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Some Related Oscillation Problems

-by-

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SUMMARY

Two simple means for establishing a relation between a pair of oscillation problems are briefly discussed. In the first, the displacements are connected by use of a differential operator. The set of natural frequencies is identical for the two problems and results of interest are obtained when the transformed boundary conditions can be physically interpreted. In this manner it is shown, for example, that a flywheel on a uniform shaft can be transformed into a flexible coupling and a mass carried on a uniform beam into a flexible hinge. In the second, the connection is established by use of the concept of mechanical admittance. Here the frequency equations are simply related but the frequencies are not.

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CONTENTS

1. Introduction

In this paper attention is drawn to two simple ways in which problems of free oscillation can be connected. For the first, the displacement in problem B is derived from that in problem A by means of a linear differential operator and the two problems have the same set of natural frequencies. The boundary conditions for B differ from those for A and in some instances it is possible to give them a simple physical interpretation. When this is so, results of interest can be obtained. For the second, the connection is established by use of the concept of admittance[1,2] and here the equations which determine the natural frequencies are simply related but the natural frequencies themselves are not simply related.

The first method of connection is illustrated by some applications to the free oscillations of uniform shafts and beams. By use of very simple operators it is shown that the end conditions are transformed into others which have an easy interpretation; likewise the conditions at such discontinuities as carried masses and flexible joints are transformed into others which can be interpreted. For example, a flywheel carried on a uniform shaft can be transformed into a flexible coupling and a massive particle carried by a uniform beam can be transformed into an elastic hinge.

The connection of problems through the use of admittances is only briefly touched on here. The illustrative examples relate to shafts and beams and are thus relevant to the problems discussed by the other method.

2. Problems Related by a Differential Operator (General)

Suppose that the displacement u in some pure mode of oscillation of a body satisfies the differential equation

$$f(D)u - \omega^2 u = 0 \quad \ldots\ldots (2,1)$$

where ω is 2π times the frequency and $f(D)$ stands for a linear differential operator in the spatial coordinate or coordinates. Operate on (2,1) with the linear differential operator $\varphi(D)$. We get

$$\varphi(D)\, f(D)\, u - \omega^2 \varphi(D)\, u = 0. \quad \ldots\ldots (2,2)$$

Now suppose that $f(D)$ and $\varphi(D)$ are commutative. Then the last equation can be written

$$f(D)\, v - \omega^2 v = 0 \quad \ldots\ldots (2,3)$$

where

$$v = \varphi(D)\, u. \quad \ldots\ldots (2,4)$$

Thus v is also a solution of (2,1) but it will not, in general, satisfy the same boundary conditions* as u. However, it will be possible to obtain the boundary conditions for v from those for u and it may be possible to give these a physical interpretation. When this can be done we shall have a pair of related problems whose modal displacements are u and v respectively and which have an identical set of natural frequencies, for the same ω appears in (2,1) and (2,3).

/We ...

* We include here the conditions at points of discontinuity in the solution, such as points where loads are applied.

We may remark that $f(D)$ and $\varphi(D)$ always commute when they are polynomials in D with constant coefficients or like polynomials in D_1, D_2 etc. where these symbols represent partial differentiation with respect to corresponding independent variables. More generally, the operators will commute when they are both polynomials with constant coefficients of the same linear operator, whose coefficients need not be constants.

The examples given in §§3 and 4 show that it is in fact possible to derive interesting and useful results by means of the device described above.

3. Torsional Oscillations of Uniform Shafts.

3.1. Preliminaries

When the shaft is oscillating freely in one pure mode the angular displacement θ satisfies the equation

$$D^2 \theta + \mu^2 \theta = 0. \qquad \text{(3.1,1)}$$

where $$\mu^2 = \frac{\omega^2 j}{C} \qquad \text{(3.1,2)}$$

with j = moment of inertia of unit length of shaft about its axis,

C = torsional stiffness of unit length of shaft,

$D = \dfrac{d}{dx}$.

The value of the characteristic number μ depends on the boundary conditions and on the particular mode selected.

It follows from (3.1,1) by differentiation that

$$D^{n+2} \theta + \mu^2 D^n \theta = 0. \qquad \text{(3.1,3)}$$

Hence at any point where

$$D^n \theta = 0$$

we must have also

$$\left.\begin{aligned} D^{n+2} \theta &= 0 \\ D^{n-2} \theta &= 0 \quad (n \geq 2) \end{aligned}\right\} \qquad \text{(3.1,4)}$$

where in the last equation it is assumed that ω is not zero.

