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Analyzing region of attraction of load balancing on complex network

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Many complex engineering systems network together functional elements to balance demand spikes, but suffer from stability issues due to cascades. The research challenge is to prove the stability conditions for any arbitrarily large and dynamic network topology with any complex balancing function. Most current analyses linearize the system around fixed equilibrium solutions. This approach is insufficient for dynamic networks with multiple equilibria, e.g., with different initial conditions or perturbations. Region of Attraction (ROA) estimation is needed in order to ensure that the desirable equilibria is reached. This is challenging because a networked system of nonlinear dynamics requires compression to obtain a tractable ROA analysis. Here, we employ a master stability inspired method to reveal that the extreme eigenvalues of the Laplacian are explicitly linked to the ROA. This novel relationship between the ROA and the largest eigenvalue in turn provides a pathway to augmenting the network structure to improve stability. We demonstrate using a case study on how the network with multiple equilibria can be optimized to ensure stability.

Keywords: Complex Network; Region of Attraction; Load Balancing; Stability.

2000 Math Subject Classification: 34K30, 35K57, 35Q80, 92D25

1. Introduction

Engineering complex systems often comprise of a lot of functional nodes (e.g., transceivers, pumps, capacitors, amplifiers), connected via edges (e.g., data links, pipes, electric lines, optic fibres). Together, they serve multiple end-users' demand (e.g., multiple access communication, safe drinking water, and synchronized electric power). There is often a local demand surge, which can cause one node to be overwhelmed by demand and load balancing to nearby nodes is required. For example, in wireless communication networks, load balancing between adjacent base stations (BSs) [1], [2] is possible either by handing off edge users or coverage expansion [3].

1.1 *Review on the Stability of Complex Networked Systems*

Cascade load balancing arises when a node passes load to a neighbour which in turn becomes overloaded and need to pass some of its load to the next neighbour, etc. This does not necessarily reduce overall network outage and can lead to significant inefficiencies due to information passing. Against this background, endless load balancing should be avoided in complex system.

1.1.1 *Review on Near Equilibrium Methods*

Recent breakthroughs have developed the framework to analyze the relationship between microscopic (like local component dynamics), macroscopic (global dynamics) behaviors and network topology [4–7]. Our previous work in [1] proved that provided 3 conditions are met: any networked system of arbitrarily large size, topological complexity, and balancing function form is stable. The conditions are: (1) the network topology is static, (2) the state of the system is not far from equilibrium, and (3) knowledge of network parameters has a smaller variance than any noise in the system.

1.1.2 *Review on Region of Attraction for Non-Equilibrium Problems* Now, if we consider that perturbations occur in the networked system such as the variation of network topology (e.g., a UAV communication network [8]) and the surge of node demand (e.g., increase of users' traffic demand in wireless network), clearly conditions (1) and (2) are violated. So, the research question is to what extent load changes or network topology changes will break the stability of load balancing? Estimating the region of attraction (ROA) is needed when dealing with this problem. It is advisable to maximize the ROA for the desirable equilibrium state, since the larger the ROA the easilier it is to reach the desirable equilibrium state when purbations arise. In other words, the larger the ROA, the more robust the system [9]. The ROA is the set of initial states from which the system converges to the equilibrium point [10]. Studies on ROA of load balancing can identify how to maintain or control the stability of the networked system. There are some studies on the stability of load balancing in a networked system, e.g, [1, 11] study the stability of load balancing in wireless network. Yet, to the best of our knowledge, little attention has been paid to the estimation of ROA for load balancing of a networked system before, especially to analyze the relationship between network topology and ROA in a large-scale networked system.

The problem of estimating ROA has attracted a lot of attention in various fields, such as mechanical systems, biological systems, power systems, etc. [12–14]. The challenge is that finding the exact ROA of a nonlinear system is a very difficult problem, because of the complex dynamics of nonlinear systems [9]. Many approaches have been proposed to solve this problem and these approaches can be generally divided into two types: methods based on Lyapunov function (LF) and non Lyapunov methods. These methods can offer sufficient conditions to verify the stability of ordinary differential equations (ODEs). Maximal LF method [15] [16] can find the exact stability region but these methods are generally difficult to employ. To improve the optimization efficiency, sum of squares (SOS) optimization method [17] and linear matrix inequalities (LMI) [18][19] have been proposed. All of these methods are based on the LF method and the common limitations of LF methods limit their applicability in practice. First, these methods rely on the exisitence of suitable LF which is extremely difficult to find for nonlinear system [20]. Second, the LF methods usually lead to conservative results [21]. Another direction is non Lyapunov methods. These methods mainly include Level Set methods [20] [21], trajectory reversing methods [22] and occupation measures [23]. The non Lyapunov methods can improve the conservative problem in some degree and provide accuracy estimation. The limitation is that the computational requirements grow rapidly with the system dimension, which makes these methods appropriate in low-dimensional systems. Methods to analyze stability of nonlinear dynamics are shown in Table (1).

In a large-scale networked system, a large number of nodes couple together and affect each others' dynamics, which is typically a high-dimensional nonlinear complex system. Although the above methods have been successfully proposed to estimate ROA, it is impossible to directly use these methods to estimate the ROA of such a high-dimensional networked system with nonlinear dynamics. Therefore, to solve this problem, we employ a decomposition technique proposed by [24] inspired by the master stability function [25] method to reduce dimensionality. By doing this, we can use LF methods to estimate ROA of the system. In addition, we know that dynamics and network topology are both important to the system's behavior. Our analysis indicates that there exists a negative relationship between network topology (the extreme eigenvalue of the network) and the estimated ROA of the networked system. This provides us with an effective way to control and optimize the network topology to enlarge the ROA with

Table 1. Methods to analyse stability of nonlinear dynamic systems

	Methods	Advantage	Limitations
Near equilibrium problems	Linearize the system near equilibrium	It is easy to employ this method to analyse the stability of the equilibrium.	Can not identify to what extent perturbation the system can resist for nonlinear dynamics system
Region of attraction for non-equilibrium problem	Lyapunov functions (LF) based method: maximal LF method; linear matrix inequalities; sum of square programming	Offer sufficient conditions for verifying stability of ordinary differential equations.	Rely on the existence of suitable LF which is extremely difficult to find for nonlinear systems especially in high dimensional system; The LF methods usually lead to conservative results.
	Non LF methods: level set methods; trajectory reversing methods; occupation measures	Improve the conservative problem in some degree and provide accurate estimation	The computational requirements grow rapidly with the system dimension, which makes these methods appropriate in low-dimensional system.

determinated dynamic functions.

1.2 *Novelty & Organization*

In this paper, we propose an analysis framework to estimate the ROA for load balancing of the networked system. This informs us how to control the system to maintain in the ideal equilibrium state, which makes the system more efficient. According to our analysis, we find that there exists a negative relationship between the estimated ROA and the extreme eigenvalue of the network, which enables us to optimize the stability from the perspective of network topology by enlarging the desirable ROA. This will bridge the gap between graph theory and ROA of networked systems. The key detailed contributions are as follows: (1) we employ a master stability inspired method to compress a high-dimensional networked system with nonlinear dynamics for tractable ROA analysis, (2) we explicitly link the ROA and the extreme eigenvalue in a relationship that enables us to identify how network rewiring improves the desirable ROA, and (3) we use a case study to demonstrate how we can improve the stability of a load balancing networked system.

