Asian Journal of Control, 2014, Vol. 17, Iss. 2, pp. 417 - 431

Accepted version

Asian Journal of Control, Vol. 16, No. 6, pp. 115, November 2014
Published online in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.923

GAIN-SCHEDULED H_∞ CONTROL FOR TENSOR PRODUCT TYPE POLYTOPIC PLANTS

Sunan Chumalee
Research and Development Centre for Space and Aeronautical Science and Technology
Royal Thai Air Force, Thailand

James F Whidborne
Centre for Aeronautics
Cranfield University, United Kingdom

Abstract

A tensor product (TP) model transformation is a recently proposed technique for transforming a given linear parameter-varying (LPV) model into polytopic model form for which there are many methods that can be used for controller design. This paper proposes an alternative approach to design a gain-scheduled output feedback H_∞ controller with guaranteed L_2-gain parameter-dependent performance for a class of TP type polytopic models using parameter-dependent Lyapunov functions where the linear matrix inequalities (LMIs) need only to be evaluated at all vertices of the system state-space model matrices and the variation rate of the scheduled parameters. In addition, a construction technique of the intermediate controller variables is also proposed as a matrix-valued function in the polytopic coordinates of the scheduled parameters. The performance of the proposed approach is tested on a missile autopilot design problem. Furthermore, nonlinear simulation results show the effectiveness of these proposed techniques.

1 Introduction

Although most real plants are nonlinear, they can often be modelled as linear parameter varying (LPV) plants [29] whereby their dynamic characteristics vary, following some time-varying parameters whose values are unknown a *priori* but can be measured in real-time and lie in some set bounded by known minimum and maximum possible values. An LPV plant was first introduced by Shamma and Athans [29] from which a number of algebraic manipulation techniques exist for deriving an LPV model from the original nonlinear model such as Jacobian linearisation [26], function substitution [30] or state transformation [29].

In practice, the resulting LPV models from these methods are often nonlinearly dependent on the time-varying parameters whereby the system matrices are known functions and depend nonlinearly on the scheduled parameters [15]. To synthesize an LPV controller from such an LPV model, Becker [12] has introduced a grid LPV model [26, 37, 38] whereby the system matrices are functions of the scheduled parameters at all grid points over the entire parameter space. Hence, an infinite number of linear matrix inequality (LMI) constraints have to be evaluated at all points over the entire parameter space in order to determine a pair of positive definite symmetric matrices (X, Y) that is required for the solution for the LPV control problem. However, in practice, (X, Y) can be determined from a finite number of LMIs by gridding the entire parameter space with a non-dense set of grid points. Having determined (X, Y), a more dense grid points set can be tested with this (X, Y) to check whether the LMIs are still satisfied [37, 38]. If not, the process is repeated with a denser grid until an (X, Y) that satisfies the LMIs for all points over the entire parameter space is obtained [37, 38].

In general, the result of such a heuristic gridding technique is not necessarily reliable and the analysis result is dependent on the choice of gridding points [35]. In addition, the resulting gain-scheduled controller has high computational on-line complexity at the gain-scheduling level [36, 37]. An alternative LPV model which has been introduced by Apkarian et al. [3] is an affine LPV model [1, 2, 3] whereby the system matrices are known functions and assumed to depend affinely on the scheduled parameters. Unlike the grid LPV model case, to synthesize an LPV controller from an affine LPV model, a finite number of LMI constraints have to be evaluated only at all vertices points of the system matrices. However, as discussed earlier, many LPV models do not depend affinely on the scheduled parameters [15].

A tensor product (TP) type polytopic model, proposed by Baranyi [5], provides another approach for which a polytopic model form can be obtained from an LPV model that depends nonlinearly on the scheduled parameters. The TP model transformation [6] is a recently proposed technique for transforming given LPV models into polytopic model form in which the interesting features of a TP type polytopic model are it is applicable to represent nonlinear parameter dependent LPV models and it is a polytopic model form.

The TP model transformation is an automatically executable numerical method and has three key steps [5, 6, 11]. The first step is the discretization of the given system matrices over a large number of points. The discretized points are defined by a dense hyper-rectangular grid of the scheduled parameters. The second step extracts the linear time invariant (LTI) vertex systems of the given (discretized) system matrices from a higher order singular value decomposition (HOSVD) based canonical form [8, 9, 31]. The TP model transformation directly leads to the HOSVD based canonical form of the TP type polytopic models by decomposing a given n-dimensional tensor into a full orthonormal system in a special ordering of higher order singular values that express the rank properties of the given LPV model for each element of the parameter vector in the L_2-norm. The third step defines the continuous weighting functions to the vertex systems. In the literature, there are many approaches that have been proposed for designing a controller based on TP model transformation [5, 6, 10, 11, 32] since the TP type polytopic model has various advantages for complexity trade-off and convex hull manipulation, all relying on the power of the HOSVD [8, 9, 11, 31].

In this paper, we propose an alternative approach for designing a gain-scheduled controller with guaranteed L_2-gain parameter-dependent performance [25, 13], $\gamma(\theta)$, for a class of TP type polytopic models using parameter-dependent Lyapunov functions where the LMIs need only to be evaluated at all vertices of the system state-space model matrices and the variation rate of the scheduled parameters. In addition, the intermediate controller variables, i.e. $\hat{A}_{\mathrm{k}}(\theta)$, $\hat{B}_{\mathrm{k}}(\theta)$, $\hat{C}_{\mathrm{k}}(\theta)$ and $D_{\mathrm{k}}(\theta)$, are proposed to be constructed as a matrix-valued function in the polytopic coordinates of the scheduled parameters; this reduces the computational burden and eases controller implementation. Furthermore, they are applicable to both regular and singular problems without the need for constraints on the D_{12} and D_{21} matrices of TP type polytopic models. The performance of the proposed approach is tested on a missile autopilot design problem [37] which is suitable test problem for advanced control design due to fast and wide parameter variations during its operation.

The organisation of the paper is as follows. A mathematical background of TP model transformation is briefly introduced and elaborated in the next section. In section 3, a vertex-type stability analysis technique for TP type polytopic systems based on parameter-dependent Lyapunov functions is summarised. In section 4, a gain-scheduled output feedback H_∞ control synthesis method for TP type polytopic systems using parameter-dependent Lyapunov functions is proposed. An LPV controller synthesis technique on the missile autopilot design problem using the proposed method is presented in section 5, where nonlinear simulation results are also presented. This paper concludes with some comments.

2 Tensor Product Model Transformation

2.1 Linear parameter-varying state-space model

Following Baranyi [6, 10], a linear parameter-varying state-space model is given by

$$\begin{aligned}\dot{x}(t) &= A\left(\theta(t)\right)x(t) + B\left(\theta(t)\right)u(t),\\ y(t) &= C\left(\theta(t)\right)x(t) + D\left(\theta(t)\right)u(t),\end{aligned} \tag{1}$$

where $t \in \mathbb{R}$ is time, $x \in \mathbb{R}^p$ is the state vector, $u \in \mathbb{R}^{m_2}$ is the control input vector, $y \in \mathbb{R}^{q_2}$ is the measurement output vector, $\theta(t) = [\theta_1(t), \dots, \theta_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$ is a time-varying N-dimensional parameter vector which is assumed to be measured in real-time and N is the total number of time-varying parameters.

The system matrices $A(\theta(t))$, $B(\theta(t))$, $C(\theta(t))$ and $D(\theta(t))$ are the continuous mapping matrix functions $A : \mathbb{R}^N \to \mathbb{R}^{p\times p}$, $B : \mathbb{R}^N \to \mathbb{R}^{p\times m_2}$, $C : \mathbb{R}^N \to \mathbb{R}^{q_2\times p}$ and $D : \mathbb{R}^N \to \mathbb{R}^{q_2\times m_2}$, respectively. We also assume that each parameter $\theta_i(t),\ i = 1, \dots, N$ lies between known minimum $\underline{\theta}_i$ and maximum $\overline{\theta}_i$ possible values, $\theta_i(t) \in [\underline{\theta}_i, \overline{\theta}_i]$, and $\theta(t)$ is an element of the closed hypercube $\Theta = [\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2] \times \ldots \times [\underline{\theta}_N, \overline{\theta}_N]$, $\theta(t) \in \Theta$. $\theta(t)$ can also include some elements of $x(t)$. The rate of variation $\dot{\theta}_i(t),\ i = 1, \dots, N$ is well defined at all times and satisfies $\dot{\theta}_i(t) \in [\underline{v}_i, \overline{v}_i]$ and $\dot{\theta}_i(t)$ is an element of the closed hypercube $\Phi = [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2] \times \ldots \times [\underline{v}_N, \overline{v}_N]$, $\dot{\theta}_i(t) \in \Phi$.

Note that an LPV system (1) is considered in the class of nonlinear dynamic models.

2.2 TP type polytopic models

Following Baranyi [5, 11], the system matrix of an LPV system (1) can also be written as:

$$S(\theta(t)) = \begin{pmatrix} A(\theta(t)) & B(\theta(t)) \\ C(\theta(t)) & D(\theta(t)) \end{pmatrix} \in \mathbb{R}^{(p+q_2)\times(p+m_2)} . \tag{2}$$

Assume that the system matrix $S(\theta(t))$ can be approximated for any parameter $\theta(t) \in \Theta$ as the convex combination of LTI system matrices $S_1, S_2, \dots, S_R$ which $S_r,\ r = 1, \dots, R$ are also called vertex systems. That is

$$S(\theta(t)) \in \left\{ \sum_{r=1}^{R} \alpha_r S_r : \alpha_r \geq 0, \sum_{r=1}^{R} \alpha_r = 1 \right\} . \tag{3}$$

If values α_r become weighting functions $w_r(\theta(t)) \in [0, 1] \subset \mathbb{R},\ r = 1, \dots, R,\ \theta(t) \in \Theta$ such that matrix $S(\theta(t))$ can be expressed as convex combination of vertex systems S_r then we obtain the TP form [5]

$$S(\theta(t)) \approx \sum_{r=1}^{R} w_r(\theta(t)) S_r, \tag{4}$$

with

$$\left\{ \forall r, \theta(t) : w_r(\theta(t)) \geq 0,\ \forall \theta(t) : \sum_{r=1}^{R} w_r(\theta(t)) = 1 \right\} . \tag{5}$$

