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The Relative Accuracy of Quadrature Formulae of the Cotes' Closed Type

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- SUMMARY -

Quadrature formulae, such as those discovered by Gregory, Newton, Simpson and Cotes, which are derivable by integration of Lagrange's interpolation formula between definite limits, are classified as Cotes' Type Formulae. When the functional values at the end-points of the range of integration are used the corresponding formulae are said to be of the 'closed type'.

It is shown that, for closed type formulae, the error due to application of a 2n-strip formula is in general less than that due to a (2n + 1)-strip formula over the same range of integration when using the same tabular interval of the argument.

1. INTRODUCTION.

It is well known that in numerical integration by closed type formulae, Simpson's one-third rule (2-strip) is more accurate than Simpson's three-eighths rule (3-strip) when the tabular interval is the same. The relative accuracy, based on the leading term in the error series, is 4/9 provided that the integrand and its fourth derivative are continuous throughout and at the limits of the range of integration.

Similarly, and generally, for closed type formulae, it is shown that a 2n-strip formula is more accurate than a (2n + 1)-strip formula when the same tabular interval is used. The relative accuracy, again based on the leading error term and assuming continuity of the integrand and its (2n + 3)th derivative, is in fact always less than 3/4, however large the value of n.

The above comparisons inherently necessitate using an even number, 2n(2n + 1), of strips for the integration. This is not always convenient, particularly when the integration is to be evaluated from an even number of experimental points, yielding an odd number of strips. In these circumstances it may be expedient to use an odd-strip formula over part of the range of integration.

2. RELATIVE ERROR OF CLOSED TYPE FORMULAE.

In general we may express a Cotes' closed type quadrature formula in the form

$$\int_1^N f(x)\,dx = \int_1^N \phi(x)\,dx + R_N$$

$$= \sum_{r=1}^{N} J_r\, f(r) + R_N \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $\phi(x)$ is a polynomial of degree $N - 1$, which coincides with $f(x)$ for N values of the argument, and the multipliers J_r are Christoffel numbers whose numerical values depend upon N and are given by the equation

$$J_r = (-1)^{N-r}\, N \binom{N-1}{r-1} \int_1^N \binom{y-1}{N} \frac{dy}{y-r} .$$

The remainder, R_N , after N terms is discussed below.

We distinguish between the cases N odd or even by writing $N = 2n + 1$ or $N = 2n + 2$ respectively. Then for closed type formulae (Ref.1)

$$R_{2n+1} = -h^{2n+3}\, f^{(2n+3)}(\xi)\, C_{2n+1} \qquad \ldots\ldots\ldots\ldots(2)$$

/ and ...

and

$$R_{2n+2} = -h^{2n+3} f^{(2n+3)}(\xi)\, C_{2n+2} \quad , \quad \ldots\ldots\ldots\ldots(3)$$

where $1 < \xi < N$ and h is the tabular interval of the argument. The coefficients C_{2n+1}, C_{2n+2} are expressible in terms of Bernoulli's polynomials of order $(2n + 3)$ by the equalities

$$C_{2n+1} = \frac{B^{(2n+3)}_{2n+3}(2) + B^{(2n+3)}_{2n+3}(1)}{(2n+3)!} \quad \ldots\ldots\ldots\ldots(4)$$

and

$$C_{2n+2} = \frac{2B^{(2n+3)}_{2n+3}(2)}{(2n+3)!} \quad . \quad \ldots\ldots\ldots\ldots(5)$$

Since

$$B^{(n)}_{\nu}(x+1) = B^{(n)}_{\nu}(x) + \nu B^{(n-1)}_{\nu-1}(x) \quad , \quad \ldots\ldots\ldots\ldots(6)$$

it follows that, after putting $x = 1$ and 0 in equation (6) and substituting in equations (4) and (5), the coefficients C_{2n+1}, C_{2n+2} may also be expressed in terms of generalised Bernoulli's numbers by the equations

$$C_{2n+1} = \frac{2B^{(2n+3)}_{2n+3}}{(2n+3)!} + \frac{3B^{(2n+2)}_{2n+2}}{(2n+2)!} + \frac{B^{(2n+1)}_{2n+1}}{(2n+1)!} \ldots(7)$$

and

$$C_{2n+2} = \frac{2B^{(2n+3)}_{2n+3}}{(2n+3)!} + \frac{2B^{(2n+2)}_{2n+2}}{(2n+2)!} \quad . \quad \ldots\ldots\ldots\ldots(8)$$

These last two equations represent a convenient form for evaluating the leading term in the error series.

