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THE COLLEGE OF AERONAUTICS
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TABULATION OF SOME LAYOUTS AND VIRTUAL DISPLACEMENT FIELDS IN THE THEORY OF MICHELL OPTIMUM STRUCTURES

by

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Notation

α, β	curvilinear co-ordinates
x, y	Cartesian co-ordinates
A, B	Lamé's parameters (defined by eq. (1))
u, v	virtual displacements along α, β directions
r_1, r_2	radii
e	strain
θ	arbitrary angle between 0 and $\pi/4$
$J_k(z)$	k-th order Bessel function
$I_k(z)$	k-th order modified Bessel function
$U_k(w,z)$	k-th order Lommel function of two variables
$\Gamma(z)$	Gamma function
$x_{11}, x_{12}, y_{11}, y_{12}$	numerical quantities defined by eq. (18)*
u_{21}, u_{22}, u_{23}	numerical quantities defined by eq. (23)*
$x_{21}, x_{22}. y_{21}. y_{22}$	numerical quantities defined by eq. (24)*

Part 1 - Introduction

1. Formulation of a boundary - value problem

In the theory of two-dimensional Michell optimum structures, there arises the problem of calculating the lines of principal stresses and the virtual displacements which are analogous to slip lines and velocities in plane plastic flow. A detailed analysis of the problem has been given in reference 1 and 2. Both analytical and numerical methods of calculation are given in reference 3.

Two special problems are considered here, in which the strain fields are generated from two given orthogonal curves with negative initial curvatures. A curvilinear co-ordinate system (α, β) is defined such that the magnitudes of the parameters α, β are chosen to be the angles between the tangents of the curves and two fixed directions ox, oy, which are cartesian axes with the same origin as (α, β). (See figure 1).

If A, B are the radii of curvature of the α, β curves then the linear element ds is defined by

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \tag{1}$$

where A, B are related by

$$\frac{\partial A}{\partial \beta} = B, \quad \frac{\partial B}{\partial \alpha} = A \; ; \tag{2}$$

and in the problems considered here $A(\alpha, 0)$, $B(0, \beta)$ are known functions.

From (2), it is easy to show that

$$\left\{ \begin{aligned} \frac{\partial^2 A}{\partial\alpha\partial\beta} - A &= 0 \\ \frac{\partial^2 B}{\partial\alpha\partial\beta} - B &= 0 \end{aligned} \right. \tag{3}$$

The virtual displacement field (u, v) along α-β directions is governed by (eq. (A7) of reference 2).

$$\left\{ \begin{aligned} \frac{\partial^2 u}{\partial\alpha\partial\beta} - u &= -2eB \\ \frac{\partial^2 v}{\partial\alpha\partial\beta} - v &= 2eA \end{aligned} \right. \tag{4}$$

where e is the maximum strain. It is assumed here that the direct strain along an α-curve is $-e$ and that along a β-curve is e.

2. Method of solution

The second order partial differential equation

$$\frac{\partial^2 H}{\partial\alpha\partial\beta} - H = G \tag{5}$$

with boundary values $H(\alpha,0)$, $H(0,\beta)$ $\left(\text{or } \frac{\partial H(0,\beta)}{\partial\beta}\right)$ can be solved by means of Riemann's method, which gives

$$H(\alpha,\beta) = H(0,0)I_o(2\sqrt{\alpha\beta}) + \int_o^{\alpha} I_o(2\sqrt{(\alpha-\xi)\beta})\,\frac{\partial H(\xi,0)}{\partial\xi}d\xi + \int_o^{\beta} I_o(2\sqrt{\alpha(\beta-\eta)})\,\frac{\partial H(0,\eta)}{\partial\eta}d\eta + \int_o^{\alpha}\int_o^{\beta} I_o(2\sqrt{(\alpha-\xi)(\beta-\eta)})G(\xi,\eta)\,d\xi\,d\eta \tag{6}$$

Generally, no explicit solution will be available. If, however, $G(\xi,\eta)$ is of the form

$$\left(\frac{\xi}{\eta}\right)^{\frac{k}{2}} I_k(2\sqrt{\xi\eta}),$$

the double integral in (6) can always be reduced to a single integral thanks to

$$\frac{\partial}{\partial\eta}\int_o^{\alpha} I_o(2\sqrt{(\alpha-\xi)(\beta-\eta)})\cdot\left(\frac{\xi}{\eta}\right)^{\frac{k}{2}}\cdot I_k(2\sqrt{\xi\eta})\,d\xi = 0, \tag{7}$$

the proof of which is given in appendix A. Then it may be possible to find explicit expressions for the integrals in terms of Bessel functions.