When the weighting functions $w_r(\theta(t))$ are decomposed for all dimensions of $\theta(t)$, we get a higher order structure as [11]

$$w_r(\theta(t)) = \prod_{n=1}^{N} w_{n,i_n}(\theta_n(t)) . \tag{6}$$

That is

$$\begin{aligned} w_1(\theta(t)) &= w_{1,1}(\theta_1(t)) \times \cdots \times w_{N,1}(\theta_N(t)), \\ w_2(\theta(t)) &= w_{1,1}(\theta_1(t)) \times \cdots \times w_{N,2}(\theta_N(t)), \\ &\vdots \\ w_R(\theta(t)) &= w_{1,I_1}(\theta_1(t)) \times \cdots \times w_{N,I_N}(\theta_N(t)), \end{aligned}$$

where $R = I_1 \times I_2 \times \cdots \times I_N$ is the total number of vertex systems, $I_n,\ n = 1, \dots, N$, is the index upper bounds of the weighting functions used in the n-th dimension of the parameter vector $\theta(t)$, $w_r : \mathbb{R}^N \to \mathbb{R}$ is a continuous mapping weighting function, $w_{n,j}(\theta_n(t)),\ j = 1, \dots, I_n,\ n = 1, \dots, N$ is the j-th one-dimensional parameter weighting function defined on the n-th dimension of Θ, and $\theta_n(t)$ is the n-th element of the parameter vector $\theta(t)$. The equation (4) becomes [5]

$$S(\theta(t)) \approx \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(\theta_n(t)) S_{i_1,i_2,\dots,i_N}, \tag{7}$$

with

$$\left\{\forall i, n, \theta_n(t) : w_{n,i}\left(\theta_n(t)\right) \geq 0, \forall n, \theta_n(t) : \sum_{i_n=1}^{I_n} w_{n,i_n}\left(\theta_n(t)\right) = 1\right\}. \tag{8}$$

Note that the subscripts $r = 1, \ldots, R,\ R = \prod_{n=1}^{N} I_n$ and *ordering*$(i_1, i_2, \ldots, i_N)$ of vertex systems S in (4) and (7) are the same, respectively [5, 11]. That is

$$S_r = \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} = S_{i_1, i_2, \ldots, i_N}. \tag{9}$$

The explicit form (7) can be reformulated in terms of tensor algebra as [5]

$$S\left(\theta(t)\right) \approx \left(\mathcal{S} \mathop{\boxtimes}_{n=1}^{N} W_n\left(\theta_n(t)\right)\right), \tag{10}$$

where the row vector $W_n\left(\theta_n(t)\right) \in \mathbb{R}^{I_n}$ contains the one-dimensional parameter weighting functions $w_{n,i_n}\left(\theta_n(t)\right),\ i_n = 1, \ldots, I_n,\ n = 1, \ldots, N$. The $(N+2)$ dimensional tensor $\mathcal{S} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times (p+q_2) \times (p+m_2)}$ is constructed from vertex systems $S_{i_1, i_2, \ldots, i_N} \in \mathbb{R}^{(p+q_2) \times (p+m_2)}$

Therefore, an LPV system (1) can be approximated for any parameter $\theta(t) \in \Theta$ as a TP type polytopic model in terms of tensor product as [11]

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} \approx \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} W_n\left(\theta_n(t)\right) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \tag{11}$$

for which

$$\left\| S\left(\theta(t)\right) - \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} W_n\left(\theta_n(t)\right) \right\| \leq \epsilon, \tag{12}$$

where ϵ symbolizes the approximation error.

Note that the TP model transformation is capable of finding the exact TP type polytopic model representation, and in well-defined cases (when the exact representation does not exist), the TP type polytopic model is an approximation and a trade-off is supported via the singular values [34, 33].

For convenience, in the following sections, we will henceforth often drop the dependence of θ on t.

3 Stability Analysis Using Parameter-Dependent Lyapunov Functions

Following Gahinet et al. [20], the TP type polytopic system (11) is said to be parameter-dependent stable if there exists a continuously differentiable parameter-dependent Lyapunov function $V(x, \theta) = x^{\mathrm{T}} P(\theta) x$ whose derivative, $\dot{V}(x, \theta)$, is negative along all state trajectories and is given by $\dot{V}(x, \theta) = x^{\mathrm{T}} \left(A^{\mathrm{T}}(\theta) P(\theta) + P(\theta) A(\theta) + \dot{P}(\theta)\right) x$. This is equivalent to the existence of a $P(\theta) = P^{\mathrm{T}}(\theta)$ such that [20]

$$P(\theta) > 0,\ A^{\mathrm{T}}(\theta) P(\theta) + P(\theta) A(\theta) + \dot{P}(\theta) < 0, \forall (\theta, \dot{\theta}) \in \Theta \times \Phi. \tag{13}$$

Although an exact parameter-dependent function for a continuously differentiable parameter-dependent Lyapunov variable $P(\theta)$ is still not established, a basis parameter-dependent function for the parameter-dependent Lyapunov variable is suggested in [2, 37, 38] and is to copy the plant's parameter-dependent function. Therefore, we can constrain the basis parameter-dependent function for the parameter-dependent Lyapunov variable to vary in a TP type polytopic fashion

$$\begin{aligned} P(\theta) &= \sum_{r=1}^{R} w_r\left(\theta\right) P_r \\ &= w_1(\theta) P_1 + w_2(\theta) P_2 + \cdots + w_R(\theta) P_R \end{aligned} \tag{14}$$

Note that the weighting functions $w_r(\theta)$ can be determined using (6) and P_r are the matrix vertices. And also, note that obtaining $\dot{P}(\theta)$ by differentiating (14) with respect to time gives an expression with a large number of terms. Hence we propose to convert (14) into an affine form using an equation-error method (least-squares method) [24, 16] in order to achieve a simpler form of $\dot{P}(\theta)$.

The least-squares problem for the $P(\theta)$ in an affine form is formulated as

$$w_{\rm tp} P_{\rm tp} = \theta_{\rm aff} P_{\rm aff} + \upsilon \tag{15}$$

where

$$\begin{aligned}
w_{\rm tp} &= \begin{bmatrix} w_1\left(\theta_1(1),\theta_2(1),\ldots,\theta_N(1)\right) & w_2\left(\theta_1(1),\theta_2(1),\ldots,\theta_N(1)\right) & \cdots & w_R\left(\theta_1(1),\theta_2(1),\ldots,\theta_N(1)\right) \\ w_1\left(\theta_1(2),\theta_2(2),\ldots,\theta_N(2)\right) & w_2\left(\theta_1(2),\theta_2(2),\ldots,\theta_N(2)\right) & \cdots & w_R\left(\theta_1(2),\theta_2(2),\ldots,\theta_N(2)\right) \\ \vdots & & & \\ w_1\left(\theta_1(L),\theta_2(L),\ldots,\theta_N(L)\right) & w_2\left(\theta_1(L),\theta_2(L),\ldots,\theta_N(L)\right) & \cdots & w_R\left(\theta_1(L),\theta_2(L),\ldots,\theta_N(L)\right) \end{bmatrix} \\
P_{\rm tp} &= \begin{bmatrix} P_1 & P_2 & \cdots & P_R \end{bmatrix}^{\rm T} \qquad (16) \\
\theta_{\rm aff} &= \begin{bmatrix} 1 & \theta_1(1) & \theta_2(1) & \cdots & \theta_N(1) \\ 1 & \theta_1(2) & \theta_2(2) & \cdots & \theta_N(2) \\ \vdots & & & & \\ 1 & \theta_1(L) & \theta_2(L) & \cdots & \theta_N(L) \end{bmatrix} \\
P_{\rm aff} &= \begin{bmatrix} \check{P}_0 & \check{P}_1 & \cdots & \check{P}_N \end{bmatrix}^{\rm T} \\
\upsilon &= \begin{bmatrix} \upsilon(1) & \upsilon(2) & \cdots & \upsilon(L) \end{bmatrix}^{\rm T} \qquad (17)
\end{aligned}$$

The best estimator of $P_{\rm aff}$ minimizes the sum of squared error $\upsilon^T\upsilon$, and is given by [24, 16]

$$P_{\rm aff} = \left(\theta_{\rm aff}^{\rm T}\theta_{\rm aff}\right)^{-1}\theta_{\rm aff}^{\rm T} w_{\rm tp} P_{\rm tp} \tag{18}$$

That is

$$\begin{aligned} P(\theta) &= w_1(\theta)P_1 + w_2(\theta)P_2 + \cdots + w_R(\theta)P_R \\ &\approx \check{P}_0 + \theta_1\check{P}_1 + \theta_2\check{P}_2 + \cdots + \theta_N\check{P}_N \end{aligned} \tag{19}$$

Note that $\check{P}_i$, $i = 0,\ldots,N$, map to P_r, $r = 1,\ldots,R$, using (18). Differentiating (19) with respect to time gives

$$\dot{P}(\theta) \approx \dot{\theta}_1\check{P}_1 + \dot{\theta}_2\check{P}_2 + \cdots + \dot{\theta}_N\check{P}_N \tag{20}$$

The affine equation (20) can also be written in terms of a convex combination of the matrix vertices as

$$\dot{P}(\theta) \approx \beta_1(\dot{\theta})\tilde{P}_1 + \beta_2(\dot{\theta})\tilde{P}_2 + \cdots + \beta_M(\dot{\theta})\tilde{P}_M, \tag{21}$$

where $M = 2^N$ is the total number of matrix vertices and

$$\begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \\ \vdots \\ \tilde{P}_M \end{bmatrix} = \begin{bmatrix} 0 & \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_{N-1} & \underline{v}_N \\ 0 & \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_{N-1} & \overline{v}_N \\ 0 & \underline{v}_1 & \underline{v}_2 & \cdots & \overline{v}_{N-1} & \underline{v}_N \\ \vdots & & & & & \\ 0 & \overline{v}_1 & \overline{v}_2 & \ldots & \overline{v}_{N-1} & \overline{v}_N \end{bmatrix} \begin{bmatrix} \check{P}_0 \\ \check{P}_1 \\ \vdots \\ \check{P}_N \end{bmatrix}. \tag{22}$$

Following [28], in order to compute $\beta_i(\dot{\theta})$, $i = 1,\ldots,M$, we first compute the normalised coordinates

$$\beta(\dot{\theta}_j) = \frac{\overline{v}_j - \dot{\theta}_j}{\overline{v}_j - \underline{v}_j}, \quad j = 1,\ldots,N\,. \tag{23}$$

Then, for each vertex Φ_i, $i = 1,\ldots,M$, the corresponding polytopic coordinates are calculated by

$$\beta_i(\dot{\theta}) = \prod_{j=1}^{N}\tilde{\beta}(\dot{\theta}_j), \tag{24}$$

where

$$\tilde{\beta}(\dot{\theta}_j) = \begin{cases} \beta(\dot{\theta}_j), & \text{if } \underline{v}_j \text{ is a coordinate of } \Phi_i; \\ 1-\beta(\dot{\theta}_j), & \text{if } \overline{v}_j \text{ is a coordinate of } \Phi_i. \end{cases}$$

Lemma 3.1 *[14, 17] A given symmetric matrix polytope, $N(\theta) \in \mathbb{R}^{p\times p}$, for which $N(\theta) = \sum_{i=1}^{M} \alpha_i(\theta)\hat{N}_i$, where $\alpha_i(\theta)$ is determined using* (23) *and* (24)*, is a negative definite symmetric matrix for all possible parameter trajectories, $N(\theta) < 0$, $\forall\theta \in \Theta$, if and only if $\hat{N}_i < 0$, $i = 1, \ldots, M$.*

The plant state matrix $A(\theta)$ of the TP type polytopic system (11) can also be written in terms of a convex combination of the matrix vertices as

$$\begin{aligned} A(\theta) &= \sum_{r=1}^{R} w_r(\theta) A_r \\ &= w_1(\theta)A_1 + w_2(\theta)A_2 + \cdots + w_R(\theta)A_R, \end{aligned} \tag{25}$$

