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Note on the Application of the Linearised Theory for Compressible Flow to Transonic Speeds

- By -

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SUMMARY.

IT IS SHOWN THAT FOR FINITE ASPECT RATIO THE LINEARISED THEORY OF COMPRESSIBLE FLOW REMAINS THEORETICALLY CONSISTENT IN THE REGION OF TRANSONIC SPEEDS, ALTHOUGH ITS PREDICTIONS MAY DEVIATE APPRECIABLY FROM EXPERIMENTAL RESULTS IN THAT REGION. THE VARIATION OF THE THEORETICAL LIFT CURVE SLOPE OF AN AEROFOIL OF FINITE SPAN IS CONSIDERED AS THE MACH NUMBER INCREASES FROM BELOW UNITY TO ABOVE UNITY, AND IT IS SHOWN THAT THE LIFT CURVE SLOPE REMAINS FINITE AND CONTINUOUS.

INTRODUCTION.

It is well known that on the basis of the linearised theory for subsonic speeds (i.e. the Prandtl-Glauert theory) the lift curve slope of an aerofoil in two dimensions becomes infinite as the speed of sound is approached. Similarly, the linearised theory for supersonic speeds (i.e. the Ackeret theory) shows the lift curve slope in two dimensions to become infinite as the speed of sound is approached from above. This has led to the belief that the linearised theory breaks down in the transonic region. However, a recent application of the theory to Delta wings in supersonic flow (Ref.1) showed that with the aspect ratio finite the lift curve slope tended to a determinate finite value as the Mach number tended to unity. Further R.T.Jones (Ref.2) has shown that for aerofoils of vanishingly small aspect ratio the lift curve slope is independent of Mach number (and hence is continuous from subsonic to supersonic speeds).

In this note it is shown that the lift curve slope of a wing of finite aspect ratio remains finite in subsonic flow as the Mach number of unity is approached from below. This is shown to be true even on the very simple basis of lifting line theory. However, as the speed of sound is approached the pressure distribution on a wing of finite aspect ratio can be related to that on a wing of vanishingly small aspect ratio in incompressible flow (see para.3). Hence we must reject the quantitative results given by the lifting line theory and examine the results given by lifting surface theory. These latter results are shown to agree at a Mach number of unity with the

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results given by the analysis of Delta wings in supersonic flow mentioned above. It is concluded that the linearised theory gives both finite and continuous values of the lift curve slope of wings of finite aspect ratio as the Mach number passes through unity.

This conclusion does not, of course, imply that the linearised theory will necessarily give results in close agreement with experiment through the transonic region. The development of a shock stall in that region may be associated with disturbances which are far too large for the linearised theory to remain applicable.

2. NOTATION

A - aspect ratio

2γ - apex angle

C_L - lift coefficient

α incidence (in radians)

M - Mach number

μ - Mach angle

u - longitudinal velocity

U_o - free stream velocity

p - pressure

p_o - free stream pressure

ρ_o - free stream density

ϕ - velocity potential.

The suffices i and c indicate incompressible, and compressible flow respectively.

3. SUBSONIC FLOW.

3.1. Linearised Theory - General.

The linearised theory for subsonic speeds has been developed in some detail in Ref.3, where it is shown that if

$$\phi_i = U_o x + f(x, y, z)$$

is the potential function for the flow round a body in a uniform incompressible flow of velocity U_o streaming in the direction of the x asis, then

$$\phi_c = U_o x + \frac{1}{\beta} f(x, \beta y, \beta z)$$

is the potential function for the flow round the same body in compressible flow, the Mach number M being related to β by the equation

$$M = \sqrt{1 - \beta^2}$$

Consider a flat plate of any plan form and of aspect ratio A set at a small angle of incidence. Suppose the x axis is along the main stream direction, and the y axis is in the plane of the plate. We note that, on the surface of the plate when z is small, in incompressible flow

$$u_i = \frac{\partial \phi_i}{\partial x} = U_o + f_x(x, y, o) \qquad - \quad - \quad - \quad -(1)$$

to the order of accuracy of the theory, and, in compressible flow

/U_c ...

