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THE COLLEGE OF AERONAUTICS
CRANFIELD

TWO-PORT NETWORK REPRESENTATION OF D.C. ELECTRO-MECHANICAL TRANSDUCERS

by

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Two-port Network Representations of D.C. Electro-Mechanical Transducers

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SUMMARY

Two port network representations are derived for the general linear magnetic and electric field transducers. The constraints imposed by linearity requirements are discussed.

It is shown that for the most general form of transducer, the conversion of energy leads to non-linear relationships, and a method of solving these equations is suggested.

Typical applications are included to illustrate the analysis procedure and in particular the case of the d.c. motor is discussed in detail.

CONTENTS

1. Introduction

The topic was first suggested to the author by S. R. Deards, but it subsequently appeared that it was treated in 1957 by Rideout and Swift in papers presented at a conference of the American Institute of Electrical Engineers. Reference is made to this work, which deals specifically with d.c. motors, in a later paper by Swift (Ref. 1), but the papers were not published in the Transactions. As details of this work are not generally available, it is considered that there may be some virtue in the publication of this note.

Some later work on a similar topic has been the subject of a thesis (Ref. 2) which, after further work, will form the subject matter of a future paper.

In this note, force or torque and current, are regarded as flow (through) variables and hence are analogous. Similarly velocity and potential difference (voltage) are considered as potential (across) variables. The adoption of this procedure results in topological equivalence in analogous electrical and mechanical networks and this is considered to be an advantage compared with the dual analogous systems resulting from the classical force voltage analogy.

The material presented represents a preliminary study into the possibilities of applying some electrical network concepts, to the solution of generalized electro-mechanical and mechanical networks. (Ref. 3).

2. The basic d.c. electro-mechanical transducer

The purpose of any electro-mechanical transducer is to convert electrical energy to mechanical energy or vice versa. The coupling may be magnetic or electric. In the most general form of transducer there may be n electrical ports and m mechanical ports.

In this note attention is confined to the response of the mechanical variables at one mechanical port to the excitation at one electrical port. The reciprocal case is also considered.

2.1. Electro-magnetic coupling

(a) Rectilinear motion

If current i and co-ordinate x are the independent variables, then

$$\text{Force } F_x = -\frac{\partial W_m}{\partial x} + i\,\frac{\partial \phi}{\partial x} \tag{2.1.1}$$

where W_m is the stored energy in the magnetic field, and ϕ is the flux linkage.

Also,
$$W_m = \tfrac{1}{2}\int_{\text{volume}} B.H.\,dv \tag{2.1.2}$$

where B is the flux density, H is the magnetic force, v is the volume of the field.

We have also

$$\text{electro-motive force } e(t) = \frac{d}{dt} \phi(i, x) \qquad (2.1.3)$$

$$\text{i.e. } e(t) = \frac{\partial \phi(i, x)}{\partial i} \frac{di}{dt} + \frac{\partial \phi}{\partial x} (i, x) \frac{dx}{dt} \qquad (2.1.4)$$

(b) Angular Motion

If current i and co-ordinate θ are the independent variables then,

$$\text{Torque } F_\theta = \frac{\partial W_m}{\partial \theta} + i \frac{\partial \phi}{\partial \theta}, \qquad (2.1.5)$$

$$\text{and} \quad e(t) = \frac{d}{dt} \phi(i, \theta) \qquad (2.1.6)$$

$$= \frac{\partial \phi(i,\theta)}{\partial i} \frac{di}{dt} + \frac{\partial \phi(i,\theta)}{\partial \theta} \frac{d\theta}{dt}. \qquad (2.1.7)$$

2.2. Electric field coupling

(a) Rectilinear motion

If voltage e and co-ordinate x are the independent variables, then

$$\text{Force } F_x = - \frac{\partial W_e}{\partial x} + e \frac{\partial q}{\partial x}, \qquad 2.2.1)$$

where W_e is the energy stored in the electric field and,

$$W_e = \tfrac{1}{2} \int_{\text{volume}} D.E.\ dv \qquad (2.2.2)$$

where D is the electric flux density, E is the electric field force, v is the volume of the field, and q is the total charge.

$$\text{Also} \quad i(t) = \frac{d}{dt} q(e, x) \qquad (2.2.3)$$

$$\text{and} \quad \therefore \quad i(t) = \frac{\partial q(e,x)}{\partial e} . \frac{de}{dt} + \frac{\partial q(e,x)}{\partial x} . \frac{dx}{dt} . \qquad (2.2.4)$$

(b) Angular motion

If voltage e and co-ordinate θ are the independent variables, then

$$\text{Torque } F_\theta = - \frac{\partial W_e}{\partial \theta} + e \frac{\partial q}{\partial \theta} \qquad (2.2.5)$$

$$\text{and} \quad i(t) = \frac{\partial q(e,\theta)}{\partial e} . \frac{de}{dt} + \frac{\partial q(e,\theta)}{\partial \theta} . \frac{d\theta}{dt} . \qquad (2.2.6)$$

3. Two-port network representations

For these representations to be valid, certain assumptions regarding parameters, have to be imposed. In other words, the assumptions must be such, that there is a linear relationship between the mechanical variables and the electrical variables.

3.1. Magnetic coupling

As an example of these constraints consider the case of rectilinear motion.

Let us impose constraints as follows: -

$$\frac{\partial W_m}{\partial x} = 0 \tag{3.1.1}$$

$$\frac{\partial \phi}{\partial x} = k_m \text{ (constant)} \qquad 3.1.2)$$

$$\frac{\partial \phi}{\partial i} = 0 \tag{3.1.3}$$

From 2.1.1

$$F_x(t) = i\frac{\partial \phi}{\partial x} = k_m\, i(t) \tag{3.1.4}$$

$$e(t) = \frac{\partial \phi}{\partial x} \cdot \frac{dx}{dt} = k_m\, \omega(t) \tag{3.1.5}$$

where $$\omega(t) = \frac{dx}{dt} \tag{3.1.6}$$

Thus

$$\begin{bmatrix} e(s) \\ i(s) \end{bmatrix} = \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \omega(s) \\ i(s) \end{bmatrix} \tag{3.1.7}$$

Two port network representation.

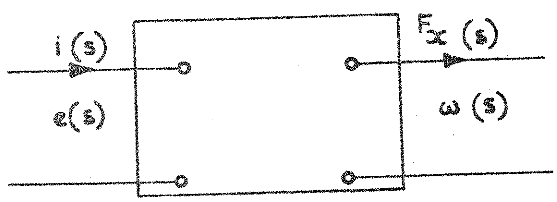

Fig. 3.1.1.

3.2. Electric field coupling

Consider again rectilinear motion with imposed constraints.

$$\frac{\partial W_e}{\partial x} = 0 \qquad (3.2.1)$$

$$\frac{\partial q}{\partial x} = k_m \qquad (3.2.2)$$

$$\frac{\partial q}{\partial e} = 0 \qquad (3.2.3)$$

From (2.2.1)

$$F_x(t) = e(t)\frac{\partial q}{\partial x} = k_m\, e(t) \qquad (3.2.4)$$

From (2.2.4)

$$i(t) = \frac{\partial q}{\partial x} \cdot \frac{dx}{dt} = k_m\, \omega(t) \qquad (3.2.5)$$

where $$\omega(t) = \frac{dx}{dt} \qquad (3.2.6)$$

Thus

$$\begin{bmatrix} e(s) \\ i(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{k_m} \\ k_m & 0 \end{bmatrix} \begin{bmatrix} \omega(s) \\ F_x(s) \end{bmatrix} \qquad (3.2.7)$$

i.e.

$$\begin{bmatrix} e(s) \\ i(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \omega(s) \\ F(s) \end{bmatrix} \qquad (3.2.8)$$

Note

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents the transmission matrix of a gyrator.

