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Abstract

This paper investigates the issue of linear stability analysis for two and three level explicit and implicit one dimensional finite different numerical schemes. A new approach which simplifies the Von Neumann method is presented. It has been proved that the new technique is efficient and effective for linear stability study. This is especially true for high-order and complicated numerical schemes.

1 Introduction

As is well known, a numerical method is useless if this method will not converge to the differential equation. To prove convergence, there is a fundamental equivalence theorem for linear finite difference methods, which declares that for a consistent linear method stability is necessary and sufficient for convergence [1]. Although linear stability is not a sufficient condition for guaranteeing nonlinear stability, it is still a necessary condition for achieving nonlinear stability. Therefore, linear stability analysis plays a significant role in the development of a numerical method.

At present, there are several techniques available to analyse linear stability. This includes the discrete perturbation method, the Hirt method, the matrix method and the Von Neumann method [2] [3]. Comparing with other techniques the Von Neumann method is the most widely applied technique. However it is by no means an easy task using these methods to analyse linear stability even for constant coefficient initial value problems. For a complex numerical scheme the linear stability analysis can be extremely complicated applying these techniques. Normally quite tedious and complicated algebraic functions or matrices will be encountered, which are very difficult to analyse, or even impossible to manipulate. From time to time, numerical schemes cannot be performed because of lack of stability information. Obviously a simple and reliable method for proving linear stability is desired.

In this paper we investigate a new approach to the linear stability analysis using the Von Neumann method. The format of this paper is organized as follows: section 2 presents a simplified Von Neumann method for linear stability analysis. Section 3 illustrates the method by applying the technique to some numerical schemes. Section 4 is conclusions.

2 Simplified Von Neumann Method for Linear Stability Analysis

For one dimensional linear finite difference numerical schemes with smooth initial value problem

$$\sum_k B_k^{n+1} U_{j+k}^{n+1} = \sum_k B_k^n U_{j+k}^n + \sum_k B_k^{n-1} U_{j+k}^{n-1} \tag{1}$$

if the amplification coefficient $|\lambda(\theta)|$ *of the scheme is a monotone (increase or decrease) function, i.e.* $(\lambda(\theta)\bar{\lambda(\theta)})' \geq 0$ *(or* ≤ 0*), with respect to the phase angle* θ *in the interval [0,π], then the linear stability conditions of the scheme can be determined by*

$$\lambda = \frac{\sum_k (-1)^{|k|}(B_k^n + \frac{1}{\lambda} B_k^{n-1})}{\sum_k (-1)^{|k|} B_k^{n+1}} \tag{2}$$

$$\lambda = \frac{-\sum_k B_k^{n-1}}{\sum_k B_k^n + \sum_k B_k^{n-1}} \tag{3}$$

For pure odd grid point or pure even grid point finite difference numerical schemes if the amplification coefficient is a concave or convex function, i.e. $|\lambda(\theta)|'' \geq 0$ *(or* ≤ 0*), in the interval [0, π], then an additional stability conditions of the scheme is required*

$$\lambda = \frac{\sum_k sign\left(B_k^n + \frac{1}{\lambda} B_k^{n-1}\right)}{\sum_k sign B_k^{n+1}} \tag{4}$$

here

$$\begin{cases} sign = sin\frac{k\pi}{2} & \forall\ odd\ \ number\ \ k \\ sign = cos\frac{k\pi}{2} & \forall\ even\ \ number\ \ k\ \ and k = 0 \end{cases} \tag{5}$$

where k are the integer grid point numbers at time level n; B_k^n *are constant coefficients;* $\lambda(\theta)$ *is the amplification factor of the numerical scheme;* $\bar{\lambda(\theta)}$ *is the harmonic function of the* $\lambda(\theta)$*. To obtain stability*

$$|\lambda| \leq 1 \tag{6}$$

If the amplification coefficient is not consistent with the conditions above, then the stability conditions can be defined by investegation of those phase angles at which the amplification coefficient has extreme values.