The conditions to be satisfied at the supports and at discontinuities of various kinds are as follows, where the displacements to the right and left of a discontinuity are denoted by θ_r and θ_l respectively.

(A) Fixed End

$$\theta = 0. \qquad \text{(3.1,5)}$$

(B) Free End

$$D\theta = 0. \qquad \text{(3.1,6)}$$

/(C) ...

(C) Spring Support

$$\left.\begin{aligned} CD\theta_r &= CD\theta_l + \sigma\theta_l \\ \theta_r &= \theta_l \end{aligned}\right\} \qquad \ldots\ldots (3.1,7)$$

where σ is the stiffness (restoring moment per radian) of the support.

(D) Carried Flywheel

$$\left.\begin{aligned} CD\theta_r &= CD\theta_l - J\omega^2\theta_l \\ \theta_r &= \theta_l \end{aligned}\right\} \qquad \ldots\ldots (3.1,8)$$

where J is the moment of inertia of the flywheel.

(E) Spring Coupling

$$\left.\begin{aligned} s\theta_r &= s\theta_l + CD\theta_l \\ D\theta_r &= D\theta_l \end{aligned}\right\} \qquad \ldots\ldots (3.1,9)$$

where s is the stiffness of the coupling (moment per radian of relative twist) which is assumed to have a negligible moment of inertia.

(F) Change of Torsional Stiffness of Shaft

Displacement and torque are continuous. Hence

$$\left.\begin{aligned} \theta_r &= \theta_l \\ C_r D\theta_r &= C_l D\theta_l. \end{aligned}\right\} \qquad \ldots\ldots (3.1,10)$$

3.2. Application of the Operator D to Torsional Oscillations

Let us take the new variable to be

$$\varphi = D\theta$$

which implies

$$D\varphi = D^2\theta = -\mu^2\theta \qquad \ldots\ldots (3.2,1)$$

and

$$D^2\varphi = D^3\theta = -\mu^2 D\theta$$

Example 1. Apply Transformation to Free-Free Shaft without Discontinuities.

The free ends become fixed ends and we see that a uniform shaft with both ends free has the same spectrum of non-vanishing frequencies as an equal shaft with fixed ends. In fact the frequency equation is

$$\sin \mu l = 0 \qquad \ldots\ldots (3.2,2)$$

in both cases, where l is the length of the shaft, but the zero root does not apply to the shaft with fixed ends.

/Example 2. ...

Example 2. Apply Transformation to Carried Flywheel.

Equations (3.1,8) become when transformed and reduced

$$\varphi_r = \varphi_l + \frac{J}{j} D\varphi_l$$

$$D\varphi_r = D\varphi_l.$$

These are identical with equations (3.1,9) provided that

$$s = \frac{Cj}{J}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2,3)$$

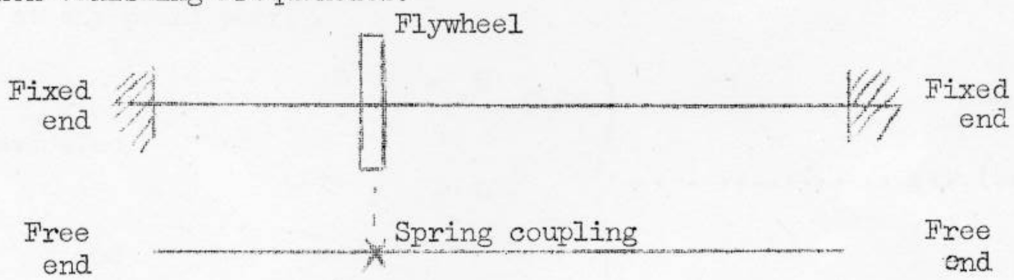
Thus a carried flywheel is transformed into a spring coupling whose stiffness is given by the last equation. The relation is a reciprocal one for we find that a spring coupling is transformed into a flywheel whose moment of inertia is