This is necessary in this case, since electrostatic energy is converted into electro-magnetic energy.

Network configuration

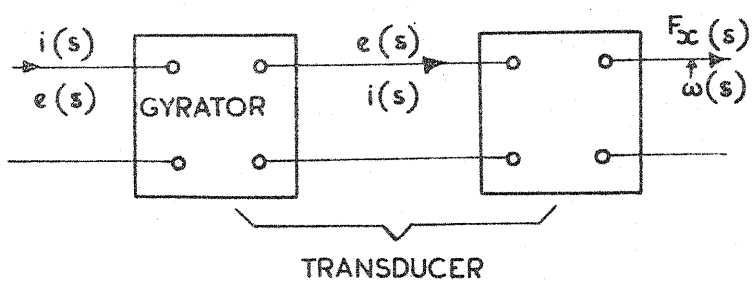

Fig. 3.2.1

3.3. Mechanically loaded transducer

Consider now that the mechanical port variables are subject to mechanical constraints. In the general case these will comprise :-

(a) Rectilinear case

Mass M including mass of transducer

Viscous friction coefficient f including that of transducer.

Spring constant k.

Total force $F_x(s) = sM\,\omega(s) + f\omega(s) + \frac{k}{s}\,\omega(s)$. (3.3.1)

∴ Mechanical transfer impedance

$$Z_m(s) = \frac{\omega(s)}{F_x(s)} = \frac{1}{sM + f + \frac{k}{s}} \quad . \qquad (3.3.2)$$

Mechanical transfer admittance

$$Y_m(s) = sM + f + \frac{k}{s} \quad . \qquad (3.3.3)$$

Thus

$$\begin{bmatrix} \omega_1(s) \\ F_{x_1}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \bar{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \omega_2(s) \\ F_{x_2}(s) \end{bmatrix} \qquad (3.3.4)$$

Network configuration

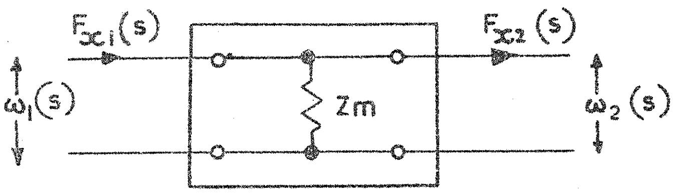

Fig. 3.3.1

3.4. Electrical circuit impedance

Consider the series impedance of the electrical circuit of transducer to be $Z_a(s)$.

The linear transducer may then be represented as shown in Fig. 3.4.1.

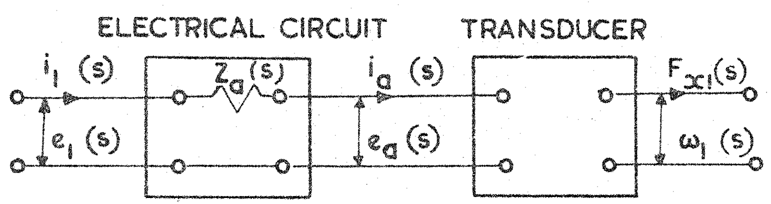

Fig. 3.4.1

Thus

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} 1 & \overline{Z}_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \omega_1(s) \\ F_{x1}(s) \end{bmatrix} \qquad (3.4.1)$$

For electro magnetic transducer.

Now let admittance of electrical circuit be $Y_e(s)$

We have now,

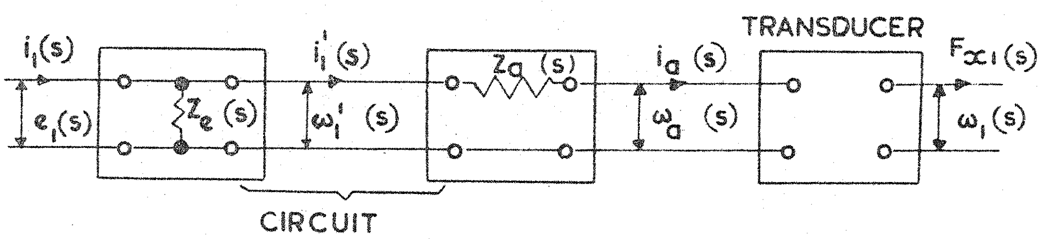

Fig. 3.4.2

Thus

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \overline{Y}_e & 1 \end{bmatrix} \begin{bmatrix} 1 & \overline{Z}_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \omega_1(s) \\ F_{x1}(s) \end{bmatrix} \tag{3.4.2}$$

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} k_m & \frac{\overline{Z}_a}{k_m} \\ k_m \overline{Y}_e & \frac{1+\overline{Y}_e \overline{Z}_a}{k_m} \end{bmatrix} \begin{bmatrix} \omega_1(s) \\ F_{x1}(s) \end{bmatrix} \tag{3.4.3}$$

For electro-mechanical transducer.

Note: The electrical circuit configuration may be more complicated than that considered above, but such cases may be treated by conventional network analysis, in the manner illustrated above.

4. Typical rectilinear transducers

Two examples are included to illustrate the application of the techniques discussed.

4.1. Electro-magnetic loud-speaker

The basic configuration is shown in Fig. 4.1.1.

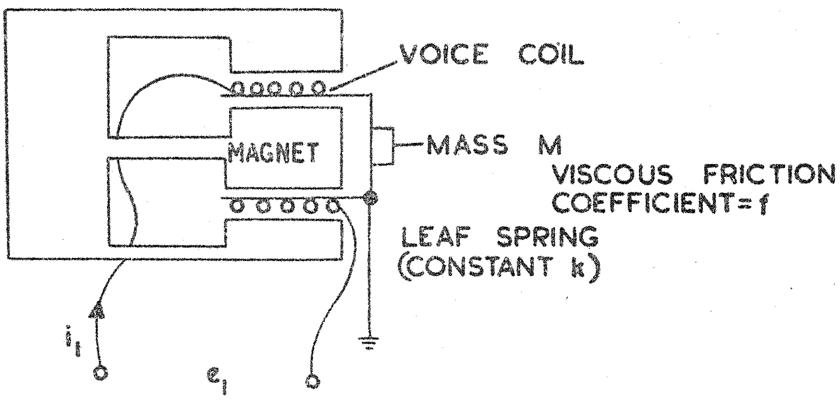

Fig. 4.1.1.

$$\frac{\partial W_m}{\partial x} = 0 \tag{4.1.1}$$

If B is the flux density,

ℓ is the length of wire in coil,

and x is the translational movement,

then $\phi = B\ell x$, (4.1.2)

and $\dfrac{\partial \phi}{\partial x} = B\ell = k_m$ (constant) , (4.1.3)

also $\dfrac{\partial \phi}{\partial i} = 0$. (4.1.4)