The remainder of this paper is organised as follows. Section II builds the dynamics model for load balancing in complex networked system. In SectionIII, we present the analytical framework to estimate the ROA of this system and analyze the relationship between the network topology and the ROA. In Section IV, we show the simulation results of our method. Finally, the conclusion of our work is drawn and the future direction is foreseen in Section V.

2. System Setup

In this section, we will build the dynamics model for load balancing of networked system. Symbols used in this paper are shown in Table (2). We assume that there are N nodes in the network and two time scales: long-term demand variations (demand variation time scale T) and short-term load dynamics

Table 2. List of symbols used in this paper

Symbol	**Describtion**
l_i	load of node i
$c_i(t)$	capacity of node i
$d_i(T)$	demand of node i
$f(\cdot)$	self-dynamics function
$g(\cdot)$	inner coupling function
c	coupling strength
Λ	Laplacian matrix of the network
λ_i	eigenvalue of Λ
$\mathbf{D}$	degree matrix
$\mathbf{A}$	adjacent matrix
$\mathbf{A}_{ij}$	element of matrix A
η_i	the variation on node i
k_i	degree of node i
$\mathbf{J}(\cdot)$	the Jacobian matrix
s	equilirium of function $f(\cdot)$
$\mathbf{s}$	set of s
$\mathbf{I}_N$	the identity matrix of dimension N
$\|\cdot\|$	2-norm of a vector or matrix
superscrit $\top$	transpose of a vector or matrix
$O(\cdot)$	big O notation bounding asymptotic behavior

under some constant demand (symbol period time scale t). In this paper, the short-term load dynamics is mainly concerned. Each node i has a load defined by $l_i(t) = d_i(T)/c_i(t)$, which is the ratio between the quasi-static long-term demand aggregated across all users u in node i, $d_i(T) = \sum_u d_{i,u}(T)$, and the node aggregated area capacity over all users u in node i, $c_i(t) = \sum_u c_{i,u}(t)$. It is assumed that the capacity of each node is stationary, which means that the capacity will not be dramatically affected by the change of load balancing.

For the single node, the capacity $c_i(t)$ reacts to the demand $d_i(T)$ (shown in Fig. 1). The self control of each node tends to drive the load to a desirable equilibrium. When the capacity of node i saturates, node i needs to offload to neighbour nodes to avoid outrage. Therefore, there exists coupling dynamics between nodes. Dynamics of the overall networked system is the linear combination of the self-dynamics and coupling dynamics (shown in Fig. 2). They are coupled through the complex network, characterized by the connectivity matrix $\mathbf{A}$. If connection exists between node i and j, $\mathbf{A}_{ij} = 1$. Otherwise, $\mathbf{A}_{ij} = 0$. The networked dynamics model has three main characteristics: (1) The number of nodes in the network is arbitrarily large. (2) The network is arbitrarily complex. (3) All nodes in the networked system are nonlinear dynamics embedded. The dynamics of node i in the network is described by

$$\dot{l}_i = f(l_i) + c\sum_{j=1}^{N} \mathbf{A}_{ji}(g(l_j) - g(l_i)), \tag{2.1}$$

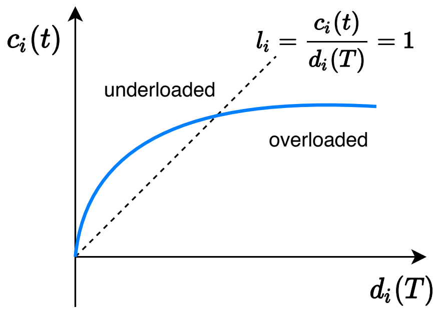


FIG. 1. It shows the capacity $c_i(t)$ reacts to the demand $d_i(T)$. At first, node i is underloaded. With the increase of demand, the capacity of node i increases to meet the demand. When $c_i(t)$ saturate and $l_i > 1$, node i needs to offload traffic to neighbour nodes to avoid outrage.

c is the coupling strength and $c > 0$.

Firstly, let us consider a simple and ideal linear system. The load dynamics of node i can be described as

$$f(l_i) = \beta(1 - l_i), \tag{2.2}$$

where $\beta > 0$. If $\beta < 0$, then node i attempts to continue to attract load demand from other nodes when node i is overloaded. When node i is underloaded, it still offloads to neighbour nodes. This scenario is not common in most systems. As a result, we do not consider this situation in this paper. In this system, $l_i = 1$ is a desirable equilibrium. Therefore, what we are interested in is to estimate the ROA of this equilibrium. We mainly consider two kinds of unstable behavior as follows. First, the introduction of users within the node leads to the variation of its load l_i. Second, the different distribution of nodes causes the change of network topology $\mathbf{A}$.

3. Method to Estimate ROA

In a stationary complex network, we usually linearize the system near the equilibrium to analyze the stability. However, in a dynamic complex network, the change of network topology or perturbation on dynamics will cause the state of node far away from the equilibrium (shown in Fig. 3). As a result, it is not sufficient to analyze the stability of the equilibrium. It is more important to estimate the ROA. Here, we propose an analysis framework to estimate ROA of load balancing in a networked system with dynamics.

3.1 *Stability Analysis for the System*

We consider the general case of equation (2.1) and assume that $f(\cdot)$ and $g(\cdot)$ are twice differentiable functions. $g(x)$ can be written as the Taylor expansion

$$g(x) = \sum_{n=0}^{N} \frac{g^{(n)}(x_0)}{n!}(x - x_0)^n + O[(x - x_0)^n], \tag{3.1}$$

and only consider the first order of $g(x)$. So $g(x) = g^{(0)}(x_0) + g'(x_0)(x - x_0) + O((x - x_0)^2)$, where $g'(x_0) > 0$ and $O(\cdot)$ is the big O notation which bounds asymptotic behavior at $x = x_0$. So $g(l_j) - g(l_i) =$

