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On the Analysis of Statically Indeterminate Structures

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SUMMARY

THIS REPORT DEVELOPS A GENERAL METHOD FOR THE ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES. IT CONCERNS ITSELF BOTH WITH A RIGOROUS DEMONSTRATION OF THE VALIDITY OF THE METHODS AND WITH RECOMMENDATIONS FOR THEIR SUCCESSFUL PRACTICAL APPLICATION. THE ABSTRACTIONS NECESSARY FOR THE SUCCESSFUL ANALYSIS OF STRUCTURES ARE CONSIDERED IN PARA. 1. THE NATURE OF THE SOLUTION OF THE EQUILIBRIUM EQUATIONS IN PARA. 2. THE FUNDAMENTAL THEOREM OF THE METHOD IS ENUNCIATED IN PARA. 3 AND APPLIED PARA. 4. FORMULAE FOR SPECIAL CASES ARE GIVEN IN PARA. 5 AND RECOMMENDATIONS FOR THE CHOICE OF UNKNOWN PARAMETERS IN PARA. 6. THE RELATION OF THE METHOD TO THAT OF CASTIGLIANO IS DISCUSSED IN PARA. 7, AND THE PROCEDURE FOR PRACTICAL CALCULATION IS SUMMARIZED IN PARA. 8. AN APPENDIX GIVES A PROOF OF THE FUNDAMENTAL THEOREM.

INTRODUCTION

The problem treated in this paper is that of the determination of the load distribution in a structure for the general case in which the principles of statics alone are inadequate for its solution. The methods developed are applicable both to the case where the internal loads are produced by externally applied forces and to the case of self strain.

Actual structures are so complex that considerable abstraction is necessary before mathematical analysis becomes possible. Familiar examples of this abstraction are the use of "pinned joints" in the analysis of frameworks and the limitation of load carrying possibilities of the members of a stressed skin structure, whereby it is assumed that stringers, longerons, spar flanges, etc., carry only direct loads, while skin panels, spar webs, etc., carry only shear loads. In what follows we assume that abstraction of this nature has been carried out and that the problem which faces us is the analysis of a structure consisting of a finite number of members, which carry loads that are completely specified by a finite or at worst enumerably infinite set of quantities. An example of such a structure is afforded by a three-dimensional framework with stiff joints. Here the loads carried by the members are completely specified by six quantities, e.g. the end load, two shears, two bending moments and the torque defined at a single point on the axis of the member. It is clear, furthermore, that when the quantities defining the load are known the stresses and strains at all points of the member can be calculated.

2. THE EQUILIBRIUM EQUATIONS AND THEIR SOLUTION

The equations of equilibrium are of two kinds. In the first place there are relations defining the loads applied to the members at places of juncture with other members, in terms of the quantities defining the loads carried by the members and any external forces which may be applied to them. In the second place, there are relations which express the statical equivalence (action and reaction) of the forces acting across places of juncture between the members.* In the example of a stiff-jointed three-dimensional framework cited above relations of the first kind define, in the case where the load is specified at one end of the member, the values of the six corresponding quantities at the other. Relations of the second kind define, in this case, the equations of equilibrium of the nodes.

The equations of equilibrium, as specified above, are, in general, a set of indeterminate equations, less in number than the number of unknown quantities contained in them. Their general solution consists therefore in expressions defining the unknowns in terms of the externally applied forces and a number of arbitrary parameters R_1, R_2, R_3, ... R_r. The parameters R_i may be infinite in number, but for definiteness we consider only the finite case. The number of R_i, i.e. r is then called the "order of redundancy" of the structure. These quantities R_i are often taken to be "loads" in members or reaction components at supports. This, however, is not necessary and as we shall see later a judicious choice of the R_i can simplify the calculations.

If the parameters R_i are given definite values (say $R_i = 0$) then we obtain a "particular solution" of the equilibrium equations which is a "statically correct" diffusion of the externally applied forces through the structure. We denote typical stress and strain components at a point of the structure corresponding to this particular solution by σ_0, ε_0 respectively.

* A "support" is a "member" of the structure.

If the external forces and all the $R_i (i \neq k)$ are written equal to zero and R_k is written unity, then we obtain an internal system of loading which can be described by the title - $(R_k = 1)$. Typical stress and strain components of the system $R_k = 1$ are written σ_k, ε_k respectively.

