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THE REALIZATION OF FOURTH-ORDER RATIONAL TRANSFER FUNCTIONS WITH ADJUSTABLE COEFFICIENTS

by

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June 1963

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SUMMARY

Design techniques are described for the simulation of third-order and fourth-order rational transfer functions with adjustable coefficients. The realization procedure, which is based on the use of an ideal computing amplifier associated with linear passive R.C. two-port networks, is an extension of the methods described previously[1,2] for quadratic functions.

Contents

1. Introduction

Design techniques are outlined in previous papers[1,2] for the realization of networks having a forward open-circuit voltage transfer function, of quadratic form, with the facility of independent control of each coefficient.

These techniques are extended in this note to give a realization of third-order and fourth-order rational transfer functions with adjustable coefficients.

Typical applications of the latter include the realization of adaptive filter networks in which the pass-band, or alternatively, the cut-off frequency may be required to be adjusted by manual or automatic means.

The basic approach is the synthesis of minimum-phase transfer functions by the use of one computing amplifier and passive R.C. networks. The non minimum phase case, for the general case, may be synthesized by the use of an additional sign reversing amplifier.

2. The Generalized two-port active-network arrangement

The derivation of all rational transfer functions given in the paper is based on the arrangement shown in Fig. 1.

Networks A and B are both passive and comprise combinations of linear resistors and capacitors. Network C represents a computing or operational amplifier having an infinite driving-point input impedance at port 2 and a zero driving point output impedance. The forward open-circuit-voltage transfer function is of the form K $G_1(s)$, where K is a positive constant and s is the complex frequency $\sigma + j\omega$.

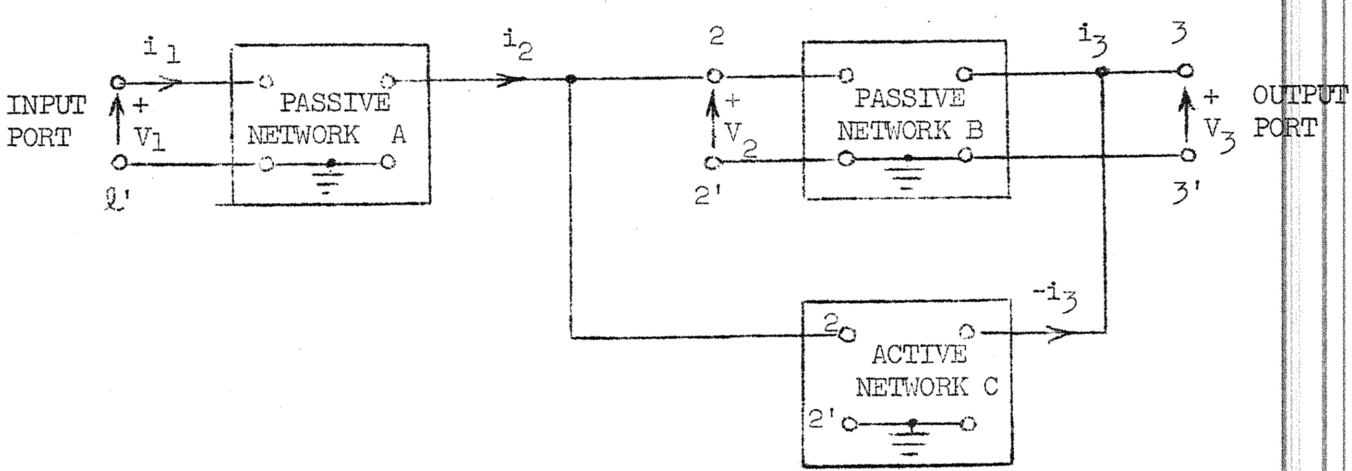


Fig. 1 The Basic Arrangement

A detailed analysis of the above arrangement is given in Ref.1, where the following result is derived:

$$\frac{V_3(s)}{V_1(s)} = \frac{y^A_{21}(s)}{y^B_{12}(s)} \qquad (1)$$

where

$$y^A_{21}(s) = \left.\frac{I_2(s)}{V_1(s)}\right|_{V_2(s)=0} = \text{forward short-circuit transfer admittance of network A}$$

and

$$y^B_{12}(s) = \left.\frac{I_2(s)}{V_3(s)}\right|_{V_2(s)=0} = \text{minus reverse short-circuit transfer admittance of network B}$$

The derivation of eqn. 1 is based on the following assumptions:

(a) Networks A and B are initially relaxed (i.e. zero charge on all capacitors).

(b) $KG_1(s) \rightarrow \infty$ over the frequency range of interest.

(c) $y^C_{11}(s) = 0$

(d) $y^C_{22}(s) = -\infty$

(e) $y^B_{12}(s) \gg \left[y^A_{22}(s) - y^B_{11}(s)\right]$ /K for the frequency range of interest.

(f) $y^B_{21}(s)$ and $y^B_{22}(s)$ are both finite over the frequency range of interest.

It should be noted that these assumptions are implied in conventional analogue-computing circuit applications, and, in practice, may be achieved without undue difficulty.

3. <u>Basic Design Procedure</u>

If $y^A_{21}(s)$ can be expressed in the form $\frac{1}{D(sT)} \sum_{q=0}^{n} k_q (sT)^q$

and $y^B_{12}(s)$ can be expressed in the form $\frac{1}{D(sT)} \sum_{r=0}^{m} k_r (sT)^r$

where D(sT) is a rational function of sT and T is a time constant chosen to be a convenient integer value, then from (1)

$$\frac{V_3(s)}{V_1(s)} = \frac{y^A_{21}(s)}{y^B_{12}(s)} = \frac{\sum_{q=0}^{n} k_q (sT)^q}{\sum_{r=0}^{m} k_r (sT)^r} \qquad (2)$$

D(sT) must be chosen so that each short circuit transfer admittance conforms with the physical realizability conditions for linear discrete R.C. networks.

Network A may be realized, in the above form, by the parallel connection of networks having short-circuit transfer admittances of the form $\frac{k_q (sT)^q}{D(sT)}$ for $q = 0 \ldots n$.

Similarly network B may be realized by the parallel connection of networks having short circuit transfer admittances of the form $\frac{k_r (sT)^r}{D(sT)}$ for $r = 0 \ldots m$.

There is considerable freedom in the choice of D(sT) since the only restriction is that its form should be such that the short circuit transfer admittances having D(sT) as the denominator are physically realizable, i.e. the roots of D(s) are negative real and the highest degree of sT in (2) is equal to, or exceeds by one, the degree of D(s).

A particular form for D(sT) which controls the spread of component values is given by the expression

$$D(sT) = a_o + a_1 (sT)^1 + a_2 (sT)^2 + a_3 (sT)^3 \ldots a_n (sT)^n$$

where for the nth degree polynomial

$$a_n = a_o = 1$$

$$a_{n-1} = a_1$$

$$a_{n-2} = a_2 \qquad (3)$$

or for general case, $a_{n-k} = a_k$ for $k = 0$ to n

Thus for n = 2

$$D(sT) = 1 + a_1 sT + s^2T^2 \tag{4}$$

and for n = 3

$$D(sT) = 1 + A_1 sT + A_1 s^2T^2 + s^3T^3 \tag{5}$$

These forms have the advantage that

$$\left.\begin{aligned} &\text{for } n = 2, \quad D\left(\frac{1}{sT}\right) = 1 + \frac{a_1}{sT} + \frac{1}{s^2T^2} \\ &\qquad\qquad = s^2T^2(1 + a_1 sT + s^2T^2) = s^2T^2 D(sT) \\ &\text{and for } n=3 \quad D\left(\frac{1}{sT}\right) = s^3T^3\, D(sT) \\ &\text{or for the general case of nth degree} \\ &\qquad D\left(\frac{1}{sT}\right) = s^nT^n\, D(sT) \end{aligned}\right\} \tag{6}$$

4. Third-order Rational Transfer Functions

4.1. Unbalanced R.C. ladder network

An unbalanced ladder network realization is considered for networks A and B in Fig. 1.

