - 2\eta')\gamma + 2}{(3 - 2\eta')\gamma + 1} \qquad \ldots\ldots\ldots \quad (28)$$

/ The Mach ....

The Mach number at the throat follows from equation (19) viz:

$$M_m = \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}} \left(\frac{T_1}{T_m} - 1\right)^{\frac{1}{2}}$$

$$= \left\{\frac{1}{(1 - \eta') + 1}\right\}^{\frac{1}{2}} \quad \ldots\ldots\ldots \quad (29)$$

M_m is less than unity because η' must be less than 1 (see Fig. 2). The temperature drop and pressure drop required to give this Mach number will be less than for the true critical flow. Thus it appears that the maximum value of W calculated from equation (26) fails to agree with the true condition of Mach number unity. A similar discrepancy occurs if the expansion efficiency is based on the ratio of the actual to the isentropic heat drop[1] and in this respect the method of reckoning efficiency given here can claim no advantage.

6. Method of estimating choking mass flow

In view of the discrepancy found above, the critical or "choking" mass flow should be estimated by the method given in paragraph (4). Sonic speed at the throat requires the conditions given in equations (23) and (24) and the proper mass flow follows from equation (25). A value for the efficiency η' must be assumed from practice and also an allowance made in the area A for the boundary layer.

If the method of paragraph (5) is followed, for the same efficiency, a greater mass flow will be predicted, and the pressure drop to the throat will be less than that actually required. With a given mass flow, this will lead to an underestimate of the throat area of the nozzle and of the rate of momentum discharge.

The discrepancy arises from inaccuracies introduced by the use of simplifying assumptions of one dimensional flow. The continuity equation (22) assumes that the velocity across any section is a constant thus neglecting the slower moving boundary layers. It is also unlikely that the rate of exchange between enthalpy and frictional reheat is constant along the flow path. Strictly speaking one should expect a variation in η' due to Mach number effects near the throat. The peak point on the flow curve (Fig. 3) will only agree with the true critical for M unity if η' approaches 1 near the throat. This cannot be true, for one must expect the frictional losses to be proportionately greater at the higher speeds.

REFERENCES

(1) Moyes, S. J. "Critical Conditions in a Convergent Nozzle." R.A.E. report E 3923, February, 1942. (R & M 2045).

(2) Hudson, G. "The Physical Implications of representing Frictional Flow in Convergent Nozzles by a Law of Constant Adiabatic Efficiency." P. J. Memo. M 1202, March, 1947.

(3) Frossel. "Flow in Smooth Straight Pipes at Velocities above and below Sound Velocity." N.A.C.A. Technical Memorandum No. 844.

(4) Pabst. "Application of the Methods of Calculation Used for Single Stage to Multi Stage Blowers." Report G.D.V. No. 4. Trans. G.D.C. 16/18 T.

(5) Pabst. "Frictional Heat in Turbo-Machinery". Luftfahrtforschung Vol. 19, No. 8, August, 1942. (Available as R.T.P. Translation No. 1732.)

(6) Woodrow. "Steady Flow of a Coolant Gas through a Channel". VIIth International Congress of Applied Mechanics, London, 1948.