PROOF

The Von Neumann method based on assuming that

$$U_j^n = A_L^n e^{iLj\Delta x} \tag{7}$$

where A_L^n is the amplitude at time level n; L is the wave number in x-direction, $L = \frac{2\pi}{\tau}$; τ is the wavelength; i is the complex number, $i = \sqrt{-1}$.

Considering the general form of equation (1), from equation (7) we have

$$\begin{cases} U_{j+k^{n+1}}^{n+1} = A_L^{n+1} e^{iL(j+k^{n+1})\Delta x} \\ U_{j+k^n}^{n} = A_L^{n} e^{iL(j+k^{n})\Delta x} \\ U_{j+k^{n-1}}^{n-1} = A_L^{n-1} e^{iL(j+k^{n-1})\Delta x} \end{cases} \tag{8}$$

Substituting equation (8) into equation (1) we have

$$\begin{aligned} \sum_{k^{n+1}} B_{k^{n+1}}^{n+1} A_L^{n+1} e^{iL(j+k^{n+1})\Delta x} &= \sum_{k^n} B_{k^n}^{n} A_L^{n} e^{iL(j+k^{n})\Delta x} \\ &+ \sum_{k^{n-1}} B_{k^{n-1}}^{n-1} A_L^{n-1} e^{iL(j+k^{n-1})\Delta x} \end{aligned} \tag{9}$$

Dividing both sides of equation (9) by $A_L^n e^{iLj\Delta x}$ and reorganizing it we get the amplification factor at the new time level:

$$\begin{aligned} \lambda(\theta) &= \frac{A_L^{n+1}}{A_L^n} \\ &= \frac{\sum_k (B_{k^n}^n e^{ik^n\theta} + \frac{1}{\lambda} B_{k^{n-1}}^{n-1} e^{ik^{n-1}\theta})}{\sum_{k^{n+1}} B_{k^{n+1}}^{n+1} e^{ik^{n+1}\theta}} \\ &= \gamma_r + i\gamma_i \end{aligned} \tag{10}$$

here $\theta = L\Delta x$ is the phase angle.

$$\gamma_r = \frac{\left(\sum_k b_1 + \frac{1}{\lambda}\sum_k c_1\right)\sum_k a_1 + \left(\sum_k b_2 + \frac{1}{\lambda}\sum_k c_2\right)\sum_k a_2}{\left(\sum_k a_1\right)^2 + \left(\sum_k a_2\right)^2} \tag{11}$$

$$\gamma_i = \frac{\left(\sum_k b_2 + \frac{1}{\lambda}\sum_k c_2\right)\sum_k a_1 - \left(\sum_k b_1 + \frac{1}{\lambda}\sum_k c_1\right)\sum_k a_2}{\left(\sum_k a_1\right)^2 + \left(\sum_k a_2\right)^2} \tag{12}$$

where

$$\begin{aligned} a_1 &= B_{k^{n+1}} cos\ k^{n+1}\theta \\ a_2 &= B_{k^{n+1}} sin\ k^{n+1}\theta \\ b_1 &= B_{k^n} cos\ k^n\theta \\ b_2 &= B_{k^n} sin\ k^n\theta \\ c_1 &= B_{k^{n-1}} cos\ k^{n-1}\theta \\ c_2 &= B_{k^{n-1}} sin\ k^{n-1}\theta \end{aligned}$$

Note that since the Courant number is constant in equation (10) the amplification factors at different time levels are identical.

The absolute value of the amplification factor $|\lambda|$ is called amplifier coefficient. Obviously if $|\lambda| > 1$, the numerical method will not be stable, otherwise, it is stable. Therefore, for stability

$$|\lambda| = \sqrt{\gamma_r^2 + \gamma_i^2} \leq 1 \qquad (13)$$

for all phase angles ranging from $\theta = 0$ to $\theta = \pi$.