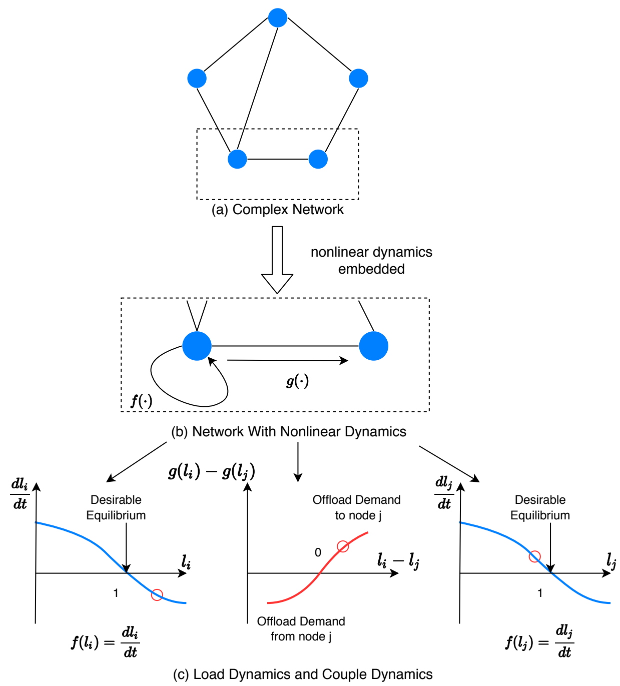


FIG. 2. Complex Network with Nonlinear Dynamics. (a) The network contains a large number of nodes. All nodes are coupled together forming the complex network. (b) shows the coupled dynamics of node i in complex network. The dynamics of node i is a linear combination of self dynamics $f(l_i)$ and coupling dynamics $g(l_i) - g(l_j)$. (c) shows the load dynamics of node i, j $f(l_i), f(l_j)$ and the coupling dynamics $g(l_i) - g(l_j)$. The left one characterizes the self load control of node i. The middle one shows the coupling dynamics between node i and j. The right one shows the dynamics of node j. When node i is overloaded, it will offload to the neighbour node j with light load to avoid outrage.

$g'(x_0)(l_j - l_i)$ If $g'(x_0) < 0$, node i attempts to offload demand to neighbor nodes, even though the load demand of node i is less than other nodes, which happens when node i is not active (e.g. the base station in wireless network is in sleep mode). However, such situation is not considered in this paper based on the assumption that all nodes are in active mode. Accordingly, we do not consider this situation. When we analyze the system's stability around $l_i = l_0$, $\dot{l}_i$ can be characterised as

$$\dot{l}_i = f(l_i) + c \sum_{j=1}^{N} \mathbf{A}_{ji} g'(l_0)(l_j - l_i). \tag{3.2}$$

Then the stability of the equilibrium, $l_i = 1$, can be determined according to the eigenvalue of the Jacobian of the system [1] as

$$\mathbf{J}(l_i = 1) = f'(1)\mathbf{I}_N - cg'(1)\mathbf{D} + cg'(1)\mathbf{A}^T = f'(1)\mathbf{I}_N + cg'(1)\Lambda^\top, \tag{3.3}$$

where $\mathbf{J}$ represents the Jacobian of the system, $\mathbf{I}_N$ is the identity matrix, $\mathbf{D}$ is the in-degree matrix, Λ represents the Laplacian of the graph, and $\Lambda^\top$ denotes the transpose matrix of Λ. $\Lambda = \mathbf{A} - \mathbf{D}$. $\mathbf{D}$ is

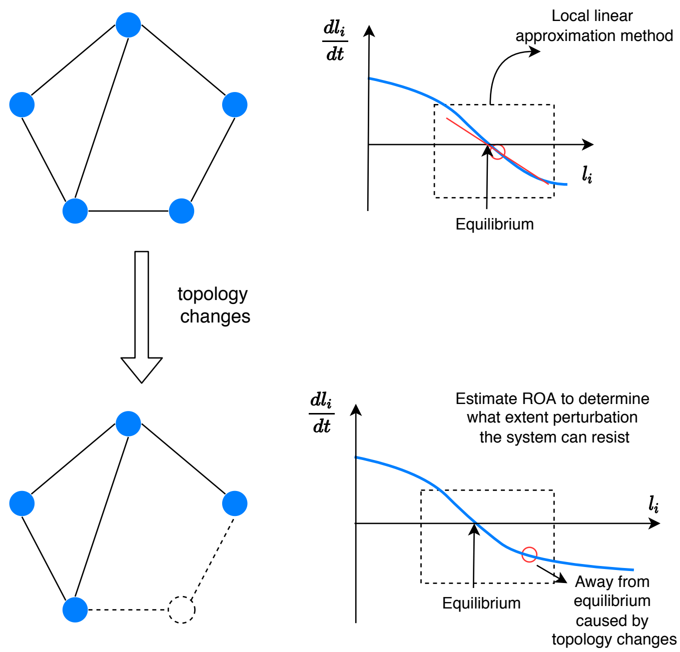


FIG. 3. In stationary complex network, it is usually to linearize the system around the equilibrium. When the network topology changes, the node may move away from the equilibrium. We need to estimate the ROA to know what extent perturbation will break the stability.

defined as

$$\mathbf{D}_{ij} = \begin{cases} \sum_{j=1}^{N} \mathbf{A}_{ji} & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Obviously, $\mathbf{D}$ is a diagonal matrix, and diagonal elements are positive. Then we can get the Laplacian matrix Λ,

$$\Lambda_{ij} = \begin{cases} -\mathbf{D}_{ij} & \text{if} \quad i = j \\ \mathbf{A}_{ij} & \text{otherwise.} \end{cases} \tag{3.5}$$

The diagonal elements of Λ are negative. Let the eigenvalues of Λ be $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$ and for the Laplacian matrix Λ, $\lambda_1 \leqslant \lambda_2 \leqslant, \ldots \leqslant \lambda_N = 0$. As discussed above, if $f'(l_i) > 0$, then node i attempts to continue to attract load demand from other nodes when node i is overloaded. When node i is underloaded, it still offloads to neighbour nodes. This scenario is not common in most systems and we only consider $f'(l_i) < 0$. Let μ_i denote the eigenvalues of $\mathbf{J}(l_i = 1)$. The relationship between μ_i and λ_i is $\mu_i = f'(1) + cg'(1)\lambda_i$. Since $g'(l_0) > 0, \lambda_i \leqslant 0, c > 0$, we have $\mu_i < 0$. According to the Lyapunov indirect method [26], if all real parts of the eigenvalues of the Jacobian matrix of a system around the equilibrium are negative, the equilibrium is asymptotically stable. Therefore, the equilibrium $l_i = 1$ is asymptotically stable.

For this linear system, $l_i = 1$ is the only equilibrium. We can prove it as below:

Proof. Consider the linear system $f(l_i) = \beta(1 - l_i), g(l_i) = \alpha l_i + b$, then

$$\dot{l}_i = \beta(1 - l_i) + c \sum_{j=1}^{N} \mathbf{A}_{ji} \alpha (l_j - l_i). \tag{3.6}$$

The equilibrium of the system is the root of $\dot{l}_i = 0$, which can be converted to solve the linear equations

$$-\beta \mathbf{l} + c\alpha \Lambda \mathbf{l} = -\beta[1, 1, ..., 1]^{\top}, \tag{3.7}$$

where $\mathbf{l} = [l_1, l_2, ..., l_N]^{\top}$. We define matrix $\mathbf{B} = -\beta \mathbf{I}_N + c\alpha\Lambda$. Equation (3.7) can be rewritten as

$$\mathbf{B}\mathbf{l} = -\beta[1, 1, ..., 1]^{\top}. \tag{3.8}$$

Eigenvalues of matrix $\mathbf{B}$ are defined as ξ_i. The relationship between ξ_i and λ_i is $\xi_i = -\beta + c\alpha\lambda_i$. Since $\beta > 0, c > 0, \alpha > 0, \lambda_i \leqslant 0$, we can get $\xi_i < 0$. Therefore, $\det(\mathbf{B}) > 0$. The number of solutions of equation (3.8) is no more than 1. Therefore, $l_i = 1$ is the only equilibrium of the linear system. □

For a linear system, if the only equilibrium is asymptotically stable, then the ROA of this equilibrium is the whole space, which means that for any initial state of $l_i(0)$, $\lim_{t\to\infty} l_i(t) = 1$.