For a third order rational transfer function to be realized by the arrangement of Fig. 1 the maximum number of zeros of the component networks of A and B is therefore three. A network to satisfy this requirement is shown in Fig. 2(a) where Y represents the admittance of one or more resistors connected in series, and/or in parallel, with one or more capacitors. Z represents the impedance of the parallel and/or series connections of resistors and capacitors.

The short circuit transfer admittance of the network in Fig. 2(a) is given by:

$$y_{21}(s) = - y_{12}(s) = \frac{Y_1Y_2Y_3Z_1Z_2}{1+Z_1(Y_1+Y_2)+Z_2(Y_2+Y_3)+Z_1Z_2(Y_1Y_2+Y_1Y_3+Y_2Y_3)} \tag{7}$$

A particular form of (7) is given when $Y_1=Y_2=Y_3=Y$

i.e. $$y_{21}(s) = \frac{Y^3 Z_1 Z_2}{1 + 2Y(Z_1+Z_2) + 3Y^2 Z_1 Z_2} \qquad (8)$$

The requirement for (3) may be fulfilled if YZ is identified as sT and $Z_1 = \frac{Z}{3}$ and $Z_2 = Z$. Alternative forms are obviously possible but will not be considered here.

With $Z_1 = \frac{Z}{3}$ and $Z_2 = Z$

$$y_{21}(s) = \frac{Y^3 \frac{Z^2}{3}}{1 + \frac{8}{3}YZ + Y^2Z^2} \qquad (9)$$

The corresponding network is shown in Fig. 2(b).

For $Y = sC$; $Z = R$ and $T = RC$

$$y_{21}(s) = \frac{1}{3R} \frac{s^3T^3}{1 + \frac{8}{3}sT + s^2T^2} \qquad (10)$$

The network with this short circuit transfer admittance is shown in Fig. 2(c).

For $y = \frac{1}{R}$; $Z = \frac{1}{sC}$; $YZ = \frac{1}{sT}$

$$y_{21}(s) = \frac{1}{3R} \frac{1}{1 + \frac{8}{3}sT + s^2T^2} \qquad (11)$$

The corresponding network is shown in Fig. 2(g).

The network of Fig. 2(c) may be modified to give $y_{21}(s)$ with two zeros by interchanging C and R as shown in Fig. 2(d). A short circuit transfer admittance with one zero is given by network Fig. 2(e) which is obtained by interchanging $\frac{R}{3}$ and C in Fig. 2(d). The short circuit transfer admittance of the network in Fig. 2(d) is $\frac{1}{R} \cdot \frac{sT}{1 + \frac{8}{3}sT + s^2T^2}$. This may be scaled by a factor of $\frac{1}{3}$ by multiplying each resistor value in Fig. 2(d) and dividing each capacitor value by 3 to give the network of Fig. 2(e). Finally the network of Fig. 2(h) is derived from Fig. 2(g) by interchanging C and R. This network may be used as an alternative to Fig. 2(e).

4.2 Active Network Realization of Third Order Rational Transfer Functions (with Adjustable Coefficients)

An active network arrangement is shown in Fig. 3 where networks A and B of Fig. 1 consist of parallel connections of networks of the form shown in Fig. 2.

The forward transfer function of the network shown in Fig. 3 is given by

$$\frac{V_3(s)}{V_1(s)} = - \frac{k_1 + k_2 sT + k_3 s^2T^2 + k_4 s^3T^3}{k_5 + k_6 sT + k_7 s^2T^2 + k_8 s^3T^3} \tag{12}$$

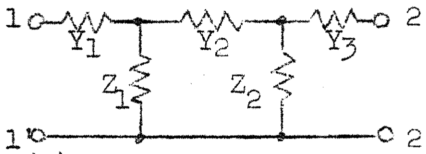
where the k factors are obtained by means of the potentiometers.

The time constant T is chosen to be a convenient integer so that the transfer function is realized with all k's $\leqslant 1$. Particular forms of 12 are obtained by setting particular coefficients to zero. For such simplified uses, only the relevant component networks need to be incorporated in Fig. 3.

NETWORK

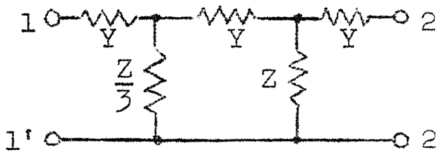
SHORT CIRCUIT TRANSFER ADMITTANCE

$y_{21}(s) = - y_{12}s$



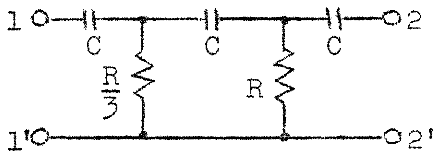
(a) Basic RC ladder network

$$\frac{Y_1Y_2Y_3Z_1Z_2}{1 + Z_1(Y_1+Y_2) + Z_2(Y_2+Y_3) + Z_1Z_2(Y_1Y_2+Y_1Y_3+Y_2Y_3)}$$



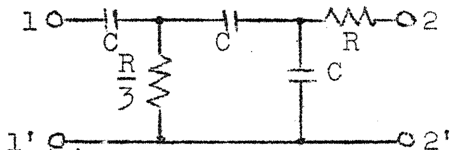
(b) Particular form.