FIGURES ATTACHED

FIG 1

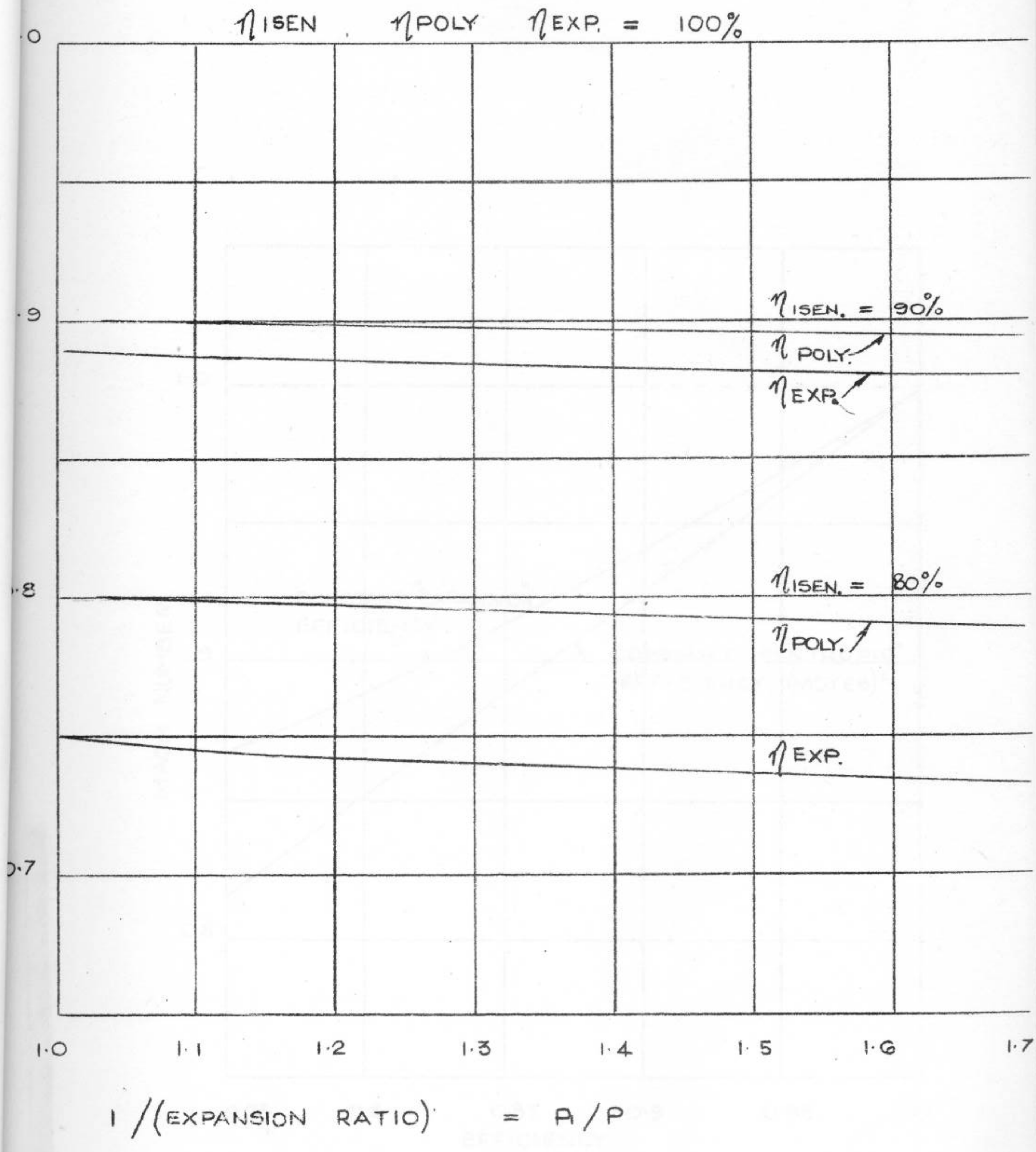

FIG 1. COMPARISON OF EFFICIENCIES FOR ADIABATIC EXPANSION.
EXAMPLE: FOR A PRESSURE RATIO OF 1·4 : 1 ; IF THE "ISENTROPIC" EFFICIENCY IS 80%. THE "POLYTROPIC" EFFICIENCY IS 79·4% AND "EXPANSION" EFFICIENCY, (AS CONSIDERED IN THE REPORT), 73·9% DRAWN FOR $\gamma = 1{\cdot}33$.