Thus
$$F_x(t) = B\ell\, i(t) = k_m\, i(t) \; . \tag{4.1.5}$$

$$e(t) = B\ell\, \omega(t) = k_m\, \omega(t) \; . \tag{4.1.6}$$

Also
$$Z_a(s) = R + sL \; , \tag{4.1.7}$$

$$Y_e(s) = 0 \; . \tag{4.1.8}$$

$$Y_m(s) = sM + f + \frac{k}{s}. \tag{4.1.9}$$

Thus the network configuration

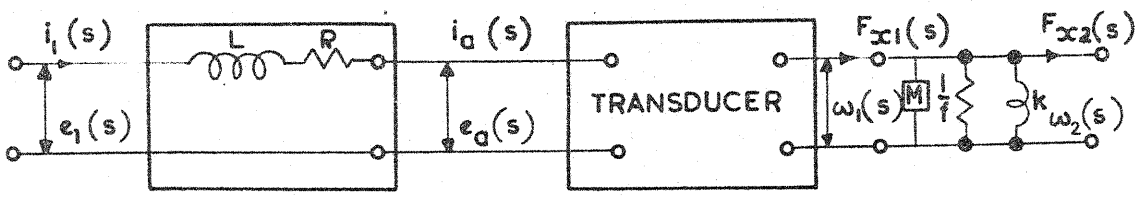

Fig. 4.1.2

We have

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} 1 & \bar{Z}_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \bar{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \omega_2(s) \\ F_{x_2}(s) \end{bmatrix} \qquad (4.1.10)$$

$$= \begin{bmatrix} k_m & \frac{\bar{Z}_a}{k_m} \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \bar{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \omega_2(s) \\ F_{x_2}(s) \end{bmatrix} \qquad (4.1.11)$$

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} \left(k_m + \frac{\bar{Z}_a \bar{Y}_m}{k_m}\right) & \frac{\bar{Z}_a}{k_m} \\ \frac{\bar{Y}_m}{k_m} & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \omega_2(s) \\ F_{x_2}(s) \end{bmatrix} \qquad (4.1.12)$$

If $Z_m(s)$ constitutes the total mechanical constraint on the coil former, then

$$F_{x_2}(s) = 0$$

and

$$e_1(s) = \left(k_m + \frac{\bar{Z}_a \bar{Y}_m}{k_m}\right) \omega_2(s) . \qquad (4.1.13)$$

Thus the reciprocal transfer function under these conditions is shown in Fig. 4.1.3.

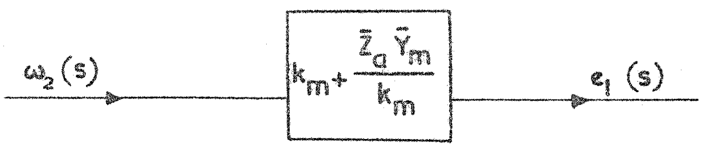

Fig. 4.1.3

And $$k_m + \frac{\bar{Z}_a \bar{Y}_m}{k_m} = B\ell + \frac{(R + sL)(sM + f + \frac{k}{s})}{B\ell} \qquad (4.1.14)$$

$$\therefore \quad \frac{1}{B\ell} \cdot \frac{e_1(s)}{\omega_2(s)} = 1 + (R + sL)\left[\frac{sM}{B^2\ell^2} + \frac{f}{B^2\ell^2} + \frac{1}{s}\left(\frac{k}{B^2\ell^2}\right)\right] \qquad (4.1.15)$$

Analogous electrical representation

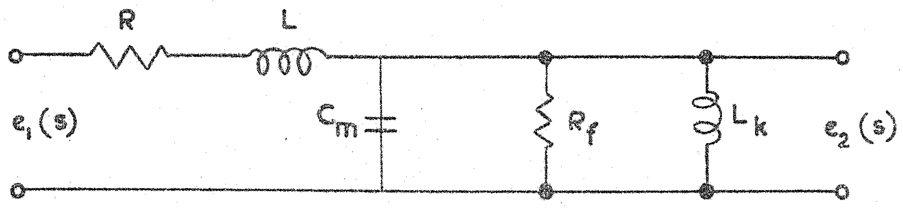

Fig. 4.1.4

$$C_M = \frac{M}{B^2\ell^2} \; ; \; \frac{1}{R_f} = \frac{f}{B^2\ell^2} \; ; \; \frac{1}{L_k} = \frac{k}{B^2\ell^2} \; ; \; e_2(s) = B\ell\omega(s) \, . \qquad (4.1.16)$$

4.2. Capacitor microphone

The basic configuration is illustrated in Fig. 4.2.1.

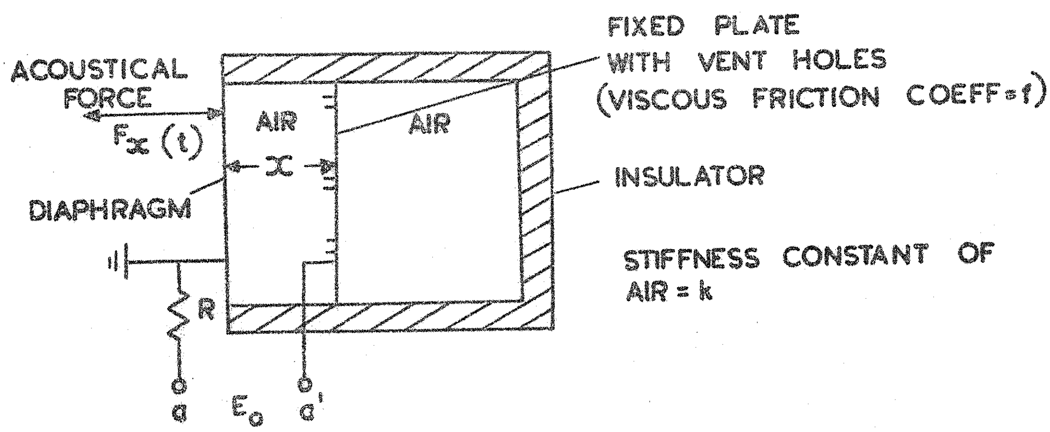

Fig. 4.2.1.
Capacitor Microphone

Assumption

Diaphragm has a translational movement x across whole surface (i.e. no bowing) when subjected to force F_x.

Stiffness of air assumed constant = k.

From 2.2.1

$$F_x = -\frac{\partial W_e}{\partial x} + e\frac{\partial q}{\partial x} . \qquad (4.2.1)$$

For this application

$$\frac{\partial q}{\partial x} = 0 \qquad (4.2.2)$$

$$\therefore \quad W_e = +\tfrac{1}{2}\int_{\text{volume}} D\,E\,dv$$

$$= +\tfrac{1}{2}\int_{\text{volume}} \frac{q(t)}{A} \cdot \frac{q(t)}{\epsilon_o A}\,dA\,dx \qquad (4.2.3)$$

$$W_e = +\tfrac{1}{2}\frac{q(t)^2}{\epsilon_o A}\,x \qquad (4.2.4)$$

$$F_2(t) = -\frac{\partial W_e}{\partial x} = -\tfrac{1}{2}\frac{q(t)^2}{\epsilon_o A} , \qquad (4.2.5)$$

where ϵ_o is the permitivity of air, and A is the area of the plate.