$$J = \frac{Cj}{s}$$

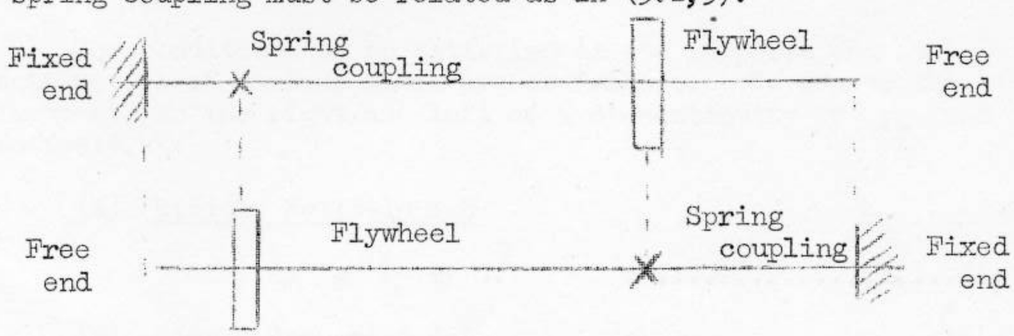
in accordance with (3.2,3). It is to be remarked that the relation (3.2,3) is independent of the frequency.

3.3. Some Specific Examples of Related Torsional Problems

In each case the two systems sketched have identical spectra of non-vanishing frequencies.



The moment of inertia of the flywheel and the stiffness of the spring coupling must be related as in (3.2,3).



Corresponding flywheels and spring couplings are related as in (3.2,3). As a special case, a shaft with an end anchored through a spring coupling is related to one carrying a flywheel at a free end

/4. Flexural ...

4. Flexural Oscillations of Uniform Beams

4.1. Preliminaries

The differential equation satisfied by the normal displacement y of the beam when oscillating in a single pure mode is

$$D^4 y - \beta^4 y = 0 \qquad \text{(4.1,1)}$$

where

$$D \equiv \frac{d}{dx}$$

x = abscissa, measured along the axis of the beam

and

$$\beta^4 = \frac{m\omega^2}{EI} \qquad \text{(4.1,2)}$$

with m = mass of beam per unit run (constant).

EI = constant flexural rigidity of beam.

The value of the characteristic number β depends on the boundary conditions and on the particular free mode considered.

It follows from (4.1,1) by differentiation that

$$D^{n+4} y - \beta^4 D^n y = 0. \qquad \text{(4.1,3)}$$

Thus at any point where

$$D^n y = 0$$

we have also

$$D^{n+4} y = 0$$

and

$$D^{n-4} y = 0 \quad (n \geqslant 4) \qquad \text{(4.1,4)}$$

where in the last equation it is assumed that the frequency is not zero.

The conditions to be satisfied at the supports and at discontinuities of various kinds are as follows. We denote the displacements to the right and left of a discontinuity by y_r and y_l respectively.

(A) Rigidly Built-in End

$$y = Dy = 0. \qquad \text{(4.1,5)}$$

(B) Simply Supported End

$$y = D^2 y = 0. \qquad \text{(4.1,6)}$$

(C) Free End

$$D^2 y = D^3 y = 0. \qquad \text{(4.1,7)}$$

/(D) Spring ...

(D) Spring Support giving Restoring Force Proportional to Normal Deflexion

Let the stiffness of the spring support be σ. Then

$$\left.\begin{aligned} y_r &= y_l \\ Dy_r &= Dy_l \\ D^2y_r &= D^2y_l \\ EI\,D^3y_r &= EI\,D^3y_l - \sigma y_l \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1,8)$$

(E) Spring Support giving Restoring Couple Proportional to Change of Slope

Let s be the stiffness (moment per radian). Then

$$\left.\begin{aligned} y_r &= y_l \\ Dy_r &= Dy_l \\ EI\,D^2y_r &= EI\,D^2y_l + s\,Dy_l \\ D^3y_r &= D^3y_l. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1,9)$$

(F) Carried Mass

The mass M is supposed to be rigidly connected to the beam at the neutral axis, and not to influence the local flexural rigidity. The moment of inertia of the carried mass about the point of attachment is J. Then

$$\left.\begin{aligned} y_r &= y_l \\ Dy_r &= Dy_l \\ EI\,D^2y_r &= EI\,D^2y_l - J\omega^2 Dy_l \\ EI\,D^3y_r &= EI\,D^3y_l + M\omega^2 y_l. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1,10)$$

For a mere particle we may make J zero. We could take J to be finite with M negligible if the radius of gyration were sufficiently large.