In what follows, two transformation formulae taken from references 4 and 5 are of great value.

$$\int_o^{\frac{\pi}{2}} J_\lambda(q\sin\zeta)J_\nu(p\cos\zeta).\sin^{2k+1}\zeta.\cos^{2\mu+1}\zeta.d\zeta =$$

$$= 2^{k+\mu-\frac{\lambda}{2}-\frac{\nu}{2}}\Gamma(k+1+\tfrac{\lambda}{2})\Gamma(\mu+1+\tfrac{\nu}{2})\sum_{r=o}^{\infty}\sum_{s=o}^{\infty}\frac{(\frac{\lambda}{2}-k)_r(\frac{\nu}{2}-\mu)_s q^{\lambda+2r}p^{\nu+2s}J_{k+\mu+1+r+s+(\lambda+\nu)/2}(\sqrt{p^2+q^2})}{2^{r+s}r!\,s!\,\Gamma(\lambda+r+1)\Gamma(\nu+s+1)(p^2+q^2)^{\frac{1}{2}(k+\mu+1+\frac{\lambda}{2}+\frac{\nu}{2}+r+s)}} \tag{8}$$

where $(a)_n = a(a+1)(a+2)....(a+n-1)$, $(a)_o = 1$; and

$$(z+w)^{-\frac{\nu}{2}} . J_\nu(\sqrt{z+w}) = \sum_{m=o}^{\infty} \frac{(-\frac{w}{2})^m}{m!} . z^{-\frac{1}{2}(\nu+m)} . J_{\nu+m}(\sqrt{z}) \tag{9}$$

3. Transformation formulae for the co-ordinate systems

The co-ordinate system (α, β) is related to the Cartesian (x, y) by (Eq. (22) of reference 2).

$$\begin{cases} \cos(\beta-\alpha) = \frac{1}{A}\frac{\partial x}{\partial\alpha} = \frac{1}{B}\frac{\partial y}{\partial\beta} \\ \sin(\beta-\alpha) = \frac{1}{A}\frac{\partial y}{\partial\alpha} = -\frac{1}{B}\frac{\partial x}{\partial\beta} \end{cases} \tag{10}$$

or in integral form

$$(x-x_o) + i(y-y_o) = \int_o^\alpha A.\mathrm{Exp}\{i(\beta-\alpha)\}\,d\alpha, \tag{11}$$

where

$$x_o+iy_o = x(0,\beta) + iy(0,\beta) = i\int_o^\beta B(0,\beta).\mathrm{Exp}\{i\beta\}\,d\beta \tag{12}$$

When A is the Bessel function $I_o(2\sqrt{\alpha\beta})$, the integral (11) can be represented by means of Lommel's function of two variables.

$$\int_o^\alpha I_o(2\sqrt{\alpha\beta}).\mathrm{Exp}\{i(\beta-\alpha)\}\,d\alpha = \mathrm{Exp}\{i(\beta-\alpha)\}\sum_{k=1}^{\infty}(i)^{k-1}.\left(\frac{\alpha}{\beta}\right)^{k/2} . I_k(2\sqrt{\alpha\beta})$$

$$= \mathrm{Exp}\{i(\beta-\alpha)\}.\left[U_1(2\alpha,2i\sqrt{\alpha\beta}) + i\,U_2(2\alpha,2i\sqrt{\alpha\beta})\right] \tag{13^1}$$

---

[1] See, for instance, reference 5, pp. 537/543. The Lommel function is defined by:

$$U_k(w,z) = \sum_{m=o}^{\infty}(-1)^m.\left(\frac{w}{z}\right)^{k+2m} . J_{k+2m}(z)$$

4. Particular solutions

Two particular solutions of strain fields are considered here, the geometrics of which are presented in figures 2 and 3. Detail calculations of the second case are given in Appendix B so as to demonstrate the procedures.