Proof. Let $\hat{l}_i = l_i - 1$. Then the equilibrium of the system is $\hat{l}_i = 0$. Let $\dot{\hat{\mathbf{l}}} = [\dot{\hat{l}}_1, \dot{\hat{l}}_2, ..., \dot{\hat{l}}_N]^{\top}$ and $\hat{\mathbf{l}} = [\hat{l}_1, \hat{l}_2, ..., \hat{l}_N]^{\top}$. The dynamic function of the system is

$$\dot{\hat{\mathbf{l}}} = \mathbf{B}\hat{\mathbf{l}}. \tag{3.9}$$

According to the linear system theory [27], the solution of equation (3.9) for a given initial state $\hat{\mathbf{l}}(0)$ is

$$\hat{\mathbf{l}}(t) = \exp(\mathbf{B}t)\hat{\mathbf{l}}(0). \tag{3.10}$$

If all of eigenvalues of matrix $\mathbf{B}$ have negative real part, the linear system of equation (3.9) is asymptotic stable and the equilibrium is $\hat{l}_i = 0$ [26]. In this situation, for any initial state $\hat{l}_i(0)$, $\lim_{t\to\infty} \hat{l}_i(t) = 0$. □

However, in a high-dimensional nonlinear networked system, usually, there exist several equilibria. If we know how to estimate ROA, we can maintain the system at the ideal equilibrium to keep the system's high efficiency.

3.2 *ROA for Nonlinear System*

Let s be the equilibrium of the self-dynamic function $\dot{l}_i = f(l_i)$. Then $l_1 = l_2 = ... = l_N = s$ is an equilibrium of the system. We set $\mathbf{l}(t) = [l_1(t), l_2(t), ..., l_N(t)]^T \in \mathbb{R}^N$ and $\mathbf{s} = [s, s...s]^T \in \mathbb{R}^N$. $\mathbf{l}(t_0) = [l_1(t_0), l_2(t_0), ..., l_N(t_0)]^T$ is the initial state at $t = 0$. Then the ROA of the equilibrium can be defined as

$$R_f = \left\{ \mathbf{l}(t_0) \in \mathbb{R}^N \,\middle|\, \lim_{t\to\infty} \mathbf{l}(t, \mathbf{l}(t_0)) = \mathbf{s} \right\}. \tag{3.11}$$

Due to the high dimension and nonlinear dynamics of the system, determination of the exact ROA is nearly impossible. Therefore, the purpose of this section is to estimate R_f by finding the subset of R_f as large as possible. The small perturbation δl_i can be obtained by $\delta l_i = l_i - s$. Since s is the equilibrium of the system, $f(s) + c\sum_{j=1}^{N} \Lambda_{ji} g(s) = 0$. Perturbation δl_i of the system (2.1) can be written as [28]

$$\dot{\delta l_i} = f(l_i) - f(s) + c \sum_{j=1}^{N} \Lambda_{ji} (g(l_j) - g(s)), \tag{3.12}$$

Expanding the functions $f(\cdot)$ and $g(\cdot)$ to the first-order Taylor expansion like the equation (3.1), we can get $f(l_i) = f(s) + f'(s)\delta l_i$ and $g(l_i) = g(s) + g'(s)\delta l_i$. Then $\dot{\delta l_i}$ can be transformed to

$$\dot{\delta l_i} = f'(s)\delta l_i + c\sum_{j=1}^{N}\Lambda_{ji}g'(s)\delta l_j + r_i^{(1)} + c\sum_{j=1}^{N} r_j^{(2)}, \tag{3.13}$$

where $r_i^{(1)} = f(l_i) - f(s) - f'(s)\delta l_i$ and $r_i^{(2)} = g(l_i) - g(s) - g'(s)\delta l_i$. Let $\boldsymbol{\delta}\mathbf{l} = [\delta l_1, \delta l_2, ...\delta l_N]^\top, \mathbf{r}^{(k)} = [r_1^{(k)}, r_2^{(k)}, ..., r_N^{(k)}]^\top$. Rewrite equation (3.13) as

$$\dot{\boldsymbol{\delta}\mathbf{l}} = (\mathbf{I}_N f'(s) + c\Lambda g'(s))\boldsymbol{\delta}\mathbf{l} + \mathbf{r}^{(1)} + c\Lambda\mathbf{r}^{(2)}. \tag{3.14}$$

Since Λ is symmetric, $\Lambda = \mathbf{U}^{-1}\mathbf{Q}\mathbf{U}$, where $\mathbf{U}$ is an orthogonal matrix, and $\mathbf{Q} = \mathrm{diag}\{\lambda_1, \lambda_2, ..., \lambda_\mathrm{N}\}$. Inspired by the master stability function, let $\boldsymbol{\eta} = [\eta_1, \eta_2, ..., \eta_N]^\top = \mathbf{U}\boldsymbol{\delta}\mathbf{l}$, $\mathbf{h}^{(k)} = [h_1^{(k)}, h_2^{(k)}, ..., h_N^{(k)}]^\top = \mathbf{U}\mathbf{r}^{(k)}$. Then equation (3.14) can be transformed as

$$\dot{\boldsymbol{\eta}} = (\mathbf{I}_N f'(s) + c\mathbf{Q}g'(s))\boldsymbol{\eta} + \mathbf{h}^{(1)} + c\mathbf{Q}\mathbf{h}^{(2)}, \tag{3.15}$$

which equals to

$$\dot{\eta}_i = (f'(s) + c\lambda_i g'(s))\eta_i + h_i^{(1)} + c\lambda_i h_i^{(2)}. \tag{3.16}$$

In [24] they proved the Lemma :

LEMMA 3.1 Let $x \in \mathbb{R}^n$ and differentiable functions satisfy $a_1\|x\|^2 \leqslant V(x) \leqslant a_2\|x\|^2$, where a_1 and a_2 are positive constants. For any positive constant $\gamma < \gamma_0$, suppose that there exist a positive constant β such that $\dot{V}(x(t)) \leqslant -\beta\|x(t)\|^2 \forall t \in \{t \,|\, \|x(t)\| < \gamma\}$. If $x(t_0) \in \{x \in \mathbb{R}^n \,|\, \|x\| < \sqrt{\frac{a_1}{a_2}}\gamma_0\}$, then $\lim_{t\to\infty}\|x(t)\| = 0$.

