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VARIOUS OPTIMISATION METHODS FOR PRELIMINARY COST AND MASS DISTRIBUTION ASSESSMENT FOR MULTISTAGE ROCKET VEHICLES.

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Abstract:

Preliminary staging studies of multi-stage space launchers are described. Both propellant and thrust scaling factors are used, and also cost parameters for propellant, propellant scaling units and thrust scaling units. The general minimum cost study, including reusable stage(s), turns out to be a generalisation of the optimisation method of the payload ratio, for specified mission requirement. Both studies lead to a velocity increment distribution between the stages, based upon optimum initial acceleration for each stage. The similarity between this analysis and that developed by Vertrégt, Hall and Zambelli, and others is pointed out. Penalties for non-optimum solutions are also considered.

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Nomenclature

a	=	vehicle absolute acceleration, m/s^2.
c	=	effective exhaust velocity, m/s.
C	=	cost criterion $(\lambda_P + \lambda_T T)/\lambda_E$
$\bar{D}$	=	mean drag over flight N.
E	=	engine scaling constant $M_E/\dot{m}_P$ seconds.
$\bar{F}$	=	mean thrust over flight N.
G	=	mean effective component of gravitational field opposing vehicle acceleration, m/s^2.
g	=	local gravitational acceleration.
L	=	stage payload mass ratio M_L/M_α
M	=	mass, Kgm.
$\dot{m}$	=	dM/dt, Kgm/s.
P	=	stage propellant mass ratio M_P/M_α
R	=	reactor scaling constant $M_R/\frac{1}{2}\dot{m}_P c^2$, s^3/m^2
T	=	"tank" scaling constant M_T/M_P
v	=	velocity, m/s.
Δv	=	achieved velocity increment $v_\omega - v_\alpha$, m/s.
X	=	velocity loss factor due to drag $(1 - \bar{D}/\bar{F})$
λ	=	stage specific cost per unit mass of payload, £/Kgm.
λ_E	=	specific cost per unit mass of engine, £/Kgm.
λ_{ER}	=	specific cost per unit mass of reactor + engine, £/Kgm.
λ_P	=	specific cost, propellant, £/Kgm.
λ_T	=	specific cost, propellant container, £/Kgm.

Suffices

E	=	engine or solid propellant nozzle
L	=	stage payload
P	=	propellant
T	=	tank or solid propellant case
R	=	reactor
α	=	stage state at launch
ω	=	stage state at all burnt

Specific Cost Standard Exchange Rate (Rounded Values)

£1/Kgm. = \$ 1.25/lbm = 9 shilling/lbm.

\$ 1/lbm = £0.8/lbm.

1. The Criterion of Maximum Payload Ratio.

1.1 Introduction.

Up to 1962, published information on staging optimization of multistaged rocket vehicles has been based on the criterion of maximum overall payload ratio, L_o, for a given overall velocity increment, V_o, or vice versa. By assuming flight in a drag-free, gravitationless environment, the problem may be reduced to closed form formulas, and the ordinary theory of maxima and minima can be applied. Malina and Summerfield started this kind of work. Vertregt continued by defining three stage ratios which he considered sufficient for all basic calculations, the structure ratio, the payload ratio and the mass ratio. All these ratios can be defined in such a way that they are greater or smaller than one: this is only a matter of choice. Several authors have brought refinements to the method. Bell (ref.1) gives a good survey of their efforts and improvements and a good list of references to the subject. In more recent publications, Froehlich (ref.2) brings up to date the original paper by Malina and Summerfield. Coleman (ref.3) introduces a more complete representation for the variation of structural mass with stage size.

1.2 A single structural factor.

The optimization method based on a single structural factor seems to have reached its useful development with the comprehensive publication by Hall and Zambelli (ref.4). It should be mentioned that the gravity losses are taken into account in a more realistic way in the paper written by Weisbord (ref.5). He introduces the acceleration into the optimization criterion, but he prefixes its value. It is so treated in several papers reviewed by Bell. These authors accept that a part of the total structure scales proportionally with the thrust. A scaling constant proportional to thrust is used by Cohen (ref.6). Nevertheless no attempt is made to find an optimum thrust level: the latter is also predetermined before starting the optimization study, which results in the calculation of a set of mass ratios, called optimum when they make the overall payload ratio maximum.

