Cranfield

College of Aeronautics Report No.9207
March 1992

RESTORING MONOTONICITY OF SLOWLY-MOVING SHOCKS COMPUTED WITH GODUNOV-TYPE SCHEMES

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ABSTRACT

We tackle, using the Isothermal Gas Equations, the problem of loss of monotonicity behind slowly moving shock waves as computed by Godunov-type schemes. A parameter by which slow-shocks can be detected within a flow-field is presented, along with a modification of Godunov's scheme that introduces numerical dissipation into the flow to damp the oscillations. We also extend the scheme to second order in smooth flow.

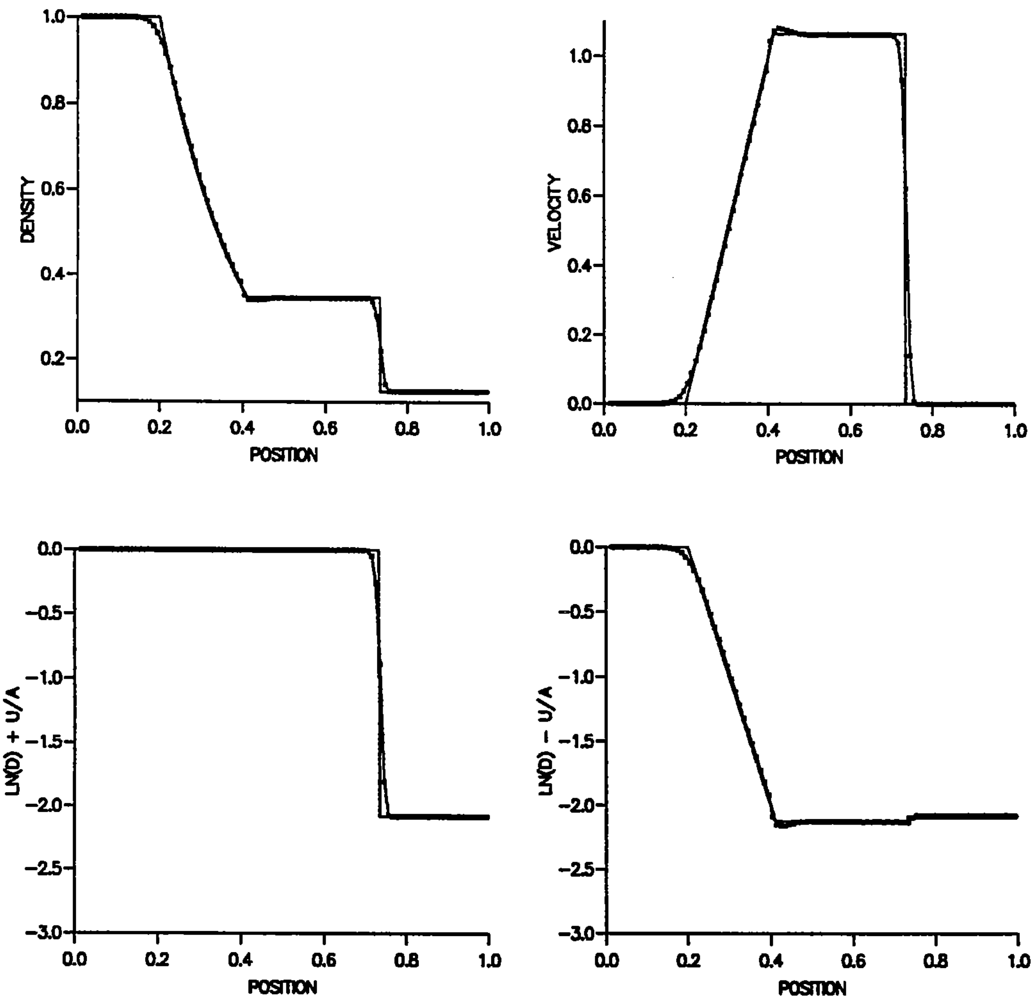

Figure 14 *Our second-order solution to a non-slow-shock problem. The solid line represents the exact solution; the symbols show the numerical solution.*