$$\frac{Y^3\,\frac{Z^2}{3}}{1 + \frac{8}{3}YZ + Y^2Z^2}$$



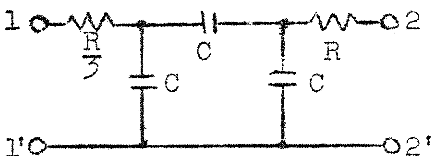
(c) Three finite zeros

$$\frac{1}{3R} \cdot \frac{s^3T^3}{1 + \frac{8}{3}sT + s^2T^2}$$



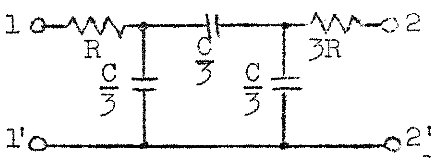
(d) Two finite zeros

$$\frac{1}{3R} \cdot \frac{s^2T^2}{1 + \frac{8}{3}sT + s^2T^2}$$



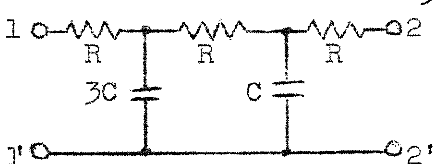
(e) One finite zero

$$\frac{1}{R} \cdot \frac{sT}{1 + \frac{8}{3}sT + s^2T^2}$$



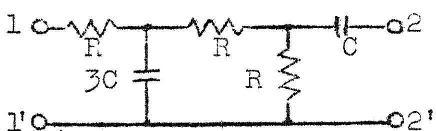
(f) (e) scaled by factor $\frac{1}{3}$

$$\frac{1}{3R} \cdot \frac{sT}{1 + \frac{8}{3}sT + s^2T^2}$$



(g) No finite zeros

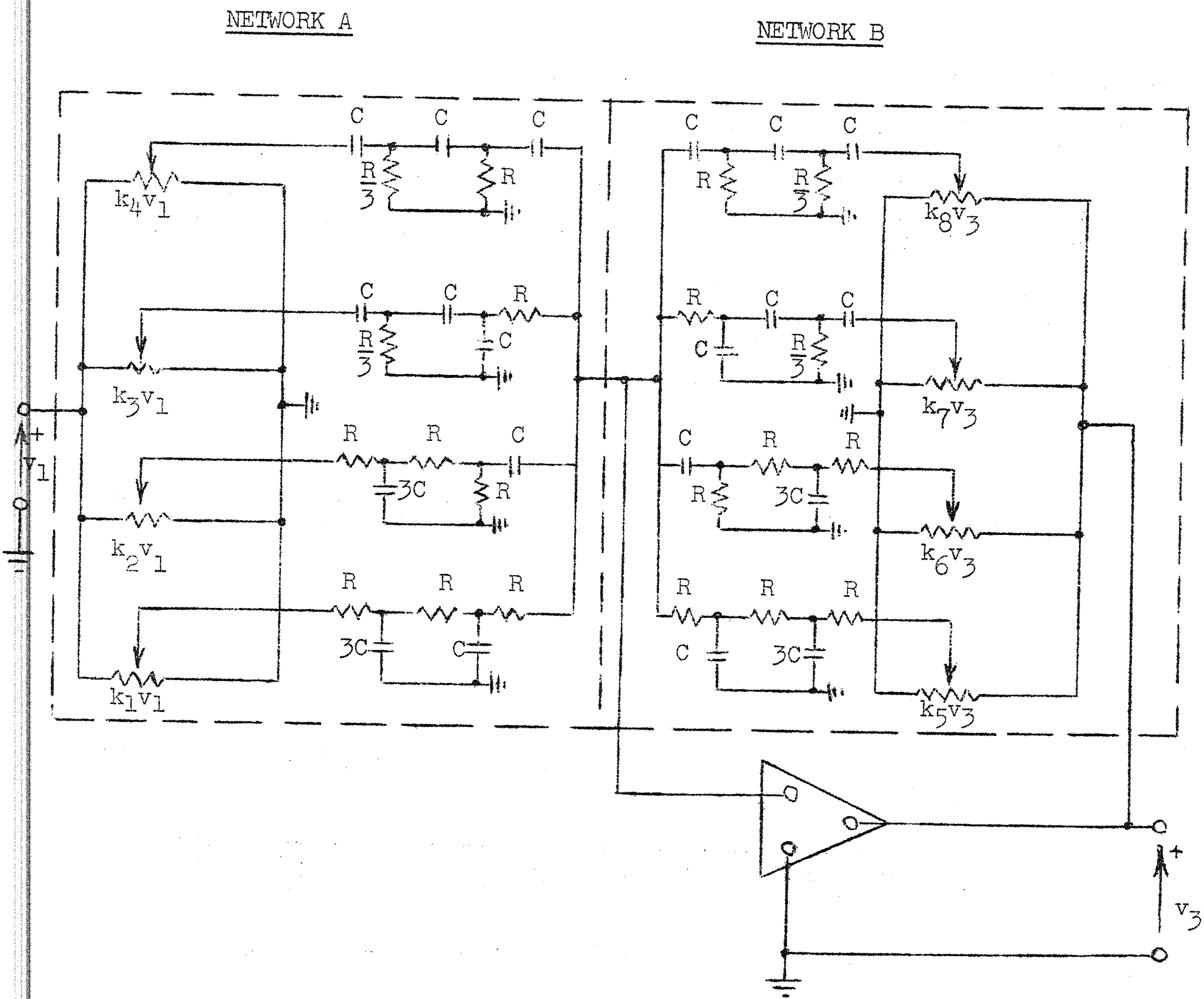
$$\frac{1}{3R} \cdot \frac{1}{1 + \frac{8}{3}sT + s^2T^2}$$



(h) One finite zero
(Alternative to (f))

$$\frac{1}{3R} \cdot \frac{sT}{1 + \frac{8}{3}sT + s^2T^2}$$

Fig. 2 - Ladder Networks for use in Fig. 1
(Third order rational transfer functions)



▷ denotes an operational amplifier

$T = RC$; $k's \leq 1$

Assumption - output resistance of each potentiometer is negligible.

$$\frac{V_3(s)}{V_1(s)} = - \frac{k_1 + k_2sT + k_3s^2T^2 + k_4s^3T^3}{k_5 + k_6sT + k_7s^2T^2 + k_8s^3T^3}$$

Fig. 3 - Active Network Arrangement for Third-order Transfer Functions

5. Fourth-Order Rational Transfer Functions

5.1. Basic RC Ladder Network

The development for this case follows closely that described in Section 4. The basic RC ladder network is shown in Fig. 4(c) which is capable of providing four zeros in the short circuit admittance function.

For this network

$$y_{21}(s) = \frac{Y_1Y_2Y_3Y_4Z_1Z_2Z_3}{D} \tag{13}$$

where

$$D = 1 + Z_1Z_2(Y_1Y_2+Y_1Y_3+Y_2Y_3) + Z_1Z_3(Y_1Y_3+Y_1Y_4+Y_2Y_3+Y_2Y_4) + Z_2Z_3(Y_2Y_3+Y_2Y_4+Y_3Y_4)$$

$$+ Z_1Z_2Z_3(Y_1Y_2Y_3+Y_1Y_2Y_4+Y_2Y_3Y_4+Y_1Y_3Y_4)$$

$$+ Z_1(Y_1+Y_2) + Z_2(Y_2+Y_3) + Z_3(Y_3+Y_4)$$

As in section 4 we may simplify this expression by putting $Y = Y_1 = Y_2 = Y_3 = Y_4$, with the result that

$$y_{21}(s) = \frac{Y^4Z_1Z_2Z_3}{1 + Y^2(3Z_1Z_2+4Z_1Z_3+3Z_2Z_3) + 4Y^3Z_1Z_2Z_3 + 2Y(Z_1+Z_2+Z_3)} \tag{14}$$

Identifying YZ as sT ; $Z_1 = Z_3 = \frac{Z}{2}$ and $Z_2 = Z$, the form of equation 5 results for the denominator.

i.e. $$y_{21}(s) = \frac{Y^4Z^3/4}{1 + 4YZ + 4Y^2Z^2 + Y^3Z^3} \tag{15}$$

The corresponding network is shown in Fig. 4(b).

For $Y = sC$; $Z = R$; $T = RC$,

$$y_{21}(s) = \frac{1}{4R}\,\frac{s^4T^4}{1 + 4sT + 4s^2T^2 + s^3T^3} \tag{16}$$

Also for $Y = \frac{1}{R}$; $Z = \frac{1}{sC}$

$$y_{21}(s) = \frac{1}{4R} \; \frac{1}{1 + 4sT + 4s^2T^2 + s^3T^3} \tag{17}$$

The corresponding network is shown in Fig.4(f).

Deviations of network (c) of Fig. 4 are given as networks (d) and (e) respectively which are obtained by interchanging resistors and capacitors and scaling in a similar manner to that described in Section 4. Similarly networks (g) and (h) of Fig. 4 are derivations of network f.

5.2 Active Network Realization of Fourth-order Rational Transfer Functions (With adjustable coefficients)

The active network arrangement is shown in Fig. 5 where networks A and B of Fig.1 consist of parallel connections of networks of the form shown in Fig. 4.

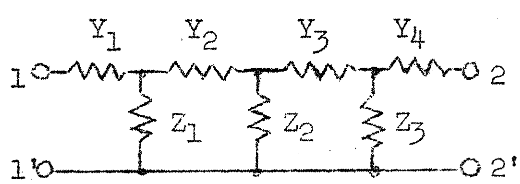
The forward transfer function of this network is given by

$$\frac{V_3(s)}{V_1(s)} = - \frac{K_1 + K_2sT + K_3s^2T^2 + K_4s^3T^3 + K_5s^4T^4}{K_6 + K_7sT + K_8s^2T^2 + K_9s^3T^3 + K_{10}s^4T^4} \tag{18}$$

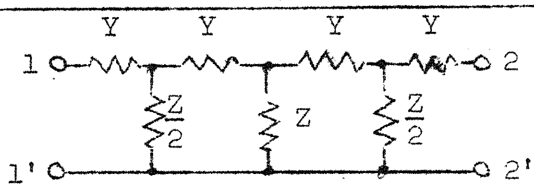
As in section 4 the time constant T is chosen to be a convenient integer so that all k's ≤ 1.

