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A note on the decay of aircraft trailing vortices

- by -

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Summary

An elementary theory of aircraft trailing vortex decay is presented based on an assumed law for the variation of the mean eddy viscosity with distance from the wing. This law is based on the experimental data of Rose and Dee (1963). The analysis gives results, as might be expected, in agreement with their data. The justification for the analysis must however be in doubt until more data are available covering a wide range of variables such as aircraft size, distance, incidence, etc.

1. Introduction

Rose and Dee (1963) present experimental flight data for the structure of vortex wakes behind a series of aircraft at two altitudes. The data are difficult to obtain, and therefore are sparse and subject to the usual errors, but the results presented do show some consistency in the magnitude of the peak circumferential velocity around the vortex and the gradient of the circumferential velocity across the vortex core. These data establish that the vortex structure is grossly different from that predicted by a laminar theory. An attempt is made to compare the experimental results with those calculated from a theory in which the viscosity is augmented by the presence of an eddy viscosity. It is found that fair agreement is obtained if the latter has a value some 2000 times the kinematic viscosity. From the few results of the experiments with varying wing circulation it is suggested that the eddy viscosity is proportional to the wing circulation and independent of the distance downstream of the wing. The aim of this paper is to review the data and analysis presented by Rose and Dee and to show that an alternative empirical formulation fits their data equally well. It is clear however that further data over a wide range of Reynolds numbers and at a number of distances downstream of wings at various incidences are required before this new theory can be used with confidence.

Only the case of a simple trailing vortex pair behind an isolated wing will be treated.

2. Trailing vortex system

It is well known that far downstream of a lifting aerofoil a trailing vortex pair exists. The rolling up the vortex sheet shed by the aerofoil, from which the vortex pair is derived, takes place within a few chord lengths downstream of the aerofoil, although the rolling up is possibly not complete within a very large number of chord lengths. The existence of a weak turbulent wake stretching between the vortices far downstream of the aerofoil is not considered in the theory, although its presence may account for the form of the eddy viscosity law used in the analysis.

The structure of the trailing vortices close to the wing has been considered at low Reynolds numbers by Dosanjh, Gasparek and Eskinazi (1962) and at higher Reynolds numbers by Newman (1959). These data for the circumferential and radial velocity distributions show good agreement with the laminar vortex theory when the viscosity was replaced by an eddy viscosity, but in the low Reynolds number experiments of Dosanjh et al a value only 6 to 10 times that of the viscosity was required with somewhat higher values in the experiments of Newman. In Newman's experiments the wake was turbulent but Dosanjh et al report that in their experiments the wake possessed a considerable amount of periodicity. However, in both sets of experiments the eddy viscosity was considerably less than reported by Rose and Dee (loc. cit).

Now it has been shown by Townsend (1956) that a three-dimensional wake flow becomes stable beyond a certain distance, which depends on the flow Reynolds number around the body, since the wake turbulence no longer receives sufficient energy from the main flow to maintain its intensity. This is the result of the local flow Reynolds number in the wake decreasing with distance downstream to a sufficiently low value to produce stable flow. If this result were applied to the trailing vortex wake the turbulent vortex cores would be replaced by laminar vortex cores at a certain distance downstream of the wing. However, it would appear that this distance to stable vortex flows, if it exists, is probably beyond the distances of current interest and beyond the distance at which reliable measurements could be made.

The structure of a wing trailing vortex system of course differs appreciably from that of a turbulent wake, say behind a bluff body such as a sphere. A further factor is the presence of the induced downwash which propels the trailing vortex pair downwards relative to the lifting wing at a velocity

$$\bar{w} = \frac{2\Gamma_o}{\pi^2 b} \qquad (1)$$

where Γ_o is the circulation of the wing of span b.

The presence of strong atmospheric turbulence on the vortex pair break-up is well known but it is assumed that under ordinary atmospheric conditions the effect of the atmosphere can be neglected.