This is a non linear equation since W_e is a function of x.
Also

$$e(t) = \frac{q(t)}{c(t)} \qquad (4.2.6)$$

$$= \frac{q(t)\,x(t)}{\epsilon_o A} , \qquad (4.2.7)$$

where e(t) is the p.d. across the plates.

$$x(t) = \epsilon_o A\frac{e(t)}{q(t)} = \epsilon_o A\,\alpha(t) , \qquad (4.2.8)$$

where $\alpha(t)$ represents the across variable.

$$F_x(t) = \frac{1}{\epsilon_o A} \cdot \tfrac{1}{2}q(t)\;q(t) = \frac{1}{\epsilon_o A}\,\beta(t) , \qquad (4.2.9)$$

where $\beta(t)$ represents the through variable.

Note positive sign because of network variable convention below.

Network representation

$$\begin{bmatrix} x(t) \\ F_x(t) \end{bmatrix} = \begin{bmatrix} \epsilon_o A & 0 \\ 0 & \frac{1}{\epsilon_o A} \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} \qquad (4.2.10)$$

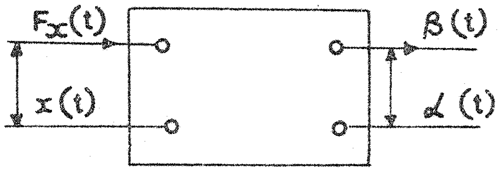

Fig. 4.2.2

If we now consider the mechanical constraints,

$$F_x = \left(sM + f + \frac{k}{s} \right) s\, x(s) \qquad (4.2.11)$$

$$= Y_m(s)\, x(s) \qquad (4.2.12)$$

$$\therefore \quad Y_m(s) = (s^2M + fs + k). \qquad (4.2.13)$$

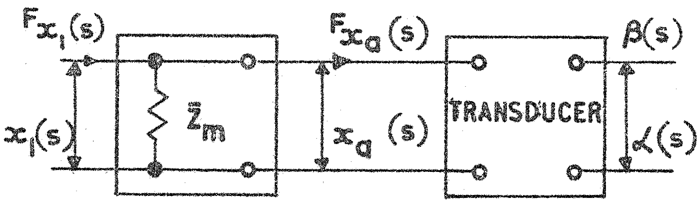

Fig. 4.2.3

$$\begin{bmatrix} x_1(s) \\ F_{x_1}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \overline{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \epsilon_o A & 0 \\ 0 & \frac{1}{\epsilon_o A} \end{bmatrix} \begin{bmatrix} \alpha_1(s) \\ \beta_1(s) \end{bmatrix} \quad (4.2.14)$$

$$= \begin{bmatrix} \epsilon_o A & 0 \\ \overline{Y}_m \epsilon_o A & \frac{1}{\epsilon_o A} \end{bmatrix} \begin{bmatrix} \alpha_2(s) \\ \beta_2(s) \end{bmatrix} \quad (4.2.15)$$

Linear incremental relationships

If we assume small perturbations about fixed values of parameters, we may determine a linear mathematical model.

Let $q(t) = q_o + q_1(t)$ (4.2.16)

and $x(t) = x_o + \delta$ (4.2.17)

where q_o and x_o correspond to bias operating point.

From (4.2.5)

$$F_x(t) = \frac{1}{2} \frac{q(t)^2}{\epsilon_o A} = \frac{1}{2} \frac{(q_o + q_1(t))^2}{\epsilon_o A} \quad (4.2.18)$$

$$= \frac{(q_o^2 + 2q_o q_1(t) + q_1(t))^2}{2\epsilon_o A} .$$

$$F_x(t) = \frac{q_o^2}{2\epsilon_o A} + \frac{q_o q_1(t)}{\epsilon_o A} \quad (4.2.19)$$

if $q_1(t) << q_o$.

$$F_{xo}(t) = \frac{1}{2} \frac{q_o^2}{\epsilon_o A} . \quad (4.2.20)$$

$\therefore$ incremental force $F_\delta(t) = \dfrac{q_o q_1(t)}{\epsilon_o A}$. (4.2.21)

Now $q_o = E_o C_o = \dfrac{E_o \epsilon_o A}{x_o}$.

$$F_\alpha(t) = \frac{E_o q_1(t)}{x_o} . \quad (4.2.22)$$

From (4.2.6)

$$e(t) = \frac{q(t)}{C(t)} = \frac{q(t)\,x(t)}{\epsilon_o A} \tag{4.2.23}$$

$$= \frac{(q_o + q_1(t))(x_o + \delta(t))}{\epsilon_o A}$$

$$= \frac{q_o\, x_o + q_o\delta(t) + x_o\, q_1(t) + q_1(t)\delta(t)}{\epsilon_o A} \tag{4.2.24}$$

$$\simeq E_o + \frac{q_o\,\delta(t) + x_o\, q_1(t)}{\epsilon_o A} \tag{4.2.25}$$

$$e_o = E_o$$

$\therefore$ Incremental voltage $e_1(t) = \frac{q_o\,\delta(t)}{\epsilon_o A} + \frac{x_o}{\epsilon_o A}\, q_1(t)$ (4.2.26)

$$\text{i.e.}\quad e_1(t) = \frac{E_o}{x_o}\,\delta(t) + \frac{q_1(t)}{C_o} \tag{4.2.27}$$

From equation (4.2.22) and (4.2.27) we have the matrix relationship.

$$\begin{bmatrix} \delta(t) \\ F_\delta(t) \end{bmatrix} = \begin{bmatrix} \frac{x_o}{E_o} & -\frac{x_o}{C_o E_o} \\ 0 & \frac{E_o}{x_o} \end{bmatrix} \begin{bmatrix} e_1(t) \\ q_1(t) \end{bmatrix} \tag{4.2.28}$$

For the complete transducer with mechanical constraints.

$$\begin{bmatrix} \delta(s) \\ F_\delta(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \overline{Y}_m & 1 \end{bmatrix} \begin{bmatrix} -\frac{x_o}{E_o} & -\frac{x_o}{C_o E_o} \\ 0 & \frac{E_o}{x_o} \end{bmatrix} \begin{bmatrix} e_1(s) \\ q_1(s) \end{bmatrix} \tag{4.2.29}$$

$$\begin{bmatrix} \delta(s) \\ F_\delta(s) \end{bmatrix} = \begin{bmatrix} \left(\frac{x_o}{E_o}\right) & \left(-\frac{x_o}{C_o E_o}\right) \\ \left(\frac{x_o \overline{Y}_M}{E_o}\right) & \left(-\frac{x_o \overline{Y}_M}{C_o E_o} + \frac{E_o}{x_o}\right) \end{bmatrix} \begin{bmatrix} e_1(s) \\ q_1(s) \end{bmatrix} \tag{4.2.30}$$

$$F_\delta(s) = \frac{x_o \overline{Y}_M}{E_o}\, e_1(s) + q_1(s) \left[\frac{-x_o \overline{Y}_M}{C_o E_o} + \frac{E_o}{x_o} \right] \tag{4.2.31}$$

$$\delta(s) = \frac{x_o}{E_o} e_1(s) - \frac{x_o}{C_o E_o} q_1(s) \quad (4.2.32)$$

$$F_\delta(s) = \frac{x_o \overline{Y}_M}{E_o}\left[\frac{E_o}{x_o}\delta(s) + \frac{q_1(s)}{C_o}\right] + q_1(s)\left[\frac{-x_o \overline{Y}_M}{C_o E_o} + \frac{E_o}{x_o}\right] \quad (4.2.33)$$

$$F_\delta(s) = Y_m \delta(s) + \frac{E_o}{x_o} q_1(s) \quad (4.2.34)$$

$$F_\delta(t) = M\frac{d^2}{dt^2} + f\frac{d}{dt} + k\delta + \frac{E_o}{x_o} q_1(t) \quad (4.2.35)$$

Finally note that

$$E_o = R\frac{dq_1}{dt} + \frac{q_o}{C_o} + \frac{q_1(t)}{C_o} + \frac{E_o}{x_o}\delta \quad (4.2.36)$$

i.e.

$$0 = R\frac{dq_1}{dt} + \frac{q_1(t)}{C_o} + \frac{E_o}{x_o}\delta \, . \quad (4.2.37)$$

But from 4.2.32

$$e_1(t) = \frac{E_o}{x_o}\delta(t) + \frac{q_1(t)}{C_o} \quad (4.2.38)$$

$$\therefore \quad e_1(t) = -R\frac{dq_1}{dt} \, , \quad (4.2.39)$$

which is obvious from the circuit configuration.