This is the normal approach of analysing the stability in practice using the Von Neumann method. However generally equation (13) is a very complicated triangular algebra. For a complex numerical method $|\lambda|$ is very difficult to work out, or even impossible to manipulate.

Here we adopt a new approach.

The difficulty of analysing equation (13) lies in the phase angle θ which covers the whole domain from 0 to π associated with all wave numbers. The question here is that as far as the stability of a numerical scheme is concerned is it necessary to analyse the whole range of the phase angles? If not, which phase angle do we need to analyse?

From the physical point of view, the instability of a numerical method is caused by the unbounded fast accumulated amplitude error with the time evolution. To limit the amplitude error overgrowing we need first to find out at which phase angles the amplification coefficient have extreme values in the interval $[0, \pi]$, and then it is sufficient to analyse stability condition at these phase angles.

In order to find the angles at which the $|\lambda|$ has extreme values, first we need the first derivative of $|\lambda|$ with respect to θ, i.e. $|\lambda|'$, then by setting $|\lambda|'$ equals to zero we can define the extreme value angles.

From equation (13) we have

$$\begin{aligned} |\lambda|' &= \frac{\gamma_r\ \gamma_r' + \gamma_i\ \gamma_i'}{\sqrt{\gamma_r^2 + \gamma_i^2}} \\ &= 0 \end{aligned} \tag{14}$$

Equation (14) is equivalent to

$$(\lambda(\theta)\bar{\lambda(\theta)})' = 0 \tag{15}$$

or

$$(\gamma_r^2 + \gamma_i^2)' = 0 \tag{16}$$

here

$$\gamma_r^2 + \gamma_i^2 = \frac{(\sum_k b_1 + \frac{1}{\lambda}\sum_k c_1)^2 + (\sum_k b_2 + \frac{1}{\lambda}\sum_k c_2)^2}{(\sum_k a_1)^2 + (\sum_k a_2)^2} \tag{17}$$

and

$$\begin{aligned} (\gamma_r^2 + \gamma_i^2)' &= 2\left[(\Sigma\ a_1)^2 + (\Sigma\ a_2)^2\right]\left[(\Sigma\ b_2 + \frac{1}{\lambda}\Sigma\ c_2)(\Sigma k^n b_1 + \frac{1}{\lambda}\Sigma\ k^{n-1} c_1)\right. \\ &\quad \left. - (\Sigma\ b_1 + \frac{1}{\lambda}\Sigma\ c_1) + (\Sigma\ k^n b_2 - \frac{1}{\lambda}\Sigma\ k^{n-1} c_2)\right] \\ &\quad -2\left[(\Sigma\ b_1 + \Sigma\frac{1}{\lambda}c_1)^2 + (\Sigma\ b_2 + \frac{1}{\lambda}\Sigma\ c_2)^2\right]\left[\Sigma\ a_2\ \Sigma\ k^{n+1} a_1\right. \\ &\quad \left. - \Sigma\ a_1\ \Sigma\ k^{n+1} a_2\right] / \left[(\Sigma\ a_1)^2 + (\Sigma\ a_2)^2\right]^2 \end{aligned} \tag{18}$$

For 3-level explicit schemes equation (18) is reduced to

$$\begin{aligned} (\gamma_r^2 + \gamma_i^2)' &= 2\left[(\Sigma\ b_2 + \frac{1}{\lambda}\Sigma\ c_2)(\Sigma\ k^n b_1 + \frac{1}{\lambda}\Sigma\ k^{n-1} c_1)\right. \\ &\quad \left. - (\Sigma\ b_1 + \frac{1}{\lambda}\Sigma\ c_1)(\Sigma\ k^n b_2 - \frac{1}{\lambda}\Sigma\ k^{n-1} c_2)\right] \end{aligned} \tag{19}$$