Substituting (14), (21) and (25) into (13), and recalling that $\sum_{r=1}^{R} w_r(\theta) = 1$ and $\sum_{j=1}^{M} \beta_j(\dot{\theta}) = 1$, we get

$$\sum_{r=1}^{R} w_r(\theta) P_r > 0, \tag{26}$$

$$\begin{aligned} &\sum_{r=1}^{R}\sum_{k=1}^{M} w_r^2(\theta)\beta_k(\dot{\theta}) \left(A_r^{\mathrm{T}} P_r + P_r A_r + \tilde{P}_k\right) \\ &\quad + 2\sum_{i=1}^{R-1}\sum_{j=i+1}^{R}\sum_{k=1}^{M} w_i(\theta)w_j(\theta)\beta_k(\dot{\theta}) \left(\frac{1}{2}\left(A_i^{\mathrm{T}} P_j + P_j A_i + A_j^{\mathrm{T}} P_i + P_i A_j + 2\tilde{P}_k\right)\right) < 0, \\ &\forall \left(\theta, \dot{\theta}\right) \in \Theta \times \Phi \end{aligned} \tag{27}$$

As

$$w_r^2(\theta)\beta_k(\dot{\theta}) \in [0,1], \quad r = 1,\ldots,R, \quad k = 1,\ldots,M, \tag{28}$$

and

$$2w_i(\theta)w_j(\theta)\beta_k(\dot{\theta}) \in [0, 0.5], \quad i = 1,\ldots,R-1, \quad j = i+1,\ldots,R, \quad k = 1,\ldots,M, \tag{29}$$

and

$$\sum_{r=1}^{R}\sum_{k=1}^{M} w_r^2(\theta)\beta_k(\dot{\theta}) + 2\sum_{i=1}^{R-1}\sum_{j=i+1}^{R}\sum_{k=1}^{M} w_i(\theta)w_j(\theta)\beta_k(\dot{\theta}) = 1, \tag{30}$$

by Lemma 3.1 solving (27) for parameter-dependent Lyapunov variable $P(\theta) = \sum_{r=1}^{R} w_r(\theta)P_r$ need only to be done at all vertices. Hence we get the following proposition.

Proposition 3.2 *The TP type polytopic system* (11) *is parameter-dependent stable whenever there exist a positive definite symmetric matrix P_r, $r = 1, 2, \ldots, R$, such that the following LMI conditions hold*

$$P_r > 0, \tag{31}$$

$$A_r^{\mathrm{T}} P_r + P_r A_r + \tilde{P}_k < 0, \tag{32}$$

$$A_i^{\mathrm{T}} P_j + P_j A_i + A_j^{\mathrm{T}} P_i + P_i A_j + 2\tilde{P}_k < 0, \tag{33}$$

for $r = 1, \ldots, R$, $k = 1, \ldots, M$ and $1 \le i < j \le R$.

Note that $\tilde{P}_k$, $k = 1, \ldots, M$, map to $\check{P}_i$, $i = 0, \ldots, N$, using (22) and $\check{P}_i$, $i = 0, \ldots, N$, map to P_r, $r = 1, \ldots, R$, using (18). In addition, the numbers of LMIs for (31)-(33) are R, RM and $RM(R-1)/2$, respectively. Therefore, the total number of LMIs to be solved is $R(RM + M + 2)/2$.

4 Controller Synthesis Using Parameter-Dependent Lyapunov Functions

In the previous section, a sufficient condition to guarantee the stability property of the TP closed-loop system using parameter-dependent Lyapunov functions has been presented in which the analysis conditions can be represented in the form of a finite number of LMIs. Next, we consider the problem of designing a gain-scheduled output feedback H_∞ control with guaranteed L_2-gain parameter-dependent performance [25, 13], $\gamma(\theta)$, for a class of TP type polytopic systems for which the proposed techniques in the previous section can be directly extended to synthesizing a gain-scheduled H_∞ controller.

Consider a generalised form of an LPV system (1) with state-space realisation taken from Apkarian et al. [3]:

$$\begin{aligned} \dot{x} &= A(\theta)x + B_1(\theta)w + B_2u, \\ z &= C_1(\theta)x + D_{11}(\theta)w + D_{12}u, \\ y &= C_2x + D_{21}w, \end{aligned} \tag{34}$$

where $x \in \mathbb{R}^p$ is the state vector, $w \in \mathbb{R}^{m_1}$ is the generalised disturbance vector, $u \in \mathbb{R}^{m_2}$ is the control input vector, $z \in \mathbb{R}^{q_1}$ is the controlled variable or error vector, $y \in \mathbb{R}^{q_2}$ is the measurement output vector, $\theta \in \Theta$, $\dot{\theta} \in \Phi$, and continuous mapping matrix functions $A : \mathbb{R}^N \to \mathbb{R}^{p\times p}$, $B_1 : \mathbb{R}^N \to \mathbb{R}^{p\times m_1}$, $C_1 : \mathbb{R}^N \to \mathbb{R}^{q_1\times p}$ and $D_{11} : \mathbb{R}^N \to \mathbb{R}^{q_1\times m_1}$.

Assume the LPV system (34) can be approximated for any parameter $\theta(t) \in \Theta$ as a TP type polytopic model in terms of the tensor product as

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} \approx \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} W_n(\theta_n) \begin{pmatrix} x \\ w \\ u \end{pmatrix}, \tag{35}$$

where the approximation error ϵ can be determined using (12), and

$$\begin{pmatrix} A(\theta) & B_1(\theta) & B_2 \\ C_1(\theta) & D_{11}(\theta) & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix} \approx \sum_{r=1}^{R} w_r(\theta) \begin{pmatrix} A_r & B_{1_r} & B_2 \\ C_{1_r} & D_{11_r} & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix}, \tag{36}$$

where the weighting functions $w_r(\theta)$ are determined using (6). The gain-scheduled output feedback H_∞ control problem using the parameter-dependent Lyapunov functions is to compute a dynamic LPV controller, $K(\theta)$, with state-space equations

$$\begin{aligned} \dot{x}_{\mathrm{k}} &= A_{\mathrm{k}}(\theta,\dot{\theta})x_{\mathrm{k}} + B_{\mathrm{k}}(\theta)y, \\ u &= C_{\mathrm{k}}(\theta)x_{\mathrm{k}} + D_{\mathrm{k}}(\theta)y, \end{aligned} \tag{37}$$

which stabilises the closed-loop system, (35) and (37), and minimises the closed-loop quadratic H_∞ parameter-dependent performance [25, 13], $\gamma(\theta)$, ensures the induced L_2-norm of the operator mapping the disturbance signal w into the controlled signal z is bounded by $\gamma(\theta)$

$$\int_0^{t_1} z^{\mathrm{T}} z dt \leq \gamma^2(\theta) \int_0^{t_1} w^{\mathrm{T}} w dt, \quad \forall t_1 \geq 0, \tag{38}$$

along all possible parameter trajectories, $\forall(\theta,\dot{\theta}) \in \Theta \times \Phi$. The assumed dimensions of the controller matrices are $A_{\mathrm{k}} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^{p\times p}$, $B_{\mathrm{k}} : \mathbb{R}^N \to \mathbb{R}^{p\times q_2}$, $C_{\mathrm{k}} : \mathbb{R}^N \to \mathbb{R}^{m_2\times p}$, and $D_{\mathrm{k}} : \mathbb{R}^N \to \mathbb{R}^{m_2\times q_2}$. Note that A and A_{k} have the same dimensions, since we restrict ourselves to the full-order case.

The closed-loop system, (35) and (37), is described by the state-space equations

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_{\mathrm{k}} \end{bmatrix} &= A_{\mathrm{cl}}(\theta,\dot{\theta}) \begin{bmatrix} x \\ x_{\mathrm{k}} \end{bmatrix} + B_{\mathrm{cl}}(\theta)w, \\ z &= C_{\mathrm{cl}}(\theta) \begin{bmatrix} x \\ x_{\mathrm{k}} \end{bmatrix} + D_{\mathrm{cl}}(\theta)w, \end{aligned} \tag{39}$$

where

$$
\begin{aligned}
A_{\mathrm{cl}}(\theta,\dot\theta) &= \begin{bmatrix} A(\theta)+B_2D_{\mathrm{k}}(\theta)C_2 & B_2C_{\mathrm{k}}(\theta) \\ B_{\mathrm{k}}(\theta)C_2 & A_{\mathrm{k}}(\theta,\dot\theta) \end{bmatrix}, \\
B_{\mathrm{cl}}(\theta) &= \begin{bmatrix} B_1(\theta)+B_2D_{\mathrm{k}}(\theta)D_{21} \\ B_{\mathrm{k}}(\theta)D_{21} \end{bmatrix}, \\
C_{\mathrm{cl}}(\theta) &= \begin{bmatrix} C_1(\theta)+D_{12}D_{\mathrm{k}}(\theta)C_2 & D_{12}C_{\mathrm{k}}(\theta) \end{bmatrix}, \\
D_{\mathrm{cl}}(\theta) &= D_{11}(\theta)+D_{12}D_{\mathrm{k}}(\theta)D_{21} \; .
\end{aligned} \tag{40}
$$

Based on the parameter-dependent Lyapunov functions, $V(x,\theta)=x^{\mathrm{T}}P(\theta)x$, there is an LPV controller $K(\theta)$ of the form of (37) that stabilises the closed-loop system, (35) and (37), if and only if there exists $P(\theta)=P^{\mathrm{T}}(\theta)$ such that [20]

$$
P(\theta)>0,\;\; \frac{d}{dt}\left(x^{\mathrm{T}}P(\theta)x\right)+z^{\mathrm{T}}z-\gamma^2(\theta)w^{\mathrm{T}}w<0, \tag{41}
$$

along all possible parameter trajectories, $\forall(\theta,\dot\theta)\in\Theta\times\Phi$. The inequality (41) leads to [20]

$$
\begin{pmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(\theta,\dot\theta)P(\theta)+P(\theta)A_{\mathrm{cl}}(\theta,\dot\theta)+\dot P(\theta) & P(\theta)B_{\mathrm{cl}}(\theta) & C_{\mathrm{cl}}^{\mathrm{T}}(\theta) \\ B_{\mathrm{cl}}^{\mathrm{T}}(\theta)P(\theta) & -\gamma(\theta)I & D_{\mathrm{cl}}^{\mathrm{T}}(\theta) \\ C_{\mathrm{cl}}(\theta) & D_{\mathrm{cl}}(\theta) & -\gamma(\theta)I \end{pmatrix} < 0 \; . \tag{42}
$$