(4a) In the case when the strain field is generated by two orthogonal circular arcs of different radii, (figure 2):

$$\begin{cases} A_1(\alpha,0) = r_1 \\ B_1(0,\beta) = r_2 \end{cases} \tag{14}$$

then

$$\begin{cases} A_1(\alpha,\beta) = r_1\, I_o(2\sqrt{\alpha\beta}) + r_2 \sqrt{\dfrac{\beta}{\alpha}}\; I_1(2\sqrt{\alpha\beta}) \\ B_1(\alpha,\beta) = r_1\sqrt{\dfrac{\alpha}{\beta}}\, I_1(2\sqrt{\alpha\beta}) + r_2 I_o(2\sqrt{\alpha\beta}) \end{cases} \tag{15}$$

which reduces, when $r_1 = r_2$, to equation (44) of reference 2. If the virtual displacements on the boundary are given as

$$\begin{cases} u_1(\alpha,0) = -\, e r_1(2\alpha+1) \\ v_1(\alpha,0) = e r_1 \end{cases}$$
$$\begin{cases} u_1(0,\beta) = -\, e r_2 + e(r_2-r_1)(\cos\beta - \sin\beta) \\ v_1(0,\beta) = e r_2(2\beta+1) - e(r_2-r_1)(\cos\beta + \sin\beta), \end{cases} \tag{16}$$

and A, B are given by (15), then

$$\begin{cases} u_1(\alpha,\beta) = -\, e r_1(1+2\alpha) I_o(2\sqrt{\alpha\beta}) - 2 e r_2\sqrt{\alpha\beta}\; I_1(2\sqrt{\alpha\beta}) - \\ \qquad -\, e(r_2-r_1)\, [U_1(2\beta,2i\sqrt{\alpha\beta}) + U_2(2\beta,2i\sqrt{\alpha\beta})] \\ v_1(\alpha,\beta) = e r_1[I_o(2\sqrt{\alpha\beta}) + 2\sqrt{\alpha\beta}\; I_1(2\sqrt{\alpha\beta})] + 2 e r_2\beta I_o(2\sqrt{\alpha\beta}) - \\ \qquad -\, e(r_2-r_1)\, [U_1(2\beta,2i\sqrt{\alpha\beta}) - U_2(2\beta,2i\sqrt{\alpha\beta})]\;, \end{cases} \tag{17}$$

which reduces again, when $r_1 = r_2$ and e is replaced by $-$ e, to equation (63) of reference 2.

The co-ordinates (α,β) are then related to the Cartesian (x_1,y_1) by

$$(x_1 - x_{1o}) + i(y_1 - y_{1o}) = \int_o^\alpha A_1(\alpha,\beta) \,.\, \mathrm{Exp}\{i(\beta-\alpha)\}\, d\alpha$$

$$= r_1\, \mathrm{Exp}\{i(\beta-\alpha)\}.[U_1(2\alpha,2i\sqrt{\alpha\beta}) + i\, U_2(2\alpha,2i\sqrt{\alpha\beta})] \qquad (18)$$

$$+ r_2\left\{\mathrm{Exp}\{i(\beta-\alpha)\}\;[i\, U_1(2\alpha,2i\sqrt{\alpha\beta}) + U_o(2\alpha,2i\sqrt{\alpha\beta})] - \mathrm{Exp}\{i\beta\}\right\}$$

and

$$x_{1o} + i\, y_{1o} = i\int_o^\beta r_2\, \mathrm{Exp}\{i\beta\}\, d\beta = r_2[\mathrm{Exp}\{i\beta\} - 1] \qquad (19)$$

Combining (18) and (19) gives the following form for x_1, y_1,

$$x_1 + i\, y_1 = (r_1\, x_{11} + r_2\, x_{12}) + i\, (r_1\, y_{11} + r_2\, y_{12}) \qquad (18)^*$$

Numerical values of x_{11}, x_{12}, y_{11}, y_{12} are tabulated in Part II.