In these procedures the stage initial mass is divided into the mass of the stage payload, which includes upper stages if any, M_L, the mass of the to be burnt propellants, M_P, and the mass of the wet empty stage, M_e.

$$M_\alpha = M_L + M_P + M_e .$$

The mass of the empty stage is assumed to scale proportionally to the initial mass less the mass of the payload or proportionally to the to be burnt propellants.

$$M_e = \beta(M_\alpha - M_L) = \frac{\beta}{1-\beta} M_P = \sigma M_P$$

The stage mass ratio is defined as the ratio of initial to burnout mass of the stage.

$$\mu = \frac{M_\alpha}{M_\alpha - M_P} = \frac{1}{1-P}$$

Its optimum value is for each stage a function of the effective exhaust velocity and the structural factor of that stage and of a constant "multiplier". (Appendix I)

$$\mu_i = \frac{1 + Ic_i}{Ic_i \beta_i}$$

I is a constant for all stages, whose value determines the overall velocity increment. It has the dimensions of $[vel]^{-1}$.

1.3 Two scaling constants.

In reference 7, Carton and the present author have divided the mass of the structure into two parts. The first scales in proportion to the mass of the propellants, and the second to the mass flow rate of propellants.

$$M_\alpha = M_L + M_P + M_T + M_E$$

The assumed scaling laws are

$$M_T = T\, M_P \quad \text{and} \quad M_E = E \frac{F}{c}$$

It follows that the structural factor will vary with stage size and thrust level.

$$\beta = \frac{1 - L - P}{1 - L} = \frac{TP + {}^{Ea_\alpha}/_c}{(1+T)P + Ea_\alpha/_c}$$

At a first sight, there seems to be little advantage in considering the mass of the engines separately, because they represent only a few percents of the stage mass. However, the advantage of this can be appreciated when we apply the new mass distribution to the optimization procedure. As explained in Appendix 2, an optimum engine size can now be determined for each stage by choosing the optimum initial acceleration. The maximum overall payload ratio will be reached for a set of optimum stage velocity increments distributed between the stages, for each of which the optimum thrust level and propellant fraction are calculated.

The proposed method may be shown to be a generalisation of the method based on a single structural factor and can be easily applied by those who are used to that method.

This point is developed in Appendix 3.

Analysis of non-optimum multistage vehicles has not been discussed elsewhere. This is necessary in order to appreciate the penalty incurred when departing from the best solution for some reason other than the optimization criterion. An example of this is presented in figure one. The scaling constants used are listed in Table 1. They are conservative.

Stage	c (m/sec)	T	E (sec)	G (m/sec²)	$\frac{D}{F}$	λ_E £/kgm	λ_T £/kgm	λ_P £/kgm
1	2500	0.02	8	8	0.5	100	10	0.1
2	2500	0.02	6	6	0	100	10	0.1
3	2800	0.10	4	2	0	100	10	0.2

Table 1.

For the purpose of this paper they represent sufficiently realistic values for present day pump-fed first and second stages and a pressure fed third stage.

In figure 1, non optimum velocity distribution between the stages is considered, but for each point, the optimum initial acceleration and propellant fraction have been computed. It can be seen that it is possible to depart to a certain extent from the optimum velocity increments: when changing the velocity increment of a stage by 10%, there will be a penalty of less than 2% in overall payload ratio. When we distribute the overall velocity increment equally between the stages (point B in figure 1), we have a loss in payload ratio of 4.5%.

The envelope curve (1) of figure 1 has been computed and is presented in figure 2, together with the envelope curve (2) obtained when systematically keeping the initial acceleration constant in the first stage. In the region of the optimum solution (point A in figure 1) the initial accelerations are of the order of 2.5 g_o for the first stage, 2.1 g_o for the second stage and 1.8 g_o for the third one.

A non optimum acceleration has been chosen for the first stage equal to 1.3 g_o and the results of this choice can be appreciated in figure 2. First the velocity distribution between the stages is affected, but by less than 10%. [The number after the letter C in the graph (maximum payload ratio for $a_{\alpha_1} = 1.3\ g_o$) indicates the third stage velocity increment.] Secondly, there is a loss of 18% in payload ratio.