NETWORK

$y_{21}(s) = - y_{12}(s)$



(a) Basic RC Network

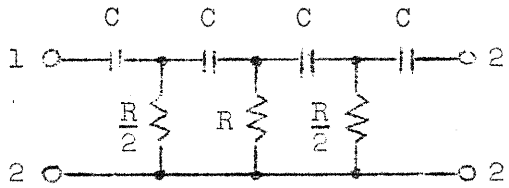


$$\frac{Y^4Z^3/4}{1 + 4YZ + 4Y^2Z^2 + Y^3Z^3}$$

(b) Particular Form

Fig. 4 - Ladder Networks for use in Fig. 1 (Fourth Order Rational Transfer Functions)

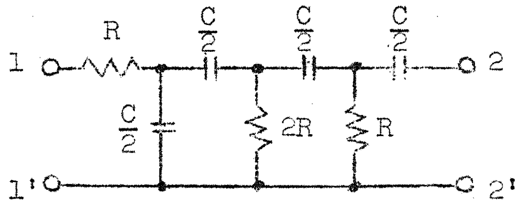
NETWORK	$y_{21}(s) = -y_{12}(s)$



(c) Four Finite Zeros

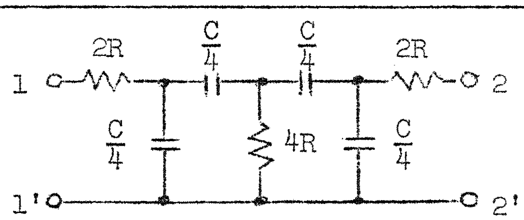
$$\frac{1}{4R} \cdot \frac{s^4T^4}{1 + 4sT + 4s^2T^2 + s^3T^3}$$

where T = RC



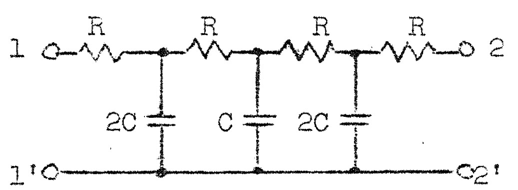
(d) Three Finite Zeros

$$\frac{1}{4R} \cdot \frac{s^3T^3}{1 + 4sT + 4s^2T^2 + s^3T^3}$$



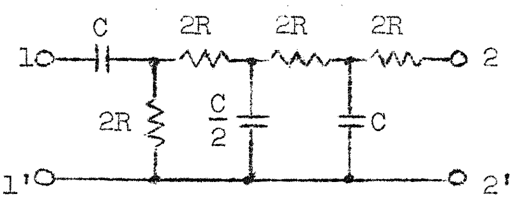
(e) Two Finite Zeros

$$\frac{1}{4R} \cdot \frac{s^2T^2}{1 + 4sT + 4s^2T^2 + s^3T^3}$$



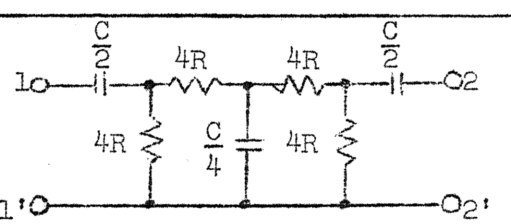
(f) No Finite Zeros

$$\frac{1}{4R} \cdot \frac{1}{1 + 4sT + 4s^2T^2 + s^3T^3}$$



(g) One finite Zero

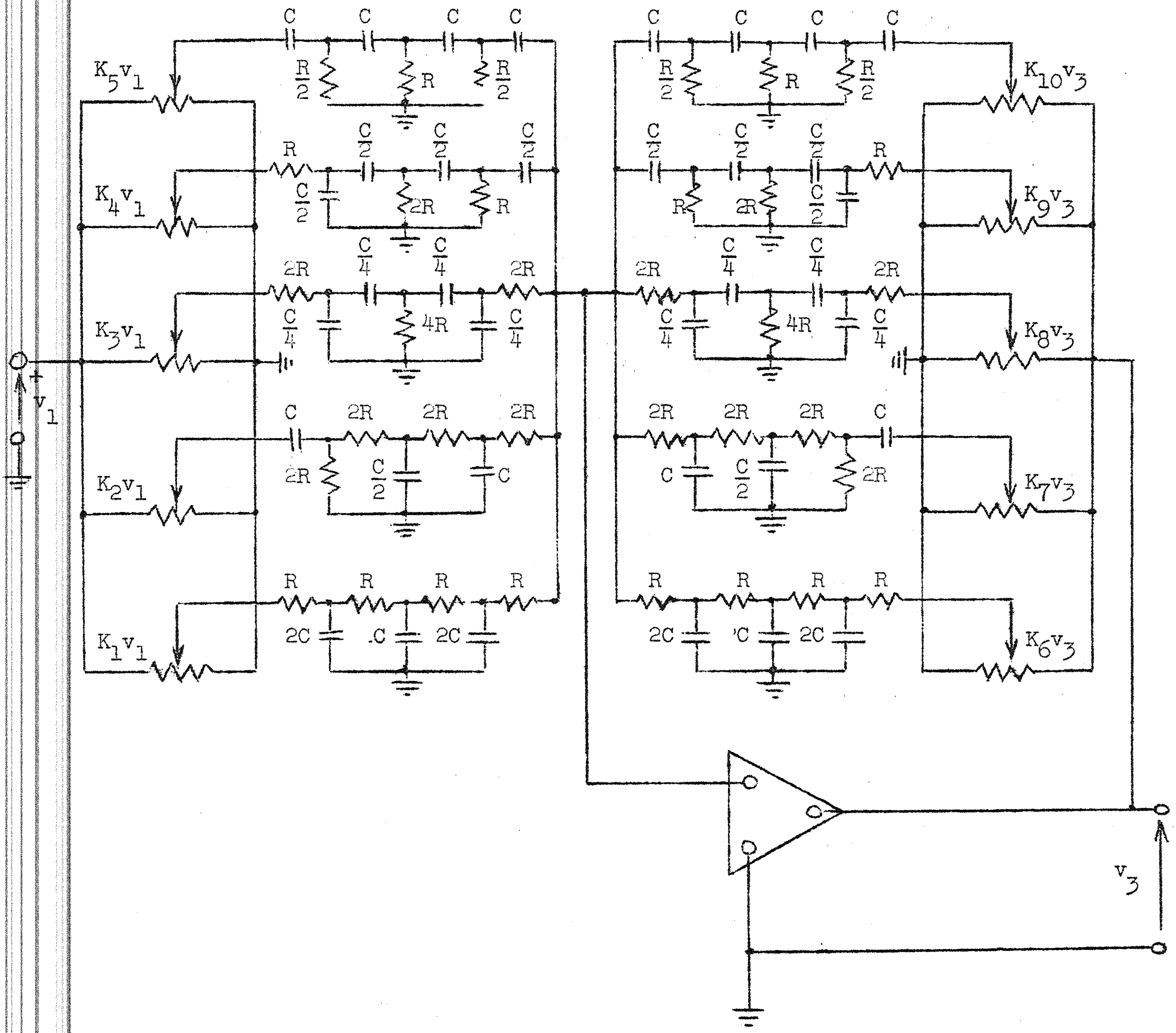
$$\frac{1}{4R} \cdot \frac{sT}{1 + 4sT + 4s^2T^2 + s^3T^3}$$