The structure of the vortex at sufficiently low Reynolds numbers is a slender central viscous core having a solid body rotation, outside of which exists a potential flow free vortex. Lamb (1932) shows that an isolated infinite vortex of strength Γ_o initially concentrated in the z-axis diffuses outwards with increase in time according to the law

$$\Gamma_{(r,t)} = \Gamma_o \left(1 - e^{-r^2/4\nu t}\right) \qquad (2)$$

and its circumferential velocity distribution is given by

$$v_\theta(r,t) = \frac{\Gamma_o}{2\pi r} \left(1 - e^{-r^2/4\nu t}\right) \qquad (3)$$

If we define the vortex core as the value of $r = r^*$ where $v_\theta = (v_\theta)_{max}$ then

$$r^* \approx \sqrt{5.03\nu t} \qquad (4)$$

showing that the core increases as $\sqrt{t}$. When $\nu = 2 \times 10^{-4}$ ft^2/sec. (the kinematic viscosity of air at 10,000 ft. altitude) the value of r^* after $t = 100$ secs. is according to (4) $r^* = 0.32$ ft. while the experimental value reported by Rose and Dee (loc. cit.) is nearer 6 ft. This is surely proof indeed of the inapplicability of the extension of the laminar flow formula to the high Reynolds number case. But as already noted Dosanjh et al found it necessary to replace ν in equation (3) by $\nu + \nu_T$, where ν_T/ν was taken to be a constant having a value of about 10, when the Reynolds number of the flow based on wing chord and freestream velocity was only 10,000. However, no proof was given to justify such a simple relation and there exists the possibility that the agreement between the experimental and calculated values of v_θ, together with those of the radial velocity component v_r, may be largely fortuitous.

There appear to be good reasons to doubt the existence of a constant eddy viscosity across a vortex. One is that the circumferential and radial velocity components vanish on the axis while the perturbation axial velocity component, for the cases tested by Dosanjh et al. was vanishingly small beyond a few chord lengths downstream of the wing. Thus near the centre of the core of the vortex we might expect the eddy viscosity to be vanishingly small. A further reason is that in free turbulent shear flows it is shown by Townsend (1956) that a virtual eddy viscosity is set up when turbulence is strained by a mean rate of strain, whereas in a simple vortex such a mean rate of strain is absent.

But against these arguments lies the undisputed fact that the trailing vortex pair is created by the rolling up of the shed turbulent boundary layers from the wing edges and the existence of a vortex structure far downstream which is certainly not laminar. We are therefore led to believe that the turbulence existing initially between the successive turns of the vortex sheet and the remaining turbulence which is convected and diffused into it along its path is dissipated very slowly by the presence of the strong circulatory motion around the vortex core. Clearly experimental verification for this hypothesis is lacking at the present time. However, if we accept its plausibility we see that the turbulence existing across the entire vortex diffuses the mean vorticity outwards more rapidly than do the viscous stresses. Although, by such an hypothesis, it would appear reasonable that the eddy viscosity ν_T might be nearly constant across a radius of the vortex, there is no justification for the assumption that ν_T remains constant with increase in distance downstream or that ν_T is proportional to the strength of the vortex. Since dissipation of the turbulence must take place continuously along the vortex we see that the eddy viscosity should fall from a very high value, at sufficiently high Reynolds numbers, to a value close to the viscosity ν far enough downstream. For want of a more simple relation between ν_T/ν and s, the distance downstream of the wing, we assume that

$$\nu_T/\nu = 1 + 1/(s/s_o)^m \qquad (5)$$

where s_o is a very large distance, m is positive and less than unity, while in the range of interest $s/s_o \ll 1$.

The quantity s_o and the index m will be determined from the results of Rose and Dee (loc. cit.). It will be shown later that $m \approx {}^1/_3$ while $s_o \approx 10^{12}$ ft. Thus for distances of the order 10^5 ft. we can replace (5) by

$$\nu_T(s)/_\nu = (s/_{s_o})^{-m} \qquad (6)$$

The following well-known relations between the wing circulation, wing geometrical parameters and freestream velocity are also required. If Γ_o is the wing circulation then for a wing of elliptic planform having a span b and a geometric mean chord $\bar{c}$ then

$$\frac{\Gamma_o}{\nu} = \frac{2C_L}{\pi} \frac{V_\infty \bar{c}}{\nu} \qquad (7)$$

where C_L is the wing lift coefficient and ν is the kinematic viscosity. Since the mean downwash velocity $\bar{w}$ is given by (1) we find from (7) that alternatively

$$\frac{\Gamma_o}{\nu} = \frac{\pi^2}{2}\left(\frac{\bar{w}b}{\nu}\right) \qquad (8)$$

where $\frac{\bar{w}b}{\nu}$ is the Reynolds number of the vortex pair far downstream of the wing and is constant. (We note that for the experiments of Rose and Dee (loc. cit.) $\frac{2}{\pi^2}\frac{\Gamma_o}{\nu}$ is of order 2×10^6. Although the typical Reynolds number of each of the vortices of the trailing vortex pair will be many times smaller, we should not exclude the possibility that it is due to this high downwash Reynolds number that turbulence is being created continuously near the outer regions of the vortex core and it is the diffusion of this turbulence across the vortex which gives rise to its turbulent structure).