The most general internal system of loads possible in the structure is given by a summation of arbitrary amounts of the systems $R_k = 1$. A general solution of the equilibrium equations is obtained by adding this general internal system to the "particular solution" defined above. If σ, ε are typical stress and strain components determined by this general solution we can write

$$\left.\begin{aligned} \sigma &= \sigma_0 + \sum_{i=1}^{r} \sigma_i R_i \\ \varepsilon &= \varepsilon_0 + \sum_{i=1}^{r} \varepsilon_i R_i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1)$$

3. THE FUNDAMENTAL THEOREM

The central problem in the theory of the load distribution in structures is the determination of the equations for the quantities R_i. This can be achieved by means of the following theorem:-

"The work done by the loads of an internal system taken over the dislocations* and the sinking of supports associated with any general system of load distribution is equal to the integral through the volume of the structure of the sum of the stress components of the internal system multiplied by the corresponding strains of the general system." .... (2)

This theorem will be proved in the Appendix. It is a special case of the Reciprocal Theorem (suitably generalised and extended). We confine ourselves here to a few words of explanation of its meaning.

An internal system will involve loads in the members and reactions at points of support and so if the displacements of the general system involve alteration of the dimensions of members by cutting and adding or removing thin slices of material (i.e. dislocations), and besides involve sinking of supports, then the loads of the internal system will do work when taken over these displacements. On the other hand if there is no self strain and all the supports are rigid, then the "work" will be zero and likewise the integral defined in the theorem.

The meaning of the volume integral can perhaps be made more precise by a little symbolism. Referring the stresses and strains to a cartesian coordinate system, then if X_x, Y_y, Z_z, Y_z, Z_x, X_y are the stresses of the internal system and e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy} are the strains of the general system, the required volume integral I is given by:-

$$I = \iiint (X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + Y_z e_{yz} + Z_x e_{zx} + X_y e_{xy})\,dx\,dy\,dz$$

4. EQUATIONS FOR THE UNKNOWN PARAMETERS

The equations for the R_i are obtained from (2) as follows. For the internal system of (2) we take $R_i = 1$. For the general system

* Dislocation:- An arbitrary small change in the dimensions of a member interpreted as a discontinuity of the displacement. See Love's Mathematical Theory of Elasticity, Appendix to Chaps. VIII and IX.

we take the actual system which we desire to calculate. This system is given by ascribing certain definite values to the R_i and for the moment we interpret (1) in this sense. The actual stresses and strains which occur are given by (1). The stress due to $R_i = 1$ is σ; and so substituting in (2) we obtain:

$$a_{io} + \sum_j a_{ij}R_j = \text{Work done by the loads of } R_i = 1 \text{ taken over the actual dislocations and sinking of supports} \qquad \ldots\ldots\ldots\ldots(3)$$

$$\text{where } a_{io} = \int\sum(\sigma_i \ \varepsilon_o)dV \qquad \ldots\ldots\ldots\ldots(4)$$

$$a_{ij} = \int\sum(\sigma_i \ \varepsilon_j)dV \qquad \ldots\ldots\ldots\ldots(5)$$

Here dV is an element of volume and $\sum$ sums over the several stress and strain components. The stresses and strains σ_i, ε_o, ε_j are known and so a_{io}, a_{ij} are known constants for the structure and its external force system. Likewise the loads and reactions of the system $R_i = 1$ are known. The equations (3) thus determine the unknown parameters R_i when the dislocations and sinking of the supports are given. If the supports are elastic with typical known flexibilities K, then if f_o and f_i are typical reaction components in the "particular solution" and $R_i = 1$ respectively, the actual reaction is $f_o + \sum_j f_jR_j$ and the sinking of the support $-K(f_o + \sum_j f_jR_j)$. The term on the R.H.S. of (3) becomes:-

$$-\sum f_iK(f_o + \sum_j f_jR_j)$$

and so introducing:-

$$b_{io} = \sum Kf_if_o \ , \qquad b_{ij} = \sum Kf_if_j \qquad \ldots\ldots\ldots\ldots(6)$$

where $\sum$ is taken over all the reaction components, we find from (3):

$$(a_{io} + b_{io}) + \sum_j (a_{ij} + b_{ij})R_i = \text{Work done by the loads of } R_i = 1 \text{ taken over the actual dislocations} \qquad \ldots\ldots(7)$$