For 2-level explicit schemes equation (18) is further reduced to

$$(\gamma_r^2 + \gamma_i^2)' = 2\,(\Sigma\ b_2\ \ \Sigma\ k^n b_1 - \Sigma\ b_1\ \ \Sigma\ k^n b_2) \tag{20}$$

By solving equation (16) the obvious Courant-number-independent extreme value angles can be easily defined. They are:

$$\theta_1 = 0 \tag{21}$$

$$\theta_2 = \pi \tag{22}$$

$$\theta_3 = \frac{\pi}{2} \quad \forall\ either\ \ odd\ \ or\ \ even\ \ k \tag{23}$$

since when $\theta = 0, \pi$ a_2, b_2, and c_2 equal to zeros, therefore $(\gamma_r^2 + \gamma_i^2)' = 0$; when $\theta = \frac{\pi}{2}$ a_1, b_1, c_1 are zeros $\forall$ odd k and a_2, b_2, c_2 are zeros $\forall$ even k, resulting in $(\gamma_r^2 + \gamma_i^2)' = 0$.

Equation (23) means that for pure odd or even number grid point schemes the amplification coefficient $|\lambda|$ has a extrerme value at phase angle $\theta = \frac{\pi}{2}$. For example, the Lax-Friedrichs scheme which is a odd number point scheme has a extreme value at the angle.

There may be other Courant-number-dependent extreme value angles between $\theta = 0$ and $\theta = \pi$ depenting on the solution of equation (16). Normally the solutions are implicit function of Courant number, i.e. $f(\theta, c)$.

However there is one important category of schemes for which the amplification coefficinet is a monotone function, that means $(\lambda(\theta)\bar{\lambda}(\theta))' \geq 0$ (or ≤ 0) $\forall$ $[0,\pi]$. In this case the extreme value angles must be either at $\theta = 0$ or at $\theta = \pi$. Actually as we will see later large number of finite differece numerical schemes fall into this category.

For pure odd or even grid point schemes if $|\lambda(\theta)|'' \geq 0$ (or ≤ 0) $\forall$ $[0,\pi]$, i.e. the function curve of the amplification coefficient is either concave or convex, then the maximum value angle may appear at $\theta = \frac{\pi}{2}$.

Hence we have the following theorem:

THEOREM

For finite different numerical schemes with smooth initial value problem it is necessary and sufficient to investegate the linear stability at phase angles at which $|\lambda(\theta)|$ *has extreme values in the interval [0,*π*].*

If $(\lambda(\theta)\bar{\lambda}(\theta))' \geq 0$ *(or* ≤ 0*) in the phase angle interval [0,*π*], it is necessary and sufficient to investegate the linear stability at the phase angle* $\theta = 0$ *and* $\theta = \pi$*.*

For pure odd or even grid point finite difference schemes if $|\lambda(\theta)|'' \geq 0$ *(or* ≤ 0*), it is necessary and sufficient to investegate the linear stability at phase angle* $\theta = 0$*,* $\theta = \pi$ *and* $\theta = \frac{\pi}{2}$*.*

Based on this theorem substituting $\theta = 0,\ \pi$ and $\theta = \frac{\pi}{2}$ into equation (10) the new method is established.

Therefore the procedure of linear stability analysis can be summarized as follows:

1. Apply equations (2), (3), or (4) to find out the stability conditions at $\theta = 0, \pi$ or $\frac{\pi}{2}$.

2. Check whether or not these stability conditions conform to the monotone function requirements.

3. If satify the requirements then the stability conditions are defined. If not,

4. Calculate the extreme angles $\theta(c)$ from equation (16), define stability condition using equation (13), compare with the stability condition obtained from stage 1, and modify the linear stability condition.

In next section we will use some examples to demonstrate the procedure.

