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GROUND LEVEL DISTURBANCE FROM LARGE AIRCRAFT FLYING AT SUPERSONIC SPEEDS

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On the Ground Level Disturbance from Large Aircraft Flying at Supersonic Speeds

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SUMMARY

The Whitham-Walkden theory for the estimation of the strength of shock waves at ground level from aircraft flying at supersonic speeds is applied to the case of a typical projected supersonic civil transport aeroplane.

If a figure of 2 lb/sq.ft. (including a factor of 2 for ground reflection) is taken as an upper limit for the acceptable strength of the bow wave from such an aircraft it is shown that restrictions on the climb and flight plan will be involved. The advantage of the employment of larger engines with or without afterburning is discussed, with reference also to the penalties involved owing to the increase in weight of the aircraft and its direct operating costs.

Finally it is suggested that an aircraft of given volume could be designed, by suitable choice of thickness and lift distribution, to minimise the strength of the shock waves in the far field.

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NOTATION

a	speed of sound; radius; half width (smaller) of rectangular plate
A	area
b	half width (larger) of rectangular plate
B	$\sqrt{M^2 - 1}$
C_L	lift coefficient
d	distance between shock waves
D	flexural rigidity
E	Young's modulus
$F(\tau)$	Whitham F-function
g	acceleration due to gravity
h	thickness of thin plate; altitude
k	$= K'/A\Delta p$
K'	equivalent total stiffness
l	length of aircraft
L	lift
m	$= M'/A\Delta p$
M	Mach number; total mass
M'	equivalent total mass
p	pressure
p_h	air pressure at altitude h
p_g	air pressure at ground level
Δp	pressure jump across shock wave; local load per unit area
$R(x)$	body radius distribution
s	semi-span
S	wing area; body cross-sectional area

Notation (Continued)

S_1, S_2, S_3, S_t	equivalent body areas
t	time; distance aft of nose
T	passage time of shock waves
T_o	(see text)
u	velocity component
w	deflection of plate
w_s	static deflection
w'_s	static deflection according to large deflection theory
x, y, z	co-ordinates
Z	total wing semi-thickness
γ	ratio of specific heats
δ	amplification factor
ν	Poisson's ratio
τ	describes the characteristic curves
ρ	density
ρ_m	material density
$\delta\rho(\equiv z)$	density difference
σ_m	maximum stress
θ	polar angle
ω	circular frequency

Suffixes

g	ground level
f	front shock wave
r	rear shock wave
c	centre

Bars denote quantities made non-dimensional by the aircraft length l (except in Appendix 1). Other symbols are defined where they occur in the text.

1. Introduction

The advent of aircraft flying at supersonic speeds during the past decade has brought with it the problem of the supersonic "bangs". Early experiences showed that for aircraft going supersonic in a dive or supersonic in straight and level flight some minor damage (or major damage in some isolated cases for flight at low altitudes) to structures was incurred. The damage was usually slight, such as the displacement of a roof tile or a cracked window pane. In all these cases the aircraft involved were military aircraft and up to date, the damage mentioned above has been limited by restricting such flights to regions over the sea or over sparsely populated areas.

However in view of the possibility of bringing into service in the near future supersonic civil transport aeroplanes, whose flight paths could not easily be restricted to avoid at least some closely populated areas, it would seem desirable to investigate the strength of the shock waves from such aircraft and to assess the likelihood of their causing damage to structures, such as the cracking of plate glass windows.

It will be shown below that for a typical supersonic civil transport of about 300,000 lb. all up weight the strength of the shock waves (including a ground reflection factor of 2) on the ground will be of the order of 2 lb./sq.ft. during the climb and in level flight at 60,000 ft. Current experiences with smaller aircraft have indicated that such a shock pressure can cause very slight damage to buildings. However little or no experience exists of detailed supersonic flights over heavily populated centres and so it cannot be said with any certainty that shock waves of this intensity or greater will be acceptable. It would therefore seem important at this interim stage to look closely into the design of supersonic civil aircraft and to see if there is any way of reducing the strength of the bow or tail shock waves, or both, which would at the same time not impair its low drag and lifting characteristics. These aspects of the problem are all rather tentatively discussed below but no firm conclusions are drawn.

It should be pointed out that this report is in the form of a progress report and is introduced to merely highlight certain aspects of the problem.

2. Strengths of Shock Waves

Whitham (1,2) has developed a theory for the determination of the strengths of shock waves at large distances from aircraft or missiles travelling at steady supersonic speeds. Whitham's papers deal with the cases of a body of revolution and a thin symmetrical wing, whereas Walkden (3) has extended the method to the case of a general wing-body combination provided the body is axi-symmetrical and at zero incidence.

Whitham's theory is based on the determination of the pressure at large distances from the body by linear theory. The co-ordinates of the characteristic on which this pressure must act are then found and from the geometry of the characteristics the paths of the shock waves are calculated. For non-circular bodies the flow in each plane, θ = constant is treated separately.

The case of non-uniform motion can be treated in a similar way but will not be treated here. (See Ref. 4 by Rao).

The calculations for the strength of the bow wave from a body of revolution are straightforward, even when the cross-sectional area is not an analytical function of the axial distance. On the other hand, the calculations for the case of the lifting wing with thickness are not so simple, and few numerical results have been obtained.

If however, only the strengths of the shock waves in the vertical plane below the aircraft are required the calculation is no more difficult than that of the body of revolution provided the lift distribution as well as the thickness distribution is known. It can be shown that the effects of wing thickness, body thickness and wing-body interference are additive to the lifting effect of the wing.