We introduce intermediate controller variables, i.e. $\hat A_{\mathrm{k}}(\theta)$, $\hat B_{\mathrm{k}}(\theta)$, $\hat C_{\mathrm{k}}(\theta)$, as [2, 18]

$$
\begin{aligned}
A_{\mathrm{k}}(\theta,\dot\theta) = N^{-1}(\theta)\Big(&X(\theta)\dot Y(\theta)+N(\theta)\dot M^{\mathrm{T}}(\theta)+\hat A_{\mathrm{k}}(\theta) \\
&- X(\theta)\left(A(\theta)-B_2D_{\mathrm{k}}(\theta)C_2\right)Y(\theta) - \hat B_{\mathrm{k}}(\theta)C_2Y(\theta)-X(\theta)B_2\hat C_{\mathrm{k}}(\theta)\Big)M^{-\mathrm{T}}(\theta),
\end{aligned} \tag{43}
$$

$$
B_{\mathrm{k}}(\theta) = N^{-1}(\theta)\left(\hat B_{\mathrm{k}}(\theta)-X(\theta)B_2D_{\mathrm{k}}(\theta)\right), \tag{44}
$$

$$
C_{\mathrm{k}}(\theta) = \left(\hat C_{\mathrm{k}}(\theta)-D_{\mathrm{k}}(\theta)C_2Y(\theta)\right)M^{-\mathrm{T}}(\theta), \tag{45}
$$

where $N(\theta)=-X(\theta)+Y^{-1}(\theta)$, $\dot N(\theta)=-\dot X(\theta)-Y^{-1}(\theta)\dot Y(\theta)Y^{-1}(\theta)$, $M(\theta)=Y(\theta)$ and $\dot M(\theta)=\dot Y(\theta)$. A pair of positive definite symmetric matrices $(X(\theta),Y(\theta))$ are taken from the structure of the parameter-dependent Lyapunov variable, $P(\theta)$, which is defined as [35]

$$
\begin{aligned}
P(\theta) &= \begin{bmatrix} X(\theta) & -\left(X(\theta)-Y^{-1}(\theta)\right) \\ -\left(X(\theta)-Y^{-1}(\theta)\right) & X(\theta)-Y^{-1}(\theta) \end{bmatrix} \\
&= \begin{bmatrix} I_p & X(\theta) \\ 0_{p\times p} & -\left(X(\theta)-Y^{-1}(\theta)\right) \end{bmatrix} \begin{bmatrix} Y(\theta) & I_p \\ Y(\theta) & 0_{p\times p} \end{bmatrix}^{-1},
\end{aligned} \tag{46}
$$

$$
\dot P(\theta) = \begin{bmatrix} \dot X(\theta) & -\dot X(\theta)-Y^{-1}(\theta)\dot Y(\theta)Y^{-1}(\theta) \\ -\dot X(\theta)-Y^{-1}(\theta)\dot Y(\theta)Y^{-1}(\theta) & \dot X(\theta)+Y^{-1}(\theta)\dot Y(\theta)Y^{-1}(\theta) \end{bmatrix}, \tag{47}
$$

$$
\begin{aligned}
P^{-1}(\theta) &= \begin{bmatrix} Y(\theta) & Y(\theta) \\ Y(\theta) & \left(X(\theta)-Y^{-1}(\theta)\right)^{-1}X(\theta)Y(\theta) \end{bmatrix} \\
&= \begin{bmatrix} Y(\theta) & I_p \\ Y(\theta) & 0_{p\times p} \end{bmatrix} \begin{bmatrix} I_p & X(\theta) \\ 0_{p\times p} & -\left(X(\theta)-Y^{-1}(\theta)\right) \end{bmatrix}^{-1},
\end{aligned} \tag{48}
$$

where the positive definite symmetric matrices

$$
(X(\theta),Y(\theta)) \;\in\mathbb{R}^{p\times p},\quad X(\theta)-Y^{-1}(\theta)\geq 0, \tag{49}
$$

and [27]

$$
\mathrm{rank}\left(X(\theta)-Y^{-1}(\theta)\right)\leq p. \tag{50}
$$

Following Chumalee and Whidborne [17], the intermediate controller variables, i.e. $\hat{A}_{\mathrm{k}}(\theta)$, $\hat{B}_{\mathrm{k}}(\theta)$, $\hat{C}_{\mathrm{k}}(\theta)$ and $D_{\mathrm{k}}(\theta)$, including $(X(\theta), Y(\theta))$ and $\gamma(\theta)$, are proposed to be varied in a TP type polytopic fashion as

$$\hat{A}_{\mathrm{k}}(\theta) = w_1(\theta)\hat{A}_{\mathrm{k}_1} + \cdots + w_R(\theta)\hat{A}_{\mathrm{k}_R}, \tag{51}$$

$$\hat{B}_{\mathrm{k}}(\theta) = w_1(\theta)\hat{B}_{\mathrm{k}_1} + \cdots + w_R(\theta)\hat{B}_{\mathrm{k}_R}, \tag{52}$$

$$\hat{C}_{\mathrm{k}}(\theta) = w_1(\theta)\hat{C}_{\mathrm{k}_1} + \cdots + w_R(\theta)\hat{C}_{\mathrm{k}_R}, \tag{53}$$

$$D_{\mathrm{k}}(\theta) = w_1(\theta)D_{\mathrm{k}_1} + \cdots + w_R(\theta)D_{\mathrm{k}_R}, \tag{54}$$

$$\begin{aligned} X(\theta) &= w_1(\theta)X_1 + \cdots + w_R(\theta)X_R \\ &\approx \check{X}_0 + \theta_1\check{X}_1 + \cdots + \theta_N\check{X}_N, \end{aligned} \tag{55}$$

$$\begin{aligned} Y(\theta) &= w_1(\theta)Y_1 + \cdots + w_R(\theta)Y_R \\ &\approx \check{Y}_0 + \theta_1\check{Y}_1 + \cdots + \theta_N\check{Y}_N, \end{aligned} \tag{56}$$

$$\begin{aligned} \dot{X}(\theta) &\approx \dot{\theta}_1\check{X}_1 + \dot{\theta}_2\check{X}_2 + \cdots + \dot{\theta}_N\check{X}_N \\ &\approx \beta_1(\dot{\theta})\tilde{X}_1 + \cdots + \beta_M(\dot{\theta})\tilde{X}_M, \end{aligned} \tag{57}$$

$$\begin{aligned} \dot{Y}(\theta) &\approx \dot{\theta}_1\check{Y}_1 + \dot{\theta}_2\check{Y}_2 + \cdots + \dot{\theta}_N\check{Y}_N \\ &\approx \beta_1(\dot{\theta})\tilde{Y}_1 + \cdots + \beta_M(\dot{\theta})\tilde{Y}_M, \end{aligned} \tag{58}$$

$$\gamma(\theta) = w_1(\theta)\gamma_1 + w_2(\theta)\gamma_2 + \cdots + w_R(\theta)\gamma_R\ . \tag{59}$$

Note that $\tilde{X}_i$ and $\tilde{Y}_i$, $i = 1, \ldots, M$, map to $\check{X}_j$ and $\check{Y}_j$, $j = 0, \ldots, N$, using (22) and $\check{X}_j$ and $\check{Y}_j$, $j = 0, \ldots, N$, also map to X_r and Y_r, $r = 1, \ldots, R$, using (18). In addition, $\beta_i(\dot{\theta})$, $i = 1, \ldots, M$ can be determined using (23) and (24).

In contrast with the explicit controller formulae [2, 18], this proposed technique offers obvious advantages in reducing computational burden and ease of controller implementation because the intermediate controller variables can be constructed as a matrix-valued function in the polytopic coordinates of the scheduled parameters. Define

$$P_1(\theta) = \begin{bmatrix} Y(\theta) & I_p \\ Y(\theta) & 0_{p\times p} \end{bmatrix}\ . \tag{60}$$

Following Apkarian and Adams [2], by pre-multiplying the first row and post-multiplying the first column of (42) by $P_1^{\mathrm{T}}(\theta)$ and $P_1(\theta)$ respectively and substituting (40) and (43)–(47) in (42), we get

$$\begin{pmatrix} \dot{X}(\theta) + \Big(X(\theta)A(\theta) + \hat{B}_{\mathrm{k}}(\theta)C_2(\theta) + (\star)\Big) & \star & \star & \star \\ \hat{A}_{\mathrm{k}}^{\mathrm{T}}(\theta) + A(\theta) + B_2(\theta)D_{\mathrm{k}}(\theta)C_2(\theta) & -\dot{Y}(\theta) + \Big(A(\theta)Y(\theta) + B_2(\theta)\hat{C}_{\mathrm{k}}(\theta) + (\star)\Big) & \star & \star \\ B_1^{\mathrm{T}}(\theta)X(\theta) + D_{21}^{\mathrm{T}}(\theta)\hat{B}_{\mathrm{k}}^{\mathrm{T}}(\theta) & B_1^{\mathrm{T}}(\theta) + D_{21}^{\mathrm{T}}(\theta)D_{\mathrm{k}}^{\mathrm{T}}(\theta)B_2^{\mathrm{T}}(\theta) & -\gamma I & \star \\ C_1(\theta) + D_{12}(\theta)D_{\mathrm{k}}(\theta)C_2(\theta) & C_1(\theta)Y(\theta) + D_{12}(\theta)\hat{C}_{\mathrm{k}}(\theta) & D_{11}(\theta) + D_{12}(\theta)D_{\mathrm{k}}(\theta)D_{21}(\theta) & -\gamma I \end{pmatrix} < 0, \tag{61}$$

where the notation $\star$ represents a symmetric matrix block. Moreover, substituting (36) and (51)–(59) in (61), we have (62), shown at the top of the next page, in which the inequality (62) can be also rewritten as

$$\sum_{r=1}^{R}\sum_{k=1}^{M} w_r^2(\theta)\beta_k(\dot{\theta})\left(\Psi_{\mathrm{cl}_r} + \mathcal{Q}^{\mathrm{T}}\hat{K}_r^{\mathrm{T}}\mathcal{P} + \mathcal{P}^{\mathrm{T}}\hat{K}_r\mathcal{Q}\right) + 2\sum_{i=1}^{R-1}\sum_{j=i+1}^{R}\sum_{k=1}^{M} w_i(\theta)w_j(\theta)\beta_k(\dot{\theta})\left(\frac{1}{2}\left(\Psi_{\mathrm{cl}_{ij}} + \mathcal{Q}^{\mathrm{T}}\hat{K}_j^{\mathrm{T}}\mathcal{P}\mathcal{P}^{\mathrm{T}}\hat{K}_j\mathcal{Q} + \Psi_{\mathrm{cl}_{ji}} + \mathcal{Q}^{\mathrm{T}}\hat{K}_i^{\mathrm{T}}\mathcal{P} + \mathcal{P}^{\mathrm{T}}\hat{K}_i\mathcal{Q}\right)\right) < 0 \tag{63}$$