To complete this analysis, the stage characteristics have been increased by 10% in order to determine the sensitivity of the results to these changes. These sensitivities, expressed as a percentage of the optimum payload ratio (point A), are presented in table 2, as a function of the characteristic which has been changed, and of the stage in which the change has occurred.

Characteristic	Stage 1	Stage 2	Stage 3
C	+ 20	+ 29	+ 26.5
T	-0.65	-1.05	-2.10
E	-2.0	-3.05	-1.50

Table 2.

An increase of 10 percent by the engine scaling constant in a stage reduces the stage optimum initial acceleration by 5%. The variation of the tank and engine scaling constants did not change the pattern of the distribution of the overall velocity increment seriously. The increase of the effective exhaust velocity in a stage however completely changed this pattern, increasing the velocity increment contributed by the stage.

In order to appreciate the influence of G, the mean value of gravity opposing the thrust a new three stage vehicle has been taken, the characteristics of each stage being identical to the previous second stage, G being kept as the only variable between the stages. The optimum velocity distribution (1940, 2590, 4290) shows that this factor has an important influence. This points out the need of a careful assessment for this variable.

2. Other criteria.

So far this analysis leads to a minimum starting mass launcher for specified payload mass, and velocity increment. In payload ratio optimised stages initial accelerations tend to be high. Engines will therefore tend to be - comparatively - a large part of each stage. Sometimes in order to make best use of an engine which is immediately available, an investigation may be made to maximise the payload to engine mass ratio for given specifications. Engines used thus tend to be quite a small part of a stage. Low optimum initial accelerations are obtained, sometimes lower than 1.2 g_o for liquid propellant first stages.

It is also shown in Appendix 4, that if the price of propellants is the deciding factor (this seems to be improbable and merely of academical interest), the maximisation of the payload to propellant mass ratio, for given specification, would save propellant, burning time being reduced.

3. Cost Minimization Criterion; A Generalisation

It appears that Builder (ref.8) was the first to use a cost factor per stage to optimise the relative size of the stages or the velocity distribution between the stages. He used a very simple method of describing the stage cost, and also a very simple method of reducing the cost of a recoverable first stage. Subsequent cost optimisation publications have not improved on this very much.

3.1 Non recoverable stages.

The results of the cost optimisation are more revealing when the costs per unit mass are considered separately for engines, tanks and propellants. For this purpose a cost per unit mass for propellants, λ_P , for tanks, λ_T and for engines λ_E is defined.

The cost per unit mass of payload is a function of these latter cost parameters, of the scaling constants and of the initial acceleration and propellant fraction of the stages. The algebraical treatment is given in Appendix 4. For a first stage, the launching cost per unit mass of payload is written

$$\lambda = \frac{P\,(\lambda_P + \lambda_T T\ ) + \lambda_E \frac{E}{c}\ a_\alpha}{L}$$

In the optimisation analysis, a criterion is developed which has to be satisfied to make the variable λ minimum for given specifications. It is of interest to note that this optimisation criterion contains in itself several particular cases.

If it is assumed that none of the parts of the stage has a predominant cost, the engines, tanks and propellant cost factors can be set equal to one and the physical meaning of this is that the maximum payload ratio is derived.

$$\lambda_E = \lambda_T = \lambda_P = 1$$

$$\lambda = \frac{1 - L}{L} = \frac{1}{L} - 1$$

Assume that the cost of the engines is orders of magnitude larger than the cost of the rest of the parts of the stage. Set that cost equal to one and the tanks and propellants cost factors equal to zero. The maximum of the payload to engine mass ratio is then obtained.

$$\lambda_E = 1\ , \qquad \lambda_T = \lambda_P = 0$$

$$\lambda = \frac{M_E}{M_L}$$

Finally, the maximization of the payload to propellant mass ratio is obtained when the propellant cost factor is equal to one and the tanks and engines cost factor to zero.

$$\lambda_P = 1 \,, \quad \lambda_T = \lambda_E = 0$$

$$\lambda = \frac{M_P}{M_L} \,.$$

For a specified stage velocity increment, and scaling constants, the greater the engine cost factor λ_E compared with the tank or propellant cost, the lower the optimum initial acceleration.

The lower $C = \dfrac{\lambda_P + T\,\lambda_T}{\lambda_E}$, the lower a_α.

In use, it may be necessary to keep the first stage initial acceleration above a certain value. When the $\dfrac{\Delta v}{c}$ and C are low, the minimum cost optimised value may be less than 1.2 g_o.