$$U_c = \frac{\partial \phi_c}{\partial x} = U_o + \frac{f_x}{\beta}(x,\beta y,o) \qquad - \quad - \quad - \qquad (2)$$

Hence, in incompressible flow, the pressure coefficient on the surface is given by

$$\frac{(p - p_o)_i}{\frac{1}{2}\rho_o U_o^2} = - \frac{2 f_x (x,y,o)}{U_o} \qquad - \quad - \quad - \qquad (3)$$

and in compressible flow the pressure coefficient is

$$\frac{(p - p_o)_c}{\frac{1}{2}\rho_o U_o^2} = - \frac{2 f_x}{\beta U_o}(x,\beta y,o) \qquad - \quad - \quad - \qquad (4)$$

It follows that at any point on the plate the pressure coefficient in compressible flow is $\frac{1}{\beta}$ times the coefficient at the point $(x,\beta y)$ in incompressible flow with the lateral ordinate y of the plate reduced in the ratio $\beta:1$. Therefore <u>the lift coefficient and lift curve slope at a Mach number M of a flat plate of aspect ratio A are $\frac{1}{\beta}$ times the lift coefficient and lift curve slope in incompressible flow of the flat plate with its aspect ratio reduced to $A\beta$.</u>

3.2. Lifting Line Theory.

If we apply the above conclusion to the formula given by lifting line theory for the lift curve slope of a wing of elliptic plan form in incompressible flow, we find that in compressible flow

$$\left(\frac{d\, C_L}{d\alpha}\right)_c = \frac{a_\infty}{\beta + \frac{a_\infty}{\pi A}} \qquad - \quad - \quad - \quad - \qquad (5)$$

This formula has already been deduced in Ref.4.

We note that as $M \rightarrow 1.0$ and $\beta \rightarrow o$,

$$\left(\frac{d\, C_L}{d\,\alpha}\right)_c \rightarrow \pi A \qquad - \quad - \quad - \quad - \qquad (6)$$

Hence $\left(\frac{d\, C_L}{d\,\alpha}\right)_c$ remains finite for finite aspect ratio. However, as $M \rightarrow 1.0$ the equivalent aspect ratio $A\beta \rightarrow o$, and hence we may expect the formula given by equation (6) to become increasingly invalid as $M = 1.0$ is approached.

We must therefore consider the results given by lifting surface theory.

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3.3 Lifting surface Theory.

The only available comprehensive results based on the linearised lifting surface theory for incompressible flow are those derived by Krienes (Ref.5) for elliptic flat plates, which include as a special case the result obtained by Kinner (Ref.6) for the circular flat plate. These results give us values for $\left(\frac{dC_L}{d\alpha}\right)_i$ for aspect ratios of 0.637, 1.27, 2.55, and 6.37, and these values are plotted against A in Fig.1.

In addition, R. T. Jones (Ref.2) has shown that for flat plates of any plan form and of vanishingly small aspect ratio

$$\left(\frac{d\,C_L}{d\,\alpha}\right)_i \rightarrow \frac{\pi}{2} A \quad , \text{ as } A \rightarrow o.$$

It will be seen from Fig.1. that the smooth curve through Krienes' and Kinner's points is quite consistent with a tangent at the origin having a slope equal to $\frac{\pi}{2}$ as given by Jones' theory. It is, therefore, reasonable to accept the curve of Fig.1 as describing the variation of $\left(\frac{d\,C_L}{d\,\alpha}\right)_i$ with aspect ratio given by the lifting surface theory for elliptical plates. We proceed to accept this curve, with perhaps less justification, as applicable to triangular plan forms (Delta wings). In favour of this we may note that we are primarily interested in small aspect ratios when Jones' [2] argument would lead us to expect a negligible effect due to plan form, and we may also note that in practice measured differences in lift curve slope as between one plan form and another are generally within the order of experimental accuracy.

Using this curve, therefore, and the result developed in para.2.1 above we can estimate the lift curve slope of a lifting surface of any aspect ratio at any Mach number up to 1.0.

Since we have accepted the result that for incompressible flow

$$\left(\frac{d\,C_L}{d\,\alpha}\right)_i \rightarrow \frac{\pi}{2} A, \text{ as } A \rightarrow o,$$

then for compressible flow

$$\left(\frac{d\,C_L}{d\,\alpha}\right)_c \rightarrow \frac{\pi}{2} A, \text{ as } M \rightarrow 1.0 \text{ from below.}$$

This is exactly half the value given by the lifting line theory.