$$
\begin{aligned}
&\sum_{r=1}^{R}\sum_{k=1}^{M} w_r^2(\theta)\beta_k(\dot{\theta}) \begin{pmatrix} \tilde{X}_k + \big(X_r A_r + \hat{B}_{\mathrm{k}_r} C_2 + (\star)\big) & \star & \star & \star \\ \hat{A}_{\mathrm{k}_r}^{\mathrm{T}} + A_r + B_2 D_{\mathrm{k}_r} C_2 & -\tilde{Y}_k + \big(A_r Y_r + B_2 \hat{C}_{\mathrm{k}_r} + (\star)\big) & \star & \star \\ B_{1_r}^{\mathrm{T}} X_r + D_{21}^{\mathrm{T}} \hat{B}_{\mathrm{k}_r}^{\mathrm{T}} & B_{1_r}^{\mathrm{T}} + D_{21}^{\mathrm{T}} D_{\mathrm{k}_r}^{\mathrm{T}} B_2^{\mathrm{T}} & -\gamma_r I & \star \\ C_{1_r} + D_{12} D_{\mathrm{k}_r} C_2 & C_{1_r} Y_r + D_{12} \hat{C}_{\mathrm{k}_r} & D_{11_r} + D_{12} D_{\mathrm{k}_r} D_{21} & -\gamma_r I \end{pmatrix} \\
&+ 2 \sum_{i=1}^{R-1}\sum_{j=i+1}^{R}\sum_{k=1}^{M} w_i(\theta) w_j(\theta) \beta_k(\dot{\theta}) \begin{pmatrix} \tilde{X}_k + \frac{1}{2}\big(X_j A_i + \hat{B}_{\mathrm{k}_j} C_2 + X_i A_j + \hat{B}_{\mathrm{k}_i} C_2 + (\star)\big) & \star & \star & \star \\ \frac{1}{2}\Big(\hat{A}_{\mathrm{k}_j}^{\mathrm{T}} + A_i + B_2 D_{\mathrm{k}_j} C_2 + \hat{A}_{\mathrm{k}_i}^{\mathrm{T}} + A_j + B_2 D_{\mathrm{k}_i} C_2\Big) & -\tilde{Y}_k + \frac{1}{2}\big(A_i Y_j + B_2 \hat{C}_{\mathrm{k}_j} + A_j Y_i + B_2 \hat{C}_{\mathrm{k}_i} + (\star)\big) & \star & \star \\ \frac{1}{2}\Big(B_{1_i}^{\mathrm{T}} X_j + D_{21}^{\mathrm{T}} \hat{B}_{\mathrm{k}_j}^{\mathrm{T}} + B_{1_j}^{\mathrm{T}} X_i + D_{21}^{\mathrm{T}} \hat{B}_{\mathrm{k}_i}^{\mathrm{T}}\Big) & \frac{1}{2}\Big(B_{1_i}^{\mathrm{T}} + D_{21}^{\mathrm{T}} D_{\mathrm{k}_j}^{\mathrm{T}} B_2^{\mathrm{T}} + B_{1_j}^{\mathrm{T}} + D_{21}^{\mathrm{T}} D_{\mathrm{k}_i}^{\mathrm{T}} B_2^{\mathrm{T}}\Big) & -\frac{1}{2}(\gamma_i + \gamma_j) I & \star \\ \frac{1}{2}\big(C_{1_i} + D_{12} D_{\mathrm{k}_j} C_2 + C_{1_j} + D_{12} D_{\mathrm{k}_i} C_2\big) & \frac{1}{2}\big(C_{1_i} Y_j + D_{12} \hat{C}_{\mathrm{k}_j} + C_{1_j} Y_i + D_{12} \hat{C}_{\mathrm{k}_i}\big) & \frac{1}{2}\big(D_{11_i} + D_{12} D_{\mathrm{k}_j} D_{21} + D_{11_j} + D_{12} D_{\mathrm{k}_i} D_{21}\big) & -\frac{1}{2}(\gamma_i + \gamma_j) I \end{pmatrix} < 0
\end{aligned}
\tag{62}
$$

---

where

$$
\begin{aligned}
\Psi_{\mathrm{cl}_r} &= \begin{pmatrix} \tilde{X}_k + X_r A_r + (\star) & \star & \star & \star \\ A_r & -\tilde{Y}_k + A_r Y_r + (\star) & \star & \star \\ B_{1_r}^{\mathrm{T}} X_r & B_{1_r}^{\mathrm{T}} & -\gamma_r I & \star \\ C_{1_r} & C_{1_r} Y_r & D_{11_r} & -\gamma_r I \end{pmatrix}, \\
\hat{K}_r &= \begin{pmatrix} \hat{A}_{\mathrm{k}_r} & \hat{B}_{\mathrm{k}_r} \\ \hat{C}_{\mathrm{k}_r} & D_{\mathrm{k}_r} \end{pmatrix}, \\
\Psi_{\mathrm{cl}_{ij}} &= \begin{pmatrix} \tilde{X}_k + X_j A_i + (\star) & \star & \star & \star \\ A_i & -\tilde{Y}_k + A_i Y_j + (\star) & \star & \star \\ B_{1_i}^{\mathrm{T}} X_j & B_{1_i}^{\mathrm{T}} & \gamma_i I & \star \\ C_{1_i} & C_{1_i} Y_j & D_{11_i} & -\gamma_i I \end{pmatrix}, \\
\Psi_{\mathrm{cl}_{ji}} &= \begin{pmatrix} \tilde{X}_k + X_i A_j + (\star) & \star & \star & \star \\ A_j & -\tilde{Y}_k + A_j Y_i + (\star) & \star & \star \\ B_{1_j}^{\mathrm{T}} X_i & B_{1_j}^{\mathrm{T}} & \gamma_j I & \star \\ C_{1_j} & C_{1_j} Y_i & D_{11_j} & -\gamma_j I \end{pmatrix}, \\
\mathcal{Q} &= \big[\mathcal{C}, \quad \mathcal{D}_{21}, \quad 0_{(p+q_2)\times q_1}\big], \\
\mathcal{P} &= \big[\tilde{\mathcal{B}}^{\mathrm{T}}, \quad 0_{(p+m_2)\times m_1}, \quad \mathcal{D}_{12}^{\mathrm{T}}\big], \\
\tilde{\mathcal{B}} &= \begin{bmatrix} I_p & 0 \\ 0 & B_2 \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} 0 & I_p \\ C_2 & 0 \end{bmatrix}, \\
\mathcal{D}_{12} &= \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \ \mathcal{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.
\end{aligned}
\tag{64}
$$

By Lemma 3.1 and knowing the matrix vertices (X_r, Y_r), $r = 1, 2, \ldots, R$, the system matrix vertices $\hat{K}_r$ can be determined from (63), that is an LMI in $\hat{K}_r$, at all vertices for which $(\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_R)$ have to satisfy all of $R(RM + M + 2)/2$ LMIs. Furthermore, knowing $\hat{A}_{\mathrm{k}_r}, \hat{B}_{\mathrm{k}_r}, \ldots, D_{\mathrm{k}_r}$, the controller system matrices $A_{\mathrm{k}}(\theta, \dot{\theta}), \ldots, D_{\mathrm{k}}(\theta)$ can be computed on-line in real-time using (43)–(45) and (54) with instantaneous measurement values of θ and $\dot{\theta}$, where the proposed intermediate controller variables $\hat{A}_{\mathrm{k}}(\theta)$, $\hat{B}_{\mathrm{k}}(\theta)$, $\hat{C}_{\mathrm{k}}(\theta)$ and $D_{\mathrm{k}}(\theta)$, and $(X(\theta), Y(\theta))$ are varied in a TP type polytopic form and they can be computed on-line in real-time using (51)–(56). Hence, the proposed method reduces computational burden and eases controller implementation compared to the explicit controller formulae [2, 18].

However, usually, the parameter derivatives either are not available or are difficult to estimate during system operation [2]. To avoid using the measured value of $\dot{\theta}$, we can constrain either $X(\theta)$ or $Y(\theta)$ to depend on θ. This yields $\dot{X}(\theta)Y(\theta) + \dot{N}(\theta)M^{\mathrm{T}}(\theta) = -\Big(X(\theta)\dot{Y}(\theta) + N(\theta)\dot{M}^{\mathrm{T}}(\theta)\Big) = 0$ [2], hence equation

$$\begin{bmatrix} \tilde{X}_k + X_r A_r + A_r^{\mathrm{T}} X_r & X_r B_{1_r} & C_{1_r}^{\mathrm{T}} \\ B_{1_r}^{\mathrm{T}} X_r & -\gamma_r I & D_{11_r}^{\mathrm{T}} \\ C_{1_r} & D_{11_r} & -\gamma_r I \end{bmatrix} - \sigma \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \\ 0 \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix} < 0, \tag{69}$$

$$\begin{bmatrix} -\tilde{Y}_k + Y_r A_r^{\mathrm{T}} + A_r Y_r & Y_r C_{1_r}^{\mathrm{T}} & B_{1_r} \\ C_{1_r} Y_r & -\gamma_r I & D_{11_r} \\ B_{1_r}^{\mathrm{T}} & D_{11_r}^{\mathrm{T}} & -\gamma_r I \end{bmatrix} - \sigma \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \begin{bmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} & 0 \end{bmatrix} < 0, \tag{70}$$

$$\begin{bmatrix} 2\tilde{X}_k + X_j A_i + A_i^{\mathrm{T}} X_j + X_i A_j + A_j^{\mathrm{T}} X_i & X_j B_{1_i} + X_i B_{1_j} & C_{1_i}^{\mathrm{T}} + C_{1_j}^{\mathrm{T}} \\ B_{1_i}^{\mathrm{T}} X_j + B_{1_j}^{\mathrm{T}} X_i & -(\gamma_i + \gamma_j) I & D_{11_i}^{\mathrm{T}} + D_{11_j}^{\mathrm{T}} \\ C_{1_i} + C_{1_j} & D_{11_i} + D_{11_j} & -(\gamma_i + \gamma_j) I \end{bmatrix} - \sigma \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \\ 0 \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix} < 0, \tag{71}$$

$$\begin{bmatrix} -2\tilde{Y}_k + Y_j A_i^{\mathrm{T}} + A_i Y_j + Y_i A_j^{\mathrm{T}} + A_j Y_i & Y_j C_{1_i}^{\mathrm{T}} + Y_i C_{1_j}^{\mathrm{T}} & B_{1_i} + B_{1_j} \\ C_{1_i} Y_j + C_{1_j} Y_i & -(\gamma_i + \gamma_j) I & D_{11_i} + D_{11_j} \\ B_{1_i}^{\mathrm{T}} + B_{1_j}^{\mathrm{T}} & D_{11_i}^{\mathrm{T}} + D_{11_j}^{\mathrm{T}} & -(\gamma_i + \gamma_j) I \end{bmatrix} - \sigma \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \begin{bmatrix} B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} & 0 \end{bmatrix} < 0, \tag{72}$$

$$\begin{bmatrix} X_r & I \\ I & Y_r \end{bmatrix} > 0, \quad \text{for } r = 1, \ldots, R,\ k = 1, \ldots, M \text{ and } 1 \le i < j \le R. \tag{73}$$

---

(43) becomes

$$A_{\mathrm{k}}(\theta) = N^{-1}(\theta) \Big(\hat{A}_{\mathrm{k}}(\theta) - X(\theta) \left(A(\theta) - B_2(\theta) D_{\mathrm{k}}(\theta) C_2(\theta) \right) Y(\theta) \\ - \hat{B}_{\mathrm{k}}(\theta) C_2(\theta) Y(\theta) - X(\theta) B_2(\theta) \hat{C}_{\mathrm{k}}(\theta) \Big) M^{-\mathrm{T}}(\theta) \ . \tag{65}$$

Lemma 4.1 *[23, Finsler's Lemma] Given an inequality problem of the form*

$$\Psi + Q^{\mathrm{T}} K^{\mathrm{T}} P + P^{\mathrm{T}} K Q < 0, \tag{66}$$

where $\Psi \in \mathbb{R}^{m \times m}$ is a symmetric matrix, Q and P are matrices with column dimension m. Let σ be any real number $\sigma \in \mathbb{R}$; the above problem is solvable for a matrix K of compatible dimensions if and only if

$$\Psi - \sigma Q^{\mathrm{T}} Q < 0, \tag{67}$$

$$\Psi - \sigma P^{\mathrm{T}} P < 0 \ . \tag{68}$$

Note that Gahinet and Apkarian's projection lemma [19] is also equivalent to Finsler's Lemma [23].