3. Whitham-Walkden Theory

If the overall length of the aircraft is denoted by l its Mach number as M and the uniform ambient pressure as p_h, Walkden shows that the pressure jump across the bow shock wave in the vertical plane below the aircraft ($\theta = -\pi/2$) in the far field is given by

$$\frac{\Delta p_f}{p_h} = \frac{\gamma}{(\gamma + 1)^{\frac{1}{2}}} \cdot \frac{(2B)^{\frac{1}{4}}}{h^{\frac{3}{4}}} \left[\int_0^{T_0} F(\tau, -\frac{\pi}{2}) d\tau \right]^{\frac{1}{2}} \qquad (1)$$

where h is the vertical distance below the aircraft centre-line

B equals $(M^2 - 1)^{\frac{1}{2}}$

τ describes the characteristic curves

T_0 is the value of τ at which $F(\tau, -\pi/2)$ is zero and $\int_0^{T_0} F\, d\tau$ is a maximum over the length of the body and

$$F(\tau, -\frac{\pi}{2}) = \frac{1}{2\pi} \int_0^\tau \left[\frac{S''(t) - S_1''(t) + S_2''(t) + \frac{B}{2} S_3''(t)}{(\tau - t)^{\frac{1}{2}}} \right] dt \qquad (2)$$

$S(t)$ is the body cross sectional area

$S_1(t)$ is the wing body interference effect which can be of either sign

$S_2(t)$ is the wing thickness effect

$S_3(t)$ is the lifting effect

and a prime denotes differentiation with respect to t

The interference effect is found from

$$\frac{dS_1}{dx} = 4\, R(x) \frac{dZ(x)}{dx} \qquad (3)$$

where x is the distance behind the nose of the aircraft

$R(x)$ is the body radius distribution

and $2Z(x)$ is the wing thickness distribution along the body centreline

The wing thickness effect is found from

$$\frac{dS_2}{dx} = 2 \int_{-s}^{+s} Z(x, y,)\, dy \qquad (4)$$

where $s(x)$ is the local semi-span and

$2\, Z(x, y,)$ is the total wing thickness for the gross-wing (i.e. including that covered by the body)

The wing lifting effect is found from

$$\frac{dS_3}{dx} = \int_{-s}^{+s} \frac{\Delta p(x,y,)}{\frac{1}{2}\gamma M^2 p}\, dy \qquad (5)$$

where $\Delta p(x,y,)$ is the localload per unit area on the wing surface.

Equation (2) can be written

$$F(\tau, -\frac{\pi}{2}) = \frac{1}{2\pi} \int_0^{\tau} \frac{(S''_t + \frac{B}{2} S''_3)}{(\tau - t)^{\frac{1}{2}}} dt \qquad (6)$$

$$\text{where } S''_t = \frac{d}{dt}(S' - S_1' + S_2') \qquad (7)$$

In equation (6) the contributions to the integral from S_1, S_2 and S_3 must be zero upstream of the wing leading edge. In what follows it will be assumed that the origin of the body and the wing are coincident at $x = 0$. This corresponds to an integrated layout.

Since the pressure in the far field is given by

$$\frac{p - p_h}{p_h} = \frac{\gamma M^2 F(\tau, -\pi/2)}{(2\,Bh)^{\frac{1}{2}}} \qquad (8)$$

on the characteristic given by

$$\tau = x - Bh + \frac{(\gamma + 1)\, M^4\, F(\tau, -\pi/2)\, h^{\frac{1}{2}}}{(2\,B^3)^{\frac{1}{2}}} \qquad (9)$$

it can be seen that positive values of $F(\tau, -\pi/2)$ correspond to regions of compression whilst negative values of $F(\tau, -\pi/2)$ correspond to regions of expansion.

The compression and expansion waves which combine to form the bow wave in the far field are therefore confined to the region $0 \leq \tau \leq T_0$, and the tail wave results from the waves corresponding to the region $T_0 \leq \tau \leq \infty$. The pressure jump in the far field corresponding to the rear shock must be

$$\frac{\Delta p_r}{p_h} = \frac{-\gamma}{(\gamma + 1)^{\frac{1}{2}}} \cdot \frac{(2B)^{\frac{1}{4}}}{h^{\frac{3}{4}}} \left[\int_{T_0}^{\infty} (-)\; F(\tau, -\pi/2)\; d\tau\right]^{\frac{1}{2}} \qquad (10)$$

Whitham has shown that

$$\int_0^{2\pi}\int_0^{\infty} F(\tau, \theta)\, d\tau\, d\theta = 0 \qquad (11)$$

Thus for a body of revolution without lift,

$$\int_{0}^{\infty} F(\tau, -\pi/2)\, d\tau = 0 . \tag{12}$$

from which it follows that the far field pressure jumps across the bow and stern shock must be the same magnitude.

For slender bodies this result must be a close approximation even when bodies of irregular section shape are used, since the variation in S_t with θ should be small. Then the pressure signal on the ground must have the characteristic N wave shape associated with the far field disturbance. (Fig. 1a).

This result cannot be correct for the lifting body, since then the surface integral of pressure on the ground must be equal to the total lift. Thus, in the lifting case, the pressure signal on the ground must be of the form shown in Fig. 1b. It follows that, for positive lift, the pressure rise associated with the bow wave must always be greater than that associated with the stern wave. The pressure jump associated with the bow wave is not necessarily greater in the case of a lifting configuration than that arising from the same configuration at zero lift. The whole of the lifting effect can be associated with the region $T_o \leq t \leq \infty$ in which case the front pressure jump will be the same in the lifting and non-lifting cases.

The theory developed by Whitham related to an atmosphere of uniform ambient pressure p_h. In Appendix 1 it is shown that the correction factor to Whitham's results, in order to allow for the increase in ambient pressure with decrease in altitude is approximately

$$\sqrt{\frac{(p_g)}{(p_h)}} \tag{13}$$

that is, the pressure jump across the shock wave is proportional to the square root of the ratio of ambient pressure.

Thus, for a non-homogeneous atmosphere the pressure jump across the front shock at the ground is, from equation (1),

$$\left(\Delta p_f\right)_g = \frac{\gamma}{(\gamma + 1)^{\frac{1}{2}}} \frac{(2B)^{\frac{1}{4}}}{h^{\frac{3}{4}}} \cdot (p_g\, p_h)^{\frac{1}{2}} \left[\int_{0}^{T_o} F(\tau, -\pi/2)\, d\tau \right]^{\frac{1}{2}} \qquad \ldots\ldots \quad (14)$$

The rear shock pressure jump at the ground, is from equation (10)

$$(\Delta p_r)_g = (-) \frac{\gamma}{(\gamma+1)^{\frac{1}{2}}} \frac{(2B)^{\frac{1}{4}}}{h^{\frac{3}{4}}} (p_g\, p_h)^{\frac{1}{2}} \left[\int_{T_o}^{\infty} (-)\, F(\tau, -\pi/2)\, d\tau \right]^{\frac{1}{2}} \qquad \ldots\ldots (15)$$

Note: for $\gamma = 1.4$

$$\frac{\gamma}{(\gamma+1)^{\frac{1}{2}}} \cdot 2^{\frac{1}{4}} = 1.075$$

4. The Response of a Thin Panel to an N-wave

It is shown in Appendix 2 that for the N-type shock waves from a supersonic aircraft, the pressure distribution near the ground is given by

$$p - p_g = \Delta p\left(1 - \frac{2t}{T}\right) \qquad 0 \leqslant t \leqslant T \qquad (16)$$

$$= 0 \qquad t > T$$

where p_g is the ambient pressure at ground level

Δp is the pressure rise across the bow and tail waves

and T is the passage time for the bow and tail waves.