Equations (2) and (3) are the general form of amplification function which is valid for two and three time levels, explicit and implicit numerical schemes. If we consider

3-level explicit schemes

$$U_j^{n+1} = \sum_k (B_k^n U_{j+k}^n + B_k^{n-1} U_{j+k}^{n-1}) \tag{24}$$

then the equation (2) and equation (3) becomes the following form:

$$\lambda = \sum_k (-1)^{|k|} (B_k^n + \frac{1}{\lambda} B_k^{n-1}) \tag{25}$$

$$\lambda = -\sum_k B_k^{n-1} \tag{26}$$

If we consider 2-level explicit schemes

$$U_j^{n+1} = \sum_k B_k^n U_{j+k}^n \tag{27}$$

then equation (2) is further simplified to

$$\lambda = 1 - 2 \sum_{k=\pm 1, \pm 3, \ldots} B_k^n \tag{28}$$

since $\sum_k B_k^n = 1$ for consistency.

For 2-level implicit schemes

$$\sum_k B_{j+k}^{n+1} U_{j+k}^{n+1} = \sum_k B_k^n U_{j+k}^n \tag{29}$$

the amplification factor of equation (2) becomes

$$\lambda = \frac{\sum_k (-1)^{|k|} B_k^n}{\sum_k (-1)^{|k|} B_k^{n+1}} \tag{30}$$

For fully implicit schemes

$$\sum_k B_k^{n+1} U_k^{n+1} = U_j^n \tag{31}$$

the amplification factor has the following simple form

$$\lambda = \frac{1}{1 - 2 \sum_{k=\pm 1, \pm 3, \ldots} B_k^{n+1}} \tag{32}$$

since $\sum_k B_k^{n+1} = 1$ for consistency.

3 Stability Analysis of Numerical Schemes for Model Hyperbolic and Parabolic Problems

In this section we will use numerical schemes some of which the stability conditions are well known to testfy the new approach introduced in last section.

3.1 Stability Analysis for Model Hyperbolic Problems

Example 3.1.1

Consider the scheme

$$U_j^{n+1} = U_j^n - \frac{c}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{d}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \tag{33}$$

here $c = \frac{a\Delta t}{\Delta x}$ is a Courant number. a is a wave speed. d is a variable.

This is a 2-level explicit scheme. From equation (28) we have

$$\lambda = 1 - 2d \tag{34}$$

Equation (34) means that for $|\lambda| \leq 1$, $0 \leq d \leq 1$ must be satisfied. To analyse the behavior of the amplification coefficient $|\lambda|$ from equation (20) we have

$$(\lambda(\theta)\bar{\lambda}(\theta))' = 2\left[\left(c^2 - d^2\right) cos\theta - (1-d)d\right] sin\theta \tag{35}$$

Two special cases are easily defined from equation (35): when $d = c^2$ and $d = c$ the amplification coefficient is monotone, since in these cases equation (35) keeps the same sign in the phase angle interval $[0, \pi]$. In the former case equation (33) turns to the second-order Lax-Wendroff scheme. Therefore from equation (34) the stability condition is

$$|\lambda| \leq 1 \quad for \quad |c| \leq 1 \tag{36}$$

This is identical to the familiar result. In the latter case equation (33) reduces to the first-order upwind scheme. The scheme is stable for $0 \leq |c| \leq 1$.

For d being other values the $|\lambda|$ is not always a monotone function. Its behavior is determined by Courant number. For example assuming $d = \frac{1}{2}$ by analysing equation (35) we know that when $|c| \leq 0.707$ equation (35) keeps same sign, i.e. the amplification coefficient $|\lambda|$ is a monotone function, surely the scheme is stable at least for $|c| \leq 0.707$. While sine $|c| > 0.707$ the $|\lambda|$ is not monotone, we need to find out all the extreme value angles $\theta(c)$ for $|c| > 0.707$. By setting equation (35) equals 0, we get the following extreme value angles

$$cos\theta = \frac{1}{4c^2 - 1} \tag{37}$$

What we do now is to define Courant numbers of $|c| > 0.707$ which satisfy the stability condition at the extraeme angles. Bringing equation (37) into equation (13) we have

$$\begin{aligned} |\lambda| &= \sqrt{\frac{1}{4}(1 + cos\theta)^2 + c^2 sin^2\theta} \\ &= \sqrt{\frac{1}{4}\left(1 + \frac{1}{4c^2 - 1}\right)^2 + c^2\left(1 - \frac{1}{(4c^2 - 1)^2}\right)} \end{aligned} \tag{38}$$