By Lemmas 3.1 and 4.1, the LMIs of (63) are solvable at all vertices for $\hat{K}_r$, $r = 1, \ldots, R$ if and only if there exist a pair of positive definite symmetric matrices $(X(\theta), Y(\theta))$ that satisfy the following theorem.

Theorem 4.2 *There exists an LPV controller $K(\theta)$ guaranteeing the closed-loop system,* (35) *and* (37)*, quadratic H_∞ parameter-dependent performance, $\gamma(\theta)$, along all possible parameter trajectories, $\forall(\theta, \dot{\theta}) \in \Theta \times \Phi$, if and only if the LMI conditions* (69)*–*(73)*, shown at the top of the page, hold for some real number σ and some positive definite symmetric matrices $(X(\theta), Y(\theta))$, which further satisfy*

$$\operatorname{rank}\left(X(\theta) - Y^{-1}(\theta) \right) \le p.$$

Note that $\tilde{X}_k$ and $\tilde{Y}_k$, $k = 1, \ldots, M$, map to $\check{X}_j$ and $\check{Y}_j$, $j = 0, \ldots, N$, using (22) and $\check{X}_j$ and $\check{Y}_j$, $j = 0, \ldots, N$, also map to X_r and Y_r, $r = 1, \ldots, R$, using (18). The inequality (73) ensures $X(\theta), Y(\theta) > 0$ and $X(\theta) - Y(\theta)^{-1} \ge 0$.

Note also that Theorem 4.2 provides a new approach for designing a gain-scheduled H_∞ controller with guaranteed L_2-gain parameter-dependent performance [25, 13], $\gamma(\theta)$, for a class of TP type polytopic models using parameter-dependent Lyapunov functions.

5 Missile Autopilot Design

5.1 Missile pitch-axis LPV model

The quasi-LPV model of a missile pitch-axis dynamic system, taken from Wu et al. [36, 37], is:

$$\begin{bmatrix}\dot{\alpha}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}K_\alpha M\left(a_n\alpha^2 + b_n\left|\alpha\right| + c_n\left(-\frac{M}{3}+2\right)\right)\cos(\alpha) & 1\\ K_q M^2\left(a_m\alpha^2 + b_m\left|\alpha\right| + c_m\left(\frac{8M}{3}-7\right)\right) & 0\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix} + \begin{bmatrix}K_\alpha M d_n\cos(\alpha)\\ K_q M^2 d_m\end{bmatrix}\delta,$$
$$\begin{bmatrix}\eta\\ q\end{bmatrix} = \begin{bmatrix}K_z M^2\left(a_n\alpha^2 + b_n\left|\alpha\right| + c_n\left(-\frac{M}{3}+2\right)\right) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ q\end{bmatrix} + \begin{bmatrix}K_z M^2 d_n\\ 0\end{bmatrix}\delta, \tag{74}$$

where the various plant variables are:

α	Angle of attack in degrees,
q	Pitch rate in degrees per second,
δ	Actual tail deflection angle in degrees,
η	Actual normal acceleration in g's,
M	Mach number,

and also, the numerical constants in the plant model are:

$$\begin{aligned}
a_n &= 0.000103\,\mathrm{deg}^{-3}, & a_m &= 0.000215\,\mathrm{deg}^{-3},\\
b_n &= -0.00945\,\mathrm{deg}^{-2}, & b_m &= -0.0195\,\mathrm{deg}^{-2},\\
c_n &= -0.1696\,\mathrm{deg}^{-1}, & c_m &= 0.051\,\mathrm{deg}^{-1},\\
d_n &= -0.034\,\mathrm{deg}^{-1}, & d_m &= -0.206\,\mathrm{deg}^{-1},\\
K_\alpha &= 1.18587, & K_q &= 70.586,\\
K_z &= 0.6661697.
\end{aligned}$$

5.2 Missile pitch-axis TP model

One can see from the missile pitch-axis quasi-LPV system (74) that it is nonlinearly dependent on the scheduled parameters (α, M). The model is valid for the missile travelling between Mach number 2 and 4 at an altitude of 20,000 ft. The missile flight conditions, taken from Wu et al. [37], are that α and M vary from $-25°$ to $25°$ and 2 to 4 respectively. However, it is observed that the state-space entries of the quasi-LPV system (74) are symmetric in terms of parameter α, therefore only positive α values need to be considered.

Hence, we defined the transformation space as $\Theta = [0, 25] \times [2, 4]$ and let the dimension of the sampling grid be 400×200. Next, we applied the MATLAB Tensor Product Model Transformation Toolbox from Baranyi et al. [7] to determine the vertex systems $S_r,\ r = 1, \ldots, R$ and the weighting functions $w_r(\theta)$. In addition, we used the weighting type [11, 21, 22] of *cno* convex hull during the transformation in order to have a tight hull representation. We received the size of tensor $\mathcal{S}$ as $5 \times 4 \times 4 \times 3$ with the singular values in α dimension as 20.6260×10^3, 15.4510×10^3, 3.6796, 25.7095×10^{-3} and 227.9305×10^{-6}, and in M dimension as 21.6549×10^3, 13.5573×10^3, 3.3806×10^3 and 820.7544×10^{-3}.

Note that the weighting function type strongly influences the controller and observer design, thus, convex hull manipulation could be very important in finding the best control performance.

This means that the missile pitch-axis quasi-LPV system (74) can be approximated for any parameter $(\alpha, M) \in [-25, 25] \times [2, 4]$ as a convex combination of $5 \times 4 = 20$ vertex systems from which, for this missile autopilot design example, the 5×4 vertices is an approximate representation of the quasi-LPV system (74). After comparing the decomposed TP type polytopic model with the quasi-LPV model (74) over 4,000 test points of randomly selected parameter values (i.e. α and M), the maximum and mean errors, ϵ, in L_2 matrix norm were obtained as 7.5160×10^{-12} and 3.8402×10^{-12} respectively.

However, in practice, a small number of vertices is preferred for synthesising and implementing an LPV controller in real applications. Therefore, we kept only the three largest singular values for both α

and M dimensions. The vertex systems (S_r or $S_{i_1,i_2,\ldots,i_N}$) was reduced to $3 \times 3 = 9$, as shown below

$$S_1 = S_{1,1} = \begin{bmatrix} -1.2029 & 1.0000 & -0.1176 \\ -236.9920 & 0 & -135.6698 \\ -2.2717 & 0 & -0.2113 \\ 0 & 1.0000 & 0 \end{bmatrix}, \tag{75}$$

$$S_2 = S_{2,1} = \begin{bmatrix} -1.1471 & 1.0000 & -0.1287 \\ -174.0876 & 0 & -135.6698 \\ -1.9797 & 0 & -0.2113 \\ 0 & 1.0000 & 0 \end{bmatrix}, \tag{76}$$

$$\vdots$$

$$S_9 = S_{3,3} = \begin{bmatrix} -0.5306 & 1.0000 & -0.1610 \\ 211.1934 & 0 & -232.6514 \\ -1.2050 & 0 & -0.3624 \\ 0 & 1.0000 & 0 \end{bmatrix}. \tag{77}$$

Theoretically, the maximum error in the L_2 matrix norm approximation is the sum of the discarded small singular values, that is, $25.7095 \times 10^{-3} + 227.9305 \times 10^{-6} + 820.7544 \times 10^{-3} = 0.8467$. However, we have compared the decomposed TP type polytopic model with the quasi-LPV model (74) over 4,000 test points of randomly selected parameter values, i.e. α and M, in the ranges given by Θ. We received the maximum and mean error, ϵ, in L_2 matrix norm as 0.0198 and 0.0103 respectively. One can see from the approximation error ϵ that the ϵ from the 3×3 vertices case is much larger than the one from the 5×4 vertices case. However, the decomposed TP type polytopic model can be reduced to a system of half the complexity while it is still accurate enough for real world experiments. Hence, the missile pitch-axis TP model can be written as

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} \approx \sum_{i=1}^{3} \sum_{j=1}^{3} w_{1,i}(\alpha)\, w_{2,j}(M) \left(A_{i,j} \begin{bmatrix} \alpha \\ q \end{bmatrix} + B_{i,j}\delta \right),$$
$$\begin{bmatrix} \eta \\ q \end{bmatrix} \approx \sum_{i=1}^{3} \sum_{j=1}^{3} w_{1,i}(\alpha)\, w_{2,j}(M) \left(C_{i,j} \begin{bmatrix} \alpha \\ q \end{bmatrix} + D_{i,j}\delta \right), \tag{78}$$

where the one-dimensional parameter weighting function $w_{n,k}(\theta_n),\ k = 1, \ldots, I_n,\ n = 1, 2$, are presented in figure 1 and the system matrices $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$ are taken from the vertex systems S_r or $S_{i_1,i_2,\ldots,i_N}$, i.e. (75)–(77).

5.3 Missile LPV autopilot design

The performance requirements for designing an LPV autopilot taken from Wu et al. [36, 37] are: (i) robust stability is over the operating range $(\alpha, M) \in [-25, 25] \times [2, 4]$ (ii) the step response settling time is no more than 0.35 sec, the maximum overshoot is no greater than 10%, and the steady-state error is less than 1% and (iii) the maximum tail deflection rate should not exceed 25 deg/sec for a $1g$ normal acceleration step command.

Thus, an LPV controller is synthesized with the criterion $\left\| [W_1 S,\ W_2 KS]^{\mathrm{T}} \right\|_\infty < 1$ where $S = [I + GK]^{-1}$ and $KS = K[I + GK]^{-1}$ are the sensitivity function and the control sensitivity function, respectively. The objective of this mixed-sensitivity function is to shape the sensitivity function and control sensitivity function frequency responses with performance weighting functions W_1 and robustness weighting functions W_2 respectively.