From the equation of motion for a thin panel of thickness h subjected to the above impulsive pressure distribution the response of the panel can be determined and its maximum deflection and stress found. It is shown in Appendix 2 that if the mode of vibration is equal to the static deflection mode for a uniformly distributed load, the maximum dynamic deflection is roughly twice the static deflection. This occurs when $\omega T \doteq 5.5$, where ω is the plate natural circular frequency and T is the passage time of the shock waves. For other values of ωT the dynamic deflection is reduced but in most practical cases (typically windows) it would seem that the static deflection is always exceeded.

As a typical example let us consider the case of a plate glass shop window $\frac{1}{4}$ in. thick and having sides 128 in. by 90 in. Such a window was described in Ref. 7 as having been cracked parallel to its shorter side by an N-wave of strength (including ground reflection) of 1.75 lb./sq.ft. approximately (the report states that the above pressure was measured at another station on the same flight, but it was likely that this pressure was exceeded in the vicinity of the window) and a passage time between shock waves of 0.136 secs.

The height of the aircraft was 25,000 ft. and it was flying at M = 1.22. If we use the formulae given in Appendix 2 for the rectangular plate having the following characteristics, and assume it is simply supported at the edges,

$$E = 10^9 \text{ lb./ft}^2$$

$$\rho_m = 5 \text{ slugs/ft}^3.$$

$$b \times a = 5.34 \text{ ft.} \times 3.75 \text{ ft.}$$

$$h = {}^1/48 \text{ ft.}$$

$$\nu = \tfrac{1}{4}$$

then $\omega T = 2.8$

$$w_s' \approx 1 \text{ in.}$$

$$\delta \doteq 1.25$$

and the maximum stress

$$\sigma_m \approx 1000 \text{ lb./in}^2 .$$

Now for plate glass windows a maximum static working stress of 1000 lb./in² is recommended. This provides a nominal factor of safety of about 10 but since it allows for inhomogeneities and stress concentrations set up in fixing, the true factor of safety may be very much less. It should be noted also that the fatigue strength of glass (Ref. 8) is good and the maximum breaking stress for impulsive loading exceeds that for static loading by a factor of about 1.2. Nevertheless it would not appear impossible for such a plate glass window to have been damaged by this shock wave of only moderate strength.

This example together, with general deductions from the above theory as applied to structures other than glass windows, confirms the experience to date on the operation of supersonic aircraft over land that only minor structural damage will result with shock waves of less than 2 lb./sq.ft. (including ground reflection). It would seem that when damage to a structure has occurred the structure was either over-stressed or on the point of failure. The present calculations however do not give us a lead as to what maximum shock-wave amplitude can be accepted without fear of any damage to typical houses and buildings. Certainly current experience indicates that a 1 lb./sq.ft. (including a reflection factor of 2) shock wave would not cause structural damage*,

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* The possibility of multiple reflection of the shock wave in its passage over closely spaced buildings, such as in a large town or city, should not be overlooked.

at least not of any magnitude. This is not to say, however, that incidents of a secondary nature would not arise, such as accidents to people, who are momentarily startled, of which many examples could be enumerated. The upper limit of shock wave amplitude which is acceptable for no appreciable damage to buildings can only be arrived at by trial and experiment.

5. Effect of Limiting the Ground Pressure Rise on the Climb Path

In order to investigate the magnitude of the pressure rises on the ground associated with the far field disturbances caused by a supersonic aircraft, the above theory has been applied to the typical slender airliner of integrated layout shown in Fig. 3.

The planform assumed was of 'ogee' shape given by

$$\bar{y} = 0.1615\,\bar{x} - 0.173\bar{x}^2 + 1.509\,\bar{x}^3 - 1.980\,\bar{x}^4 + 0.7155\,\bar{x}^5 \qquad \ldots\ldots \quad (17)$$

where $(\bar{x}, \bar{y})$ are measured in terms of the aircraft length l.

Slender wing theory suggests that the distribution of lift along such a configuration is proportional to $\bar{y}\,\frac{d\bar{y}}{d\bar{x}}$. This leads to the condition that the local chordwise loading is given by

$$\frac{dL(\bar{x})}{d\bar{x}} = 36.83\ \tfrac{1}{2}\gamma M^2\ p_h\ S\ C_L\ g(\bar{x}) \qquad (18)$$

where

$$g(\bar{x}) = \left\{ \begin{array}{l} 0.0261\,\bar{x} - 0.0838\,\bar{x}^2 + 1.0346\,\bar{x}^3 - 2.9045\,\bar{x}^4 + 9.5796\,\bar{x}^5 \\ - 21.7812\,\bar{x}^6 + 24.3192\,\bar{x}^7 - 12.7503\,\bar{x}^8 + 2.5595\,\bar{x}^9 \end{array} \right\}$$

and the total lift : $= \int_0^1 \frac{dL}{d\bar{x}}\, d\bar{x}$.

The thickness distribution along the length of the aircraft was of Lord V shape (see Ref. 6) given by

$$S(\bar{x}) = 0.0324\,\bar{x}\,(1 - \bar{x})(1 - [1 - \bar{x}]^4)\ l^2 \qquad (19)$$

Since the aircraft was of very slender form, S_t, as defined by equation (7) was taken to be $S(\bar{x})$. The values of $F(\tau, -\frac{\pi}{2})$ as given by equation (6) were calculated over the length of the aircraft for varied values of B S C_L. From the distribution of $F(\tau, -\pi/2)$ with τ, the value of T_o

for each value of B S C_L was evident, and allowed $\int_0^{T_0} F(\tau, -\frac{\pi}{2}) d\tau$

to be determined for each case. For a series of Mach numbers, the altitude of the aircraft was calculated for each value of B S C_L and hence the pressure rise across the bow wave could be evaluated from equation (14). Fig. 4 is a cross plot of these results showing the pressure rise, with no allowance for ground reflection, as a function of the aircraft height, and speed. These results correspond to level flight at constant speed at each point.