It is noticed that for stages having identical propellants and cost factors a very nearly equal distribution of velocity results in minimum cost.

3.2 Recoverable stage(s).

For recoverable boosters, the scaling constants for engines and tanks will be increased by a certain percentage to account for the added parts and improved structural strength, while the cost factors will have to be modified. The engine and tank cost factors of a non recoverable stage may be multiplied by a function of the number of usages, Nu, and of a refurbishing cost $\Delta\lambda$ expressed as a fraction of the new item cost. For instance,

$$\lambda_{E\,r} = \lambda_E \times \frac{1 + \Delta\lambda_E\,(Nu - 1)}{Nu}$$

In figure 2, point R shows the optimum solution obtained for a vehicle with a recoverable first stage, the characteristics of which are as listed in table 3. The upper stages characteristics are as listed in table 1.

stage	c	T	E	G	$\frac{D}{F}$	λ_E	λ_T	λ_P	Nu
1	2500	0.024	8.4	8	0.05	14.5	1.45	0.1	10

Table 3.

The solution represented by point R , shows a tendency of reducing the first stage importance because the stage is structurally heavier. There is an overall payload penalty of 2.4%: this can be expected when using table 2 and knowing that the tanks scaling factor has been increased by 20% and the engines scaling factor by 5%.

The trends however are completely different when the vehicle is reconsidered with a minimum cost target. In figure 3 , a plot of the launching cost per unit mass of payload with respect to the first stage velocity increment for constant values of the third stage velocity increment is presented. It can be seen that there is a clear tendency to let the first reusable stage carry the largest part of the burden. The minimum launching cost per unit mass of payload of the vehicle with reusable first stage is nearly four times less than that of the vehicle with non reusable stages. The optimum cost solution with a reusable first stage involves a very large penalty in overall mass ratio, of the order of 40% with respect to point R. Because this may seem large as compared to the results of figure 1, it is to be remembered that the overall payload ratio curves in figure 3 are drawn for initial accelerations which are optimum with respect to the cost minimization criterion, i.e. which are non optimum with respect to the payload ratio criterion.

4. Conclusions

Different kinds of staging criteria have been shown, and the possibility of arriving at diverging conclusions has been made clear. It is very important therefore to know precisely which kind of criterion is being considered when discussing the subject. It is felt that the optimisation of the payload ratio now has only a limited area of application. Future optimisation of new launchers will most certainly use the cost optimisation as the more valid approach. This seems to be without argument for recoverable stages.

Nevertheless, the maximum payload ratio criterion can still be useful. It is used in the optimisation studies of the upper parts of the ELDO vehicle. It is interesting to note that ELDO stage optimisation studies use very elaborate trajectory descriptions. The stage is described using very similar scaling constants as described in this paper. The closed forms discussed here can contribute most useful insight at the beginning of performance analysis.

The method is certainly useful first to appreciate trends both in cost and payload ratio optimization studies, and also penalties in non optimum analysis.

APPENDICES

1. Maximum Payload Ratio : Single Structural Factor

A given mission, or specification is defined here by a characteristic velocity or overall velocity increment, V_0.

$$\phi = V_0 - \sum_{i=1}^{N} \Delta v_i = V_0 - \sum_{i=1}^{N}\left[-c_i\left(1-\frac{D_i}{F_i}\right)ln(1-P_i) - \frac{G_i P_i c_i}{a_{\alpha i}}\right] = 0 \qquad (1)$$

The variables c, effective exhaust velocity, G gravity acceleration opposing the thrust, $X = \frac{D}{F}$, drag to thrust ratio have constant mean values for each stage.

The mass of a stage is divided into payload, structure and propellant mass:

$$M_\alpha = M_L + M_S + M_P \qquad (2)$$

The structural factor is defined as:

$$\beta = \frac{M_S}{M_S + M_P}$$

More exactly we must consider the wet structure or empty stage less payload, M_e ; M_P is the mass of the propellants which will effectively be burnt.

Dividing both sides of (2) by M_α, and rearranging we have

$$L = 1 - P\left(1 + \frac{\beta}{1-\beta}\right) = 1 - \left(1 + \frac{\beta}{1-\beta}\right)\left(1 - \frac{1}{\mu}\right) \qquad (3)$$

where $P = \frac{M_P}{M_\alpha}$ and $\mu = \frac{M_\alpha}{M_\omega}$.