Equation (38) is ploted in Figure 1. It indicates that when $|c| > 0.707$ the scheme is not stable. Therefore the stability condition of the scheme for $d = \frac{1}{2}$ is

$$|\lambda| \leq 1 \quad for \quad |c| \leq 0.707 \tag{39}$$

In the same manner the stability conditions can be defined for different values of d.

Example 3.1.2 Leapfrog Scheme

The leapfrog scheme for the scalar advection equation has the following form

$$U_j^{n+1} = U_j^{n-1} - cU_{j+1}^n + cU_{j-1}^n \tag{40}$$

This is a 3-level explicit odd number point scheme. It is easy to prove that the scheme has extreme values at $\theta = 0,\ \pi$, and $\frac{\pi}{2}$. Using equation (25) we have

$$\lambda = \frac{1}{\bar{\lambda}} \tag{41}$$

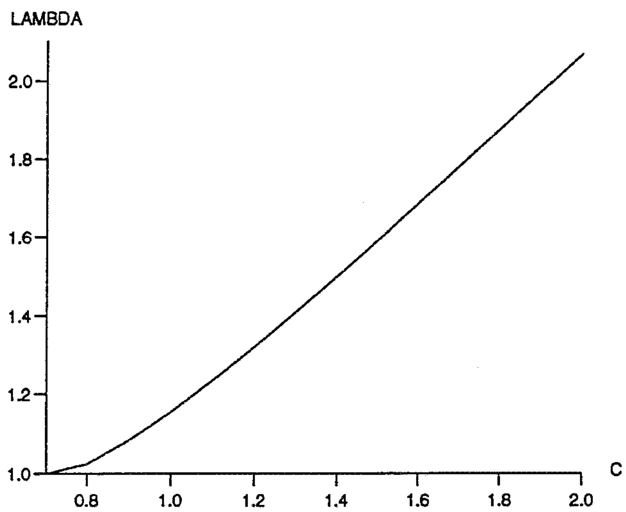

Figure 1: Stability with Different Courant number

That is

$$\lambda^2 = 1 \tag{42}$$

From equation (26) we have

$$\lambda = -1 \tag{43}$$

From equation (4) we have

$$\lambda = -c \pm \sqrt{1 + c^2} \tag{44}$$

Equations (42) and (43) mean $|\lambda| = 1$; equation (44) means unconditionally stable. Actually this scheme is neutrally stable for $|c| \leq 1$.

Example 3.1.3 Crank-Nicolson Scheme

$$U_j^{n+1} + \frac{1}{4}cU_{j+1}^{n+1} - \frac{1}{4}cU_{j-1}^{n+1} = U_j^n - \frac{1}{4}cU_{j+1}^n + \frac{1}{4}cU_{j-1}^n \tag{45}$$

This is a 2-level implicit scheme. The scheme has a monotone amplification coefficient. From equation (30)

$$\lambda = \frac{1 + \frac{1}{4}c - \frac{1}{4}c}{1 - \frac{1}{4}c + \frac{1}{4}c} = 1 \tag{46}$$

i.e. $|\lambda| = 1$. This scheme is unconditionally stable.

Example 3.1.4 Lax-Friedrichs Scheme

$$U_j^{n+1} = (\frac{1}{2} - \frac{c}{2})U_{j+1}^n + (\frac{1}{2} + \frac{c}{2})U_{j-1}^n \qquad (47)$$

This is a 2-level explicit odd point scheme. The scheme has a concave amplification coefficient function, therefore we need to check both equations (28) and (4). From equation (28) we have

$$|\lambda| = 1 \qquad (48)$$

From equation (4) we have

$$|\lambda| = |c| \qquad (49)$$

Therefore this scheme is stable if $|c| \leq 1$.