If a value of the pressure rise, with no ground reflection factor, of 1.0 lb./ft^2 is accepted as a value not to be exceeded, then the aircraft must not exceed a Mach number of 1.06 at 40,000 ft. altitude and a Mach number of 1.23 at 50,000 ft. and can only accelerate from M = 1.5 to M = 2.0 above 54,000 ft. The pressure signal on the ground at these points, if the aircraft flies at zero lift, would be about 0.5 lb./ft^2. without reflection. A calculation for a slender configuration of similar weight with an aspect ratio of 1.25 and a wing loading of 36 lb./ft^2., the length and cross-sectional area adjusted to suit, gave a 1.0 lb./ft^2. boundary very similar to that shown in Fig. 4.

The high altitude at which it is necessary to fly at low supersonic Mach number suggests that this region is likely to correspond to very low rates of climb and acceleration. Fig. 4 shows the calculated rates of change of energy height with time available for the example aircraft. The engine size was determined from the optimum cruise condition. It is apparent that, even at maximum all out r.p.m. the aircraft cannot exceed a Mach number of 1.2 to 1.3 without giving rise to a pressure increment on the ground greater than 1.0 lb./ft^2. (no reflection factor).

The available acceleration at low supersonic speeds can be increased by increasing the size or number of the engines with or without a decrease in wing area. Both these modifications will lead to an increase in cross-sectional area and weight which are likely to increase the altitude at which supersonic flight is acceptable. It follows that, since layout, weight and performance are so inter-related, the weight and direct operating costs of a supersonic airliner are likely to increase considerably as a result of imposing a limitation on the pressure rise across the bow wave at ground level.

It may be that the most attractive solution is to use some degree of afterburning during the supersonic part of the climb. Certainly the scheme is attractive if the cruise thrust and fuel consumption are not affected by the presence of the afterburner.

Some degree of alleviation of the magnitude of the pressure rises in the N wave near the ground can be attained, for a standard atmospheric condition, by the aircraft climbing during the low supersonic speed regions. This means that there must be an excess of thrust over drag, which in any case is not likely to be large. However, the alleviation depends upon the refraction of the pressure disturbance rays which is very dependent upon the temperature distribution with altitude and upon the wind gradient. Consequently the amount of alleviation would vary from day to day and is not very amenable to precise calculation even if accurate atmospheric data is available.

6. The Pressure Signal on the Ground

Whitham (1) has shown that in the far field, the front shock pressure rise precedes, and the rear shock pressure rise follows the Mach line corresponding to the point $\tau = T_o$ by a distance approximately equal to

$$x = \left[\sqrt{2} \; (\gamma + 1) \frac{M^4}{B^{3/2}} \; r^{\frac{1}{2}} \int F(\tau, \theta) \; dt \right]^{\frac{1}{2}} \qquad (20)$$

where the limits of integration are 0 to T_o for the front wave and T_o to infinity for the rear wave, and r is the distance normal to the flight path of the aircraft in the plane defined by θ.

Since the pressure jump is also proportional to $\left[\int F(\tau, \theta) dt \right]^{\frac{1}{2}}$ with the same respective limits of integration as equation (20) it follows that the distance between the shock waves is approximately

$$d(\theta) = \frac{M^2}{B} \; \frac{(\gamma + 1)}{\gamma} \; \frac{r}{(P_h . P_g)^{\frac{1}{2}}} \; (\Delta P_f + \Delta P_r)_\theta \qquad (21)$$

where ΔP_f and ΔP_r are the front and rear pressure jumps at a distance r from the aircraft in the plane defined by θ.

The ground reaction due to the reflected wave, will have the N type distribution in the x direction and will have a pressure intensity twice the far field pressure signal.

Since the surface integral of the ground reaction must equal the lift the transverse lift grading on the ground is given by

$$\frac{dL}{dy} = \frac{r}{(P_h \; P_g)^{\frac{1}{2}}} \cdot \frac{M^2}{B} \; \frac{(\gamma + 1)}{\gamma} \; (\Delta P_f^2 - \Delta P_r^2)_\theta \qquad (22)$$

where y is the lateral distance along the ground between the plane at the angle θ and the vertical plane containing the aircraft flight path.

The Whitham-Walkden theory depends upon the assumption that the pressure distribution in the far field associated with the lifting effect is determined by $\sin\theta F(\tau,\theta)$ and therefore, for the slender configuration we have considered, $\frac{dL}{dy}$ must vary approximately as $\sin\theta$, if the aircraft is flying straight and level.

Thus the total lift is given by

$$L = \left(\frac{dL}{d\theta}\right)_{\theta = -\frac{\pi}{2}} \int_0^{\pi} \sin\theta \; d\theta$$

$$= 2h\left(\frac{dL}{dy}\right)_{\theta = -\frac{\pi}{2}} \qquad (23)$$

Hence from (22) and (23),

$$L = 2\frac{M^2}{B}\frac{(\gamma+1)}{\gamma}\frac{h^2}{(P_h P_g)^{\frac{1}{2}}}\left(\Delta P_f^2 - \Delta P_r^2\right)_{\theta = -\frac{\pi}{2}} \qquad (24)$$

It follows that once the intensity of the front shock has been calculated, that of the rear shock can be obtained directly from equation (24). It can be seen that, for a given lift the maximum pressure jump on the ground decreases as ΔP_r decreases. If the influence of the wing can be contained in the region

$$(T_o)_{\text{zero lift}} \leqslant r \leqslant \infty$$

then ΔP_f will be dependent only upon the thickness distribution. This suggests that the lift should be towards the rear part of the configuration. It seems probable that, for a given length and thickness distribution, the value of T_o will increase with L, which means that the lift must be concentrated in an even smaller length. Since the centres of lift and gravity must be coincident for trim, there is obviously a limit to the aft position of the lifting region. The further aft the centre of gravity the higher the lift at which Δp_f can be kept equal to Δp_f with no lift. This suggests another reaon why engines at the back of the aircraft are attractive.

It seems probably, that by choosing a suitable thickness distribution, the pressure intensities in the far field can be minimised for a given lift distribution and a limited range of lift.

7. Conclusions

(a) The pressure intensities in the far field can be calculated for an aircraft with lift by adding to the thickness distribution an additional equivalent thickness distribution which is a function of the lift distribution, total lift and Mach number.