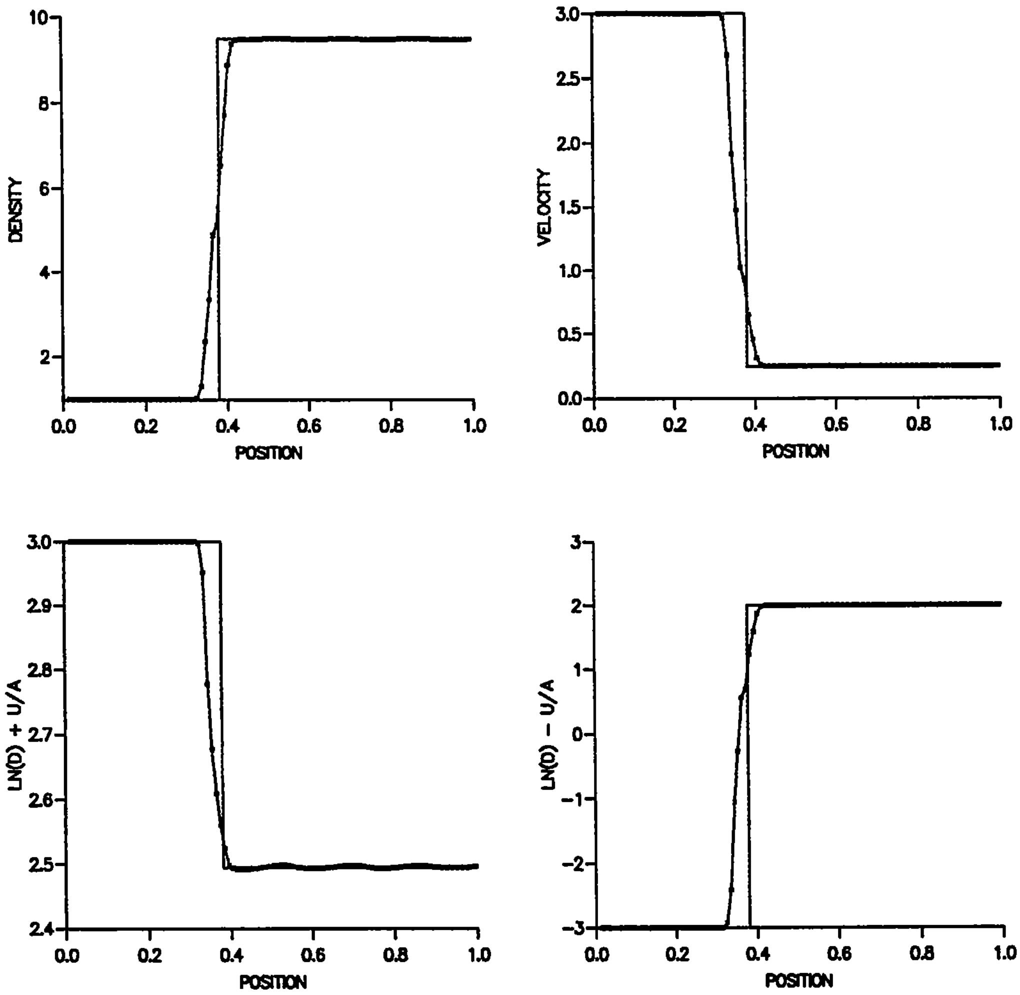

Figure 13 *Our "second-order" solution to Roberts' model problem. The solid line represents the exact solution; the symbols show the numerical solution.*

1 Introduction

An important class of schemes in Computational Fluid Dynamics (CFD) today is that of Godunov-type schemes — those in which the intercell (numerical) flux is calculated from either an exact or approximate solution of the Riemann problem at cell interfaces. These have been found in the past to be an accurate and robust class of schemes. However it has been reported by Roberts [1] and Woodward and Collela [2,3] that these schemes display numerical noise behind slowly-moving shocks, where 'slowly - moving' means that the ratio of shock speed to maximum wave speed in the domain is $\ll$1. This noise takes the form of low- frequency oscillations, and is present in the first-order versions of these schemes, i.e it is not due to modifications for obtaining second-order accuracy. The oscillations cannot be damped by resorting to the usual TVD procedures. It should be pointed out that the problem only arises when solving nonlinear systems of equations and scalar equations, such as the invicid Burgers equation, do not display this error.

There are many reasons for resolving this problem. For example, the oscillations may have an adverse effect when using these methods to compute steady flows in a time asymtotic fashion, in that the rate of convergence will be reduced. Alternatively, when computing chemically active flows, the oscillations may cause premature detonations. Both types of computations are of interest of the authors.

It is the aim of this paper to present an adaptation of Godunov's scheme that is both as accurate at Godunov's scheme in smooth parts of flow and at fast shocks, but in the vincinity of slow-shocks, produces no oscillations, or at minimum, oscillations vastly reduced in amplitude. This will involve two stages: (i) the detection of a slow shock and (ii) adding sufficient numerical dissipation in the appropriate places to damp the unwanted oscillations. It will then be shown how to extend the scheme to second-order accuracy in smooth parts of the flow.

The remainder of this paper is set out as follows. In section 2 the equations used for studying numerical methods are described. In section 3 we describe Godunov's method in detail along with the Weighted Average Flux (WAF) method, a second order extension of Godunov's scheme. In section 4 Roberts paper is briefly reviewed, and some of his results are repeated for illustration of the problem. In section 5 we present our ideas and numerical results, and in section 6 the paper is summarized.