For a multi-staged rocket vehicle, we have

$$L_0 = \prod_{i=1}^{N} L_i \quad \text{or} \quad ln\, L_0 = \sum_{i=1}^{N} ln\, L_i \qquad (4)$$

L_0 or alternatively $ln\, L_0$ is the variable to be maximised for a specified V_0. Using the Lagrange multipliers method, the variation of the variable to be optimised is set to zero, taking (1) as a constraint.

$$\delta(ln\, L_0 + \lambda\, \phi) = 0 \qquad (5)$$

Hall and Zambelli considered the following equation as the constraint:

$$\phi' = V_0' + \sum_{i=1}^{N} c_i \ln (1 - P_i) = 0 \qquad (1')$$

The gravity and drag losses were included into the specification.

Solving equation (5) with (1′) as the constraint, results in

$$\mu_i = \frac{1 + Ic_i}{Ic_i \beta_i} \qquad (6)$$

where $I = -\lambda$ is a constant, the value of which determines V_0. The mass ratios can be determined from (6) for chosen values of I. V_0 can be plotted versus I and the correct value of I interpolated.

When the initial accelerations are not free variables, and when drag and gravity losses are taken into account as stated above in equation (1), equation (5) can be written as:-

$$\frac{1 + \sigma_i}{L_i} = \lambda \left[\frac{c_i (1 - X_i)}{1 - P_i} - \frac{G_i c_i}{a\alpha_i} \right] \qquad (6')$$

or in terms of μ_i and β_i,

$$\mu_i = (1 - \mu_i \beta_i) \lambda \left[c_i (1 - X_i) \mu_i - \frac{G_i c_i}{a\alpha_i} \right] \qquad (6'')$$

For each stage the solution of this equation gives the mass ratio corresponding to a constant λ, and in turn the overall velocity increment corresponding to λ can be computed.

When the initial accelerations are considered as free variables, the condition contained in the equation (5) can be expressed by (6′) and by:

$$\frac{\lambda G_i P_i c_i}{a^2 \alpha_i} = 0 \qquad (7)$$

This means that the initial accelerations must be infinite. Analytically, there is no restriction for this, and we arrive at equation (6) which gives the condition for optimum set of stage impulses. It is well known however that this has little practical meaning.

2. Maximum Payload Ratio : Two Scaling Constants

The constraining equation (1) does not change. The stage mass is written as

$$M_\alpha = M_L + M_P + M_T + M_E \tag{8}$$

The following scaling constants are defined,

$$T = \frac{M_T}{M_P} \quad \text{and} \quad E = \frac{M_E}{\dot{m}} = \frac{c\, M_E}{F}$$

Taking these into account, equation (8) may be written for the ith stage as:

$$L_i = 1 - (1 + T_i)\, P_i - \frac{E_i}{c_i}\, a_{\alpha_i} \tag{9}$$

Equation (5) expresses the condition for the maximisation of the overall payload ratio for specified V_0, or for the maximisation of V_0 for given L_0. Propellant fractions and initial accelerations being considered as free variables, equation (5) may be written as :

$$I = -\lambda = \frac{\partial \ln L_0 / \partial P_i}{\partial \phi / \partial P_i} = \frac{\partial \ln L_0 / \partial a_{\alpha_i}}{\partial \phi / \partial a_{\alpha_i}} \tag{5'}$$

This leads to N equations of the form

$$I = \frac{(1 + T_i)}{c_i\, L_i \left[\dfrac{1 - X_i}{1 - P_i} - \dfrac{G_i}{a_{\alpha_i}} \right]} \tag{10}$$

and to N equations of the form

$$I = \frac{E_i\, a^2_{\alpha_i}}{c_i^2\, L_i\, G_i\, P_i} \tag{11}$$

For each constant value I, a_{α_i} and P_i can be calculated for each stage from equations (10) and (11) and in turn the corresponding V_0. The value of I is adjusted by a trial and error method.

3. Similarity : Appendices 1 and 2.

Having developed in Appendix 2 a more sophisticated and more complete staging optimisation, it is normal to check to what extent it can be identified with the result in appendix 1.