The b's can thus be integrated with the a's, as is likewise clear on physical grounds since the "support" is part of the structure. For the case of a structure free from self strain and supported rigidly we find:-

$$a_{io} + \sum_j a_{ij}R_j = 0 \qquad \ldots\ldots\ldots\ldots(8)$$

5. SPECIAL FORMULAE FOR THE CONSTANTS a_{io}, a_{ij}

For ease of application it is useful to have formulae for our volume integral for certain simple stress distributions which are of frequent occurrence. The formulae are expressed in terms of the loads and give the value of $I_{12} = \int\sum(\sigma_1 \ \varepsilon_2)dV = \int\sum(\sigma_2 \ \varepsilon_1)dV$ where σ_1, ε_1; σ_2, ε_2 are stress and strain components in two separate load systems, which in our application can be either the "particuler solution" or $R_i = 1$.

$$\text{Member in Tension} \qquad I_{12} = \int_0^l \frac{T_1T_2}{EA}\,dx \qquad \ldots\ldots\ldots\ldots(9)$$

T is the end load, E is Young's Modulus, A is the section area, l is the length and x is measured lengthwise.

$$\text{Member in Bending} \qquad I_{12} = \int_0^l \frac{M_1M_2}{EI}\,dx \qquad \ldots\ldots\ldots\ldots(10)$$

M is the bending moment and I the "second moment of area" of its cross-section.

Member in Shear (Rectangular plate) $I_{12} = \frac{s_1 s_2}{\mu t} bl$ ....(11)

s is the uniform shear per unit length, μ the shear modulus, t the thickness at right angles to the shear stresses, and b, l the dimensions in the plane of the shear stresses.

Member in Torsion $I_{12} = \int_0^l \frac{T_1 T_2}{C} dx$ ....(12)

T is the torque and C the torsional rigidity (for a circle radius a, $C = \pi \mu a^4/_2$).

Results for combinations of the above stress distributions can be obtained by addition of the formulae (9) - (12). The effects of shear stresses can be normally disregarded in beams.

6. THE CHOICE OF THE PARTICULAR SOLUTION AND THE PARAMETERS R_i

The "particular solution" of the equilibrium equations can be taken as any statically correct diffusion of the external forces through the structure. In practice it is best to choose a diffusion as close as possible to that which actually occurs. The system R_i then represents a small correction to the guessed load distribution.

The systems R_i constitute the most general internal system of loads possible in the structure. Their choice is quite arbitrary. In fact, given one set R_i we can derive an infinity of others R_i' by a linear substitution with arbitrary coefficients:-

$$R_i' = \sum_j c_{ij} R_j \qquad /c_{ij}/ \neq 0 \qquad ....(13)$$

where the c_{ij} are arbitrary constants, restricted only by $/c_{ij}/ \neq 0$. This infinite choice of the R_i can be utilised to simplify calculation. In choosing the systems $R_i = 1$ we should attempt to avoid "overlap" of their stress distributions. Mathematically speaking we should choose the R_i so that the mixed coefficients $a_{ij} (i \neq j)$ are as small as possible in magnitude. This will make the solution of (say) equation (8) much easier. If all the mixed coefficients were zero then (8) would yield

$$R_i = -a_{io}/_{a_{ii}}$$

and even if these mixed coefficients are merely small compared with the leading coefficients a_{ii}, the solution just written is a good first approximation which can easily be improved.

The systems $R_i = 1$ must of course be independent of one another, that is to say, it must not be possible to construct a particular system (say) $R_1 = 1$ from a linear combination of the remainder. Considerable numerical difficulties can arise in a case where one system $R_1 = 1$ can be approximately reproduced by a combination of the others. The equations for the R_i are then "arithmetically indeterminate" and require exceptional accuracy for their solution.

7. CASTIGLIANO'S THEOREM

The foregoing results constitute a generalisation of Castigliano's Second Theorem. We now express our results in terms of the strain energy U and so obtain a direct comparison.