Example 3.1.5 Fully Discrete Fourth-order Scheme (See [4]).

$$\begin{aligned} U_j^{n+1} &= \left(1 + \frac{1}{4}c^4 - \frac{5}{4}c^2\right) U_j^n + \left(\frac{1}{24}c^4 + \frac{1}{12}c^3 - \frac{1}{24}c^2 - \frac{1}{12}c\right) U_{j-2}^n \\ &+ \left(\frac{2}{3}c + \frac{2}{3}c^2 - \frac{1}{6}c^3 - \frac{1}{6}c^4\right) U_{j-1}^n + \left(\frac{1}{6}c^3 + \frac{2}{3}c^2 - \frac{1}{6}c^4 - \frac{2}{3}c\right) U_{j+1}^n \\ &+ \left(\frac{1}{12}c - \frac{1}{24}c^2 - \frac{1}{12}c^3 + \frac{1}{24}c^4\right) U_{j+2}^n \end{aligned} \qquad (50)$$

This is a 2-level explicit fully discrete fourth-order both in space and time scheme. Applying equation (28) the amplification factor is

$$\lambda = 1 - \frac{8}{3}c^2 + \frac{2}{3}c^4 \qquad (51)$$

Figure 2 shows three possiblly stable regions of the scheme for $|\lambda| \leq 1$, which are

$$\begin{aligned} -2 &\leq c \leq -1.73 \\ -1 &\leq c \leq 1 \\ 1.73 &\leq c \leq 2 \end{aligned} \qquad (52)$$

It is proved that the scheme has a monotone amplification coefficient function when $-1 \leq c \leq 1$; while other regions are not monotone and proved that in those regions the scheme is not stable. Therefore the stability conditions of the scheme is

$$-1 \leq c \leq 1 \qquad (53)$$

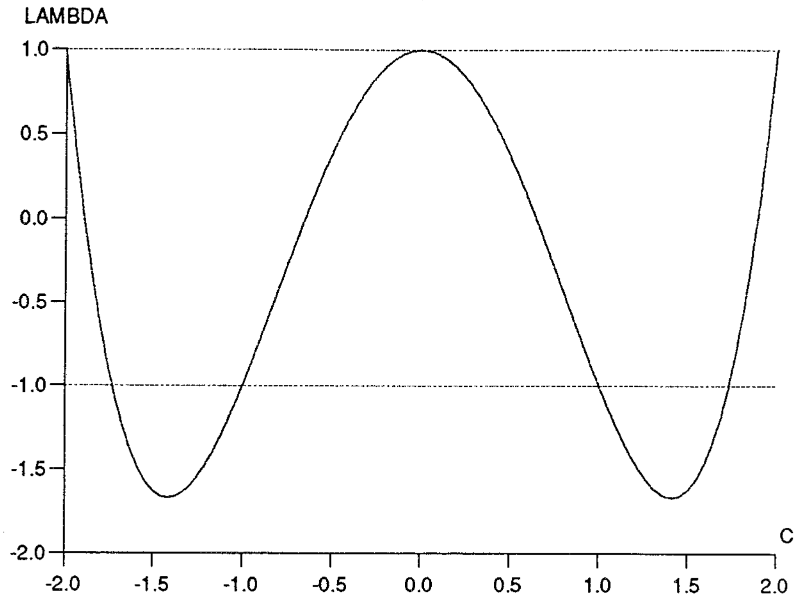

Figure 2: Amplification Factor of 5-point central Hyperbolic Scheme

3.2 Stability Analysis for Model Parabolic Problems

All the examples used here have monotone amplification coefficient functions.