(G) Elastic Flexural Hinge in Beam

Let s be the stiffness of the hinge (moment per radian of angular deflexion of hinge). Then

$$\left.\begin{aligned} y_r &= y_l \\ sDy_r &= s\,Dy_l + EI\,D^2y_l \\ D^2y_r &= D^2y_l \\ D^3y_r &= D^3y_l. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1,11)$$

/(H) Sliding ...

(H) Sliding Connection giving Relative Deflexion Proportional to Shearing Force

The connection is supposed to preserve continuity of slope. The stiffness is σ (force per unit of relative displacement).

$$\left.\begin{aligned} y_r &= \sigma y_l - EI D^3 y_l \\ Dy_r &= Dy_l \\ D^2 y_r &= D^2 y_l \\ D^3 y_r &= D^3 y_l . \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (4.1,12)$$

4.2. Application of the Operator D^2 to Flexural Oscillations

Some results of interest can be obtained by taking the new variable to be

$$\left.\begin{aligned} z &= D^2 y. \\ \text{Then} \qquad Dz &= D^3 y \\ D^2 z &= D^4 y = \beta^4 y \\ D^3 z &= D^5 y = \beta^4 Dy. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (4.2,1)$$

From (4.1,5), (4.1,6), (4.1,7) and (4.2,1) we see that the transformations of end conditions are.-

Built-in	becomes	free
Simply supported	"	simply supported
Free	"	built-in.

Example 1. Apply Transformation to Free-Free Beam without Discontinuities.

By the results just obtained we deduce that the variable z is the deflexion in a free oscillation of a doubly built-in beam. We see that the free-free and doubly built-in beams (of identical lengths and flexural rigidities) have the same spectrum of non-vanishing natural frequencies. It is well known that the frequency parameter β for both cases satisfies

$$\cos \beta l \cosh \beta l = 1. \qquad \ldots\ldots\ldots\ldots (4.2,2)$$

Example 2. Apply Transformation to Beam without Discontinuities having Built-in and Simply Supported Ends.

The transformed beam, which has the same spectrum of non-vanishing frequencies as the original, has one end free and the other simply supported. The frequency equation is

$$\tan \beta l = \tanh \beta l. \qquad \ldots\ldots\ldots\ldots (4.2,3)$$

/Example 3.

Example 3. Cantilever Beam.

This transforms into a cantilever beam, but the ends are interchanged.

We now apply the transformation to some of the discontinuities already listed. Not all of the results find obvious physical interpretations and only the useful results are recorded here.

Transformation applied to Carried Particle. We find from (4.1,10) with J made zero and (4.2,1) that

$$z_r = z_l$$

$$D^2 z_r = D^2 z_l$$

$$D^3 z_r = D^3 z_l$$

$$Dz_r = Dz_l + \frac{M\omega^2}{EI\beta^4} D^2 z_l$$

$$= Dz_l + \frac{M}{m} D^2 z_l.$$

Therefore $$sDz_r = sDz_l + \frac{sM}{m} D^2 z_l.$$

By comparison with (4.1,11) we see that these conditions appertain to a flexural hinge of stiffness s given by

$$s = \frac{m\,EI}{M}. \qquad \text{..............} (4.2,4)$$

It is notable that this relation is independent of the frequency.

Transformation applied to Flexural Hinge. We find from (4.1,11) and (4.2,1) that the flexural hinge is transformed into a carried particle of mass M given by

$$M = \frac{m\,EI}{s},$$

which accords with (4.2,4).

Transformation applied to Flywheel fixed to Beam at Simple Support. Suppose that the flywheel of moment of inertia J is fixed to the beam at its left hand end, which is simply supported. By (4.1,10)

$$y = 0$$

$$EI\,D^2 y = -J\omega^2 Dy.$$

On application of the transformation and use of (4.1,2) these equations become

$$D^2 z = 0$$

$$z = -\frac{J}{m} D^3 z.$$

/By ...