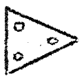


(h) Two Finite Zeros
(Alternative to (e)

$$\frac{1}{4R} \cdot \frac{s^2T^2}{1 + 4sT + 4s^2T^2 + s^3T^3}$$

(Cont.)Fig. 4 Ladder Networks for use in Fig. 1.
(Fourth Order Rational Transfer Functions)



 denotes an operational amplifier

T = RC ; K's 1

Assumption - output resistance of each potentiometer is negligible

$$\frac{V_3(sT)}{V_1(sT)} = -\frac{K_1+K_2sT+K_3s^2T^2+K_4s^3T^3+K_5s^4T^4}{K_6+K_7sT+K_8s^2T^2+K_9s^3T^3+K_{10}s^4T^4}$$

Fig. 5 <u>Active Network Arrangement for Fourth-Order Transfer Functions</u>

6. Stability Considerations

If assumption (b) with reference to (1) is expressed in its pre-limit form, i.e. the forward open-circuit voltage transfer function of active network C approximates to a positive finite constant K over the frequency range of interest, the pre-limit form of equation (1) may be shown to be of the form[2]

$$\frac{V_3(s)}{V_1(s)} = \frac{y^A_{21}(s)}{\left[y^A_{22}(s) - y^B_{11}(s)\right]/K + y^B_{12}(s)} = \frac{y^A_{21}(s)}{\theta(s)} \qquad (19)$$

To ensure stability of the active network arrangement of Fig. 1, it is necessary to check that the denominator of (19) contains no roots with positive real parts. This imposes a restriction on the combinations of the component networks, described in sections 4 and 5, which may be used. For the type of networks discussed in section 4, the overall minus reverse short circuit transfer function of network B is typically of the form

$$y^B_{12}(s) = \frac{1}{3R}\ \frac{k_5 + k_6 sT + k_7 s^2T^2 + k_8 s^3T^3}{1 + \frac{8}{3}\, sT + s^2T^2}$$

The factor $\dfrac{y^A_{22}(s) - y^B_{11}(s)}{K}$ may be expressed in the form

$\dfrac{1}{3R} \cdot \dfrac{a + bsT + cs^2T^2 + ds^3T^3}{K\phi(s)}$ where a, b, c, and d, are positive constants and $\phi(s) = 1 + \frac{8}{3}sT + s^2T^2$.

The denominator of (19) may therefore be expressed in the form

$$\theta(s) = \frac{1}{3R\phi(s)} \cdot \left[\frac{a+bsT+cs^2T^2+ds^3T^3}{K} + k_5 + k_6 sT + k_7 s^2T^2 + k_8 s^3T^3\right] \qquad (20)$$

Application of Routh's Stability Criterion indicates that for stability

$$(k_7 + \frac{c}{K})\ (k_6 + \frac{b}{K}) \geqslant \qquad (k_5 + \frac{a}{K})\ (k_8 + \frac{d}{K}) \qquad (21)$$

For a typical computing amplifier $K \gg 1$ and is of the order of 10^7, or even greater. The terms involving $\frac{1}{K}$ in (21) may therefore be neglected giving the approximate relationship

$$k_6\ k_7 \geqslant\ k_5\ k_8 \qquad (22)$$

It is therefore possible to have a stable network with k_5 and $k_8 = 0$.

However, it is not possible, for all practical purposes, to have stability with either k_6 or $k_7 = 0$.

Similar considerations apply to the fourth order functions described in section 5. For this case a typical form for $y^B_{12}(s)$ is given by

$$y^B_{12}(s) = \frac{1}{4R}\ \frac{K_6 + K_7(sT) + K_8 s^2T^2 + K_9 s^3T^3 + K_{10} s^4T^4}{1 + 4sT + 4s^2T^2 + s^3T^3}$$

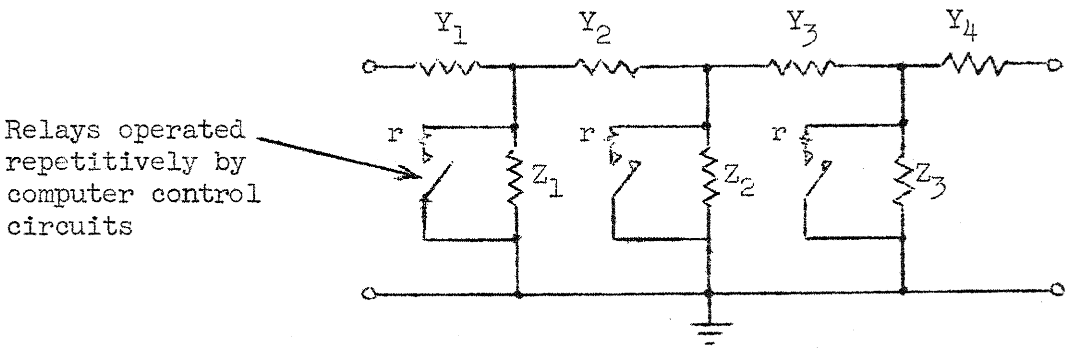
Provided all coefficients of the numerator of this expression are finite, (or in other words all terms are present)stability of the active network is assured if this numerator satisfies Routh's Stability Criterion, i.e. no roots with positive real parts, provided $K \gg 1$.

To determine the condition for stability, if not all numerator terms of the above expression are present, it is necessary to consider the denominator of (19) where, for this case, the term $\left[y^A_{22}(s) - y^B_{11}(s)\right]/K$ will have a numerator of fourth degree in s. If for example, $K_{10} = 0$ the numerator approximates to a third degree function of s and the conditions for third order functions, which have been discussed are applicable. If K_{10} is finite it is possible to have a stable network with $K_7 = 0$ and/or $K_6 = 0$. However, for the general case, it is necessary, as already stated, to investigate the stability of (19).

7. Applications

7.1. Generalized Computing Units

The active networks shown in Figs. 3 and 5, respectively, may be used as generalized computing units. The saving of amplifiers by the use of such units may result in a significant improvement in reliability for a large computer installation comprising say 100 or more computing units. Experience with such large scale computers indicates that the main source of unreliability is the computing amplifier and its associated control circuits. However, if the generalized unit is used in a repetitive computer mode, electro-mechanical relays or solid state switches must be accomodated in each component network to discharge the capacitors. Moreover, only zero initial condition can easily be accomodated. A practical approach, using electro-mechanical reed relays, which has proved to be successful is to connect a relay across each shunt arm of each component network as shown in Fig. 6.



Resistor r chosen to limit peak discharge current through relay contacts.

Fig. 6 Component Network with Relays Incorporated

Thus for a fourth order rational transfer function of the form given by (18), a total number of 30 relays is required. This relatively large number of relays would tend to offset the advantage of a lesser number of amplifiers, in respect of reliability. A better approach is the use of a solid-state switch instead of the relay but difficulties arise due to the presence of leakage currents. However, the development of a suitable solid state switch with the required restricted value of leakage current is currently under development.

7.2. Solution of a Linear Differential Equation with Constant Coefficients

As a specific application of the generalized computing unit, consider the solution of the equation

$$\frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} + y = -100 \quad \varepsilon^{-9t} \qquad (23)$$

with zero initial conditions.

Using the s-multiplied Laplace Transform, the transformed equation is

$$(s^2 + 0.2s + 1)\,\bar{y}(s) = -\frac{100s}{s+9}$$

$$\therefore \qquad \bar{y}(s) = -\frac{100s}{(s+9)(s^2+0.2s+1)}$$

$$= -\frac{100s}{s^3+9.2s^2+2.8s+9}$$

$$\text{i.e.} \qquad \bar{y}(s) = -\frac{10s}{0.9+0.28s+0.92s^2+0.1s^3}$$

$$\text{also} \qquad \bar{y}(sT) = -\frac{10sT}{0.9+0.28sT+0.92s^2T^2+0.1s^3T^3} \qquad (24)$$

where T is chosen to be a convenient integer for time scaling purposes and in the general case to ensure that all coefficients are ≤ 1.

