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Note on the Dynamics of a Slightly Deformable Body

- By -

W. S. HEMP, M.A.,

of the Department of Aircraft Design.

SUMMARY.

THE PURPOSE OF THIS NOTE IS TO DEVELOP THE EQUATIONS OF MOTION OF A SLIGHTLY DEFORMABLE BODY. APPEAL TO GENERAL PRINCIPLES SHOWS THE INDEPENDENCE OF THE TRANSLATION (Para.2). MOVING AXES ARE DEFINED IN PARA.3 WHICH CAN BE TAKEN TO DEFINE THE ROTATION. MOTION RELATIVE TO THESE AXES IS DESCRIBED BY NORMAL CO-ORDINATES (Para.4) AND THE KINETIC ENERGY OF THE MOTION RELATIVE TO THE CENTRE OF MASS IS SPLIT INTO TWO PARTS; THE ENERGY OF ROTATION AND THE ENERGY OF VIBRATION (Para.5). EQUATIONS FOR THE VIBRATION ARE THEN FORMULATED (Para. 6). ATTENTION IS DRAWN TO THE COUPLING BETWEEN ROTATION AND VIBRATION, WHICH ONLY VANISHES WHEN THE ANGULAR VELOCITIES ARE SMALL (Para.7).

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1. STATEMENT OF THE PROBLEM.

This note is concerned with the mathematical description of the motion of a slightly deformable body and with the formulation of the equations which govern this motion. The general problem of the motion of a deformable body is treated by Lamb in Art.72 of his "Higher Mechanics". The following treatment is based upon that of Lamb and develops his foundations for the special case where the deformation is small.

2. MOTION OF THE CENTRE OF MASS.

The centre of mass moves, according to known dynamical principles, as if it were a particle of mass equal to that of the body, subject to a force equal to the resultant of all the forces applied to the body. Again if moving axes are used, which pass through the centre of mass, the equations governing the variation of angular momentum can be written down without any reference to the translation of the origin. Finally the Kinetic energy can be expressed as the sum of two terms; the first is the energy of the ficticious particle which follows the path of the centre of mass, while the second is the energy of the relative motion. It is clear then, that no difficulty arises in calculating the motion of the centre of mass and further, it can be disregarded in the study of the relative motion. Only the relative motion will be considered in what follows.

3. CHOICE OF THE MOVING AXES.

Consider a system of moving axes passing through the centre of mass. The angular velocity vector of these axes is denoted by ω. If h is the angular momentum vector and M the vector moment of the applied forces about the origin the equation of motion defining the variation of h is:-

$$\frac{dh}{dt} + \omega \times h = M \quad . \quad . \quad . \quad . \quad (1)$$

Now if the body were rigid and were moving with the axes it would possess angular momentum h_r given by:-

$$h_r = (Ap-Hq-Gr,\ -Hp+Bq-Fr,-Gp-Fq+Cr) \quad . \quad (2)$$

Where A,...F,... are moments and products of inertia and $(p,q,r) = \omega$. The quantities A,F are not constants, but vary with the motion relative to our moving axes.

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We now choose our axes in such a way that the following equality holds:-

$$h = h_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The equations (1), (2) and (3) then define the motion of our axes, when the quantities A, F are known. The initial orientation is as yet undefined.

4. MOTION RELATIVE TO THE AXES.

Lamb in his treatment of this problem, chooses ω by fitting the moving axes to the actual motion by means of a principle of least squares and obtains equation (3) above. We deduce then, for our case of a slightly deformable body, that the motion relative to our moving axes is small. We remark further that this motion is without linear or angular momentum (equation (3)). It follows that the term "vibration" may with justice be applied to this motion, but the term must be used with caution since the motion is referred to moving and not to fixed axes.