(b) The effect of lift on the pressure intensities in the far field can be of the same order as that due to thickness.

(c) It is probable that, if the maximum pressure rise in the far field below an aircraft is of the order of 1 lb./ft. (2 lb./ft. with reflection), some large plate glass windows will be damaged. Since the variation of pressure intensity varies slowly with lateral distance from the vertical plane containing the path of the aircraft, such damage is likely to occur over a wide strip below the aircraft's path.

(d) A limitation on the pressure rise on the ground is likely to limit the height at which large supersonic aircraft of integrated layout can accelerate from M = 1.1 to 1.5 above 45,000 to 55,000 ft. respectively. This is likely to restrict the climb path of such an aircraft to regions where the available acceleration is small unless the thrust available and possibly the wing area are increased beyond that required for other parts of the flight plan. This must lead to an increase in weight and therefore aggravate the far field pressure rise problem. It is probably that some form of afterburning is the best solution, provided the cruise thrust and s.f.c. are not materially changed.

(e) There is some indication that an aircraft could be designed to minimise the far field pressure rises for a given volume, over limited ranges of total lift. This would involve choosing suitable thickness and lift distributions. A limitation on the method is likely to result from the necessity to trim the aircraft.

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APPENDIX 1

On the propagation of shock waves from a moving body in a non-uniform atmosphere

The propagation of shock waves towards the ground from aircraft flying at supersonic speeds at high altitude involves some consideration of the variation of the atmospheric pressure, density and temperature with height. In the general theory of Whitham, the attenuation of the shock waves from the moving body is calculated for the case of propagation in a uniform atmosphere. In this appendix an approximate estimation will be made of the correction term to be applied to Whitham's theory for the propagation in a non-uniform atmosphere.

It can easily be shown (see Lamb, Ref. 11, p.543) that the general effect of the increasing atmospheric pressure towards the ground is to increase the amplitude of the shock wave propagating towards the ground and this effect is very much greater than that associated with the temperature increase towards the ground. It can also be shown that over the distances of interest in practical cases the combined effects of viscosity and thermal conductivity as well as atmospheric turbulence in attenuating and distorting the shock wave along its path are negligible compared with the former effects. These latter effects would however need careful consideration for flight at much higher altitudes than those considered here.

If further products of small quantities are neglected, as well as the changes in entropy across the shock wave, the equation for the perturbation density associated with the motion of the shock wave is

$$\frac{\partial^2(\delta\rho)}{\partial t^2} - \frac{\partial}{\partial x_i}\left(a^2 \frac{\partial}{\partial x_i}(\delta\rho)\right) = -g\frac{\partial(\delta\rho)}{\partial x} \qquad (A.1)$$

where $\delta\rho$ is the difference between the density and the local atmospheric density, and x is measured vertically downwards in the atmosphere.

This equation is obtained from the equations of continuity and motion which are respectively

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_i}{\partial x_i} = 0 \quad ; \quad \frac{\partial\rho u_i}{\partial t} + \frac{\partial\rho u_i}{\partial x_j}u_j = -\frac{\partial p}{\partial x_i} + \rho X_i \qquad (A.2)$$

If we subtract the divergence of the second of (A.2) from the time derivative of the first, and noting that in the undisturbed atmosphere (at least in the model we have assumed)

$$\frac{\partial}{\partial x_i}\left(a^2 \frac{\partial\bar{\rho}}{\partial x_i}\right) = \frac{\partial\bar{\rho}X_i}{\partial x_i} \qquad (A.3)$$

with $X_i \equiv (g\ ,\ 0,\ 0)$ and $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$, then equation (A.1) immediately follows.

Since $a \equiv a(x)$ we can write (A.1) in the form (writing z for $\delta\rho$)

$$\frac{\partial^2 z}{\partial t^2} - a^2 \nabla^2 z = -\left(g - \frac{da^2}{dx}\right) \frac{\partial z}{\partial x} \tag{A.4}$$

In order to simplify the solution of (A.4), but not at the same time modifying its essential properties, we will replace a^2 by an average value $\bar{a}^2$ and $(g - \frac{da^2}{x})$ by an average value $\bar{g}$. Thus we have

$$z_{tt} - \bar{a}^2 \nabla^2 z = - \bar{g}\ z_x \tag{A.5}$$

Now at distances far from the body the magnitude of the perturbation density would in Whitham's solution be obtained from

$$\bar{z}_{tt} - \bar{a}^2 \nabla^2 \bar{z} = 0 \tag{A.6}$$

In principle therefore we can find the change in the amplitude function, associated with the non-uniform atmospheric conditions, by solving (A.5) from the known solution of (A.6). However it can be shown that this change in the amplitude function for a general wave motion, is similar qualitatively to the change in the amplitude of a plane wave propagating downwards through the atmosphere. It is this special case that is treated below.

The solution of (A.5) with the boundary conditions $\lim_{t \to 0} z(x,t) = 0$, $\lim_{t \to 0} \frac{\partial z}{\partial t} = 0$ and $z(0,t) = f(t)$ for $t > 0$, where the origin of x is taken at the altitude corresponding to the initial formation of the shock wave, can easily be found by the method of the Laplace transform. The initial amplitude function (or density signal) can be of any form and would be typically an N-wave. However for simplicity, we will take $f(t)$ = constant equal to unity, but it should be noted that this simplification does not affect the value of the amplification of the wave-front amplitude as the wave proceeds groundwards.

The solution of (A.5) for this weak plane shock wave problem is (see Carslaw and Jaeger Ref. 12 p.183)

$$z = \exp\left(\frac{\bar{g}\, x}{2\, \bar{a}^2}\right) \left[1 - \int_{x/\bar{a}}^{t} \frac{x\, \bar{g}}{2\, \bar{a}^2} \frac{J_1\left(\frac{\bar{g}}{2\bar{a}} \sqrt{\tau^2 - (\frac{x}{\bar{a}})^2}\right) d\tau}{\sqrt{\tau^2 - (\frac{x}{\bar{a}})^2}} \right] \quad \ldots\ldots \text{(A.7)}$$

when $(t - {}^{x}/\bar{a})\frac{g}{2\bar{a}} << 1$, noting that $J_1(y) \doteq y/2$ as $y \to 0$, we find that (A.7) becomes

$$z = \exp \eta . \left[1 + \eta^2/2 - \eta \frac{\bar{g}t}{4\bar{a}}\right] \quad (A.8)$$

where $\eta = \frac{x\bar{g}}{2\bar{a}^2}$.