The active network to simulate (24) is given in Fig. 7 where T is chosen to be 0.05 second.

The inverse transform of (24) is given by

$$y(t_1) = -\frac{100}{80.2}\left[\varepsilon^{-180t_1} + 8.95\,\varepsilon^{-2t_1}\sin(19.9t_1 - \psi)\right]u_o(t) \qquad (25)$$

where $t_1 = 0.05t$ and t is real time and $\psi = \arctan 0.1118$, and $u_o(t)$ is unit step function.

$$V_3(sT) = -\frac{20sT}{0.9+0.28sT+0.92s^2T^2+0.1s^3T^3} \equiv 2\,\bar{y}(sT)$$

where $RC = T = 0.05$; $R = 0.5 \times 10^6$ OHM ; $C = 0.1 \times 10^{-6}$ FARAD;
and $V_1(t) = 20\ u_o(t)$ VOLT.

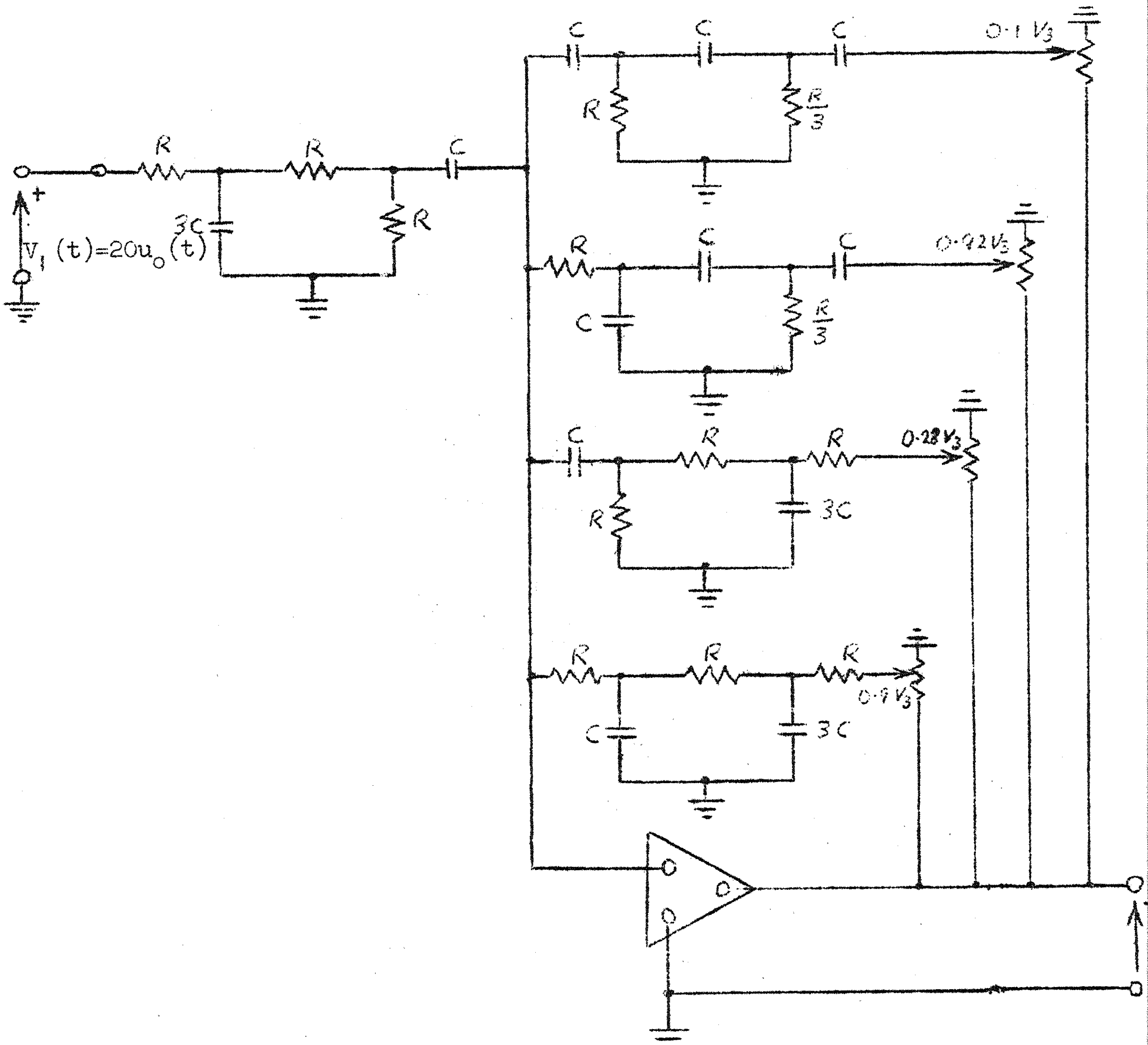
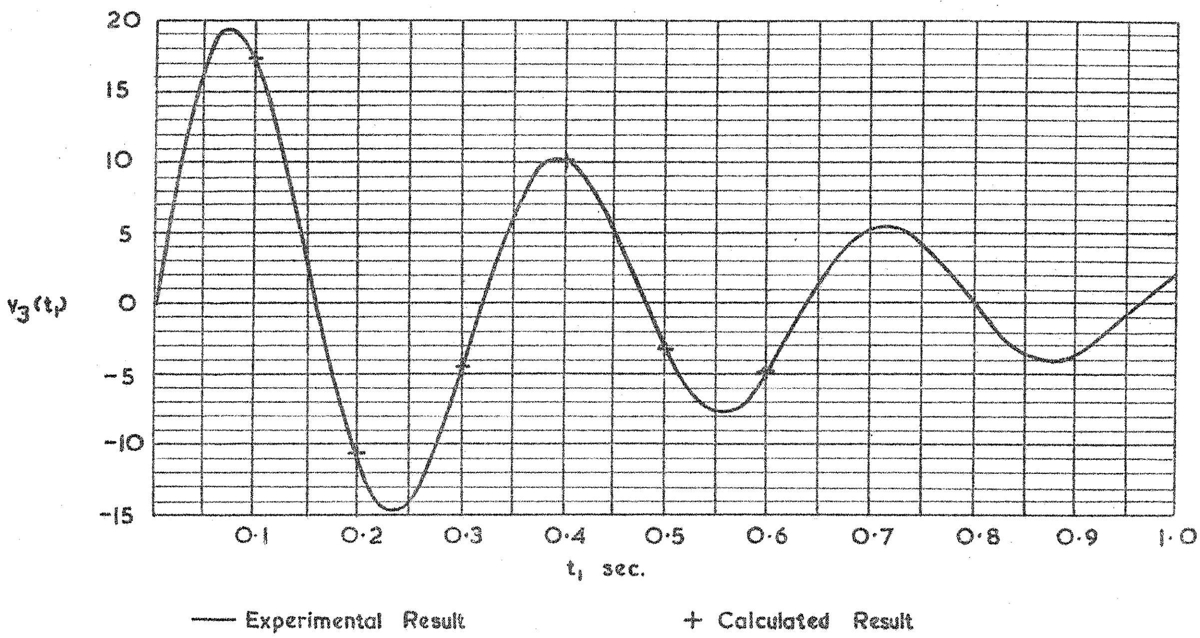
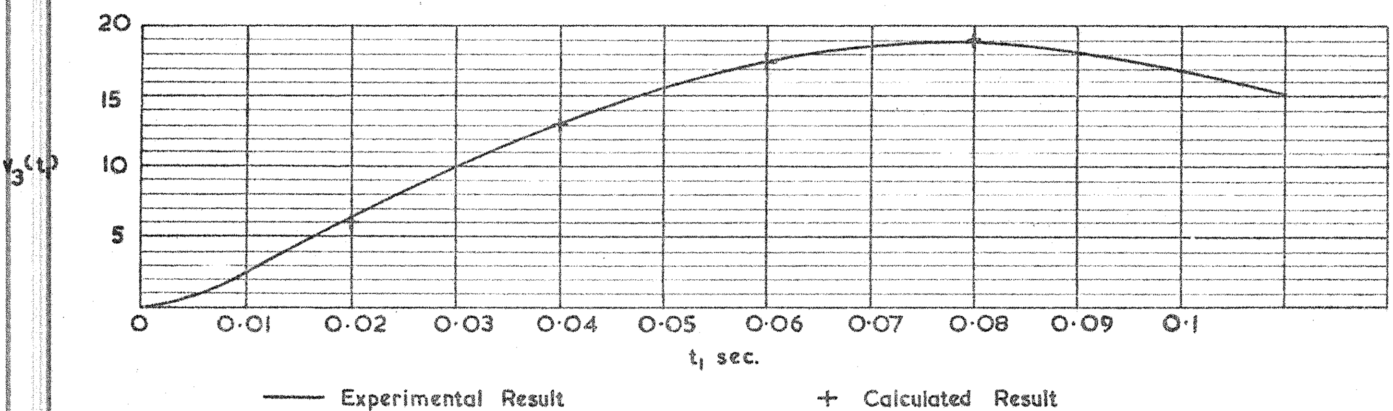