(4b) In the case when the strain field is generated by two orthogonal curves of initial radii of curvature

$$\begin{cases} A_2(\alpha,0) = r_1 + r_1 I_o(2\sqrt{2\theta\alpha}) + r_2\sqrt{\dfrac{2\theta}{\alpha}}\; I_1(2\sqrt{2\theta\alpha}) \\ B_2(0,\beta) = r_2 + r_2 I_o(2\sqrt{(\pi-2\theta)\beta}) + r_1\sqrt{\dfrac{\pi-2\theta}{\beta}}\;\; I_1(2\sqrt{(\pi-2\theta)\beta}), \end{cases} \qquad (20)$$

where θ is an arbitrary angle $(0 \leqslant \theta \leqslant \frac{\pi}{4})$, (Figure 3).

then from (B.6)

$$A_2(\alpha,\beta) = r_1\{I_o(2\sqrt{\alpha(\beta+2\theta)}) + I_o(2\sqrt{\beta(\alpha+\pi-2\theta)})\} +$$

$$+ r_2\left\{\sqrt{\frac{\beta}{\alpha+\pi-2\theta}}\;\; I_1\;(2\sqrt{\beta(\alpha+\pi-2\theta)}) + \sqrt{\frac{\beta+2\theta}{\alpha}}\; I_1(2\sqrt{\alpha(\beta+2\theta)})\right\}$$

$$B_2(\alpha,\beta;\; r_1,r_2;\; 2\theta,\pi-2\theta) = A_2(\beta,\alpha;\; r_2,r_1;\; \pi-2\theta,2\theta) \qquad (21)$$

If the virtual displacements on the boundary are given by (generalising the case considered in reference 6).

$$u_2(\alpha,0) = - er_1[(1+2\alpha+2\pi-4\theta) + 2\alpha\ I_o(2\sqrt{2\theta\alpha})] -$$
$$- er_2[I_o(2\sqrt{2\theta\alpha}) + 2\sqrt{2\theta\alpha}\ I_1(2\sqrt{2\theta\alpha})] -$$
$$- e(r_2-r_1)[U_1(2\alpha,2i\sqrt{2\theta\alpha}) - U_2(2\alpha,2i\sqrt{2\theta\alpha})]$$
$$v_2(\alpha,0) = er_1[1+2\sqrt{2\theta\alpha}\ I_1(2\sqrt{2\theta\alpha})] + er_2(1+4\theta)\ I_o(2\sqrt{2\theta\alpha}) +$$
$$+ e(r_1-r_2)[U_1(2\alpha,2i\sqrt{2\theta\alpha}) + U_2(2\alpha,2i\sqrt{2\theta\alpha})]$$

(22)

$$u_2(0,\beta) = - er_2[1+2\sqrt{(\pi-2\theta)\beta}\ I_1(2\sqrt{(\pi-2\theta)\beta})] - er_1(\overline{1+2\pi-4\theta})I_o(2\sqrt{(\pi-2\theta)\beta})$$
$$- e(r_2-r_1)[U_1(2\beta,2i\sqrt{(\pi-2\theta)\beta}) + U_2(2\beta,2i\sqrt{(\pi-2\theta)\beta})]$$
$$v_2(0,\beta) = er_2[(1+2\beta+4\theta) + 2\beta\ I_o(2\sqrt{(\pi-2\theta)\beta})] +$$
$$+ er_1[I_o(2\sqrt{(\pi-2\theta)\beta}) + 2\sqrt{(\pi-2\theta)\beta}\ I_1(2\sqrt{(\pi-2\theta)\beta}) +$$
$$+ e(r_1-r_2)[U_1(2\beta,2i\sqrt{(\pi-2\theta)\beta}) - U_2(2\beta,2i\sqrt{(\pi-2\theta)\beta}),$$