The model weighting, W_{model}, sensor noise weighting, W_{snois_η} and W_{snois_q}, performance weighting,

Table 1: Quadratic H_∞ parameter-dependent performance [25, 13]

$\gamma(\theta)$ [a]	(X, Y) [b]	$(X(\theta), Y)$ [c]	$(X, Y(\theta))$ [c]	$(X(\theta), Y(\theta))$ [c]
γ_1	1.8347	1.8691	0.8822	0.8104
γ_2	1.7621	1.7869	0.7193	0.5656
γ_3	1.7851	1.8075	0.7268	0.6007
γ_4	3.7391	3.7053	1.5119	1.4599
γ_5	3.3374	3.3092	1.5035	1.4458
γ_6	4.3685	4.3166	1.8382	1.7991
γ_7	2.6063	2.6773	0.8719	0.8175
γ_8	1.9678	2.0114	0.7134	0.5233
γ_9	2.5370	2.5833	0.9153	0.8304

[a] Equation (59).
[b] Single quadratic Lyapunov function (SQLF) [15].
[c] Parameter-dependent Lyapunov function (PDLF) (Theorem 4.2).

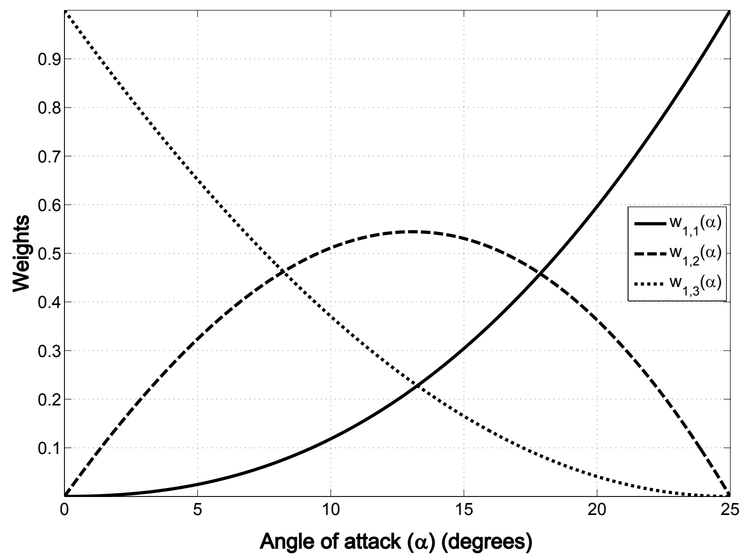
W_1, and robustness weighting, W_2, taken from Wu et al. [36, 37] are

$$\begin{aligned} W_{\text{model}}(s) &= \frac{144\,(-0.05s+1)}{s^2+2\times 0.8\times 12s+144}, & W_{\text{snois}_\eta}(s) &= W_{\text{snois}_q}(s) = 0.001, \\ W_1(s) &= \frac{0.5\,(s+34.642)}{s+0.057735}, & W_2(s) &= \frac{s}{25\,(0.005s+1)}, \\ W_{\text{pre-filter}}(s) &= \frac{1500}{s+1500}, & W_{\text{post-filter}}(s) &= \begin{pmatrix} \frac{1000}{s+1000} & 0 \\ 0 & \frac{1000}{s+1000} \end{pmatrix}. \end{aligned} \tag{79}$$

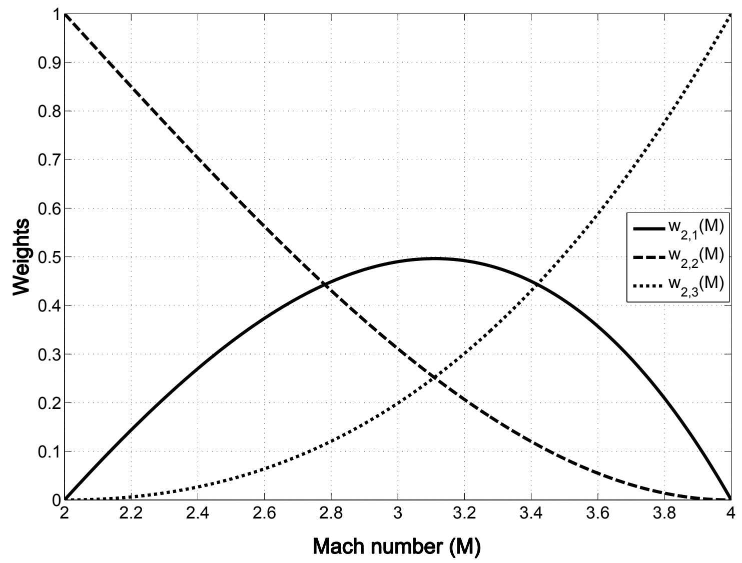
The model weighting W_{model} represents a desired ideal model for the closed-looped system for which it reflects the step response of less than 0.35 sec and also incorporates the non-minimum phase characteristics of missile plant. The right-half zero ranges from 19 to 46, then, the slowest zero ($s = 20$) was put in the desired command response filter [36, 37]. The sensor noise weightings, W_{snois_η} and W_{snois_q}, represent frequency domain models of sensor noise from which their gains 0.001 (-60 dB) were selected for all frequencies.

In addition, W_1 has a low frequency gain 300 (49 dB), a -3 dB frequency around 0.06 rad/sec, which corresponds to 1/3% tracking error, and high frequency gain 0.5 (-6 dB) to limit overshoot less than 5% while W_2 has a small low frequency gain, a high frequency gain of 8 (18 dB), and a -3 dB frequency of around 200 rad/sec, which corresponds a penalty on the high frequency response of the control signals, and 200 rad/sec desired bandwidth of control sensitivity function. Hence, we should get a controller that has good command following and disturbance attenuation, low sensitivity to measurement noise, has reasonably small control efforts and that is robustly stable to additive plant perturbations. Moreover, $W_{\text{pre-filter}}$ and $W_{\text{post-filter}}$ are required in order to make B_2, D_{12}, C_2 and D_{21} matrices of LPV models to be parameter-independent and also to make $D_{22} = 0$ [3].

After the missile pitch-axis TP model (78) was augmented with all weighting functions (79), Theorem 4.2 was applied to compute a missile LPV autopilot with the minimum and maximum variation of parameters $(\alpha,\ M)$ and $(\dot{\alpha},\ \dot{M})$, taken from Wu et al. [37], as $[0,\ 25]\times[2,\ 4]$ and $[-200,\ 200]\times[-0.5,\ 0.5]$ respectively where the LMIs are solved using the MATLAB Robust Control Toolbox function [4], `mincx`, in which the resulting parameter-dependent performance [25, 13], $\gamma(\theta)$, is presented in Table 1. One can see from Table 1 that, the obtained $\gamma(\theta)$ by using single quadratic Lyapunov function, (X, Y) case, is more conservative than by using parameter-dependent Lyapunov function, $(X(\theta), Y)$, $(X, Y(\theta))$ and $(X(\theta), Y(\theta))$ cases. In addition, the $\gamma(\theta)$ results that are obtained by $(X, Y(\theta))$ and $(X(\theta), Y(\theta))$ cases are similar. Thus, the $(X, Y(\theta))$ case was selected for synthesising a missile LPV autopilot in order to demonstrate the transient response to a sequence of step acceleration commands.



(a) $w_{1,i}(\alpha)$



(b) $w_{2,j}(M)$

Figure 1: The *cno* type convex weighting functions in one-dimensional parameter, i.e. $w_{n,k}(\theta_n)$, of the missile pitch-axis TP model.

5.4 Nonlinear simulation results

For simulation purposes, Mach number is generated by [36, 37]

$$\dot{M}(t) = \frac{1}{V_s}\left(-|\eta(t)|\sin(|\alpha(t)|) + A_x M^2(t)\cos(\alpha(t))\right),$$
$$M(0) = 3.0, \tag{80}$$

where $A_x = -6.4330$ lbs/slugs and $V_s = 1036.4$ ft/s is a speed of sound at $20,000$ ft, to provide a reasonably realistic Mach profile. The actuator dynamics model for the tail deflection, that was used for simulation, are [36]

$$\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta\omega_a \end{bmatrix} \begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_a^2 \end{bmatrix} \delta_c, \tag{81}$$

where $\zeta = 0.7$, $\omega_a = 150$ rad/s and

δ_c	Commanded tail deflection angle in degrees,
δ	Actual tail deflection angle in degrees,
ζ	Actuator damping ratio,
ω_a	Actuator undamped natural frequency.

Although the scheduled parameters of an LPV system are assumed to be measurable in real-time, an angle of attack α is not available for this missile plant. Thus, an estimated angle of attack α_e was used for simulation from which α_e was computed using [36, 37]

$$\begin{aligned} \alpha_e = &- 1.396 - 0.33421 M_N - 3.7653\delta_N - 0.91681\delta_N M_N \\ &+ \eta_N \left(-46.03 + 21.26 M_N - 8.8362 M_N^2 - 0.33564\delta_N + 0.385\delta_N M_N + 0.32892\delta_N M_N^2\right) \\ &+ \eta_N^3 \left(61.367 - 69.756 M_N + 30.44 M_N^2 + 3.9589\delta_N - 15.668\delta_N M_N + 11.498\delta_N M_N^2\right) \\ &+ \eta_N^5 \left(-54.655 + 94.381 M_N - 48.212 M_N^2 - 4.7973\delta_N + 18.807\delta_N M_N - 13.871\delta_N M_N^2\right), \end{aligned} \tag{82}$$

where $\eta_N = \eta/60$, $\delta_N = (\delta - 10)/25$ and $M_N = M - 30$.

Figure 2 shows the tracking performance of normal acceleration η, estimated angle-of-attack α_e, tail-deflection δ and tail-deflection rate $\dot{\delta}$ to a series of step commanded acceleration. One can see from the simulation results that the performance goals are satisfied and the missile LPV autopilot synthesized using parameter-dependent Lyapunov function, $(X, Y(\theta))$ case, is less conservative than the one synthesized using single quadratic Lyapunov function, (X, Y) case.

6 Conclusion

In this paper, new sufficient conditions for gain-scheduled H_∞ performance analysis and synthesis for a class of TP type polytopic systems using parameter-dependent Lyapunov functions with quadratic H_∞ parameter-dependent performance [25, 13], $\gamma(\theta)$, are as proposed in Theorem 4.2. The analysis and synthesis conditions are represented in the form of a finite number of LMIs. The intermediate controller variables, i.e. $\hat{A}_\mathrm{k}(\theta)$, $\hat{B}_\mathrm{k}(\theta)$, $\hat{C}_\mathrm{k}(\theta)$ and $D_\mathrm{k}(\theta)$, are proposed to be constructed as a matrix-valued function in the polytopic coordinates of the scheduled parameters without the need for constraints on the D_{12} and D_{21} matrices. Hence, this reduces the computational burden and eases controller implementation.

The approach was then applied to synthesise a missile LPV autopilot where it was tested with the missile pitch-axis nonlinear model taken from [37]. The nonlinear simulation results show the effectiveness of the proposed approach.