We describe the relative motion by means of generalised co-ordinates $\theta_1, \theta_2, \ldots, \theta_i, \ldots$ These may be defined as follows:- Consider the case where the body is at rest and referred to axes through its centre of mass. The body possesses an infinity of normal modes of vibration which are free from resultant momentum. Now any kind of small motion which is free from momentum can be expanded in an infinite series of these normal modes. It follows that the θ_i can be taken as normal co-ordinates. The initial orientation of our axes must be chosen to suit. For example in the case of a transient vibration, they must coincide, before the vibration begins, with the axes used above in describing normal modes of vibration.

The quantities A, F can now be expressed in terms of the θ_i. Since the θ_i are small we can write:-

$$A = A_0 + \sum_i \left(\frac{\partial A}{\partial \theta_i}\right)_0 \theta_i \quad . \quad . \quad . \quad . \qquad (4)$$

5. KINETIC ENERGY.

We now show that the Kinetic energy of the motion relative to the centre of mass can be split into two parts. The first depends upon the (p,q,r) while the second depends upon θ_i.

If m is an elementary mass, whose velocity vector is v, the Kinetic energy $T = \frac{1}{2}\sum mv^2$. If ρ is the position vector of m we can split the motion v into two parts:-

$$v = (v - \omega \times \rho) + \omega \times \rho$$

/Substituting...

Substituting in the formula for T we obtain a mixed term:-

$$\sum m\,(v-\omega\times\rho)(\omega\times\rho)=\omega\sum m\rho\times v-\sum m(\omega\times\rho)^2=\omega.h-2T_R$$

Where T_R = Kinetic energy of the masses assumed rotating with our moving axes. Using (2),(3) and the formula

$$2T_R = Ap^2 + \ldots\ldots -2Fqr - \ldots\ldots \qquad (5)$$

we see that our mixed term is zero and so we can write:-

$$T = T_R + T_V \qquad (6)$$

where T_V = Kinetic energy of the motion relative to the moving axes.

6. EQUATIONS FOR THE θ_i.

The equations of motion for the θ_i can be formulated in the Lagrangian manner. We find:-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right)-\frac{\partial T}{\partial \theta_i}+\frac{\partial U}{\partial \theta_i}=\Theta_i \qquad (7)$$

where U is the strain energy of the deformation and Θ_i is the generalised force corresponding to θ_i.

We notice that $\frac{\partial T}{\partial \dot{\theta}_i}=\frac{\partial T_V}{\partial \dot{\theta}_i}$ and $\frac{\partial T}{\partial \theta_i}=\frac{\partial T_R}{\partial \theta_i}$. T_R depends upon θ_i in virtue of the variation of A, F (equation (4)). Since θ_i are normal co-ordinates Tv and U have the forms $T_V=\frac{1}{2}\sum a_i\dot{\theta}_i^2$ and $U=\frac{1}{2}\sum c_i\theta_i^2$, where a_i and c_i are constants. Equation (1) thus transforms to:-

$$a_i\ddot{\theta}_i+c_i\theta_i=\Theta_i+\frac{1}{2}\left\{\left(\frac{\partial A}{\partial \theta_i}\right)_0 p^2+\ldots -2\left(\frac{\partial F}{\partial \theta_i}\right)_0 qr-\ldots\right\} \qquad (8)$$

The terms in the curly brackets in (8) represent the effect of centrifugal forces on the vibration.

7. CASE OF SMALL ANGULAR VELOCITIES.

The equations developed above are complex. The equations (1),(2),(3) which define the rotation involve the vibration through equation (4). Likewise the vibrational equations (8) involve the angular velocity (p,q,r). These complications disappear when the rotation is small. Neglecting terms of second order in (p,q,r) and θ_i we find using (1) (2),(3),(4) and (8):-

$$\frac{d}{dt}\left(A_0p-H_0q-G_0r,\ -H_0p+B_0q-F_0r,-G_0p-F_0q+C_0r\right)=M \qquad (9)$$

$$a_i\ddot{\theta}_i+c_i\theta_i=\Theta_i \qquad (10)$$

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We see in this case that the rotation and vibration are independent.

8. CONCLUSION

We have separated the motion of a slightly deformable body into three parts:- translation, rotation and vibration. While the translation is independent of the other two, the rotation and vibration are coupled except in the case when both are small.