then after the calculations given in Appendix B, the results are

$$u_2(\alpha,\beta) = - er_1[(1+2\alpha+2\pi-4\theta)I_o(2\sqrt{\beta(\alpha+\pi-2\theta}) + 2\alpha\ I_o(2\sqrt{\alpha(\beta+2\theta)})] -$$
$$- er_2\ I_o(2\sqrt{\alpha(\beta+2\theta}) + 2\sqrt{\alpha(\beta+2\theta)}\ I_1(2\sqrt{\alpha(\beta+2\theta}) +$$
$$+ 2\sqrt{\beta(\alpha+\pi-2\theta})\ I_1\ (2\sqrt{\beta(\alpha+\pi-2\theta})]$$
$$- e(r_2-r_1)[U_1(2\alpha,2i\sqrt{\alpha(\beta+2\theta)}) - U_2(2\alpha,2i\sqrt{\alpha(\beta+2\theta)}) +$$
$$+ U_1(2\beta,2i\sqrt{\beta(\alpha+\pi-2\theta)}) + U_2(2\beta,2i\sqrt{\beta(\alpha+\pi-2\theta)})]$$

$$v_2(\alpha,\beta;\ r_1,r_2;\ 2\theta,\pi-2\theta) = - u_2(\beta,\alpha;\ r_2,r_1;\ \pi-2\theta,2\theta) \qquad (23)$$

These depend upon u_{12}, u_{22}, u_{23} defined by

$$u_2(\alpha,\beta) = - e[r_1u_{21} + r_2u_{22} + (r_2-r_1)u_{23}] \qquad (23)^*$$

numerical values of which are tabulated in Part II.

Finally, the (x_2, y_2) co-ordinates are given by

$$(x_2-x_{20}) + i(y_2-y_{20}) = \int_0^\alpha A_2 \, \mathrm{Exp}\{i(\beta-\alpha)\} d\alpha$$

$$= r_1\{\mathrm{Exp}\{i(\beta-\alpha)\}.[U_1(2\alpha,2i\sqrt{\alpha(\beta+2\theta}) + U_1(2(\alpha+\pi-2\theta),2i\sqrt{\beta(\alpha+\pi-2\theta}) +$$

$$+ i\,U_2(2\alpha,2i\sqrt{\alpha(\beta+2\theta}) + i\,U_2(2(\alpha+\pi-2\theta),2i\sqrt{\beta(\alpha-2\theta)})]$$

$$- \mathrm{Exp}\{i\beta\}.[U_1(2(\pi-2\theta),2i\sqrt{\beta(\pi-2\theta)}) + i\,U_2(2\pi-2\theta),2i\sqrt{\beta(\pi-2\theta)})]\}$$

$$+ r_2\{\mathrm{Exp}\{i(\beta-\alpha)\}.[i\,U_1(2\alpha,2i\sqrt{\alpha(\beta+2\theta)}) + i\,U_1(2(\alpha+\pi-2\theta),2i\sqrt{\beta(\alpha+\pi-2\theta)}) +$$

$$+ U_o(2\alpha,2i\sqrt{\alpha(\beta+2\theta}) + U_o(2(\alpha+\pi-2\theta),2i\sqrt{\beta(\alpha+\pi-2\theta})]$$

$$- \mathrm{Exp}\{i\beta\}.[i\,U_1(2(\pi-2\theta),2i\sqrt{\beta(\pi-2\theta)}) + U_o(2(\pi-2\theta),2i\sqrt{\beta(\pi-2\theta)}) + 1]\} \quad (24)$$

and

$$x_{20} + i\,y_{20} = i\int_0^\beta B_2(0,\beta).\mathrm{Exp}\{i\beta\}d\beta$$

$$= r_1\{\mathrm{Exp}\{i\beta\}.[U_1(2\beta,2i\sqrt{\beta(\pi-2\theta)}) + i\,U_0(2\beta,2i\sqrt{\beta(\pi-2\theta)})] - i\} +$$

$$+ r_2\{\mathrm{Exp}\{i\beta\}.[1+i\,U_1(2\beta,2i\sqrt{\beta(\pi-2\theta)}) + U_2(2\beta,2i\sqrt{\beta(\pi-2\theta)})] - 1\} \quad (25)$$

Combining (24) and (25) gives expression of the form

$$x_2 + i\,y_2 = (r_1x_{21} + r_2x_{22}) + i(r_1y_{21} + r_2y_{22}) \quad (24)^*$$

5. Form of tabulation

Numerical quantities in equations $(18)^*$, $(23)^*$ and $(24)^*$ have been tabulated using α, β as parameters varying from 0 to 135 degrees and $\theta = \pi/4$. Two auxiliary tables of Bessel and Lommel functions are also given, so that it is possible to calculate numerically expressions (15) and (17).