Define a Lyapunov function as $V = \boldsymbol{\eta}^\top\mathbf{P}\boldsymbol{\eta}$, where $\mathbf{P} = \mathrm{diag}\{P_1, P_2, ..., P_N\}$ is a positive definite matrix with $P_i \in \mathbb{R}$. Let $P_{\max}, P_{\min}$ be the maximum and minimum values of $\{P_1, P_2, ..., P_N\}$. Then $V = \sum_{i=1}^{N}\eta_i P_i \eta_i$ and we can get

$$\dot{V} = \sum_{i=1}^{N}\dot{\eta}_i P_i \eta_i + \sum_{i=1}^{N}\eta_i P_i \dot{\eta}_i = 2\sum_{i=1}^{N}((f'(s) + c\lambda_i g'(s))\eta_i + h_i^{(1)} + c\lambda_i h_i^{(2)})P_i\eta_i. \tag{3.17}$$

We assume that there exist a positive constant θ which makes

$$P_i(f'(s) + c\lambda_i g'(s)) + \theta/2 \leqslant 0. \tag{3.18}$$

Then, we can get

$$2\sum_{i=1}^{N}(P_i(f'(s) + c\lambda_i g'(s))\eta_i^2 \leqslant -\theta\sum_{i=1}^{N}\eta_i^2 = -\theta\|\boldsymbol{\eta}\|^2. \tag{3.19}$$

Also, we have

$$\begin{aligned} &2\sum_{i=1}^{N}\eta_i P_i h_i^{(1)} + 2\sum_{i=1}^{N}c\lambda_i\eta_i P_i h_i^{(2)} = 2(\boldsymbol{\eta}^\top\mathbf{P}\mathbf{h}^{(1)} + c\boldsymbol{\eta}^\top\mathbf{P}\mathbf{Q}\mathbf{h}^{(2)}) \\ &= 2\boldsymbol{\eta}^\top\mathbf{P}(\mathbf{U}\mathbf{r}^{(1)} + c\mathbf{U}\mathbf{U}^{-1}\mathbf{Q}\mathbf{U}\mathbf{r}^{(2)}) = 2\boldsymbol{\eta}^\top\mathbf{P}\mathbf{U}(\mathbf{r}^{(1)} + c\Lambda\mathbf{r}^{(2)}). \end{aligned} \tag{3.20}$$

$\boldsymbol{\eta}^\top \mathbf{PUr}^{(1)} \leqslant |\boldsymbol{\eta}^\top \mathbf{PU}\,\mathbf{r}^{(1)}|$, $\boldsymbol{\eta}^\top \mathbf{PU}c\Lambda\mathbf{r}^{(1)} \leqslant |\boldsymbol{\eta}^\top \mathbf{PU}c\Lambda\mathbf{r}^{(1)}|$. According to Cauchy–Schwarz inequality $((\sum_{i=1}^n x_i y_i)^2 \leqslant (\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2))$, we can get

$$\begin{aligned}
\boldsymbol{\eta}^\top \mathbf{PUr}^{(1)} &= [P_1\eta_1, P_2\eta_2, ..., P_N\eta_N][\sum_{i=1}^N U_{1i}r_i^{(1)}, \sum_{i=1}^N U_{2i}r_i^{(1)}, ..., \sum_{i=1}^N U_{Ni}r_i^{(1)}]^\top \\
&= \sum_{i=1}^N P_1\eta_1 U_{1i}r_i^{(1)} + \sum_{i=1}^N P_2\eta_2 U_{2i}r_i^{(1)} + ... + \sum_{i=1}^N P_N\eta_N U_{Ni}r_N^{(1)} \\
&\leqslant \sqrt{\sum_{i=1}^N (P_i\eta_i)^2}\sqrt{\sum_{i=1}^N (U_{1i}r_i^{(1)})^2 + \sum_{i=1}^N (U_{2i}r_i^{(1)})^2 + ... + \sum_{i=1}^N (U_{Ni}r_i^{(1)})^2} \\
&\leqslant P_{\max}\sqrt{\sum_{i=1}^N \eta_i^2}\sqrt{\sum_{i=1}^N (U_{1i}r_i^{(1)})^2 + \sum_{i=1}^N (U_{2i}r_i^{(1)})^2 + ... + \sum_{i=1}^N (U_{Ni}r_i^{(1)})^2}
\end{aligned} \tag{3.21}$$

which means that

$$\boldsymbol{\eta}^\top \mathbf{PUr}^{(1)} \leqslant P_{\max}\|\boldsymbol{\eta}^\top\|\|\mathbf{Ur}^{(1)}\| \tag{3.22}$$

Similarly,

$$\boldsymbol{\eta}^\top \mathbf{PU}\Lambda\mathbf{r}^{(2)} \leqslant P_{\max}|\lambda_1|\|\boldsymbol{\eta}^\top\|\|\mathbf{Ur}^{(2)}\| \tag{3.23}$$

Therefore,

$$\dot{V} \leqslant -\theta\|\boldsymbol{\eta}\|^2 + 2P_{\max}\|\boldsymbol{\eta}\|(\|\mathbf{Ur}^{(1)}\| + c\,|\lambda_1|\,\|\mathbf{Ur}^{(2)}\|) \tag{3.24}$$

Since $\|r_i^{(k)}\| = O(\|\delta l_i\|)$, there exist positive constants γ_1, γ_2 that make

$$\begin{aligned}
&\frac{\|r_i^{(1)}\|}{\|\delta l_i\|} < \frac{\theta\rho}{2P_{\max}} \forall 0 < |\delta l_i| < \gamma_1, \\
&\frac{\|r_i^{(2)}\|}{\|\delta l_i\|} < \frac{\theta(1-\rho)}{2cP_{\max}|\lambda_1|} \forall 0 < |\delta l_i| < \gamma_2,
\end{aligned} \tag{3.25}$$

where $\rho \in (0,1)$. Then there exists a positive constant β which satisties $\frac{\|r_i^{(1)}\|}{\|\delta l_i\|} < \frac{(\theta-\beta)\rho}{2P_{\max}}$, $\frac{\|r_i^{(2)}\|}{\|\delta l_i\|} < \frac{(\theta-\beta)(1-\rho)}{2cP_{\max}|\lambda_1|} \forall 0 < |\delta l_i| < \gamma_0$, where $\gamma_0 = \min(\gamma_1, \gamma_2)$. Further, we can infer that

$$\dot{V} \leqslant -\beta\|\boldsymbol{\eta}\|^2 \leqslant -\beta\|\mathbf{U}\|^2\|\boldsymbol{\delta}\mathbf{l}\|^2 = -\beta\|\boldsymbol{\delta}\mathbf{l}\|^2, \tag{3.26}$$

where $\mathbf{U}$ is an orthogonal matrix, so $\|\mathbf{U}\| = 1$. In addition, we can infer that

$$\begin{aligned}
&V = \sum_{i=1}^N \eta_i P_i \eta_i = \sum_{i=1}^N P_i\eta_i^2; \\
&P_{\min}\sum_{i=1}^N \eta_i^2 \leqslant V \leqslant P_{\max}\sum_{i=1}^N \eta_i^2; \\
&\boldsymbol{\eta}^\top P_{\min}\boldsymbol{\eta} \leqslant V \leqslant \boldsymbol{\eta}^\top P_{\max}\boldsymbol{\eta}.
\end{aligned} \tag{3.27}$$