Consider the most general solution of our equilibrium equations. Typical stresses and strains of this system are given by (1). The strain energy U can thus be written:-

$$U = \int \tfrac{1}{2}\sum(\sigma\varepsilon)dV$$
$$= \tfrac{1}{2}\sum_i \sum_j a_{ij}R_iR_j + \sum_i a_{io}R_i + \text{constant} \qquad ....(14)$$

where we have used the reciprocal relations:-

$$a_{ij} = \int\sum(\sigma_i\,\varepsilon_j)dV = \int\sum(\sigma_j\varepsilon_i)dV = a_{ji} \qquad ....(15)$$

If we now differentiate (14) with respect to R_i we obtain:-

$$\frac{\partial U}{\partial R_i} = a_{io} + \sum_j a_{ij}R_j$$

and so our equation (3) can be rewritten:-

$\frac{\partial U}{\partial R_i}$ = Work done by the loads of $R_i = 1$ taken over the actual dislocations and sinking of supports ....(16)

Equation (16) generalises the usual Castigliano Theorem in the following ways:-

1) It is concerned with any internal system defined by a parameter R_i and not merely with the load in a member or a component reaction.

2) It permits allowance to be made for any kind of dislocation and so all types of "lack of fit" can be treated, those involving rotation and lateral displacement as well as longitudinal displacement.

3) It permits allowance to be made for sinking of supports.

8. SUMMARY OF THE RECOMMENDED PROCEDURE

The procedure recommended in a practical calculation is as follows:-

1) Calculate the loads in the members for a statically correct diffusion of the external forces. This diffusion should be as close to the actual one as possible.

2) Consider the most general internal system of load possible in the structure. Split it up into independent and exhaustive unit systems ($R_i = 1$), which overlap with one another as little as is possible.

3) Calculate the load distribution for each of the unit systems.

4) Using equations (9) - (12) or more general results if necessary, calculate the coefficients a_{io}, a_{ij} (equations (4), (5)).

5) If dislocations or sinking supports are involved, as in a problem of self strain, calculate the work done by each unit internal system of loads taken over these imposed displacements.

6) If these are elastic supports the quantities b_{io} and b_{ij} must be evaluated.

7) Formulate and solve the equations for the unknown parameters R_i. Use equations (3), (7) or (8) as the case may be.

8) By superposition of 1) and the now known multiples of 2), determine the actual load distribution in the structure.

REFERENCE AND ACKNOWLEDGMENT

Stress Distribution in Reinforced Cylindrical Shells. Ebner and Koller. A.R.C.3470 Strut 400. 29th March, 1938.

This report first induced the present writer to think upon the lines outlined above.

A P P E N D I X

PROOF OF THE FUNDAMENTAL THEOREM

Consider an elastic body subject to two separate loading systems (1) and (2). Then the Reciprocal Theorem can be enunciated as follows:-

"The work done by the forces of (1) taken over the displacements of (2) is equal to the work done by the forces of (2) taken over the displacements of (1)." ....(1)

This result is established (see Love's Mathematical Theory of Elasticity - para.121) by showing that each term of (1) is equal to identical quantities. If σ_1, ε_1 are typical stress and strain components of system (1) and σ_2, ε_2 the same for system (2), then it is shown that the first term of equation (1) is equal to

$$\int \sum (\sigma_1 \varepsilon_2)\,dV$$

and the second term equal to

$$\int \sum (\sigma_2 \varepsilon_1)\,dV$$

These last quantities are equal in virtue of the identity

$$\sum (\sigma_1 \varepsilon_2) = \sum (\sigma_2 \varepsilon_1)$$

and so the theorem is proved.

For our purposes we only require a part of these results, namely:-

"The work done by the forces of (1) taken over the displacements of (2)" = $\int \sum (\sigma_1 \varepsilon_2)\,dV$ ....(2)

Consider now the case where the load system (2) involves dislocations and sinking of the supports. It is clear that the reactions of system (1) will do work when the supports sink in the system (2) and so this term must be included in the L.H.S. of equation (2). The dislocations of system (2) are introduced by cutting portions of the body and then removing or inserting thin slices of material before welding up again. The faces of the cut thus undergo relative displacement and the stresses of the system (1), which act across the surface of the cut, will do work, which must be included in the L.H.S. of equation (2) and so we find:-

"The work done by the stresses and reactions of an internal system of stress (1) taken over, respectively, the dislocations and movements of the supports associated with a general system (2)" = $\int \sum (\sigma_1 \varepsilon_2)\,dV$ ....(3)

Equation (2) of the report is a restatement of (3) above in language more appropriate to the theory of structures.