The induced circumferential velocity near each of the semi-infinite vortices in the trailing vortex pair at a sufficient distance from the wing will be

$$v_\theta = \Gamma_o/_{4\pi r} \qquad (9)$$

where r is the radial distance from the centre of one vortex. Since the additional velocity due to the other member of the vortex pair is of the order $\Gamma_0/4\pi(b-r)$ we can neglect it when $b \gg r$.

In the next section we will develop the analysis for the decay of a single isolated vortex which is being continuously created at the wing $z = 0$, in the presence of a uniform flow of velocity V_∞.

3. Analysis

Following the work of Newman (1959) and Dosanjh et al (1962) we will assume that at a sufficiently high Reynolds number the perturbation velocities w, v_θ, and v_r measured in the axial, circumferential and radial directions respectively will be small compared with the freestream velocity V_∞. From the discussion in §2 above we will also assume the existence of an eddy viscosity $\nu_T(z) \gg \nu$ so that the linearised equation for the circumferential velocity component can be written

$$V_\infty \frac{\partial v_\theta}{\partial z} = \nu_T(z)\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right) \qquad (10)$$

The equations for v_r, and w can also be written down in a similar way, but for the purpose of this paper only the equation for the circumferential velocity component is required.

If the flow in the turbulent vortex core is self-preserving, we can define a local velocity and scale $u_o(z)$ and $l_o(z)$ respectively, (later we will equate u_o to the maximum circumferential velocity and l_o to the radius at which this maximum occurs), such that the equation of motion (10) can be written in a non-dimensional form in which the velocity distribution is written as a function of r/l_o only.

Let

$$v_\theta = u_o f(r/l_o) \qquad (11)$$

then (10) becomes

$$\frac{V_\infty l_o}{\nu_T}\left(\frac{l_o}{u_o}\frac{du_o}{dz} f - \frac{r}{l_o}\frac{dl_o}{dz} f'\right) = f'' + \frac{f'}{r/l_o} - \frac{f}{(r/l_o)^2} \qquad (12)$$

where primes denote differentiation with respect to r/l_o. The flow is therefore self-preserving if

$$\left.\begin{aligned} &\frac{V_\infty l_o}{\nu_T} \cdot \frac{l_o}{u_o}\frac{du_o}{dz} = \text{constant} \\ \text{and} \quad &\frac{V_\infty l_o}{\nu_T} \cdot \frac{dl_o}{dz} = \text{constant} \end{aligned}\right\} \qquad (13)$$

But the product $u_o l_o$ must be proportional to Γ_o * so that

$$u_o l_o \;=\; \text{constant} \tag{14}$$

Hence

$$\frac{V_\infty l_o}{\nu_T}\;\frac{dl_o}{dz} \;=\; -\,\frac{V_\infty l_o}{\nu_T}\;\frac{l_o}{u_o}\;\frac{du_o}{dz} \;=\; A \tag{15}$$

where A is a constant, and

$$A/l_o^{\,2} \;=\; \frac{V_\infty}{2\nu_T}\;\frac{dl_o^{\,2}}{dz}\;\; l_o^{\,2} \tag{16}$$

We note in passing that the local flow Reynolds number is

$$\frac{u_o l_o}{\nu} \;=\; R$$

and for turbulent flow to exist R must be sufficiently large. It is shown later that in the experiments of Rose and Dee (loc. cit.) R is of order 3×10^5 and we assume that this is sufficiently large for the existence of the type of flow postulated.