Fig. 7 Simulation of a Linear Differential Equation



RESPONSE TO 20 VOLT STEP INPUT



NOTE: $v_3(t_1) = 2\ y\ (t_1)$

FIG. 8 SOLUTION FOR LINEAR DIFFERENTIAL EQUATION.

The experimental solution, together with calculated values obtained from (25), is shown in Fig. 8. Components were chosen with a nominal accuracy of $\pm$ 0.1%. The nominal value for R was 0.5×10^6 ohms and the nominal value for C was 0.1×10^{-6} farad. The results indicate that the accuracy of the solution is well within 1% of the maximum value for y. However, as the accuracy of the recorder was no better than $\pm$ 1.0%, it is impossible to assess the inherent accuracy from this result.

7.3. Fourth-Order Butterworth Function

This approximation to the ideal low-pass filter is given by

$$F(sT) = G(sT) = G(-sT) = \frac{1}{1 + (-1)^n (sT)^{2n}}$$

$$\text{i.e.} \quad F(j\omega T) = G(j\omega T)^{2} = \frac{1}{1 + (\omega T)^{2n}} = \tfrac{1}{2} \text{ for } \omega T = 1 \qquad (26)$$

where $F(j\omega T)$ represents the square of the modulus of the frequency response function $G(j\omega T)$ and n is a positive integer.

For n = 4,

$$G(sT) = \frac{V_3(sT)}{V_1(sT)} = \frac{1}{1+2.6131sT+3.4142s^2T^2+2.6131s^3T^3+s^4T^4}$$

$$= \frac{1}{1+K_7.4sT+K_8.4s^2T^2+K_9.4s^3T^3+s^4T^4} \qquad (27)$$

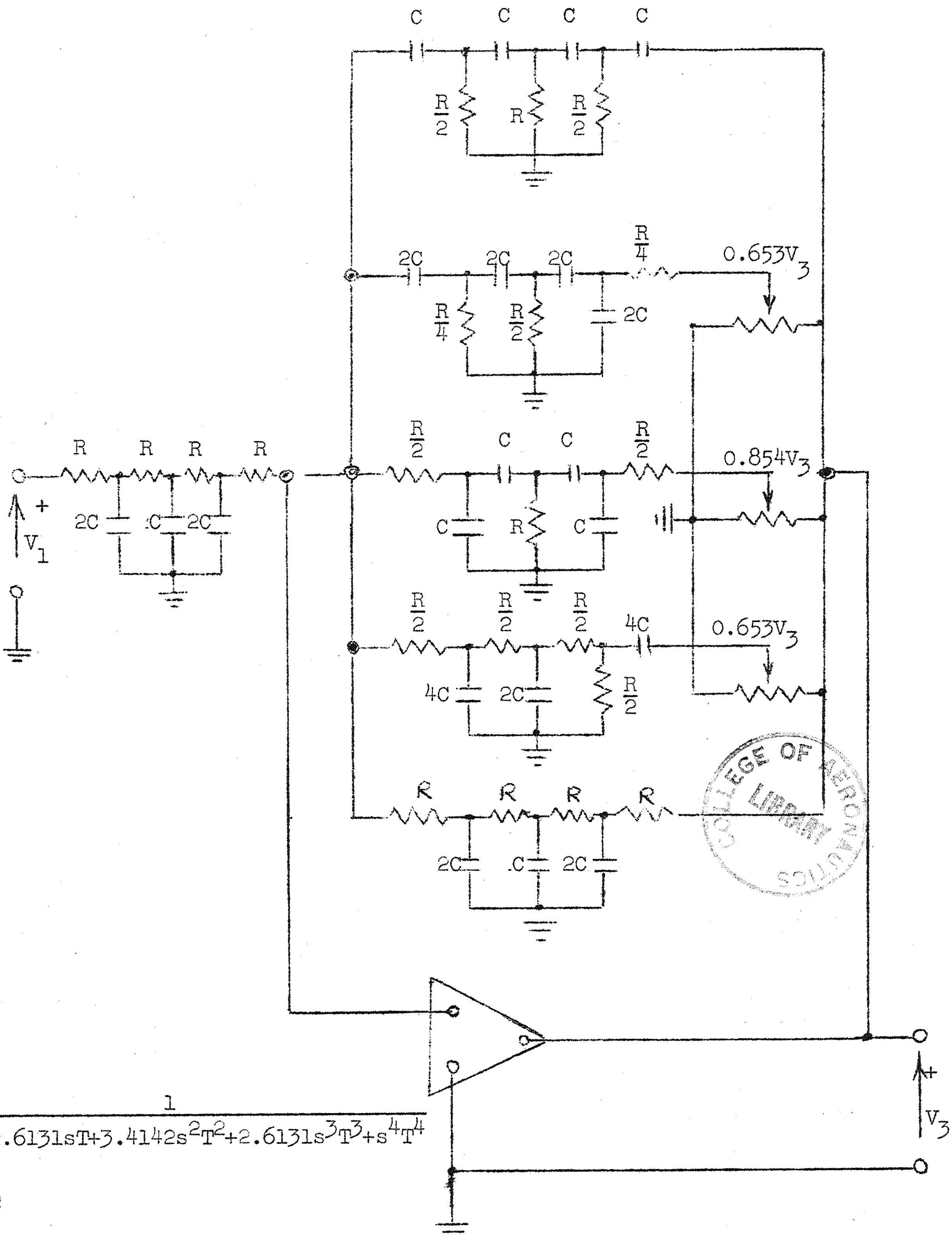
where $K_7 = 0.6533$; $K_8 = 0.8535$ and $K_9 = 0.6533$.