By (4.1,8) se see that these conditions relate to the left hand end of a beam with a spring support whose stiffness is

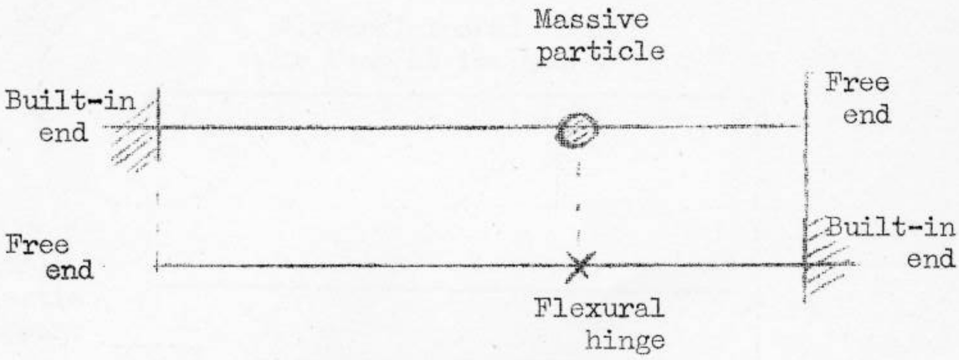
$$\sigma = \frac{m\,EI}{J} \qquad \ldots\ldots\ldots\ldots\ldots (4.2,5)$$

and with no other constraint at this end.

4.3. Some Specific Examples of Related Flexural Problems.

A large number of pairs of problems having identical spectra of non-vanishing frequencies can be constructed by use of the foregoing results. A few specimens of these are set out here.

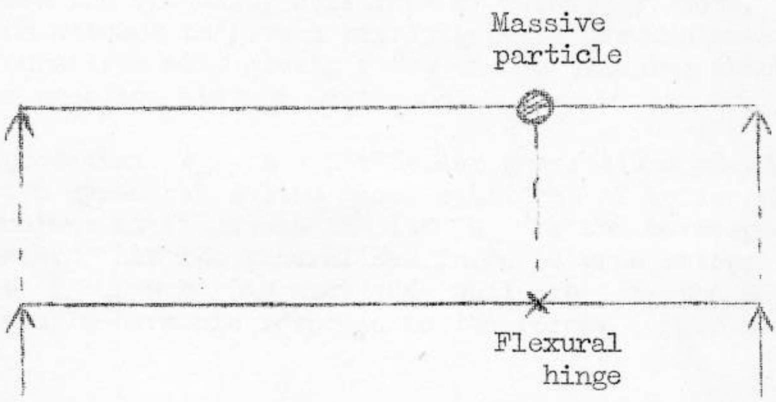
(a)



The mass of the particle and the hinge stiffness are related as in (4.2,4).

As a special case we may take the particle to be at the free end of the first cantilever. Then the flexural hinge is at the root of the second cantilever, which thus has angular elastic yield at the root.

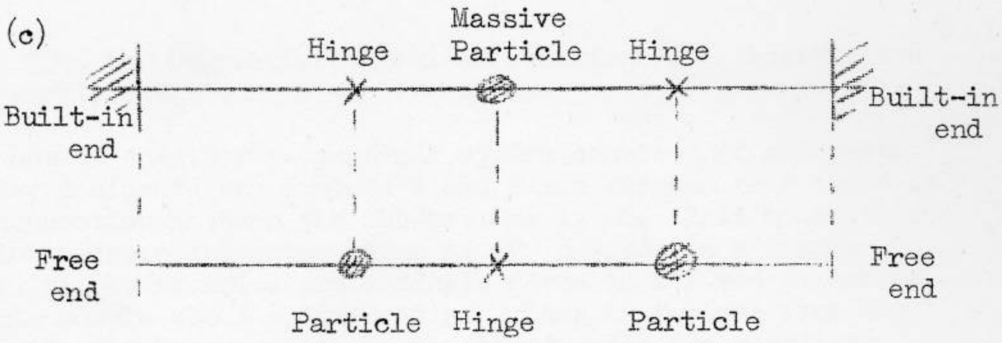
(b)



Here the beams are both simply supported and the relation (4.2,4) must again be satisfied.

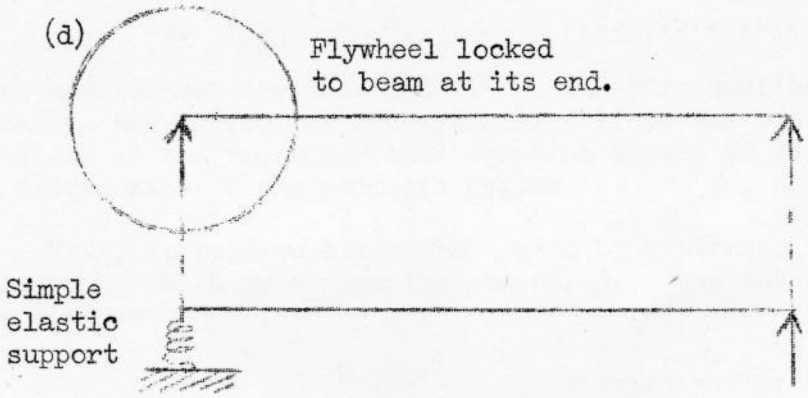
/(c) ...