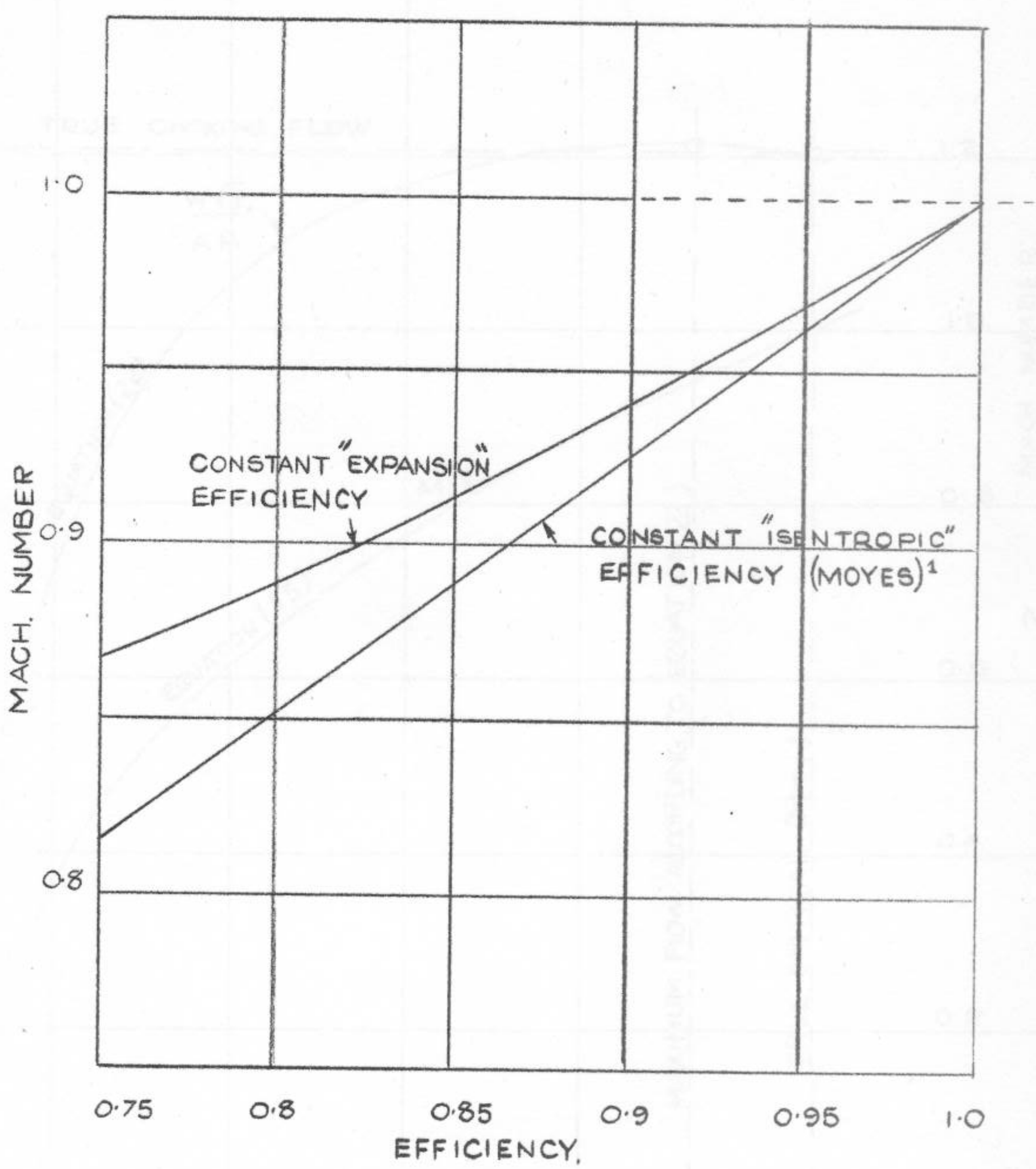

FIG. 2

CALCULATED THROAT MACH. NUMBER FOR MAXIMUM FLOW
DRAWN FOR $\gamma = 1{\cdot}33$.