From equation (11) and for an arbitrary stage, we may write:

$$\frac{a^2\alpha}{G^2} = \frac{I\ c\ L\ P}{\frac{EG}{c}} \quad \text{or} \quad \frac{a_\alpha}{G} = \frac{I\ c\ L\ P}{1-P(1+T) - L} \qquad (12)$$

From (10)

$$\frac{G}{a_\alpha} = \frac{1}{1 - P} - \frac{1 + T}{I\ c\ L} \qquad (13)$$

Eliminating $\frac{a_\alpha}{G}$ between (11) and (13) we have

$$\frac{I\ c\ L\ P}{1-P(1+T) - L} = \frac{1}{\frac{1}{1-P} - \frac{(1+T)}{I\ c\ L}}$$

$$\text{or} \quad I\ c\ L\ P = (1-P)(1-L)$$

$$\text{or} \quad P = \frac{1}{1 + \frac{I\ c\ L}{1-L}} \qquad (14)$$

Expressing this in terms of mass ratio leads to:-

$$\mu = \frac{1}{1-P} = \frac{Ic - 1}{Ic\ \frac{1-L-P}{1-L}} = \frac{Ic - 1}{Ic\ \beta}$$

Finally, if we set $I_1 = -I$

$$\mu_i = \frac{1 + I_1\ c_i}{I_1\ c_i\ \beta_i} \qquad (15)$$

Which is the same as that found in appendix 1.

It is pointed out, however, that equation (15) is only valid when the engine has an optimum size of the method of appendix 2. When non-optimum initial accelerations have been predetermined, equation (10) applies as the staging criterion, and the only variables left for optimisation are the propellant fractions or the stage mass ratios.

4. Minimum Cost

For the same constraint V_0, the launching cost per unit mass of payload is to be minimised.

For a first stage, the launching cost is

$$M_I \lambda = M_P \lambda_P + M_T \lambda_T + M_E \lambda_E \qquad (16)$$

where λ's are costs per unit mass.

Using the scaling constants defined in appendix 2, equation (16) may be written as

$$\lambda = \frac{P(\lambda_P + \lambda_T T) + \frac{E}{c} a_\alpha \lambda_E}{1 - P(1+T) - \frac{E}{c} a_\alpha} \qquad (17)$$

The minimum cost per unit mass of payload will be found when the following condition is satisfied:

$$\delta(\lambda + \theta\phi) = 0 \qquad (18)$$

where θ is Lagrange multiplier.

When P's and a_α's are left free, their optimum values are found as a function of the parameter θ, by means of the following 2N equations:

$$-\theta = \frac{\partial\lambda/\partial P_i}{\partial\phi/\partial P_i} = \frac{\partial\lambda/\partial a_{\alpha i}}{\partial\phi/\partial a_{\alpha i}} \qquad (19), (20)$$

The staging procedure is similar to that described in appendices 1 and 2, but algebraically more complicated. A stage "interaction" makes impossible the presentation of a general equation linking together the characteristics and variables of a stage only. By "interaction", it is meant that the optimum initial acceleration and propellant fraction of a stage depend on the parameters of the lower stages. Nevertheless, it is such a process which has been followed to obtain the results presented in figure 3.

As an example, the case of a single or first stage will be discussed. Eliminating the multiplier θ between equations (19) and (20) gives an equation of the second degree in a_α, the solution of which gives the optimum initial acceleration a'_α for given propellant fraction.

$$\left(\frac{a'_\alpha}{G}\right)^2 \frac{(1-X)}{(1-P)} \left\{ 1 + P\left[\frac{\lambda_P + T\lambda_T}{\lambda_E} - (1+T)\right]\right\} - \frac{a'_\alpha}{G} - \frac{cP}{EG}\,\frac{\lambda_P + T\lambda_T}{\lambda_E} = 0 \qquad (21)$$

When the optimum a_α corresponding to a chosen P has been computed, the velocity increment can be calculated and P adjusted by trial and error until the specified velocity increment is obtained.