5. Electro-magnetic d.c. rotating machines

5.1. Introduction

These are a special form of electro-magnetic transducer and form the most important class.

The d.c. machine may be represented by the network shown in Fig. 5.1.1.

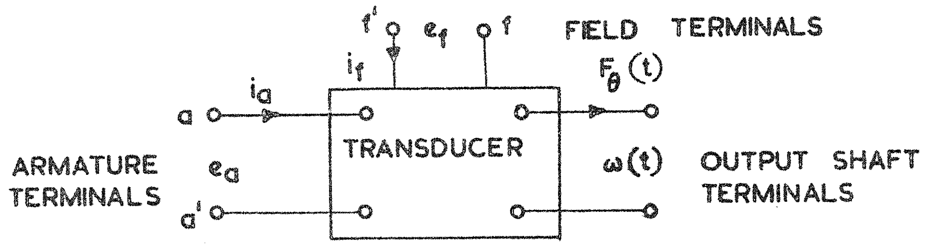

Fig. 5.1.1

From 2.1.5 we have

$$\text{Torque } F_\theta = - \frac{\partial W_m}{\partial \theta} + i \frac{\partial \phi}{\partial \theta} . \tag{5.1.1}$$

In a practical machine, the design is such that $\frac{\partial W_m}{\partial \theta} = 0$ (5.1.2)

so that

$$F_\theta = i_a \frac{\partial \phi (i_f \;\theta)}{\partial \theta} . \tag{5.1.3}$$

Let mean value of $\frac{\partial \phi}{\partial \theta} = k_m i_f$, (5.1.4)

where k_m is a constant dependent on the physical dimensions and number of poles of the machine.

Thus

$$F_\theta(t) = k_m i_f(t) i_a(t) . \tag{5.1.5}$$

From 2.1.7

$$e_a(t) = \frac{\partial \phi}{\partial i_a} (i_f, \theta) \frac{di_a}{dt} + \frac{\partial \phi}{\partial \theta} (i_f, \theta) \frac{d\theta}{dt} . \tag{5.1.6}$$

Case I

We assume i_f independent of i_a

$$\therefore \quad e_a(t) = k_m i_f \omega(t) , \tag{5.1.7}$$

where

$$\omega(t) = \frac{d\theta}{dt} . \tag{5.1.8}$$

From (5.1.5) and (5.1.7) we have

$$\begin{bmatrix} e_a(t) \\ i_a(t) \end{bmatrix} = \begin{bmatrix} k_m i_f(t) & 0 \\ 0 & \frac{1}{k_m i_f(t)} \end{bmatrix} \begin{bmatrix} \omega(t) \\ F_\theta(t) \end{bmatrix} \tag{5.1.9}$$

For i_f independent of i_a.

If we now address the field terminals as the input port we have

$$F_\theta(t) = \left[k_m i_a(t) \right] i_f(t) . \tag{5.1.10}$$

$$e_f(s) = i_f(s) Z_f(s) , \tag{5.1.11}$$

where $Z_f(s)$ is the impedance of field circuit.

$$e_a(t) = k_m i_f(t) \omega(t) . \tag{5.1.12}$$

Case II

$$i_f(t) = i_a(t) .$$

In this case

$$F_\theta(t) = k_m\, i_a(t)^2 \tag{5.1.13}$$

$$e_f(t) = L\frac{di_a}{dt} + k_m\, i_a\, \omega(t), \tag{5.1.14}$$

where L is the mean value of $\frac{\partial\phi}{\partial i_a}$.

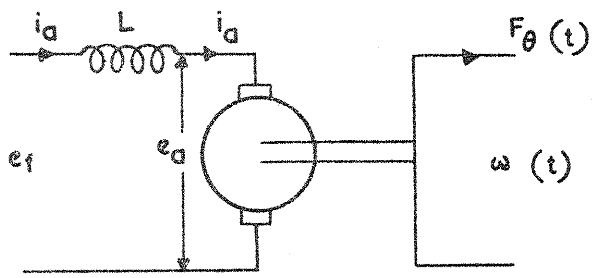

Fig. 5.1.2

For armature circuit above

Thus
$$F_\theta(t) = k_m\, i_a(t)^2 \quad \text{(i.e. non-linear)} \tag{5.1.15}$$

$$e_a(t) = k_m\, i_a(t)\, \omega(t) . \tag{5.1.16}$$

Thus
$$i_a(t) = k_m \frac{F_\theta(t)}{i_a(t)} = k_m\, \beta(t) \tag{5.1.17}$$

$$e_a(t) = k_m\, i_a(t)\, \omega(t) = k_m\, \alpha(t). \tag{5.1.18}$$

α (t) corresponds to the across variable $= i_a(t)\, \omega(t).$ (5.1.19)

β (t) corresponds to the through variable $= \frac{F_\theta(t)}{i_a(t)}$. (5.1.20)

The above equations may now be expressed as

$$\begin{bmatrix} e_a(s) \\ i_a(s) \end{bmatrix} = \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} \tag{5.1.21}$$

Representation of armature across the brushes

From the above consideration we may represent the performance of the armature shown in 5.1.3 by the following equations

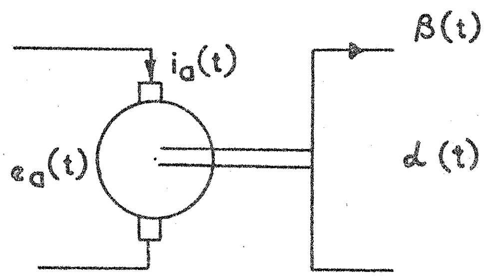

Fig. 5.1.3

$$\begin{bmatrix} e_a(t) \\ i_a(t) \end{bmatrix} = \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} \qquad (5.1.22)$$

e.g.

$$\begin{bmatrix} e_a(t) \\ i_a(t) \end{bmatrix} = \begin{bmatrix} k_m & 1 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} \omega(t),\ i_f(t) \\ \frac{F_\theta(t)}{i_f(t)} \end{bmatrix} \qquad (5.1.23)$$

This is the fundamental relationship which will be used in the following sections.