The tabulations of equation $(18)^*$ had been checked, when $r_1 = r_2$, with the table given in p. 350 of reference 3; and the tabulations of equations $(23)^*$ and $(24)^*$ had been partially checked by the special case of reference 6.

All calculations were carried out on a Pegasus digital computer, using a library code for Bessel functions; and equations $(18)^*$ and $(24)^*$ are calculated by numerical integration using the Gauss formula. It is believed that the errors of calculations nowhere exceed 0.1%.

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Appendix A Proof of formula (7)

In proving formula (7), use has been made of the following rules concerning Bessel functions:

$$d\{z^{-k}.I_k(z)\} = z^{-k}.I_{1+k}(z)dz \qquad (A1)$$

$$d\{z^{k}.I_k(z)\} = z^{k}.I_{k-1}(z)dz \qquad (A2)$$

$$I''_k(z) + \frac{1}{z} I'_k(z) - [1 + (\frac{k}{z})^2]I_k(z) = 0 \qquad (A3)$$

It follows:

$$\frac{\partial}{\partial\eta} \int_0^{\alpha} I_o(2\sqrt{(\alpha-\xi)(\beta-\eta)}).(\frac{\xi}{\eta})^{\frac{k}{2}} I_k(2\sqrt{\xi\eta})d\xi$$

$$= -\int_0^{\alpha} \left(\frac{\alpha-\xi}{\beta-\eta}\right)^{\frac{1}{2}} I_1(2\sqrt{(\alpha-\xi)(\beta-\eta)}).\ (\frac{\xi}{\eta})^{\frac{k}{2}} I_k(2\sqrt{\xi\eta})d\xi +$$

$$+\int_0^{\alpha} I_o(2\sqrt{(\alpha-\xi)(\beta-\eta)}) \left[(\frac{\xi}{\eta})^{\frac{1+k}{2}} I'_k(2\sqrt{\xi\eta}) - \frac{k}{\eta} (\frac{\xi}{\eta})^{\frac{k}{2}} I_k(2\sqrt{\xi\eta})\right]d\xi$$

$$= -\int_0^{\alpha}\left(\frac{\alpha-\xi}{\beta-\eta}\right)^{\frac{1}{2}} I_1(2\sqrt{(\alpha-\xi)(\beta-\eta)})\ (\frac{\xi}{\eta})^{\frac{k}{2}} I_k(2\sqrt{\xi\eta})d\xi -$$

$$-\int_0^{\alpha}\left[\left(\frac{\xi}{\eta}\right)^{\frac{1+k}{2}} I'_k(2\sqrt{\xi\eta}) - \frac{k}{\eta} (\frac{\xi}{\eta})^{\frac{k}{2}} I_k(2\sqrt{\xi\eta})\right] d\left\{ \left(\frac{\alpha-\xi}{\beta-\eta}\right)^{\frac{1}{2}} I_1)2\sqrt{(\alpha-\xi)(\beta-\eta)}\right\}$$

$$= \int_0^{\alpha}\left(\frac{\alpha-\xi}{\beta-\eta}\right)^{\frac{1}{2}} I_1(2\sqrt{(\alpha-\xi)(\beta-\eta)})\ (\frac{\xi}{\eta})^{\frac{k}{2}}.\left[I''_k(2\sqrt{\xi\eta}) + \frac{I'_k(2\sqrt{\xi\eta})}{2\sqrt{\xi\eta}} -\right.$$

$$\left. - (1 + \frac{k^2}{4\xi\eta})\ I_k(2\sqrt{\xi\eta})\right]d\xi$$

$$= 0. \qquad (A4)$$

Appendix B Analysis of results in (4b)

In calculating the case 4b, formula (6) gives

$$A_2(\alpha,\beta) = A_2(0,0)I_o(2\sqrt{\alpha\beta}) + \int_0^{\alpha} I_o(2\sqrt{(\alpha-\xi)\beta})\,\frac{\partial A_2(\xi,0)}{\partial\xi}\,d\xi + \int_0^{\beta} I_o(2\sqrt{\alpha(\beta-\eta)})B_2(0,\eta)\,d\eta \tag{B1}$$

where $A_2(\xi,0)$, $B_2(0,\eta)$ can be obtained from (20).