Since $\boldsymbol{\eta} = \mathbf{U}\delta\mathbf{l}$, $\boldsymbol{\eta}^\top P_{\min}\boldsymbol{\eta} = \delta\mathbf{l}^\top\mathbf{U}^\top P_{\min}\mathbf{U}\delta\mathbf{l} = P_{\min}\|\delta\mathbf{l}\|^2$. We can get $P_{\min}\|\delta\mathbf{l}\|^2 \leqslant V \leqslant P_{\max}\|\delta\mathbf{l}\|^2$. Under the condition that $\dot{V} \leqslant -\beta\|\delta\mathbf{l}\|^2 \forall 0 < \|\delta\mathbf{l}\| < \gamma_0$. Accord to 3.1 if

$$\|\delta\mathbf{l}(t_0)\| < \sqrt{\frac{P_{\min}}{P_{\max}}}\gamma_0, \tag{3.28}$$

then $\lim\limits_{t\to\infty}\|\delta\mathbf{l}(t)\| = 0$. Therefore,

$$D_f = \left\{ S+M \middle| \|M\| < \sqrt{\frac{P_{\min}}{P_{\max}}}\gamma_0 \right\} \tag{3.29}$$

is a subset of R_f.

As we analyzed in Part A of section III, $f'(s) < 0, g'(s) > 0$ and $\lambda_1 \leqslant \lambda_2 \leqslant, \ldots \leqslant \lambda_N = 0$. P_i takes the maximum value when $\lambda_i = 0$ and the minimum value when $\lambda_i = \lambda_1$. Therefore,

$$\begin{aligned} P_{\max} &= -\frac{\theta}{2f'(s)}, \\ P_{\min} &= -\frac{\theta}{2(f'(s)+cg'(s)\lambda_1)}, \\ \sqrt{\frac{P_{\min}}{P_{\max}}} &= \sqrt{\frac{f'(s)}{f'(s)+cg'(s)\lambda_1}}. \end{aligned} \tag{3.30}$$

Calculating γ_0 can be formulated as an optimization problem to find the maximum value of γ_0 under the condition of equation (3.25) and (3.30). Then we can get

$$\frac{\|f(s+\delta l_i) - f(s) - f'(s)\delta l_i\|}{\|\delta l_i\|} < -f'(s)\rho \tag{3.31a}$$

$$\frac{\|g(s+\delta l_i) - g(s) - g'(s)\delta l_i\|}{\|\delta l_i\|} < -\frac{f'(s)(1-\rho)}{c|\lambda_1|} \tag{3.31b}$$

$$\forall 0 < \|\delta l_i\| \leqslant \gamma_0; \rho \in (0,1). \tag{3.31c}$$

These equal to

$$\begin{aligned} f'(s)(1+\rho) < \frac{f(s+\delta l_i) - f(s)}{\delta l_i} < f'(s)(1-\rho) \\ \frac{f'(s)(1-\rho)}{c|\lambda_1|} < \frac{g(s+\delta l_i) - g(s)}{\delta l_i} - g'(s) < -\frac{f'(s)(1-\rho)}{c|\lambda_1|} \end{aligned} \tag{3.32}$$

Therefore, we can estimate ROA by calculating $\sqrt{\frac{P_{\min}}{P_{\max}}}$ and γ_0 according to equation (3.30) and (3.32).

3.3 *Relationship between Network Topology and ROA*

If $f(\cdot)$ and $g(\cdot)$ are linear functions and satisfy $f'(\cdot) < 0$ and $g'(\cdot) > 0$, the left-hand side of equation (3.31a) and (3.31b) equal to 0. Then γ_0 can be any positive constant. That is to say, for a linear system, the ROA is the whole space. This conclustion is consistent with our analysis in Section 3.1. In this situation, the ROA of the system will not be affected by the variation of network topology.

When the dynamics of the system $f(\cdot)$ and $g(\cdot)$ are nonlinear dynamic functions and determinated respectively, according to equations (3.30) and (3.32), it is clear that the estimation of ROA is affected by λ_1. That is to say, the network topology affects the ROA. $\|\delta l_i\|$ in equation (3.31a) is a function of ρ represented by $\phi(\rho)$. Also, $\|\delta l_i\|$ in equation (3.31b) is a function of ρ and λ_1 represented by $\psi(\rho,\lambda_1)$. $\phi(\rho)$ is positively related with ρ and $\psi(\rho,\lambda_1)$ is negatively related with ρ. The best solution of $\|\delta l_i\|$ is the crosspoint of red line and blue line in Fig. (4). If $|\lambda_1|$ increases, $\psi(\rho,\lambda_1)$ decreases and the best solution of $\|\delta l_i\|$ decreases from best solution 1 to best solution 2.

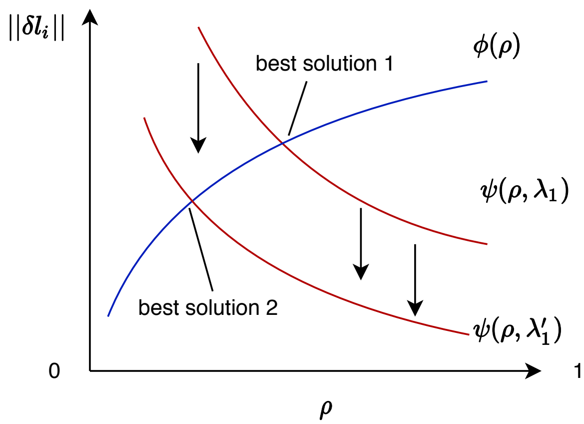


FIG. 4. This figure describes how the parameters ρ and λ_1 affect the maximum value of $\|\delta l_i\|$. Obviously, there exists a negative relationship between $|\lambda_1|$ and $\|\delta l_i\|$.

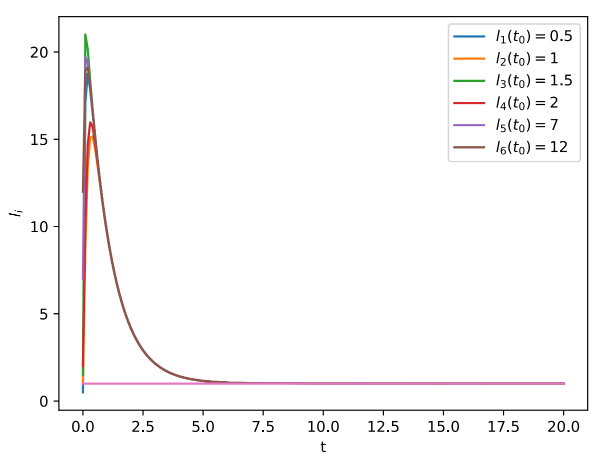


FIG. 5. The node number $N = 100$ in this network. Here we show the dynamics process of some of these nodes, $l_1, l_2, \ldots, l_6$. We can see that all of these nodes synchronize to the ideal equilibrium $l_i = 1$ at last.