(c)



Relation (4.2.4) is satisfied by each corresponding pair of particles and hinges.

(d)



The moment of inertia of the flywheel and the stiffness of the support are related by (4.2,5).

5. Oscillation Problems Related by the Admittance Concept

The method of admittances[1,2] very readily enables us to establish relations between the frequency equations of related systems. Here we shall make no attempt to give a highly general discussion and shall content ourselves with giving a few simple examples which are relevant to the problems already discussed.

Suppose that q_r is a particular generalised coordinate of a conservative dynamical system whose equations of motion are linear with constant coefficients and let Q_r be the corresponding generalised force. Let the generalised force be proportional to $\sin \omega t$ and let Q_r now be its amplitude while q_r is the amplitude of the steady simple-harmonic response to the force. Then the quantity

$$\alpha_{rr} = \frac{q_r}{Q_r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,1).$$

is called the direct admittance for the coordinate q_r and it is the dynamic flexibility for simple-harmonic motion. If in the same circumstances the amplitude of the s^{th} generalised coordinate is q_s the quantity

$$\alpha_{sr} = \frac{q_s}{Q_r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,2).$$

is called a cross admittance. For the kind of system considered the cross admittances have the reciprocal property

$$\alpha_{rs} = \alpha_{sr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,3)$$

/as ...

as follows from the Lagrangian dynamical equations[2]. Admittances are, in general, functions of the frequency.

Let us consider a dynamical system composed of two parts which we may designate sub-systems 1 and 2 and suppose that there is a simple connection between the sub-systems at P. This means that the reaction between the sub-systems at P depends on a single parameter; simple examples are a single force in a fixed direction and a single couple about a fixed axis. Then it follows from the facts that the displacement at P is shared while the reactions are equal and opposite that the sum of the admittances at P of the sub-systems is zero for any free oscillation of the complete system. Thus for any free oscillation of the system

$$ {}_1\alpha + {}_2\alpha = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,4)$$

where the admittances are measured at P and the numerical subscripts correspond to the respective sub-systems. Since the admittances are functions of the frequency this equation serves to determine the natural frequencies of the complete system.

Next, suppose we alter the system by replacing sub-system 2 by another to which we assign the numeral 3. Then the frequency equation becomes

$$ {}_1\alpha + {}_3\alpha = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,5)$$

This is of particular utility when sub-system 1 is relatively complicated or when it is a continuous elastic body.

The method briefly discussed above can easily be extended to systems whose sub-systems are multiply connected[2].

We now give some very simple examples which concern thin beams and we adopt the notation of §4.

Example 1. Oscillations of a uniform simply supported beam loaded at mid-span.

It can easily be shown that the admittance giving the normal displacement at mid-span for simple-harmonic normal loads applied there is

$$ {}_1\alpha = \frac{\tan\frac{\beta l}{2} - \tanh\frac{\beta l}{2}}{4\,\beta^3\,EI}. \qquad \ldots\ldots (5,6)$$

Now let the beam be provided with an elastic support at mid-span of stiffness σ and of negligible inertia. Then

$$ {}_2\alpha = \frac{1}{\sigma} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,7)$$

and the equation determining the frequencies of the propped beam is accordingly

$$ \frac{\tan\frac{\beta l}{2} - \tanh\frac{\beta l}{2}}{4\,\beta^3\,EI} + \frac{1}{\sigma} = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5,8)$$

/If ...

If we substitute a carried particle of mass M for the elastic support we have

$$\alpha_3 = \frac{-1}{M\omega^2}$$

and the frequency equation becomes

$$\frac{\tan\frac{\beta l}{2} - \tanh\frac{\beta l}{2}}{4\beta^3 EI} - \frac{1}{M\omega^2} = 0. \quad \ldots\ldots\ldots\ldots\ldots (5,9)$$

Similarly, if we connect the particle to the beam through a spring of stiffness σ the admittance of the spring cum particle as measured at the end of the spring is

$$\frac{1}{\sigma} - \frac{1}{M\omega^2}$$

and the frequency equation is therefore

$$\frac{\tan\frac{\beta l}{2} - \tanh\frac{\beta l}{2}}{4\beta^3 EI} + \frac{1}{\sigma} - \frac{1}{M\omega^2} = 0. \quad \ldots\ldots\ldots\ldots (5,10)$$

Example 2. A free uniform beam loaded at its middle.