By using formulae (8), (9) and (A1) (A2), it can be shown that

$$\int_0^{\beta} I_o(2\sqrt{\alpha(\beta-\eta)})\,d\eta = \sqrt{\frac{\beta}{\alpha}}\, I_1(2\sqrt{\alpha\beta}) \tag{B2}$$

$$\int_0^{\beta} I_o(2\sqrt{\alpha(\beta-\eta)})I_o(2\sqrt{(\pi-2\theta)\eta})\,d\eta = \sqrt{\frac{\beta}{\alpha+\pi-2\theta}}\, I_1(2\sqrt{\beta(\alpha+\pi-2\theta)}) \tag{B3}$$

$$\int_0^{\beta} I_o(2\sqrt{\alpha(\beta-\eta)})\; I_1(2\sqrt{(\pi-2\theta)\eta})\,\sqrt{\frac{\pi-2\theta}{\eta}}\,d\eta = I_o(2\sqrt{\beta(\alpha+\pi-2\theta)}) - I_o(2\sqrt{\alpha\beta}) \tag{B4}$$

$$\int_0^{\alpha} I_o(2\sqrt{(\alpha-\xi)\beta})\; I_2(2\sqrt{2\theta\xi})\,\frac{2\theta}{\xi}\,d\xi =$$

$$= \sum_{k=o}^{\infty} \frac{(-1)^k}{(k+1)!}\left(1-\frac{1}{k+2}\right)(2\theta)^{2+k}\left(\frac{\alpha}{\beta+2\theta}\right)^{\frac{1+k}{2}} I_{1+k}(2\sqrt{\alpha(\beta+2\theta)})$$

$$= \sqrt{\frac{\beta+2\theta}{\alpha}}\, I_1(2\sqrt{\alpha(\beta+2\theta)}) - \sqrt{\frac{\beta}{\alpha}}\, I_1(2\sqrt{\alpha\beta}) - 2\theta\, I_o(2\sqrt{\alpha\beta}) \tag{B5}$$

Substituting these results into (B1) gives

$$A_2(\alpha,\beta) = r_1[I_o(2\sqrt{\alpha(\beta+2\theta)}) + I_o(2\sqrt{\beta(\alpha+\pi-2\theta)}) +$$

$$+ r_2\left[\sqrt{\frac{\beta+2\theta}{\alpha}}\, I_1(2\sqrt{\alpha(\beta+2\theta)}) + \sqrt{\frac{\beta}{\alpha+\pi-2\theta}}\, I_1(2\sqrt{\beta(\alpha+\pi-2\theta)})\right] \tag{B6}$$

Similarly,

$$B_2(\alpha,\beta) = r_1 \left[\sqrt{\frac{\alpha+\pi-2\theta}{\beta}} \; I_1(2\sqrt{\beta(\alpha+\pi-2\theta)}) + \sqrt{\frac{\alpha}{\beta+2\theta}} \; I_1(2\sqrt{\alpha(\beta+2\theta)})\right] +$$

$$+ r_2 \, [I_o(2\sqrt{\beta(\alpha+\pi-2\theta)}) + I_o(2\sqrt{\alpha(\beta+2\theta)})]$$

The virtual displacements are also calculated from (6)

$$u_2(\alpha,\beta) = u_2(0,0)I_o(2\sqrt{\alpha\beta}) + \int_o^\alpha I_o(2\sqrt{(\alpha-\xi)\beta}) \frac{\partial u_2(\xi,0)}{\partial \xi} d\xi \, +$$

$$+ \int_o^\beta I_o(2\sqrt{\alpha(\beta-\eta)}) \frac{\partial u_2(0,\eta)}{\partial \eta} d\eta - 2e \int_o^\alpha \int_o^\beta I_o(2\sqrt{(\alpha-\xi)(\beta-\eta)}) \; B_2(\xi,\eta) d\xi \, d\eta \tag{B7}$$

From (B6), it is easily seen, by using (7), that

$$- 2e \int_o^\alpha \int_o^\beta I_o(2\sqrt{(\alpha-\xi)(\beta-\eta)}) \; B_2(\xi,\eta) d\xi \, d\eta$$

$$= - 2e \int_o^\alpha \int_o^\beta I_o(2\sqrt{(\alpha-\xi)(\beta-\beta)}) \; B_2(\xi,\beta) d\xi \, d\eta$$

$$= - 2e\beta \int_o^\alpha B_2(\xi,\beta) d\xi \tag{B8}$$

which can be integrated readily by using (A1)(A2).