5.2. Separately excited field (i_f = Constant I_f)

From 5.1.23

$$\begin{bmatrix} e_a(t) \\ i_a(t) \end{bmatrix} = \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} I_f\,\omega(t) \\ \frac{F_\theta(t)}{I_f} \end{bmatrix} \qquad (5.2.1.)$$

$$\begin{bmatrix} e_a(s) \\ i_a(s) \end{bmatrix} = \begin{bmatrix} k_m I_f & 0 \\ 0 & \frac{1}{k_m I_f} \end{bmatrix} \begin{bmatrix} \omega(s) \\ F_\theta(s) \end{bmatrix} \qquad (5.2.2)$$

If $K_m = k_m I_f$ (5.2.3)

$$\begin{bmatrix} e_a(s) \\ i_a(s) \end{bmatrix} = \begin{bmatrix} K_m & 0 \\ 0 & \frac{1}{K_m} \end{bmatrix} \begin{bmatrix} \omega(s) \\ F_\theta(s) \end{bmatrix} \qquad (5.2.4)$$

If we now include the armature circuit impedance $Z_a(s)$ and mechanical load impedance $Z_m(s)$ we have the network configuration shown in Fig. 5.2.1.

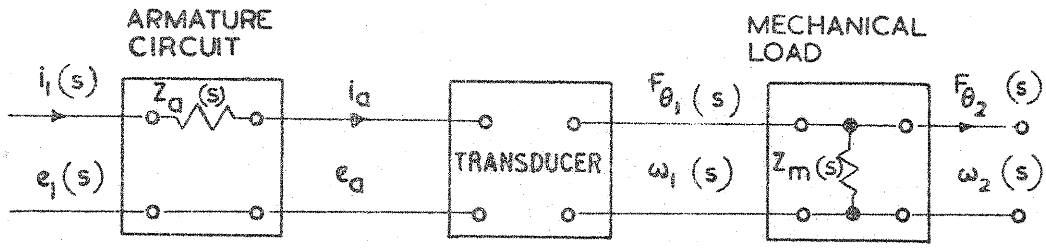

Fig. 5.2.1

where $Y_m(s) = sJ + f + \frac{k}{s} = \frac{1}{Z_m(s)}$ (5.2.5)

(See section 3.3)

Thus

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} 1 & \overline{Z}_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} K_m & 0 \\ 0 & \frac{1}{K_m} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \omega_2(s) \\ F_{\theta_2}(s) \end{bmatrix} \qquad (5.2.6)$$

Note J is the polar moment of inertia which is analogous to capacitance.

f is the viscous friction coefficient which is analogous to conductance.

k is the spring stiffness which is analogous to $\frac{1}{\text{inductance}}$.

Thus

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} K_m + \frac{\bar{Z}_a \bar{Y}_m}{K_m} & \frac{\bar{Z}_a}{K_m} \\ \frac{Y_m}{K_m} & \frac{1}{K_m} \end{bmatrix} \begin{bmatrix} \omega_2(s) \\ F_{\theta 2}(s) \end{bmatrix} \qquad (5.2.7)$$

With $F_{\theta 2}(s) = 0$

$$\frac{e_1(s)}{\omega_2(s)} = K_m + \frac{\bar{Z}_a \bar{Y}_m}{K_m} . \qquad (5.2.8)$$

With $\omega_2(s) = 0$ i.e. standstill condition

$$F_{\theta 2}(s) = K_m \, i_1(s) . \qquad (5.2.9)$$

$\therefore$ Standstill torque $= K_m i_1 = k_m i_f i_1$ (5.2.10)

If we now let

$$Y_m(s) = sJ + f \qquad (5.2.11)$$

$$Z_a(s) = R_a$$

$$\frac{e_1(s)}{\omega_2(s)} = K_m + \frac{R_a (sJ + f)}{K_m} \qquad (5.2.12)$$

$$= \frac{K_m^2 + R_a f + sR_a J}{K_m} \qquad (5.2.13)$$

i.e.

$$\frac{\omega_2(s)}{e_1(s)} = \frac{K_m}{(K_m^2 + R_a f)(1 + s \frac{R_a J}{K_m^2 + R_a f})} \qquad (5.2.14)$$

i.e.

$$\frac{\omega_2(s)}{e_1(s)} = \frac{\frac{K_m}{K_m^2 + R_a f}}{1 + sT_a} \qquad (5.2.15)$$

$$= \frac{K}{1 + sT_a} , \qquad (5.2.16)$$

where

$$K = \frac{K_m}{K_m^2 + R_a f} \quad \text{and} \quad T_a = \frac{R_a J}{K_m^2 + R_a f} . \qquad (5.2.17)$$

Under these conditions the forward transfer function of the motor is given by 5.2.16 and may be represented as shown in Fig. 5.2.2.

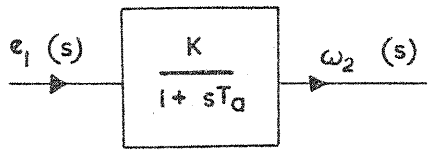

Fig. 5.2.2. Transfer function of armature controlled d.c. motor

5.3. The d.c. shunt motor

Configuration shown in Fig. 5.3.1

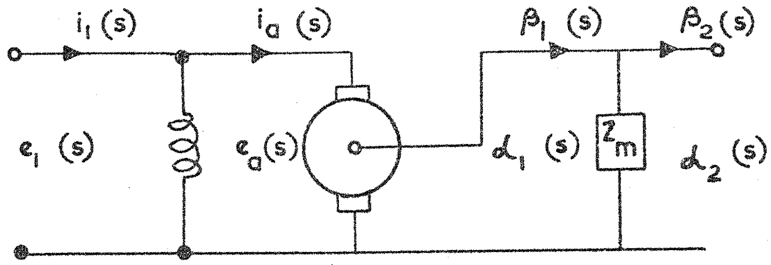

Fig. 5.3.1

Network configuration

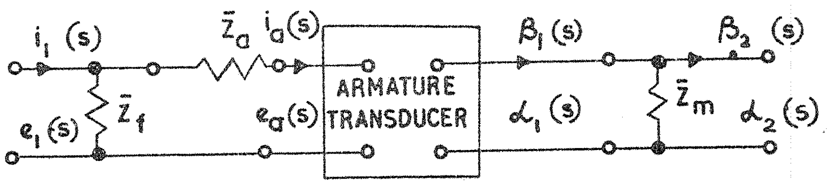

Fig. 5.3.2

$Z_f(s)$ represents field circuit impedance

$Z_a(s)$ represents armature circuit impedance

$Z_m(s) = \frac{1}{Y_m(s)}$ represents mechanical impedance

$\alpha_2(t)$ represents across variable $= \omega_2(t)\, i_f(t)$

See Section 5.1

$\beta_2(t)$ represents through variable $= \frac{F_{\theta_2}(t)}{i_\delta(t)}$

We have

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \overline{Y}_f & 1 \end{bmatrix} \begin{bmatrix} 1 & \overline{Z}_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \alpha_2(s) \\ \beta_2(s) \end{bmatrix} \quad (5.3.1)$$

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} \left(k_m + \frac{\overline{Z}_a \overline{Y}_m}{k_m}\right) & \left(\frac{\overline{Z}_a}{k_m}\right) \\ \left(\overline{Y}_f k_m + \frac{\overline{Y}_m(1 + \overline{Y}_f \overline{Z}_a)}{k_m}\right) & \left(\frac{1 + \overline{Y}_f \overline{Z}_a}{k_m}\right) \end{bmatrix} \begin{bmatrix} \alpha_2(s) \\ \beta_2(s) \end{bmatrix} \quad (5.3.2)$$

When $\alpha_2(s) = 0$ i.e. standstill condition

$$e_1(s) = \frac{\overline{Z}_a}{k_m} \beta_2(s) \quad (5.3.3)$$

Let $$Z_a(s) = R_a \quad (5.3.4)$$

$$e_1(s) = \frac{R_a}{k_m} \beta_2(s) \quad (5.3.5)$$

or $$e_1(t) = \frac{R_a}{k_m} \beta_2(t) \quad (5.3.6)$$

i.e. $$e_1(t) = \frac{R_a}{k_m} \frac{F_{\theta_2}(t)}{i_f(t)} \quad (5.3.7)$$

i.e. $$F_{\theta_2}(t) = \frac{k_m}{R_a} i_f(t)\, e_1(t) \quad . \quad (5.3.8)$$

This is the standstill torque equation.