The generalised Bernoulli's polynomials and numbers employed above were discovered by Nörlund (Ref.3) and are also described by Milne-Thomson (Ref.4). The generalised Bernoulli's polynomials of order n are given by

$$\frac{t^n e^x}{(e^t - 1)^n} = \sum \frac{t^{\nu}}{\nu!} B^{(n)}_{\nu}(x) \quad ,$$

while the generalised Bernoulli's numbers of order n are given by

$$\frac{t^n}{(e^t - 1)^n} = \sum \frac{t^{\nu}}{\nu!} B^{(n)}_{\nu} \quad .$$

/ For ...

For the purposes of comparison and to determine the relative accuracy of two quadrature formulae of the closed type, it is essential that we use both the same range of integration

$$1 \leq x \leq N$$

and the same tabular interval h in each case. We shall therefore select

$$h = \frac{N - 1}{(2n)(2n + 1)} . \qquad \text{..........(9)}$$

The relative accuracy of the numerical integrations resulting from $(2n + 1)$ applications of a 2n-strip closed type formula and from $2n$ applications of a $(2n + 1)$-strip closed type formula is then

$$\frac{E_{2n}}{E_{2n+1}} = \frac{(2n + 1)\, R_{2n+1}}{2n\, R_{2n+2}} \qquad \text{..........(10)}$$

$$= \frac{(2n + 1)\, C_{2n+1}}{2n\, C_{2n+2}} , \qquad \text{..........(11)}$$

by equations (2) and (3).

From equations (11), (4) and (5)

$$\frac{E_{2n}}{E_{2n+1}} = \frac{(2n + 1)}{2n} . \frac{\left\{B_{2n+3}^{(2n+3)}(2) + B_{2n+3}^{(2n+3)}(1)\right\}}{2B_{2n+3}^{(2n+3)}(1)}$$

$$< \frac{(2n + 1)}{2n} . \frac{1}{2}$$

since $B_{2n+3}^{(2n+3)}(2)$ is negative when $2 \leq 2n + 4$, i.e. when $n \geq 1$.

Hence

$$\frac{E_{2n}}{E_{2n+1}} < \frac{3}{4} \qquad \text{..........(12)}$$

when $n \geq 1$.

Therefore, as proved in equation (12), the error due to $(2n + 1)$ applications of a 2n-strip formula of the closed type is less than three-quarters of the error due to $2n$ applications of a $(2n + 1)$-strip formula of the closed type when applied over a given range of integration using the same tabular interval in each case.

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It is worthy of note that the error due to either of the above formulae will be decreased, although the relative accuracy will remain unchanged, if the tabular interval given by equation (9) is subdivided into an integral number, p , of sub-intervals of extent

$$\frac{N - 1}{p(2n)(2n + 1)} .$$

From equations (11), (7) and (8) we obtain an alternative formula

$$\frac{E_{2n}}{E_{2n+1}} = \frac{(2n + 1)}{2n} \left\{ \frac{2B^{(2n+3)}_{2n+3} + 3(2n + 3)B^{(2n+2)}_{2n+2} + (2n + 3)(2n + 2)B^{(2n+1)}_{2n+1}}{2B^{(2n+3)}_{2n+3} + 2(2n + 3)B^{(2n+2)}_{2n+2}} \right\} \qquad \ldots\ldots\ldots\ldots(13)$$

which is suitable for computing particular values of the relative accuracy, E_{2n}/E_{2n+1} , from a table of generalised Bernoulli's numbers such as given by Milne-Thomson (Ref.1).

If in equation (13) we substitute $n = 1$, together with the values

$$B^{(3)}_{3} = -\frac{9}{4}, \quad B^{(4)}_{4} = \frac{251}{3}, \quad B^{(5)}_{5} = -\frac{475}{12},$$

the relative accuracy of three applications of Simpson's one-third rule as against two applications of Simpson's three-eighths rule is found to be

$$\frac{E_2}{E_3} = \frac{4}{9} = 0.444 .$$

Duncan refers to this result in a recent Note in which he investigates the errors due to the use of certain quadrature formulae (Ref.4).

Similarly, when we substitute $n = 2, 3$ in equation (13), we obtain

$$\frac{E_4}{E_5} = \frac{128}{275} = 0.465$$

and

$$\frac{E_6}{E_7} = \frac{3888}{8183} = 0.475 .$$

From the above general results, it is advised that Cotes' closed type quadrature formulae using an even number of strips (odd number of ordinates) should whenever convenient be employed in preference to the corresponding odd-strip formulae with <u>one</u> more strip (or ordinate).

/ References ...

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