Hence the increase in amplitude of the wave-front, given by $t = x/\bar{a}$, as it travels groundwards increases exponentially with x.

In the more general case of pseudo-spherical waves this increase in amplitude would be more than balanced by the spherical attenuation of the waves but these two factors must always be multiplied to find the perturbation density rise at the wave-front near the ground. We see from (A.8) that the profile of the wave is also changed.

In the special case of an isothermal atmosphere

$$\frac{\bar{\rho}}{\bar{\rho}_{x=0}} = \frac{\bar{p}}{\bar{p}_{x=0}} = \exp\left(\frac{gx}{a^2}\right) \quad (A.9)$$

where (a) is the constant (isothermal) speed of sound, so that the wave-front amplification factor

$$\exp(\eta) = \sqrt{\frac{\bar{p}}{\bar{p}_{x=0}}} \quad (A.10)$$

In the general gas g and a^2 must be replaced by $\bar{g}$ and $\bar{a}^2$ as defined above but in most practical cases the approximation expressed by (A.10) will be adequate*.

Accordingly Whitham's modified formula for the strength of the bow shock wave from a body of revolution is

$$\frac{\Delta p}{p_g} = K_r \sqrt{\frac{p_s}{p_h}} \frac{\gamma}{(\gamma+1)^{\frac{1}{2}}} \frac{(2B)^{\frac{1}{4}}}{h^{\frac{3}{4}}} \left(\int_0^{T_0} F(\tau)\,d\tau\right)^{\frac{1}{2}} \quad (A.11)$$

where p_g and p_h are the pressures at ground-level and at altitude respectively and K_r is the ground reflection factor. This relation is the one used by Randall (1957) and others.

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* According to the approximations made above

$$\exp(\eta) = \sqrt{\bar{\rho}/\bar{\rho}_{x=0}} = \sqrt{\bar{p}/\bar{p}_{x=0}} \cdot \frac{a_{x=0}}{a}$$

so that when the relation expressed by (A.10) is used generally the variation of the speed of sound with altitude, which is admittedly small, will be neglected.

APPENDIX 2

The response of a plate glass window to an N-wave

The equation for the pressure distribution near the ground across the shock waves from a supersonic aircraft is

$$p - p_g = \Delta p - \frac{2\Delta p}{T} t \qquad 0 \leqslant t \leqslant T \qquad (A.12)$$
$$= 0 \qquad t > T$$

where p_g is the ambient pressure at ground level

Δp is the pressure rise across the bow and tail waves.

T is the passage time for the bow and tail waves.

The equation of motion for the vibration of a thin plate of thickness h is, if w is the normal deflection, and D is the flexural rigidity of the plate,

$$\rho_m h \frac{\partial^2 w}{\partial t^2} + D \nabla^2 w = \Delta p(1 - 2t/T) \qquad 0 \leqslant t \leqslant T$$
$$= 0 \qquad t > T \qquad \ldots\ldots \quad (A.13)$$

If we assume a mode of deflection and put w_c equal to the central deflection, then on integrating (A.13) over the area of the plate we find that

$$M'' \ddot{w}_c + K'' w_c = A \;\Delta p\,(1 - 2t/T) \qquad 0 \leqslant t \leqslant T$$
$$= 0 \qquad t > T \qquad \ldots\ldots \quad (A.14)$$

where M' is the equivalent total mass of plate weighted for the particular mode of vibration

K' is the equivalent total plate stiffness weighted for the particular mode of vibration

and A is the plate area.

For instance if the plate is circular with simply supported edges and the mode is that for static deflection under a uniformly distributed load

$$M' = \frac{M \;\; 7(1+\nu)}{3(5+\nu)} \qquad (A.15)$$

and

$$K' = \frac{16\pi^2 h^3 \;E}{3A(5+\nu)(1-\nu)} \qquad (A.16)$$

where M is the total mass, E is Young's modulus, and ν is Poisson's ratio.

If $m = M'/A\,\Delta p$ and $k = K'/A\Delta p$ and the subscript on w is dropped the equation of motion (A.14) becomes

$$m\,\ddot{w} + k\,w = 1 - 2t/T \qquad 0 \leqslant t \leqslant T$$

$$= 0 \qquad t > T \qquad \ldots\ldots \quad (A.17)$$

with the boundary conditions

$$w = \frac{\partial w}{\partial t} = 0 \quad \text{at} \quad t = 0 .$$

If we write the natural frequency of the plate as

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{K'}{M'}} \qquad (A.18)$$

then it can be shown that the solution of (A.17) is

$$m\,\omega^2\,w(t) = 1 - \cos(\omega t) - \frac{2}{\omega T}\,(\omega t - \sin(\omega t))$$
$$+ H(t - T)\left[1 - \cos\omega(t - T) + \frac{2}{\omega T}\,(\omega(t - T) - \sin\omega(t-T))\right] \qquad \ldots\ldots \quad (A.19)$$

where $H(t - T)$ is the Heaviside unit function.

It follows that

$$m\,\omega^2\,w(t) = 1 - \frac{2t}{T} + \sqrt{(1 + 4/\omega^2\,T^2)}\quad \sin\omega(t - \epsilon) \qquad (A.20)$$

$$\text{for } 0 \leqslant t \leqslant T$$

where

$$\tan\omega\epsilon = \frac{\omega T}{2} ,$$

and

$$m\,\omega^2 w(t) = 2\left(\cos\left(\frac{\omega T}{2}\right) - \frac{2\sin\left(\frac{\omega T}{2}\right)}{(\omega T)}\right)\cos\left(\omega t - \frac{\omega T}{2}\right) \qquad (A.21)$$

$$\text{for } t > T$$

In the range $0 \leqslant t \leqslant T$ the maximum value of w, written w_m, occurs when

$$\frac{\omega t}{2} = \tan^{-1}\left(\frac{\omega T}{2}\right)$$

and

$$m\,\omega^2\,w_m \;=\; 2 - \frac{4}{\omega T}\tan^{-1}\left(\frac{\omega T}{2}\right) \qquad (A.22)$$

Similarly in the range $t > T$ the maximum value of w occurs when

$$\omega t = \frac{\omega T}{2} + n\pi \qquad (n = 0,\ 1,\ 2\ \ldots)$$

and

$$m\,\omega^2\,w_m \;=\; 2\cos\left(\frac{\omega T}{2}\right) - \frac{4}{\omega T}\sin\left(\frac{\omega T}{2}\right) \qquad (A.23)$$