Example 3.2.1 Explicit Space-centred Differences Scheme

$$U_j^{n+1} = (1-2d)U_j^n + dU_{j+1}^n + dU_{j-1}^n \tag{54}$$

here $d = \frac{\nu \Delta t}{(\Delta x)^2}$ is a diffusion number; ν is a diffusive coefficient.

From equation (28) the amplification factor

$$\lambda = 1 - 4d \tag{55}$$

The stability condition is

$$|\lambda| \le 1 \qquad for \quad d \le \frac{1}{2} \tag{56}$$

Example 3.2.2 Fully Implicit Scheme

$$(1+2d)U_j^{n+1} - dU_{j+1}^{n+1} - dU_{j-1}^{n+1} = U_j^n \tag{57}$$

From equation (32) we have

$$\lambda = \frac{1}{1+4d} \tag{58}$$

Since d is positive the scheme is unconditionally stable for $d > 0$.

Example 3.2.3 Fully Discrete Seven Point Scheme (See [5]).

$$\begin{aligned} U_j^{n+1} &= \left(1 - \frac{10}{3}d^3 + \frac{14}{3}d^2 - \frac{49}{18}d\right) U_j^n + \left(\frac{15}{6}d^3 - \frac{13}{4}d^2 + \frac{3}{2}d\right)\left(U_{j-1}^n + U_{j+1}^n\right) \\ &\quad + \left(d^2 - d^3 - \frac{3}{20}d\right)\left(U_{j-2}^n + U_{j+2}^n\right) \\ &\quad + \left(\frac{1}{6}d^3 - \frac{1}{12}d^2 + \frac{1}{90}d\right)\left(U_{j-3}^n + U_{j+3}^n\right) \end{aligned} \tag{59}$$

This is a fully discrete explicit scheme which has sixth-order accuracy in space and third-order in time. From equation (28) we have

$$\lambda = 1 - \frac{32}{3}d^3 + \frac{40}{3}d^2 - \frac{272}{45}d \tag{60}$$

For $|\lambda| \le 1$, see Figure 3, the stability condition is

$$0 \le d \le 0.85 \tag{61}$$

4 Conclusions

In this paper we presented a simplified Von Neumann method for linear stability analysis of one dimensional numerical schemes. To illustrate the method linear stability of a variety of numerical schemes are analysed. This aproach offers us an efficient and effective means to deal with linear stability study.

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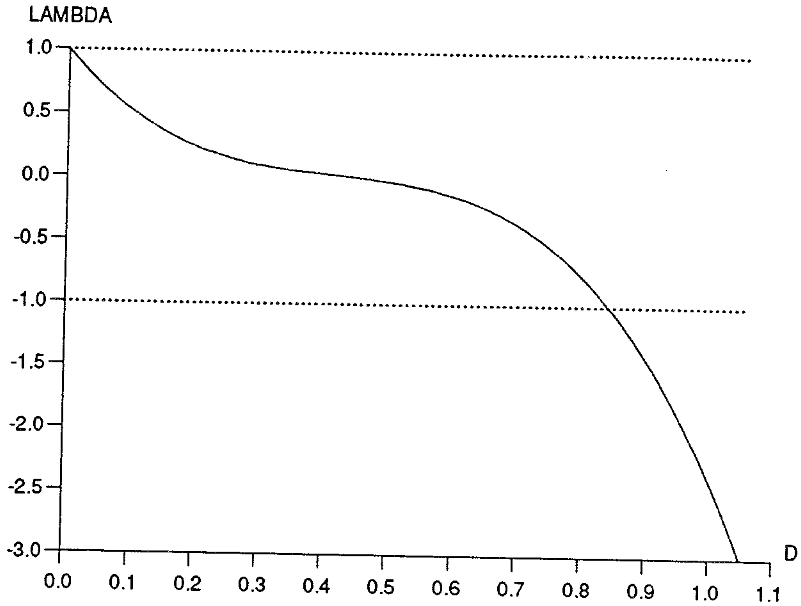

Figure 3: Amplification factor of 7-point Central Parabolic Scheme

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