It follows as above on making use of the appropriate admittance for the beam that the frequency equation when there is an elastic support at the middle of the beam is

$$\frac{-\left(1 + \cosh\frac{\beta l}{2}\cos\frac{\beta l}{2}\right)}{2\beta^3 EI\left(\cosh\frac{\beta l}{2}\sin\frac{\beta l}{2} + \cos\frac{\beta l}{2}\sinh\frac{\beta l}{2}\right)} + \frac{1}{\sigma} = 0. \quad .(5,11)$$

For a particle of mass M carried at the middle of the beam the frequency equation becomes

$$\frac{\left(1 + \cosh\frac{\beta l}{2}\cos\frac{\beta l}{2}\right)}{2\beta^3 EI\left(\cosh\frac{\beta l}{2}\sin\frac{\beta l}{2} + \cos\frac{\beta l}{2}\sinh\frac{\beta l}{2}\right)} + \frac{1}{M\omega^2} = 0. \quad .(5,12)$$

Example 3. Cantilever with angular yields at the root.

It is known[3] that the frequency equation for a uniform cantilever having elastic angular yield at the root is, in the present notation,

$$1 + \cosh\beta l\cos\beta l + \frac{EI\beta}{s}\left(\sinh\beta l\cos\beta l - \sin\beta l\cosh\beta l\right) = 0. \quad \ldots\ldots (5,13)$$

Since the angular admittance of the support, which is supposed to have no inertia, is $1/s$, it follows on

/comparison ...

comparison with (5,4) that the angular direct admittance of a simply supported-free beam for couples applied at its simply supported end is

$$_1\alpha = \frac{1 + \cosh \beta l \cos \beta l}{EI\beta(\sinh \beta l \cos \beta l - \sin \beta l \cosh \beta l)} \qquad \ldots\ldots(5,14)$$

and it is easy to verify this by direct calculation. If now we replace the elastic support by a pivoted flywheel of moment of inertia J to which the beam is attached at its root, the frequency equation is obtained from (5,13) on substituting

$-\omega^2 J$ for s.

LIST OF SYMBOLS

Note. The three quantities

$$\frac{jC}{sJ}, \quad \frac{m\,EI}{s\,M} \quad \text{and} \quad \frac{m\,EI}{\sigma\,J}$$

are non-dimensional. They all have the value unity for the related systems discussed in §§ 3 and 4.

The Greek symbols are listed after the Roman.

C	torsional stiffness of unit length of shaft
C_r, C_l	values of C to right and left of a discontinuity, respectively.
D	operator $\frac{d}{dx}$
D_1, D_2	partial differential operators with respect to x_1, x_2 respectively.
E	Young's modulus
$f(D)$	a linear differential operator
I	second moment of section of beam
J	moment of inertia of a flywheel
j	moment of inertia of unit length of shaft
l	length of shaft or beam
M	mass of particle
m	mass of unit length of beam
Q_r	generalised force corresponding to coordinate q_r
q_r	r^{th} generalised coordinate
s	stiffness of elastic coupling or hinge (moment per radian).
u, v	displacements in a pure oscillatory mode
x	abscissa, measured along shaft or beam
y	deflexion of beam
y_r, y_l	values of y to right and left, respectively, of a discontinuity
z	deflexion of beam
z_r, z_l	values of z to right and left, respectively, of a discontinuity

α an admittance

${}_1\alpha$, ${}_2\alpha$, ${}_3\alpha$ admittances of bodies 1, 2, 3, respectively

α_{rr} direct admittance for coordinate q_r

α_{rs}, α_{sr} equal cross admittances

$\beta = \sqrt[4]{\frac{m\omega^2}{EI}}$

θ angular displacement of shaft

θ_r, θ_l values of θ to right and left, respectively, of a discontinuity

$\mu = \sqrt{\frac{J\omega^2}{C}}$

σ linear stiffness of flexible support

φ angular displacement of shaft

φ_r, φ_l values of φ to right and left, respectively, of a discontinuity

$\varphi(D)$ a linear differential operator

ω 2π times frequency

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