Whether the curve of $\left(\frac{d\,C_L}{d\,\alpha}\right)_c$ against Mach number is flat topped or cusped at M = 1.0 depends on whether the curve

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of $\left(\frac{d\,C_L}{d\,\alpha}\right)_i$ against aspect ratio has a point of inflection at $A = 0$ or not, and we have not as yet sufficient evidence on this point.

4. SUPERSONIC FLOW.

The lift of a flat Delta wing moving at supersonic speed is calculated in Ref.1. on the assumptions of linearised theory. The lift coefficient is given by

$$\left.\begin{aligned} C_L &= 4\alpha \tan\mu, \text{ when } \mu < \gamma \\ \text{and by } C_L &= \frac{2\pi\alpha \tan\gamma}{E'(\cot\mu \tan\gamma)} \quad \text{when } \mu > \gamma \end{aligned}\right\} \quad - \quad - \quad - \quad (7)$$

In these formulae μ is the Mach angle, 2γ is the apex angle of the Delta wing, and $E'(u)$ is the elliptic integral defined by

$$E'(u) = \int_0^{\frac{\pi}{2}} \sqrt{1 - (1 - u^2)\sin^2\phi}\; d\phi$$

Since $A = 4\tan\gamma$ for a Delta wing, and $\cot\mu = \sqrt{M^2 - 1}$, equations (7) may be re-written

$$\left.\begin{aligned} \frac{d\,C_L}{d\,\alpha} &= \frac{4}{\sqrt{M^2 - 1}}, \text{ when } \sqrt{M^2 - 1} > \frac{4}{A} \\ \text{and} \quad \frac{d\,C_L}{d\,\alpha} &= \frac{\pi A}{2E'\frac{\sqrt{M^2-1}\;A}{4}} \quad \text{when } \sqrt{M^2 - 1} < \frac{4}{A} \end{aligned}\right\} \quad - \quad (8)$$

For a given aspect ratio, $\sqrt{M^2 - 1}$ will ultimately become smaller than $\frac{4}{A}$, as $M = 1$ is approached from above. so that the second formula in (8) will apply. Now $E'(u) \rightarrow 1$ as $u \rightarrow 0$, and so

$$\left(\frac{d\,C_L}{d\,\alpha}\right) \rightarrow \frac{\pi}{2} A, \text{ as } M \rightarrow 1.0 \text{ from above.}$$

Comparison with paragraph 3 shows that this is exactly the same value as obtained when $M = 1$ is approached from below. It follows that the lift curve slope does in fact vary continuously with M through $M = 1$ (Fig.2).

It is of some interest to calculate the slope of the curve $\left(\frac{d\,C_L}{d\,\alpha}\right)$ vs M as $M \rightarrow 1$ from above.

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We have

$$\frac{dE'(u)}{du} = \frac{u}{1-u^2} \left(K'(u) - E'(u)\right)$$

where

$$K'(u) = \int_o^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - (1-u^2)\sin\phi}}$$

Hence

$$\frac{d}{dM}\left(\frac{dC_L}{d\alpha}\right) = \frac{\pi AM\left[E'\left(\frac{\sqrt{M^2-1}\;A}{4}\right) - K'\left(\frac{\sqrt{M^2-1}\;A}{4}\right)\right]}{\left[E'\left(\frac{\sqrt{M^2-1}\;A}{4}\right)\right]^2\left[16 - (M^2-1)A^2\right]}$$

Now $E'(u) \rightarrow 1$, as $u \rightarrow o$, as mentioned above, while at the same time $K'(u) \rightarrow \infty$ as $\log \frac{4}{u}$, i.e. $(K'(u) - \log \frac{4}{u}) \rightarrow 0$, as $u \rightarrow 0$. (Ref.7, page 521). It follows that $\frac{d}{dM}\left(\frac{dC_L}{d\alpha}\right)$ tends to $-\infty$ as M tends to 1 from above, although the intensity of that infinity is "comparatively weak".