Therefore, both $\sqrt{\frac{P_{\min}}{P_{\max}}}$ and γ_0 respectively have a negative relationship with $|\lambda_1|$. By the above analysis, we can build the relationship between network topology (λ_1) and ROA. For a complex system

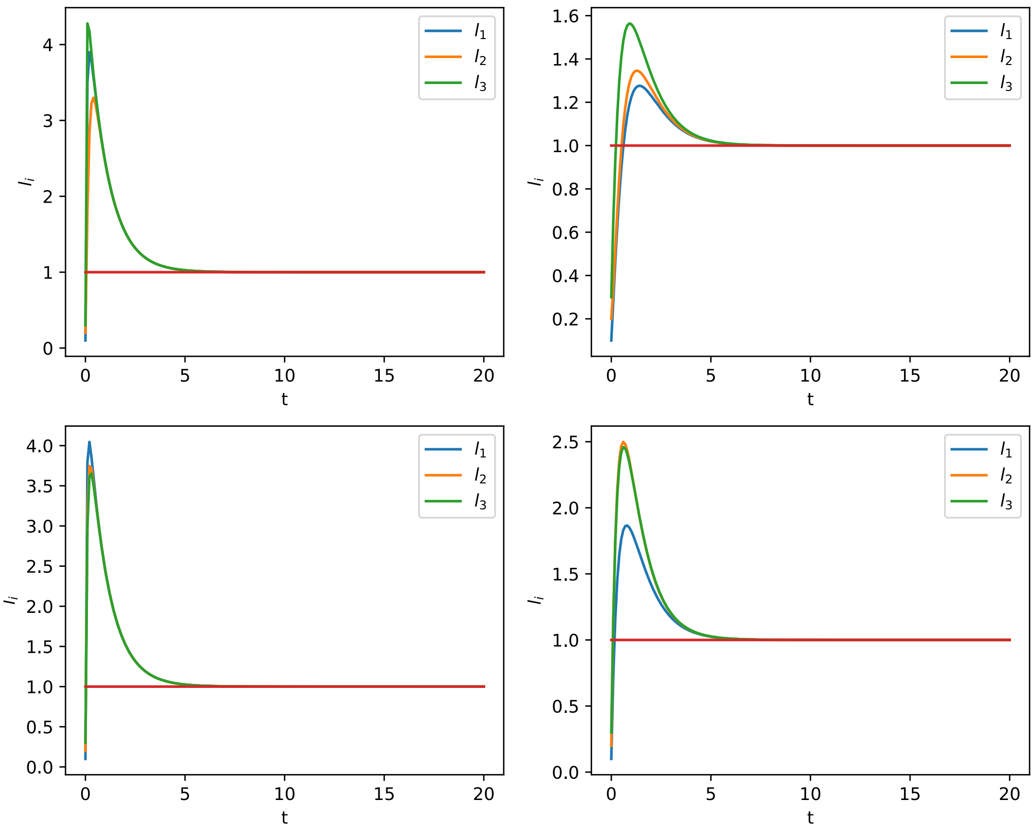


FIG. 6. This figure shows the dynamics process of the system in different network models, repectively, scale free network, small world network, random network and regular network. In different network model, the system syncrhonizes to the equilibrium $l_i = 1$ at last. That is to say, the change of network topology will not affect the staiblity of the ideal equilibrium.

consisted by N nodes, how to optimize the network topology of the system is important to enlarge the ROA. For example, consider two connectivity networks of 6 nodes, G_1 and G_2. The Laplacian matrices of G_1 and G_2 are, respectively,

$$\Lambda_1 = \begin{Bmatrix} -3 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 1 \\ 1 & 0 & -3 & 0 & 1 & 1 \\ 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 & -3 \end{Bmatrix},$$
$$\Lambda_2 = \begin{Bmatrix} -3 & 1 & 0 & 0 & 1 & 1 \\ 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 0 & 1 & -3 \end{Bmatrix}. \tag{3.33}$$

The degree of each node in these two graphs is 3, which means that each node has 3 neighbouring nodes to share loads. These two graphs have similar connectivity, but $\lambda_1 = -6$ in G_1 and $\lambda_1 = -5$ in G_2. So, the system embeded on G_2 has a larger ROA than network on G_1.

Since the ROA is related to $|\lambda_1|$, the graph theory can be used to analyze what topology characteristics mainly decide the value of $|\lambda_1|$ in different network models. This can inform us how to optimize the network topology to enlarge the ROA. Here, we present some findings in the most common classes of complex network.

For a small world network (SW), it can be adding NS links to a regular network with each node coupled to its $2K$ nearest neighbours.

$$|\lambda_1| = (2K+1)(1+2/3\pi) + \varepsilon(Np_s + \sqrt{3\pi p_s/4}), \tag{3.34}$$

where $p_s = 2S/(N-2K-1)$ and ε is a small number $\varepsilon \ll 1$ [29]. So, $|\lambda_1|$ of a SW network is mainly decided by the number of nearest neighbour nodes. Then, the ROA of SW network can be enlarged by controlling the number of neighbor nodes.

For a scale free network (SF) with N nodes, $|\lambda_1|$ can be estimated by perturbation theory.$|\lambda_1| \approx k_{\max} + 1$ [30], where $k_{\max}$ is the maximum degree of nodes. Then, $|\lambda_1|$ is decided mainly by $k_{\max}$, and we can enlarge the ROA of SF network by reconnecting links of the largest node to decrease its maximum degree.

3.4 *Effects of network perturbations on ROA*

The perturbations exist in nodes' dynamics as well as in the network structure, e.g. the link removal or node removal of the network. Here, we will discuss the effects of network structure on ROA. Since we have relates the extreme eigenvalues with ROA, the effects of perturbations of network structure on ROA can be converted to explore these effects on the extreme eigenvalues. The perturbation theory [31] can be used to estimate the variation of eigenvalues. The eigenvalues of the Laplacian matrix Λ can be calculated by $v^\top \Lambda = \lambda v^\top, \Lambda v = \lambda v$, where $v = (v_1, v_2, ..., v_N)^\top$ is the normalized eigenvector of Λ. If perturbations happen to the network, the eigenvalue equation changes to $(\Lambda + \Delta\Lambda)(v+\Delta v) = (\lambda + \Delta\lambda)(v + \Delta v)$. We can get

$$\Delta\lambda = \frac{v^\top \Delta\Lambda v + v^\top \Delta\Lambda \Delta v}{v^\top v + v^\top \Delta v}. \tag{3.35}$$

If a link between node i and j is removed, the variation of Λ is

$$\Delta\Lambda_{i,j} = \begin{cases} 1, & i = j \\ -1, & \text{else} \end{cases} \tag{3.36}$$

In a large size network, the removal of a link has little effect on the network, so $\Delta v \approx 0$. Therefore, $\Delta\lambda \approx (v_i - v_j)^2 > 0$. The removal of the link will decrease the value of $|\lambda_1|$. That is, the removal of link will enlarge the ROA of the system.