Acknowledgements

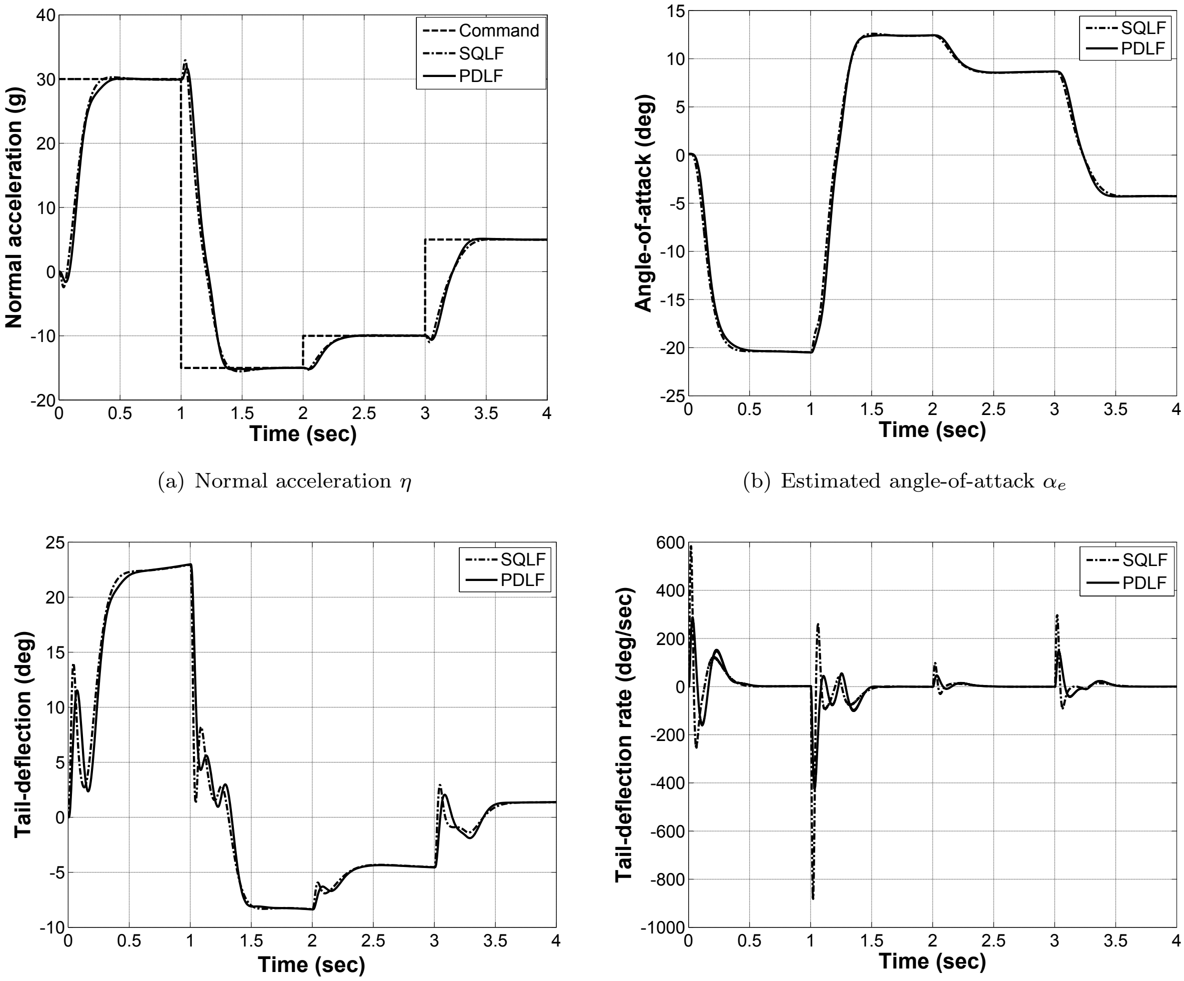
The authors acknowledge the support provided by the Royal Thai Air Force for this research project

References

[1] P. Apkarian. On the discretization of LMI-synthesized linear parameter-varying controllers. *Automatica*, 33(4):655–661, Apr 1997.

[2] P. Apkarian and R. Adams. Advanced gain-scheduling techniques for uncertain systems. *IEEE Transactions on Control Systems Technology*, 6(1):21–32, Jan 1998.

[3] P. Apkarian, P. Gahinet, and G. Becker. Self-scheduled H_∞ control of linear parameter-varying systems: A design example. *Automatica*, 31(9):1251–1261, 1995.



(a) Normal acceleration η

(b) Estimated angle-of-attack α_e

(c) Tail-deflection δ

(d) Tail-deflection rate $\dot{\delta}$

Figure 2: Missile dynamic response to a sequence of step acceleration commands.

[4] G. Balas, R. Chiang, A. Packard, and M. Safonov. *Robust Control Toolbox 3: Users Guide.* The Mathworks, Inc., Natick, MA, USA, 2007.

[5] P. Baranyi. TP model transformation as a way to LMI based controller design. *IEEE Transaction on Industrial Electronics*, 51(2):387–400, April 2004.

[6] P. Baranyi. Tensor-product model-based control of two-dimensional aeroelastic system. *Journal of Guidance, Control, and Dynamics*, 29(2):391–400, 2006.

[7] P. Baranyi, Z. Petres, and S. Nagy. TPtool Tensor Product MATLAB toolbox. In *[Online] Available: http://tptool.sztaki.hu/*, 2013.

[8] P. Baranyi, L. Szeidl, and P. Varlaki. Numerical reconstruction of the HOSVD based canonical form of polytopic dynamic models. In *Proceedings of the 10th International Conference on Intelligent Engineering Systems*, pages 196–201, London, UK, 2006.

[9] P. Baranyi, L. Szeidl, P. Varlaki, and Y. Yam. Definition of the HOSVD based canonical form of polytopic dynamic models. In *Proceedings of the 2006 IEEE International Conference on Mechatronics*, pages 660–665, Budapest, Hungary, 2006.

[10] P. Baranyi and Y. Yam. Case study of the TP model transformation in the control design of a complex dynamic model with structural non-linearity. *IEEE Transaction on Industrial Electronics*, 53(3):895–904, July 2006.

[11] P. Baranyi, Y. Yam, and P. Varlaki. *Tensor Product model transformation in polytopic model-based control*. Taylor & Francis, Boca Raton FL, CRC Press, 2013.

[12] G. Becker. *Quadratic Stability and Performance of Linear Parameter Dependent Systems*. PhD thesis, Department of Engineering, University of California, Berkeley, 1993.

[13] J. Chen, D. Gu, I. Postlethwaite, and K. Natesan. Robust LPV control of UAV with parameter dependent performance. In *Proceedings of the 17th World Congress The International Federation of Automatic Control*, pages 15070–15075, Seoul, Korea, July 2008.

[14] S. Chumalee. *Robust gain-scheduled H_∞ control for unmanned aerial vehicles*. PhD thesis, School of Engineering, Cranfield University, UK, June 2010.

[15] S. Chumalee and J. F. Whidborne. LPV autopilot design of a Jindivik UAV. In *AIAA Guidance, Navigation, and Control Conference and Exhibit*, Chicago, Illinois, August 2009.

[16] S. Chumalee and J. F. Whidborne. Unmanned aerial vehicle aerodynamic model identification from a racetrack manoeuvre. *Proceedings of the Institution of Mechanical Engineers, Part G, Journal of Aerospace Engineering*, 224(7):831–842, 2010.

[17] S. Chumalee and J. F. Whidborne. Gain-scheduled H_∞ control via parameter-dependent Lyapunov functions. *International Journal of Systems Science*, 2013. DOI:10.1080/00207721.2013.775386.

[18] P. Gahinet. Explicit controller formulas for LMI-based H_∞ synthesis. *Automatica*, 32(7):1007–1014, 1996.

[19] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, 4:421–448, 1994.

[20] P. Gahinet, P. Apkarian, and M. Chilali. Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *IEEE Transactions on Automatic Control*, 41(3):436–442, Mar 1996.

[21] P. Galambos and P. Baranyi. Representing the model of impedance controlled robot interaction with feedback delay in polytopic LPV form: TP model transformation based approach. *Acta Polytechnica Hungarica*, 10(1):139–157, 2013.

[22] P. Grof, P. Baranyi, and P. Korondi. Convex hull manipulation based control performance optimisation. *WSEAS Transactions on Systems and Control*, 5(8):691–700, August 2010.

[23] G. Herrmann, M. C. Turner, and I. Postlethwaite. Linear matrix inequalities in control. In *Mathematical Methods for Robust and Nonlinear Control*. Springer-Verlag Berlin Heidelberg, 2007.

[24] V. Klein and E. A. Morelli. *Aircraft System Identification: Theory and Practice*. AIAA Education Series, AIAA, Reston, VA, USA, 2006.

[25] L. H. Lee and M. Spillman. A parameter-dependent performance criterion for linear parameter-varying systems. In *Proceedings of the 36th Conference on Decision & Control*, pages 984–989, San Diego, California, December 1997.

[26] B. Lu, F. Wu, and S. W. Kim. Switching LPV control of an F-16 aircraft via controller state reset. *IEEE Transactions on Control Systems Technology*, 14(2):267–277, Mar 2006.

[27] A. Packard, K. Zhou, P. Pandey, and G. Becker. A collection of robust control problems leading to LMI's. In *Proceedings of the 30th Conference on Decision and Control*, pages 1245–1250, Brighton, England, December 1991.

[28] P. C. Pellanda, P. Apkarian, and H. D. Tuan. Missile autopilot design via a multi-channel LFT/LPV control method. *International Journal of Robust and Nonlinear Control*, 12:1–20, 2002.

[29] J. S. Shamma and J. Cloutier. Gain-scheduled missile autopilot design using linear parameter varying transformations. *Journal of Guidance, Control, and Dynamics*, 16(2):256–261, 1993.

[30] J.-Y. Shin, G. J. Balas, and M. A. Kaya. Blending methodology of linear parameter varying control synthesis of F-16 aircraft system. *Journal of Guidance, Control, and Dynamics*, 25(6):1040–1048, 2002.

[31] L. Szeidl and P. Varlaki. HOSVD based canonical form for polytopic models of dynamic systems. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 13(1):52–60, 2009.

[32] B. Takarics and P. Baranyi. Tensor-product-model-based control of a three degrees-of-freedom aeroelastic model. *Journal of Guidance, Control, and Dynamics*, 36(5):1527–1533, 2013.

[33] D. Tikk and P. Baranyi. Approximation properties of TP model forms and its consequences to TPDC design framework. *Asian Journal of Control*, 9(3):221–231, 2007.

[34] D. Tikk, P. Baranyi, R. Patton, and J. Tar. Approximation capability of TP model forms. *Australian Journal of Intelligent Information Processing Systems*, 8(3):155–163, 2004.

[35] F. Wang and V. Balakrishnan. Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems. *IEEE Transactions on Automatic Control*, 47(5):720–734, May 2002.

[36] F. Wu, A. Packard, and G. Balas. LPV control design for pitch-axis missile autopilots. In *Proceedings of the 34th Conference on Decision and Control*, pages 188–193, New Orleans, LA, 1995.

[37] F. Wu, A. Packard, and G. Balas. Systematic gain-scheduling control design: A missile autopilot example. *Asian Journal of Control*, 4(3):341–347, Sep 2002.

[38] F. Wu, X. H. Yang, A. Packard, and G. Becker. Induced L_2-norm control for LPV systems with bounded parameter variation rates. *International Journal of Robust and Nonlinear Control*, 6:983–998, 1996.