The solution of (12) satisfying the boundary conditions

$$v_\theta = 0 \qquad z > 0 \qquad r \to \infty$$

$$v_\theta = 0 \qquad z \to \infty \qquad \text{all } r$$

$$v_\theta = \frac{\Gamma_o}{4\pi r} \qquad z = 0$$

is

$$v_\theta = \frac{\Gamma_o}{4\pi r}\left(1 - e^{-r^2 A/2l_o^{\,2}}\right) \tag{17}$$

where from (16)

$$\frac{A}{l_o^{\,2}} \;=\; \frac{V_\infty}{2\int_0^{z} \nu_T(z')\,dz'} \tag{18}$$

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* This condition is obtained by noting that for any value of z

$$\Gamma_o \;=\; \Big[2\pi r\, v_\theta(r,z)\Big]_{r\to 0} \;=\; 2\pi u_o l_o \Big[r/l_o\, f(r/l_o)\Big]_{r\to 0}$$

$$=\; \text{constant.}$$

on putting $l_o(o) = 0$.

(Both Dosanjh et al and Newman have used the boundary condition $v_\theta = \frac{\Gamma_o}{2\pi r}$ at $z = 0$, but this would imply the single vortex is of infinite extent).

From (17) the maximum value of v_θ can be determined. If $(v_\theta)_{max} \equiv u_o$ and the radius at which it occurs $r^* = l_o$ then it can easily be shown that

$$l_o = \sqrt{\frac{5.03}{V_\infty} \int_o^z \nu_T(z')dz'} \qquad (19)$$

and

$$u_o \approx \frac{5}{88} \frac{\Gamma_o}{l_o} \qquad (20)$$

where in (20) we have replaced π by $\frac{22}{7}$ and written 5 for 5.03. From (20) we see that

$$R = \frac{u_o l_o}{\nu} = \frac{5}{88} \frac{\Gamma_o}{\nu} = \frac{5}{14\pi^2} C_L \cdot \frac{V_\infty \bar{c}}{\nu} \qquad (21)$$

which is a large number as already noted after equation (16) above.

The analysis cannot be continued without making some assumption about the variation of ν_T with distance z. As already noted in § 2 above we will assume that over the range of distances of interest

$$\frac{\nu_T}{\nu} \approx (z/z_o)^{-m} \quad \text{for } z/z_o \ll 1.$$

Thus from (19) (with 5 written for 5.03)

$$l_o^2 = \frac{5 z_o \nu}{V_\infty (1-m)} \left(\frac{z}{z_o}\right)^{1-m} \qquad (22)$$

while from (20)

$$\frac{u_o}{\Gamma_o} = \frac{\sqrt{5}}{88} \sqrt{\frac{V_\infty}{\nu z_o}} \frac{\sqrt{1-m}}{(z/z_o)^{\frac{1-m}{2}}} \qquad (23)$$

Since time rather than distance is measured in the experiments we will replace z by the relation

$$\left.\begin{aligned} z &= V_\infty t \\ \text{and} \qquad z_o &= V_\infty t_o \end{aligned}\right\} \tag{24}$$

and therefore (23) becomes

$$\frac{2u_o}{\Gamma_o} = \frac{\sqrt{5(1-m)}}{44\sqrt{\nu t_o}} \left(\frac{t_o}{t}\right)^{\frac{1-m}{2}} \tag{25}$$

This relation shows that for $t \ll t_o$ the maximum circumferential velocity component decays like $t^{-\frac{(1-m)}{2}}$, and is proportional to Γ_o and inversely proportional to $\sqrt{\nu}$.

The slope of the circumferential velocity component near $r = o$ can also be obtained from (17). It is

$$\left(\frac{\partial v_\theta}{\partial r}\right)_{r=o} = \frac{\Gamma_o(1-m)}{16\pi\nu t_o} \left(\frac{t_o}{t}\right)^{1-m} \tag{26}$$

Now Rose and Dee (loc. cit.) find that

$$\left(\frac{\partial v_\theta}{\partial r}\right)_{r=o} = 3 \text{ sec}^{-1}$$

for $\frac{\Gamma_o}{16\pi\nu} = 1.34 \times 10^5$ when $t = 89$ secs. at 10,000 ft. altitude.