4. Numerical Simulation and Results

Here, the traffic load balancing on wireless complex network [1] is demonstrated as an example to verify our method. Nodes in complex network represent base stations (BSs) in wireless network. l_i is the traffic load dynamics of BS i. Firstly, consider a linear dynamical system.

$$\dot{l}_i = \beta(1 - l_i) + c\sum_{j=1}^{N} \mathbf{A}_{ji}\alpha(l_j - l_i). \tag{4.1}$$

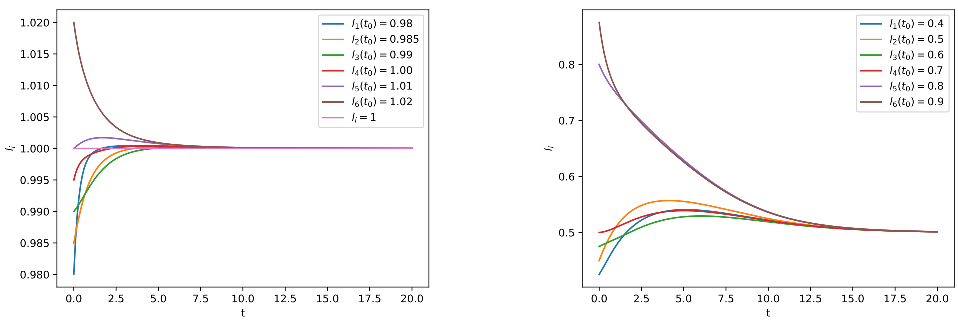
For simplicity, we assume that $\beta = 1, c = 0.5, \mu = 1$, the number of base stations $N = 100$. We will consider this dynamics system embedded in a SF network, small world network, random network, and regular network. In Fig. (5), $l_1, l_2, ...l_i$ have different initial values, finally, all converge to the ideal equilibrium $l_i = 1$. That is to say, in this situation, the increase of users' traffic demand will not affect the stability of the system. In Fig. 6, it shows that the all nodes synchronize at the ideal equilibrium at last in different network models, which means that the change of network topology will not affect the stability of the system. Therefore, from Fig. (5) and Fig. (6), we know that perturbations on the system including change of network topology or increase of traffic demand will not affect the stability of the ideal equilibrium. All of the nodes will synchronize to the ideal equilibrium at last. This simulation verifies our analysis that for linear dynamics in this networked system, the ROA of this system is the whole region.

Then we consider a nonlinear dynamic system

$$\dot{l}_i = \beta_1 l_i^3 + \beta_2 l_i^2 + \beta_3 l_i + \beta_4 + c\sum_{j=1}^{N} \mathbf{A}_{ji}\alpha(l_j^2 - l_i^2), \tag{4.2}$$

where $\beta_1 = -8/3, \beta_2 = 6, \beta_3 = -13/3, \beta_4 = 1, c = 0.5, \mu = 0.5$. The system is embedded in a small world network (Watts–Strogatz model [32]). The number of nodes is $N = 20$, the mean degree is 4, the reconnection probability of links is 0.2. According to our method, we can calculate $|\lambda_1| = 6.96$, $\sqrt{\frac{P_{\min}}{P_{\max}}} = 0.296, \gamma_0 = 0.075$, then $\|\delta \mathbf{l}\| = 0.022$. We set the initial value $0.98 < l_i(t_0) < 1.02$, the dynamic process of each node is shown in Fig. (7) (a). We can see that all nodes in this system finally converge to the quilibrium $l_i = 1$. However, when the initial value of $l_i(t_0)$ is beyond the ROA, then the dynamic process of each node is shown in Fig. (7) (b). We find that in this situation, nodes synchronize to another state which is not the ideal equilibrium. Therefore, this simulation verifies that the estimation of ROA by our method is accurate. When every node belongs to the estimation ROA, the system can keep high efficient.

According to our analysis before, the extreme eigenvalue λ_1 affects the ROA. To verify our analysis, the nonlinear dynamic system is embedded in different network model. The relationship between the



(a) converge to ideal equilibrium $l_i = 1$

(b) converge to other state

FIG. 7. These two figures show the dynamics process of each node with different initial values. In (a), the initial value of each node belongs to the ROA, then each node converges to the ideal equilibrium $l_i = 1$ at last. While, in (b), the initial values of nodes do not belong to the ROA, then these nodes converge to anthor state at last.

extreme eigenvalue λ_1 and the ROA is shown in Fig. (8). It is clear that with increasing $|\lambda_1|$, ROA decreases. Therefore, with determinated node size N, optimizing the network topology to make $|\lambda_1|$ decrease can make the system more robust.

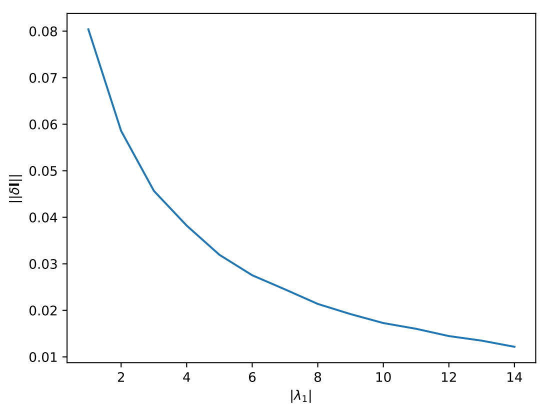


FIG. 8. This figure shows the relationship beween the extreme eigenvalue $|\lambda_1|$ and $\|\delta\mathbf{l}\|$. With the increase of $|\lambda_1|$, $\|\delta\mathbf{l}\|$ decreases.

5. Conclusion and Future Work

In this paper, we proposed an analysis framework to estimate the ROA for load balancing of networked system. A decomposition technique was employed to reduce the dimensionality of the complex networked system, then a Lyapunov function (LF) based method can be used to estimate the ROA. Furthermore, we analyzed the relationship between ROA and network topology. We found that in linear system, the network topology would not affect the ROA. In a nonlinear system, our analysis shows that there exists a negative relationship between ROA and the extreme eigenvalue of the network. By doing this, how to enlarge the ROA of a complex networked system with nonlinear dynamics could be converted to optimize the network topology to make the extreme eigenvalue decrease. This will bridge the gap between graph theory and ROA of a complex networked system. Therefore, this paper used a decoposition technique and LF method to estimate the ROA of load balancing in networked system and identified the impact of network topology on controlling its stability. The limitation of our work is that this estimation method may still be conservative and how to find the exact ROA of load balancing in networked system still need to be expored in the future. In addition, we still do not know the effects of network structure on the ROA of each node. We also believe that more work is needed to identify mesoscopic (e.g., community clusters) network structure properties on the stability of the network.

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Analysing region of attraction of load balancing on complex network

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