The position is numerised in Fig.2 which shows the variation of $dC_L/d\alpha$ with Mach number for both sub and supersonic speeds (including M =1.0) obtained for A = ∞, 4.0 2.3, and 1.07, i.e. for semi-apex angles γ = 90°, 45°, 30°, and 15° respectively.

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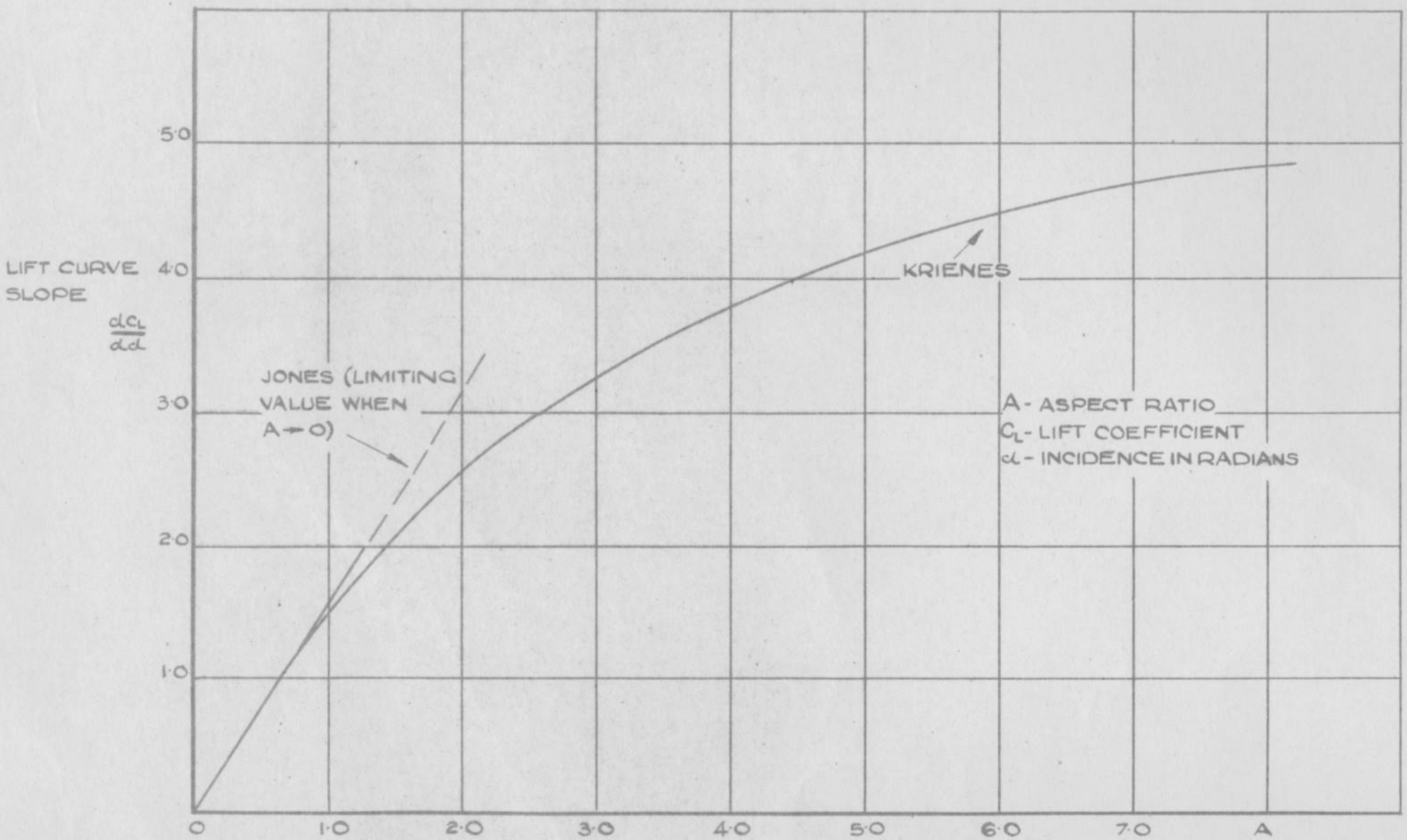
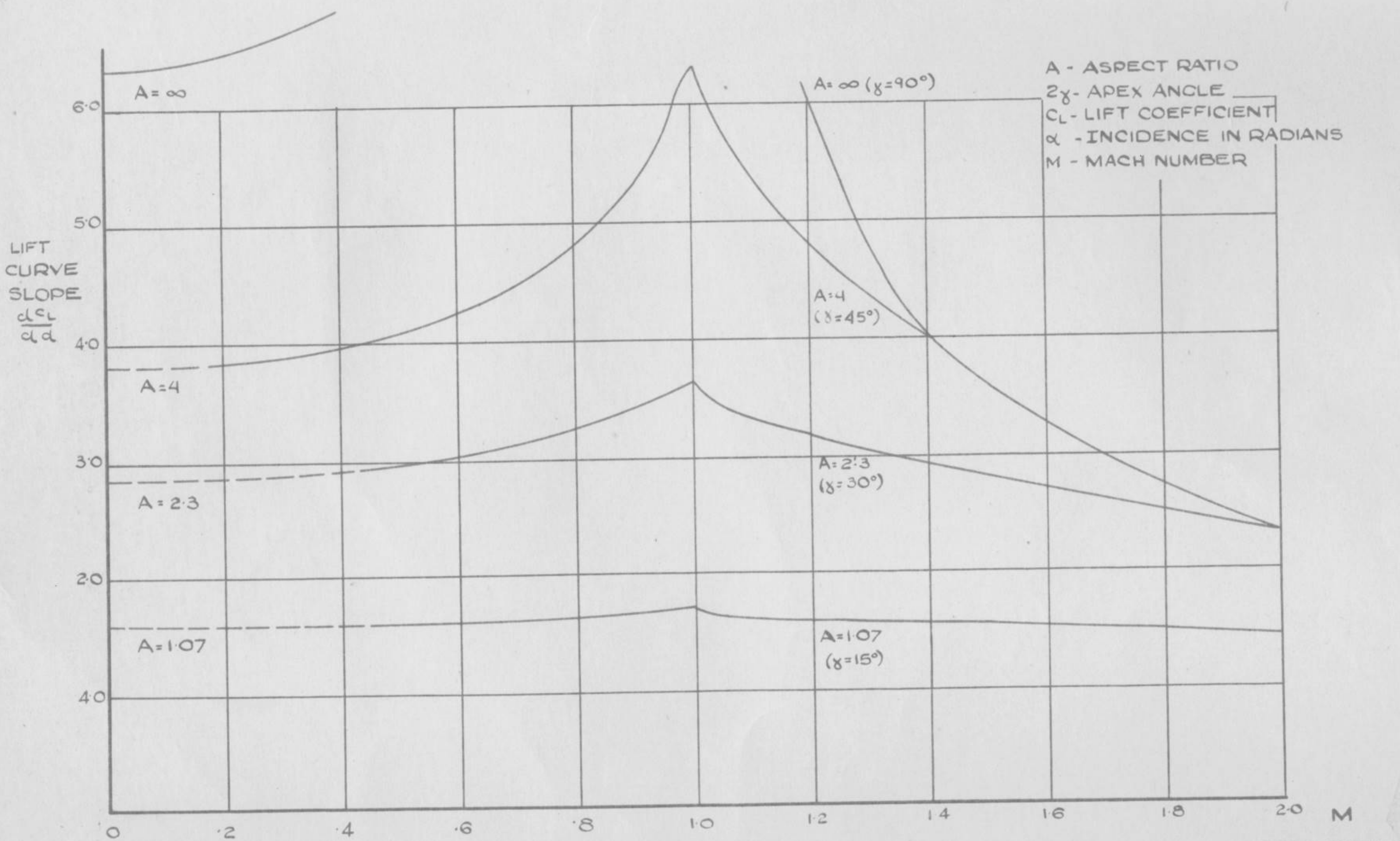


VARIATION OF LIFT CURVE SLOPE WITH ASPECT RATIO (KRIENES-JONES)

FIG. 1.



VARIATION OF THE LIFT CURVE SLOPE OF A DELTA WING WITH MACH NUMBER FOR VARIOUS ASPECT RATIOS

FIG. 2.