FIG 3

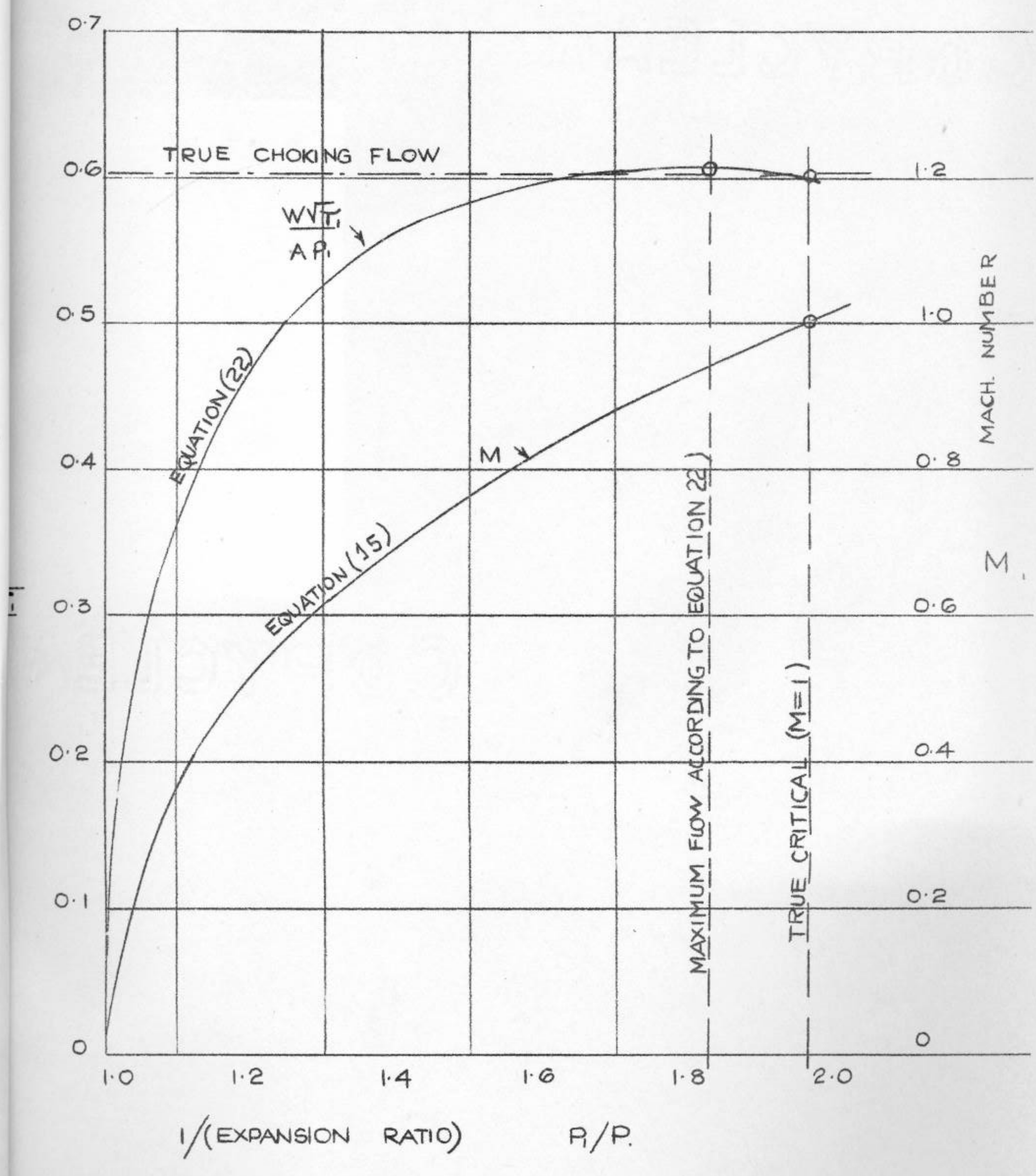

FIG 3. FLOW FUNCTION AND MACH. NUMBER DRAWN FOR $\gamma = 1{\cdot}33$ $\eta = 90\%$