Hence $\frac{1-m}{\nu t_o}\left(\frac{t_o}{t}\right)^{1-m} = 2.2 \times 10^{-5}\text{sec}^{-1}$, and therefore

$$\frac{2u_o}{\Gamma_o} = 0.017 \text{ ft.}^{-1} \text{ in agreement with the measurements.}$$

Rose and Dee next plotted $\Delta w/\Gamma_o$ (where $\Delta w \approx 2u_o$) against $\Gamma_o t$ on the basis that ν_T was independent of t but proportional to the wing circulation Γ_o. From their simplified theory they found that $\Delta w/\Gamma_o$ was inversely proportional to $\Gamma_o t$ but the experimental results do not appear to follow this law even noting the fairly large scatter.

A somewhat improved fit with their experimental data is found by putting $m = {}^1/_3$. (It is found that m lies somewhere between ${}^1/_4$ and ${}^1/_2$. The choice of $m = {}^1/_3$ is somewhat arbitrary due to the large scatter in the data).

When $m = {}^1/_3$ we find

$$\nu t_o \left(\frac{t}{t_o}\right)^{2/3} = 5.95 \text{ ft}^2.$$

for $t = 89$ secs., and since $\nu = 2 \times 10^{-4}$ ft^2/sec. at 10,000 ft. altitude $t_o = 3.4 \times 10^9$ secs. While at this altitude

$$\sqrt{\nu t_o} = 800 \text{ ft. approximately.}$$

On substitution of this value for t_o in (25) we find

$$\frac{2u_o}{\Gamma_o} = \frac{0.079}{t^{1/3}} \tag{27}$$

(in feet-second units) for the experiments of Rose and Dee (loc. cit.).

The agreement with the results of Rose and Dee (loc. cit.) is as good as might be expected allowing for the sparseness and scatter of the experimental data.

On the basis of these theoretical results we might have hoped to draw some tentative conclusions about the mutual interation between the cores of the vortex pair. However, since

$$s = \frac{s_o l_o^{\,3}}{(7.5\nu t_o)^{3/2}}$$

where l_o is the core radius, we find that for an aircraft of 120 ft. span flying at a speed of 250 ft/sec. at an altitude of 10,000 ft. that the cores would not merge until s is approximately 6×10^6 ft. The value of $2u_o$ at this distance is 3.7 ft./sec. If we place a limit of $l_o = 0.1b$ (where b is the spacing of the vortex cores or the wing span) as the limit of applicability of the above results then these calculations would cease to have much value beyond about 5×10^4 ft., or 200 secs., which is just about the upper limit for taking accurate measurements in flight.

On the choice of the two constants the value of $m = {}^1/_3$ seems reasonable but our choice of t_o (or what is equivalent s_o) needs to be treated with caution. We might well ask whether s_o is a true constant or merely proportional to a basic dimension of the wing such as $\bar{c}$.* As we might expect (27) does not apply to the experiments of Dosanjh et al** but this is not surprising since we have already noted that the flow in the vortex is not fully turbulent and in any case Dosanjh only considered the structure in the first few chords downstream of the wing.

In conclusion we note that:

(i) These tentative results such as (27) should be applicable for a wide range of full scale aircraft (the correction for altitude follows from (25)), and

(ii) for typical downstream distances the local vortex flow Reynolds number is sufficiently high for turbulent flow to exist.

References

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3. Newman, B.G. (1959) — Flow in a viscous vortex. Aeronautical Quarterly, Vol. 10, p. 149.

4. Townsend, A.A. (1956) — The structure of turbulent shear flow. C.U.P.

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* On the assumption that x_0 is proportional to $\bar{c}$ (which is justified on dimensional grounds) we can recast (25) as

$$\frac{2u_0}{\Gamma_0} = 0.00068 \frac{V_\infty \bar{c}}{\nu} \frac{1}{(x/\bar{c})^{1/3}} \qquad x \gg \bar{c}$$

** In the relation we have assumed for ν_T/ν (see equation (5)), the eddy viscosity has very large values close to the wing trailing edge. Clearly an upper limit for ν_T must exist. However for $x \gg \bar{c}$ this limiting value would not affect the values given for l_0 and u_0 in equations (19) and (20). But within a few chord lengths of the wing the value of $(\nu_T)_{max}$ cannot be ignored. In addition $(\nu_T)_{max}$ would be expected to be a function of Reynolds number at low Reynolds numbers well within the range of the experiments of Dosanjh necessitating different values for the constants m and x_0 in equations (19) and (20).