From (A.22) and (A.23) we see that the lowest value of ωT at which the maximum deflection occurs is when $\omega T \approx 5.5$ and the corresponding maximum deflection is given by

$$m\,\omega^2\,w_m \approx 2.12 \qquad (A.24)$$

If we put $m\,\omega^2\,w_m = \delta$ then δ is a number of order 2, and

$$w_m = \frac{\delta}{k} = \frac{\delta\,A\,\Delta p}{K'} \qquad (A.25)$$

showing that δ is the dynamic load magnification factor, since $\frac{A\Delta p}{K'}$ is the static deflection of the plate under the uniformly distributed load Δp per unit area. If this latter deflection is denoted by w_s then

$$w_m = \delta\,w_s \qquad (A.26)$$

The values of w_s for various shapes of plate are given below, for convenience, together with the values of the maximum stress.

Geometry	Edge Condition	Maximum Deflection $w_s D/\Delta p$	Maximum Stress $\sigma_m/\Delta p$
(Circular Radius a)	Clamped	$a^4/64$	$\frac{3a^2}{4h^2}$ (Edge)
	Simply Supported	$\left(\frac{5+\nu}{1+\nu}\right)\frac{a^4}{64}$	$\frac{3(3+\nu)a^2}{8\ h^2}$
Square (2a x 2a)	Simply Supported	$\frac{64\ a^4}{\pi^6}$ approx.	$\frac{(3+\nu)}{2}\ \frac{a^2}{h^2}$
Rectangular Short side 2a Lond side 2b	Simply Supported	$\frac{64\ a^4}{\pi^6\left(\left(\frac{a}{b}\right)^2+1\right)^2}$ approx.	$0.67\ (3+\nu)\ a^2/h^2$ when $\frac{b}{a} = 1.42$, $0.81\ (3+\nu)\ a^2/h^2$ when $b/a = 2$

It will be shown in general however that the deflection as calculated from the formulae above will not be small compared with the plate thickness. The correction to the above results when $w_{s/h} >> 1$, as given by Timoshenko, for a circular plate with clamped edges is (when $\nu = 0.25$)

$$w_s'/w_s \doteq 1.2\ (h/w_s)^{2/3} \qquad (A.27)$$

where w_s' is the central deflection according to the more exact large deflection theory.

The modified maximum stress for the clamped edge circular plate is accordingly

$$(\sigma_m)_{r=0} = -0.423\left(\frac{E(\Delta p)^2\ a^2}{h^2}\right)^{1/3}$$

and

$$(\sigma_m)_{r=a} = \frac{3}{4}\Delta p\ \frac{a^2}{h^2} + 0.328\left(\frac{E\ (\Delta p)^2\ a^2}{h^2}\right)^{1/3} \qquad (A.28)$$

at the centre and edge respectively.

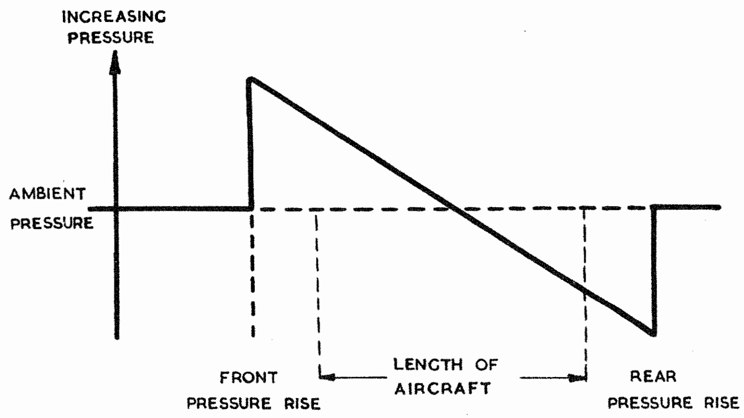


FIG 1a. FAR FIELD PRESSURE SIGNAL FOR AN AIRCRAFT WITHOUT LIFT.

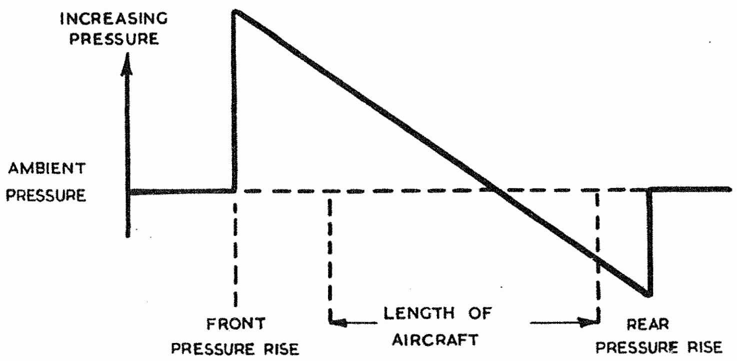
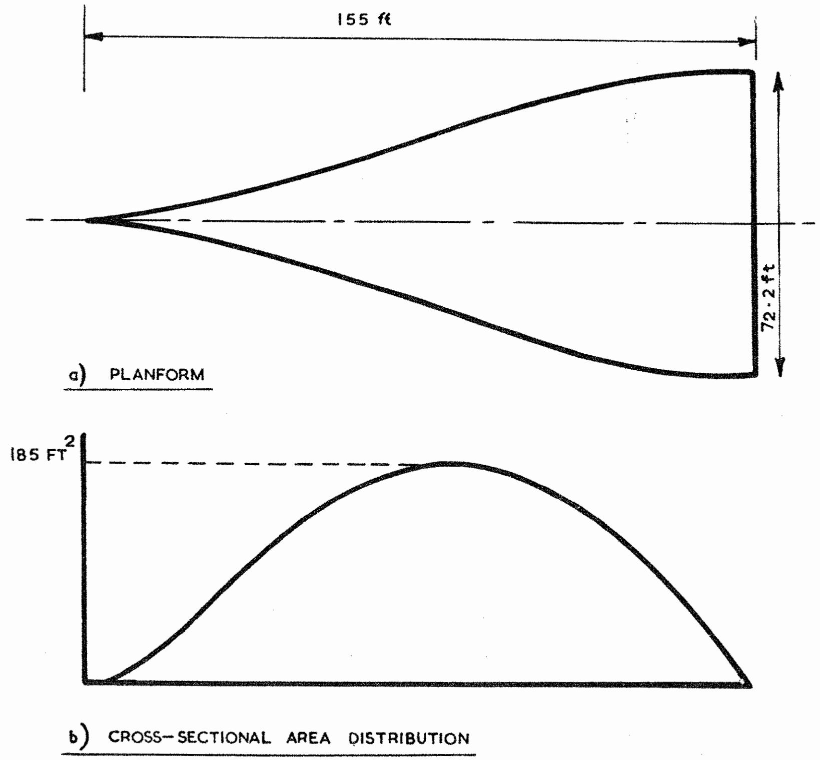


FIG. 1b. FAR FIELD PRESSURE SIGNAL FOR AN AIRCRAFT WITH LIFT.