If now $e_1(t)$ = constant E_1 (5.3.9)

$$F_{\theta 2}(t) = \frac{k_m E_1^2}{R_a R_f} , \quad (5.3.10)$$

where R_f is the field circuit resistance. (5.3.11)

Now consider case where $\beta_2(s) = 0$

$$e_1(s) = \left(k_m + \frac{\overline{Z}_a \overline{Y}_m}{k_m}\right) \alpha_2(s). \quad (5.3.12)$$

With $Z_a(s) = R_a$ and $Y_m(s) = (sJ + f)$, (5.3.13)

$$\frac{\alpha_2(s)}{e_1(s)} = \frac{K}{1 + sT_a} , \quad (5.3.14)$$

where $K = \dfrac{k_m}{k_m^2 + R_a f}$ and $T_a = \dfrac{R_a J}{k_m^2 + R_a f}$. (5.3.15)

If now $e_1(t)$ is a constant value E_1 ,

$$\alpha_2(s) = \frac{KE_1}{1 + sT_a} \quad (5.3.16)$$

$$\therefore \quad \alpha_2(t) = KE_1(1 - e^{-\frac{t}{T_a}}) \quad (5.3.17)$$

$$\alpha_2(t) = i_f(t)\ \omega_2(t) = \frac{E}{R_f}\ \omega_2(t) \quad (5.3.18)$$

$$\therefore \quad \omega_2(t) = KR_f(1 - e^{-\frac{t}{T_a}}) . \quad (5.3.19)$$

This equation indicates a method of speed control i.e. variation of R_f.

Now consider case when $\alpha_2(s)$ and $\beta_2(s)$ are finite

and $Z_m(s) = \infty$; $Z_a(s) = R_a$ (5.3.20)

$$e_1(s) = k_m \alpha_2(s) + \frac{R_a}{k_m} \beta_2(s) . \quad (5.3.21)$$

Now let $e_1(t)$ = constant E_1 (5.3.22)

$$E_1 = k_m \alpha_2(t) + \frac{R_a}{k_m} \beta_2(t) \quad (5.3.23)$$

$$= k_m \frac{E_1}{R_f} \omega_2(t) + \frac{R_a}{k_m} \frac{F_{\theta 2}(t)}{E_1} R_f \quad (5.3.24)$$

$$F_{\theta_2}(t) = \frac{k_m E_1}{R_a R_f} \left[E_1 - \frac{k_m E_1}{R_f} \omega_2(t) \right] , \qquad (5.3.25)$$

i.e. torque and speed are no longer time dependent.

$$F_{\theta_2} = \frac{k_m E_1^2}{R_a R_f} \left(1 - \frac{k_m}{R_f} \omega_2 \right) \qquad (5.3.26)$$

$$F_{\theta_2} = F_{standstill} \left(1 - \frac{k_m \omega_2}{R_f} \right) . \qquad (5.3.27)$$

This is the equation for the normal mechanically unloaded torque-speed characteristic (steady state conditions).

5.4. The d.c. series motor

Basic configuration

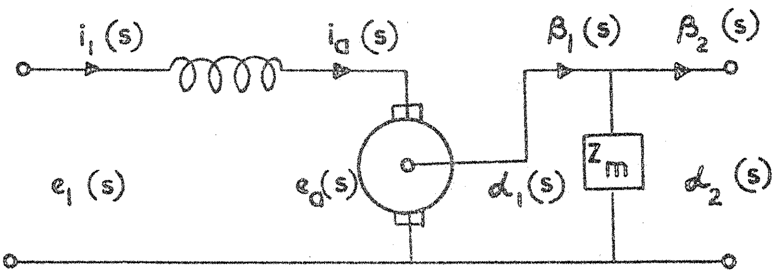

Fig. 5.4.1

Network representation

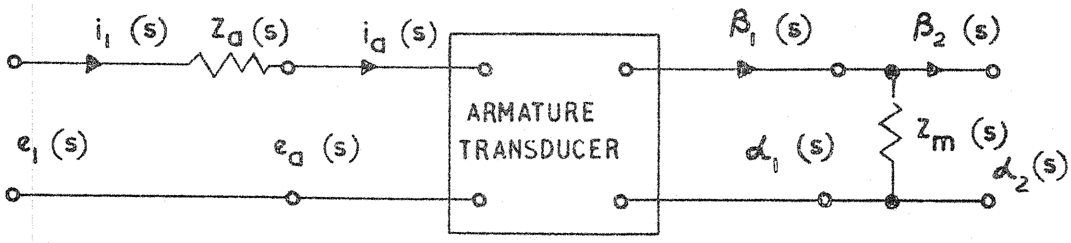

Fig. 5.4.2

We have

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} 1 & Z_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & \frac{1}{k_m} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{Y}_m & 1 \end{bmatrix} \begin{bmatrix} \alpha_2(s) \\ \beta_2(s) \end{bmatrix} \qquad (5.4.1)$$

i.e.

$$\begin{bmatrix} e_1(s) \\ i_1(s) \end{bmatrix} = \begin{bmatrix} \left(k_m + \frac{\overline{Z}_a \overline{Y}_m}{k_m} \right) & \left(\frac{\overline{Z}_a}{k_m} \right) \\ \left(\frac{\overline{Y}_m}{k_m} \right) & \left(\frac{1}{k_m} \right) \end{bmatrix} \begin{bmatrix} \alpha_2(s) \\ \beta_2(s) \end{bmatrix} \qquad (5.4.2)$$

With $\alpha_2(s) = 0$ i.e. standstill condition. (5.4.3)

$$\beta_2(s) = k_m \, i_1(s) \; . \qquad (5.4.4)$$

$$\beta_2(t) = k_m \, i_a(t) \; . \qquad (5.4.5)$$

But

$$\beta_2(t) = \frac{F_{\theta_2}(t)}{i_a(t)} \; . \qquad (5.4.6)$$

$\therefore$

$$F_{\theta_2}(t) = k_m \, (i_a(t))^2 \qquad (5.4.7)$$

i.e. standstill torque equation.