Setting $\lambda_E = \lambda_T = \lambda_P = 1$ results in an optimum initial acceleration a''_α which makes maximum the payload to initial mass ratio.

$$\frac{a''_\alpha}{G} = \frac{P}{\left(\frac{K^2}{4} + \frac{KP}{1-P}\right)^{\frac{1}{2}} - \frac{K}{2}} \qquad (22)$$

where $K = \frac{EG}{c(1+T)}$

Setting $\lambda_E = 1$ and $\lambda_T = \lambda_P = 0$ results in an optimum initial acceleration a^*_α which makes maximum the payload to engine mass ratio.

$$\frac{a^*_\alpha}{G} = \frac{1-P}{1-P(1+T)} \qquad (23)$$

Setting $\lambda_P = 1$ and $\lambda_E = \lambda_T = 0$ results in an optimum initial acceleration a^{**}_α, which makes maximum the payload to propellant mass ratio.

$$\frac{a^{**}_\alpha}{G} = \left[\frac{(1-P)\ c}{EG}\right]^{\frac{1}{2}} \qquad (24)$$

These last formulae were first derived separately in reference 9.

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FIG. 1

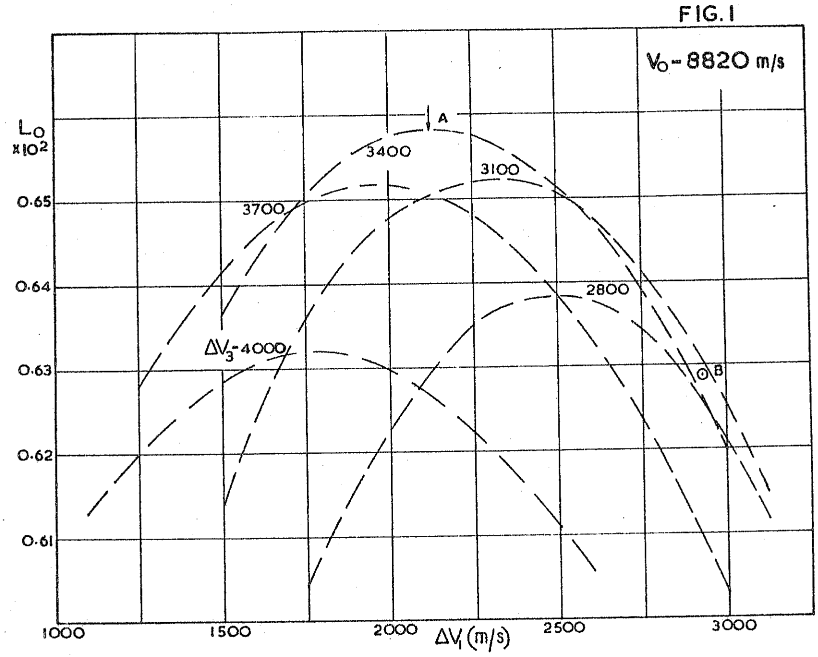

FIG. 2

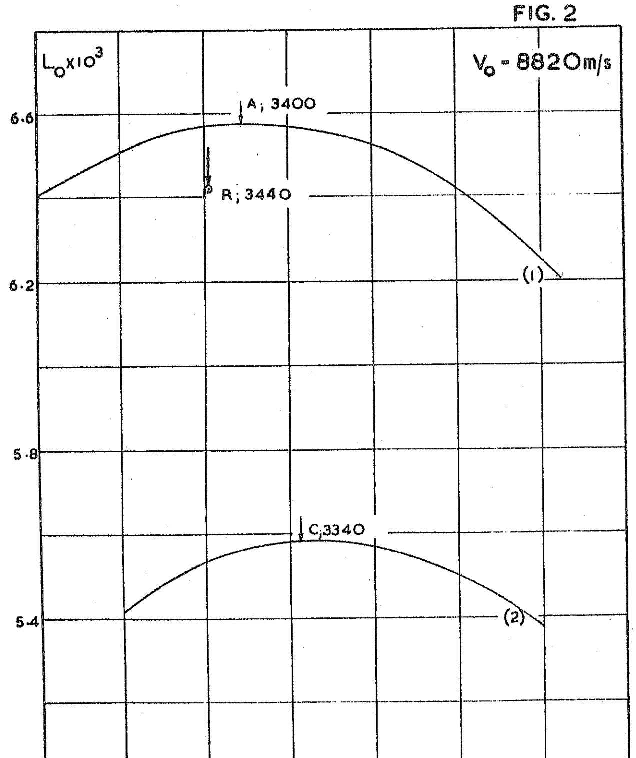

FIG. 3