The first two integrals in (B7) present only one type of integration which differs from those in (B2) - (B5) and (B8), namely,

$$\int_o^\alpha I_o(2\sqrt{(\alpha-\xi)\beta}) \left[\left(\frac{\xi}{2\theta}\right)^{\frac{k}{2}} I_k(2\sqrt{2\theta\xi}) \right]' d\xi = \left(\frac{\alpha}{\beta+2\theta}\right)^{\frac{k}{2}} I_k(2\sqrt{\alpha(\beta+2\theta)}) \tag{B9}$$

Hence, integrating term by term gives

$$\int_o^\alpha I_o(2\sqrt{(\alpha-\xi)\beta}) \; U_1'(2\xi, 2i\sqrt{2\theta\xi}) d\xi = U_1(2\alpha, 2i\sqrt{\alpha(\beta+2\theta)}) \tag{B10}$$

The result (23) can now be obtained by combining all the integrals together.

Finally, the (x,y) co-ordinates can be calculated easily by using (13) and (A1).

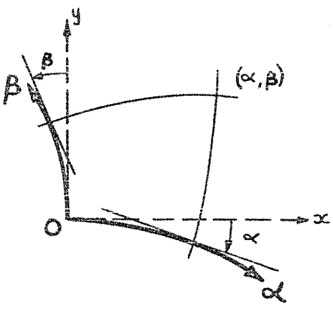

FIG. 1. CO-ORDINATE SYSTEMS.

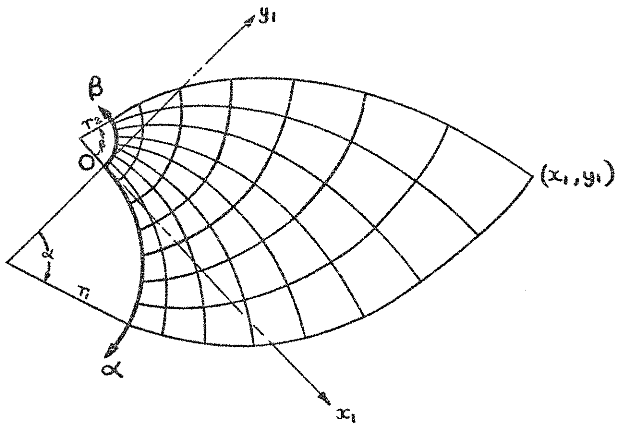

FIG. 2. FIELD GENERATED FROM TWO CIRCULAR ARCS. (RADII r_1, r_2)

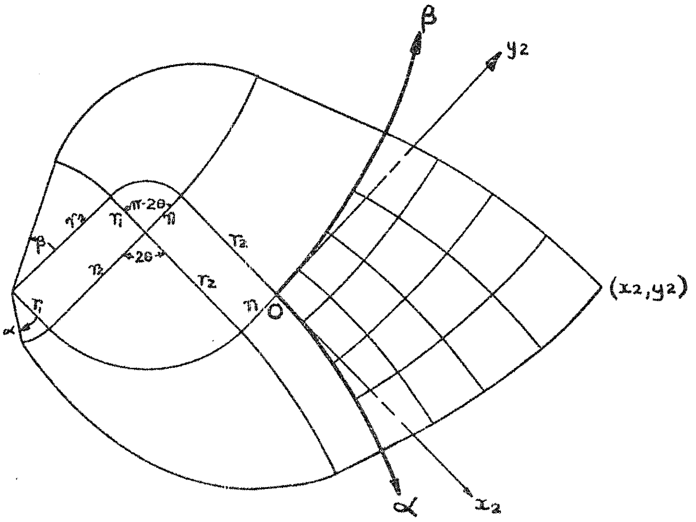

FIG. 3. A FIELD GENERATED FROM TWO ORTHOGONAL CURVES.