BASIC PARTICULARS

MEAN CLIMB WEIGHT	300,000 lb.	SEMI-SPAN-LENGTH RATIO	0·233
GROSS WING AREA.	5,710 FT^2	PLAN FORM SHAPE PARAMETER	0·51
CLIMB WING LOADING	52·5 lb/FT^2	NUMBER OF PASSENGERS	100 – 120
ASPECT RATIO	0·915	STILL AIR RANGE	4,600 n.m.

FIG. 2. DETAILS OF EXAMPLE AIRLINER.

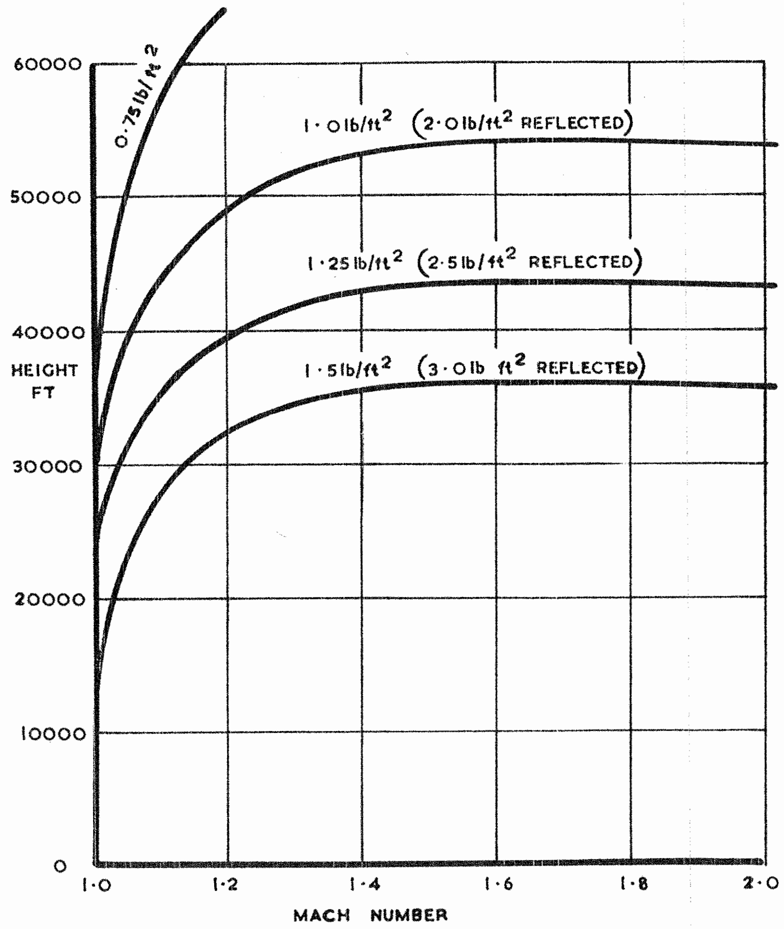


FIG. 3. PRESSURE RISE ON GROUND ACROSS BOW WAVE FROM EXAMPLE AIRLINER

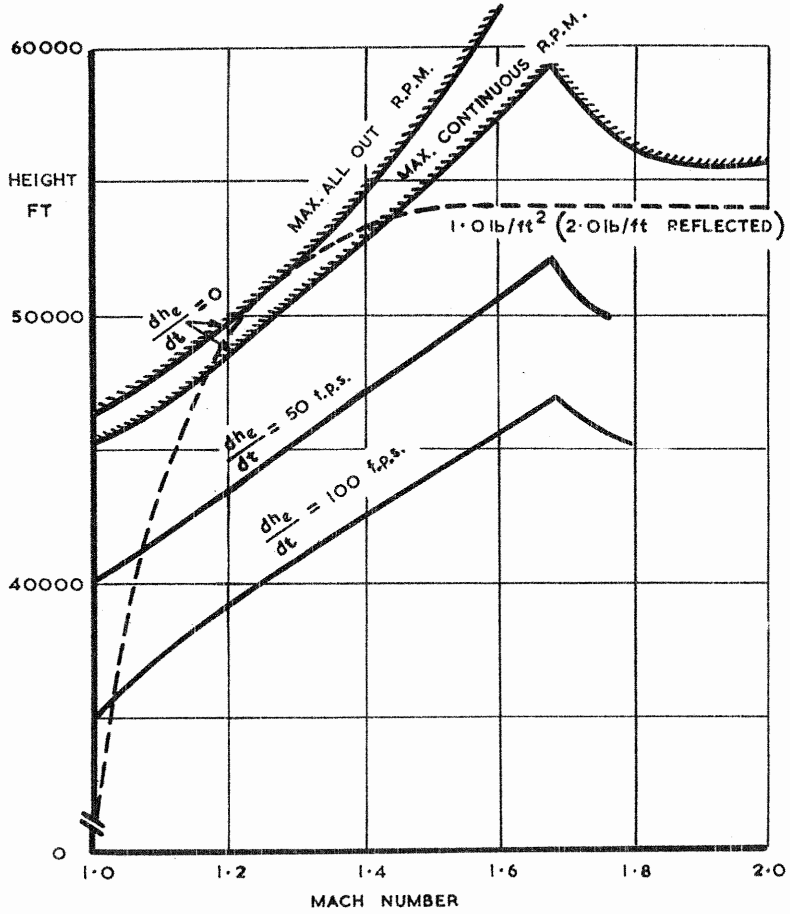


FIG. 4. CEILING, RATE OF CLIMB AND GROUND PRESSURE RISE BOUNDARIES FOR EXAMPLE AIRLINER.