Consider case when $\beta_2(s) = 0$ (5.4.8)

$$e_1(s) = \left(k_m + \frac{\overline{Z}_a \overline{Y}_m}{k_m} \right) \alpha_2(s) \; . \qquad (5.4.9)$$

Let

$$Z_a(s) = R_a + sL_a$$
$$Y_m = f + sJ \qquad (5.4.10)$$

$$e_1(s) = \frac{k_m^2 + (R_a + sL_a)(f + sJ)}{k_m} \, \alpha_2(s) \qquad (5.4.11)$$

$$e_1(s) = \frac{k_m^2 + R_a f + s(JR_a + L_a f) + s^2 L_a J}{k_m} \, \alpha_2(s) \qquad (5.4.12)$$

or

$$\alpha_2(s) = \frac{k_m \, e_1(s)}{k_m^2 + R_a f + s(JR_a + L_a f) + s^2 L_a J} \qquad (5.4.13)$$

i.e.

$$\alpha_2(s) = f(s) \qquad (5.4.14)$$

if

$$f(t) \supset f(s) \qquad (5.4.15)$$

$$\alpha_2(t) = f(t) \tag{5.4.16}$$

i.e. $$i_a(t)\,\omega_2(t) = f(t) \tag{5.4.17}$$

$\therefore$ $$\omega_2(t) = \frac{f(t)}{i_a(t)} \tag{5.4.18}$$

Now if $e_1(t)$ = constant E_1 (5.4.19)

$$\alpha_2(s) = \frac{\dfrac{k_m E_1}{L_a J}}{s^2 + s\,\dfrac{JR_a + L_a f}{L_a J} + \dfrac{k_m^2 + R_a f}{L_a J}} \tag{5.4.20}$$

i.e. $$\alpha_2(s) = \frac{KE_1}{(s+a)(s+b)} \tag{5.4.21}$$

where $$K = \frac{k_m}{L_a J} \tag{5.4.22}$$

$$a + b = \frac{JR_a + L_a f}{L_a J} \tag{5.4.23}$$

$$ab = \frac{k_m^2 + R_a f}{L_a J} \,. \tag{5.4.24}$$

$\therefore$ $$\alpha_2(t) = KE_1\left[\frac{1}{ab} + \frac{be^{-at} - ae^{-bt}}{ab(a-b)}\right] \tag{5.4.25}$$

with $$\alpha_2(t) = i_a(t)\,\omega_2(t) \tag{5.4.26}$$

$$\omega_2(t) = \frac{KE_1}{i_a(t)}\left[\frac{1}{ab} + \frac{be^{-at} - ae^{-bt}}{ab(a-b)}\right] \tag{5.4.27}$$

There is the run-up velocity equation for a step input E_1.

5.5. <u>The field controlled d.c. motor (Constant armature current).</u>

From (5.1.9)

$$F_\theta(t) = k_m\, i_a(t)\, i_f(t)\,.$$

The case of main interest is given when i_a = constant I_a.

Thus $$F(t) = k_m\, i_f(t)\,, \tag{5.5.1}$$

where $$K_m = k_m I_a \,. \tag{5.5.2}$$

If $Z_f(s)$ represents the field circuit impedance,

then $$e_f(s) = i_f(s)\, Z_f(s) \tag{5.5.3}$$

It is assumed that there is zero mutual coupling between the armature flux and the field flux.

Thus the motor action in this case is non-reciprocal, in other words we could not have a generator based on the above constraint (I_a = Const.).

If we classify $i_f(t)$ and $F_\theta(t)$ as through variables the motor action acts effectively as an unidirectional "through variable" amplifier having a factor K_m as shown in Fig. 5.5.1.

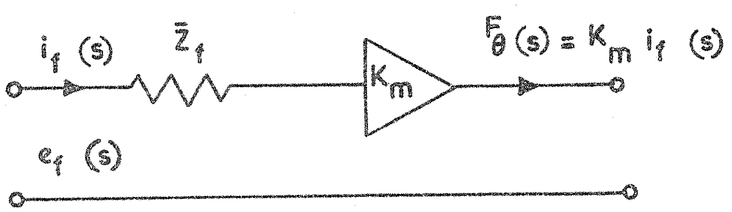

Fig. 5.5.1

With a mechanical impedance connected across the motor terminals we have the situation shown in Fig. 5.5.2.

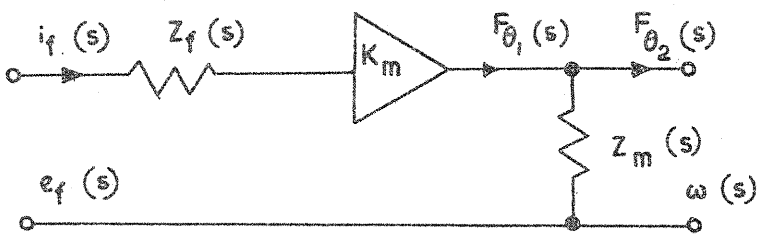

Fig. 5.5.2

If $$F_{\theta_2}(s) = 0 \;, \qquad (5.5.4)$$

$$\omega(s) = F(s)\, Z_m(s). \qquad (5.5.5)$$

$\therefore$ $$\omega(s) = K_m\, i_f(s)\, Z_m(s) \qquad (5.5.6)$$

$$= K_m\, e_f(s)\, \frac{Z_m(s)}{Z_f(s)} \qquad (5.5.7)$$

$$\omega(s) = \frac{K_m\, e_f(s)}{Z_f(s)\, Y_m(s)} \;. \qquad (5.5.8)$$

If $$Z_f(s) = R_f + sL_f \tag{5.5.9}$$

and $$Y_m(s) = sJ + f \tag{5.5.10}$$

$$\omega(s) = \frac{K_m e_f(s)}{(f + sJ)(R_f + sL_f)} \tag{5.5.11}$$

$$= \frac{\frac{K_m}{fR_f} e_f(s)}{(1 + s\frac{J}{f})(1 + s\frac{L_f}{R_f})} \tag{5.5.12}$$

i.e. $$\omega(s) = \frac{K_v e_f(s)}{(1 + sT_M)(1 + sT_f)} , \tag{5.5.13}$$

where $K_v = \frac{k_m}{fR_f}$ and is the velocity constant (5.5.14)

$T_m = \frac{J}{f}$ and is the mechanical time constant (5.5.15)

$T_f = \frac{L_f}{R_f}$ and is the field time constant . (5.5.16)

Thus

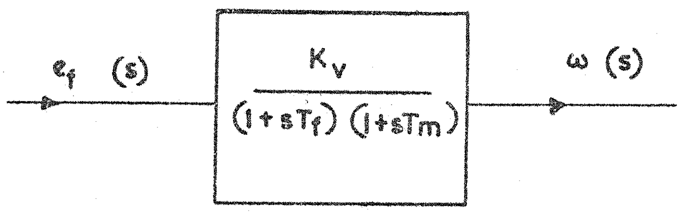

Fig. 5.5.3

Note: Since we have addressed the field terminals as the input port and the generator action of the armature is considered to have negligible effect on the armature current, two port network representations are not suitable in this case.

5.6. Linear incremental relationships

It will be obvious that we may determine linear relationships for all the motors considered if we have small perturbations about fixed operating conditions. This technique was used in the analysis of the capacitor microphone, as a particular example.

For further examples see Electro-mechanical Energy Conversion by White and Woodson (J. Wiley and Sons).

6. Conclusions

The general electro-mechanical transducer is essentially a non-linear device. Linear two port network representations may be determined for all cases provided that we postulate across variables and through variables, which in general are combinations of electrical and mechanical variables.

The network approach is considered to give a more formal procedure to the analysis of transducers.

7. References

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