With reference to Figs. 4 and 5,

$$V_3(sT)\,\frac{1}{1+4sT+4s^2T^2+s^3T^3}\left(\frac{1}{4R} + \frac{sT}{R} + \frac{s^2T^2}{R} + \frac{s^3T^3}{R} + \frac{s^4T^4}{4R}\right) = -\frac{V_1(sT)}{4R}\left(\frac{1}{1+4sT+4s^2T^2+s^3T^3}\right) \qquad (28)$$

with T = RC

Thus (27) may be realized (with a sign reversal) by the active arrangement of Fig. 1 with network A consisting of network (f) of Fig. 4 and network B comprising the following networks of Fig. 4, i.e. (c) and (f) unmodified and (d), (e) and (g) modified to give a scale factor of 4. Thus with reference to Fig. 4(d), 4(e) and (g) all resistor values are divided by 4 and all capacitor values multiplied by 4.



$$\frac{V_3(sT)}{V_1(sT)} = -\frac{1}{1+2.6131sT+3.4142s^2T^2+2.6131s^3T^3+s^4T^4}$$

$T = RC$

Fig. 9 Realization of Fourth Order Butterworth Function

The active network to suit these conditions is shown in Fig. 9 and the experimental results obtained, with $R = 10^6$ ohms and $C = 0.1 \times 10^{-6}$ farads, are given in Fig. 10 and Table 1 together with the calculated results. The potentiometers used had a value of 2×10^4 ohms with a maximum output resistance of 5.0×10^4 ohms. This finite output resistance obviously affects the accuracy of the simulation and this accounts, to some extent, for the discrepancy between calculated and observed values. The tolerance of each resistor and capacitor used was ± 0.1 % which also accounts for some discrepancy in the results. The error involved in the practical measurement of the gain of the network was less than 0.5%; this being the limit set by noise, resolution and other spurious effects. It may be seen from table 1, that the experimental and theoretical results are in good agreement in the frequency band about $\omega T = 1$.

TABLE 1

Normalized Frequency	Theoretical Attenuation	Experimental Result
0.9	- 1.553 dB	- 1.560 ±0.005 dB
1.0	- 3.010 dB	- 3.011 ±0.005 dB
1.2	- 7.346 dB	- 7.250 ±0.005 dB
1.4	- 11.97 dB	- 11.78 ±0.05 dB
1.6	- 16.43 dB	- 16.08 ±0.05 dB
2.0	- 24.10 dB	- 24.15 ±0.05 dB

Comparison of Theoretical and Experimental Results

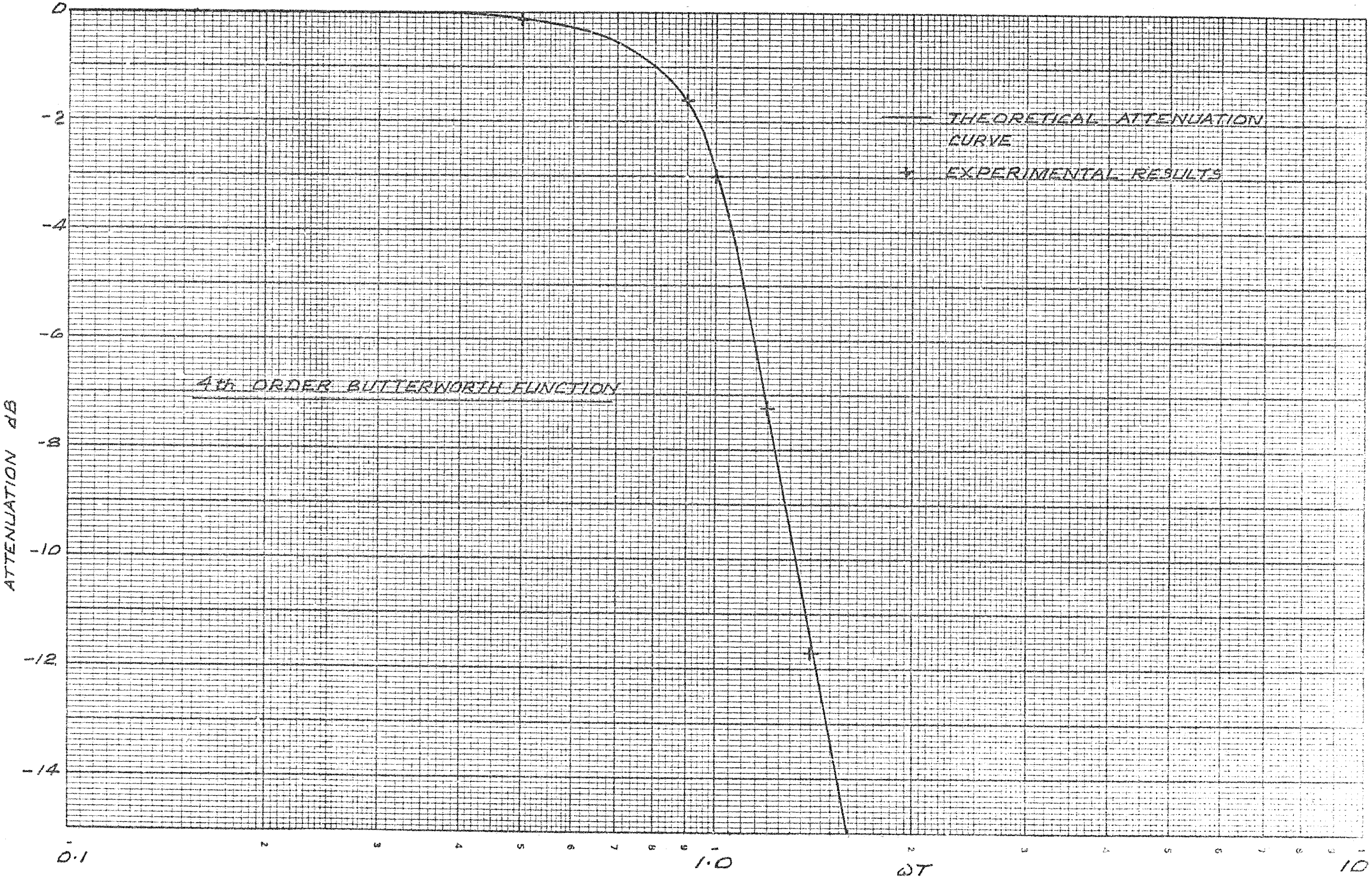
8. Conclusions

The realization may be achieved of minimum-phase third and fourth order rational transfer functions with adjustable coefficients, by the use of one operational amplifier associated with unbalanced RC ladder networks. The procedure described results in a larger number of components than that required by more conventional synthesis techniques. However, the use of redundant structures provides the facility of independent adjustment of each coefficient of the transfer functions with the added advantage that the spread of component values may be controlled to some extent.

The necessity for a relay, connected across each shunt arm of each component network, to ensure correct initial conditions for repetitive computer operation, limits the use of the technique for this application.

A further limitation of the technique may be the sensitivity of the transfer function to changes in component values. Preliminary investigations indicate that this is not a serious matter with quadratic functions but this may not be the case with higher order functions.

Perhaps the main advantage of the methods described is that the number of computing amplifiers required, for the realization of a given transfer function, may be minimized. This fact should improve reliability particularly if repetitive computer operation is not required.



4th ORDER BUTTERWORTH FUNCTION
THEORETICAL ATTENUATION CURVE
EXPERIMENTAL RESULTS
ATTENUATION dB
0
-2
-4
-6
-8
-10
-12
-14
0·1
1·0
10
ωT

FIG. 10.

Finally, the necessary conditions for stability of the active network imposes some restriction on the form of transfer functions which may be realized.

9. Acknowledgement

Thanks are due to A.J. Martin of the Electrical and Control Engineering Department for the experimental results given in the text.

REFERENCES

1. R.J.A. Paul — "Simulation of Rational Transfer Functions with Adjustable Coefficients", CofA Note No. 126, April, 1962.

2. R.J.A. Paul — "Simulation of Rational Transfer Functions with Adjustable Coefficients", Proc. I.E.E., 110, 4, 1963, pp 673-679.