2 The model equations

In this paper we are concerned with a numerical phenonema that affects only nonlinear systems of equations. In the present study, we restrict ourselves to a very simple example of such: the Isothermal Gas Equations. This is a nonlinear, hyperbolic system of two conservation laws:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0 \tag{1}$$

where $\mathbf{U}$ is the vector of conserved variables, and $\mathbf{F}(\mathbf{U})$ is the vector of fluxes:

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \rho u \\ \rho(u^2 + a^2) \end{bmatrix} \tag{2}$$

Here ρ is the density, u is the velocity and a is a non-zero constant analgous to sound speed. For this system there are two characteristic speeds (eigenvalues):

$$\begin{aligned} \lambda_1 &= u - a \\ \lambda_2 &= u + a \end{aligned} \tag{3}$$

with corresponding Riemann invariants $ln\rho \pm \frac{u}{a}$.

For the purpose of illustration we shall also refer to the nonlinear, scalar inviscid Burgers equation, which, in conservation form, reads:

$$u_t + (\frac{1}{2}u^2)_x = 0 \tag{4}$$

and has one characteristic speed u, and has no Riemann invarients.

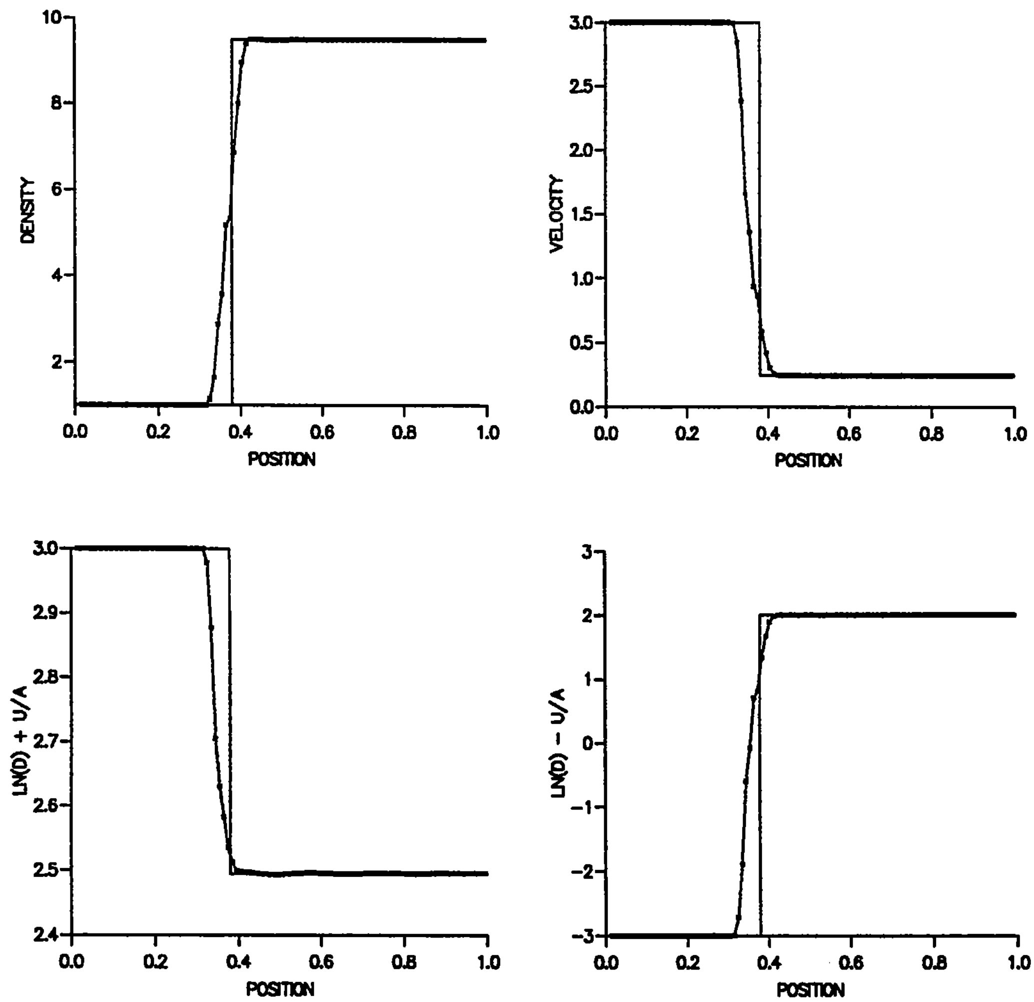

Figure 12 *Our first-order solution to Roberts' model problem. The solid line represents the exact solution; the symbols show the numerical solution.*

3 Numerical schemes

The main aim of this section is to describe the two methods of major interest in this paper: the first order Godunov's scheme, which was the original flux difference splitting scheme, and the Weighted Average Flux (WAF) scheme, a second-order extension of Godunov's scheme.

3.1 *Conservative and flux-difference splitting schemes*

Consider the computational domain, the x-t plane, descretized as depicted in Fig. 1. Evaluation of the integral

$$\oint [\mathbf{U}\delta x - \mathbf{F}(\mathbf{U})\delta t] \tag{5}$$

around the cell $ABCD$ produces the explicit conservative time-marching scheme

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\delta t}{\delta x}\left[\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n\right] \tag{6}$$

in which $\mathbf{U}_i^n$ represents the vector of conserved variables in cell i, and $\mathbf{F}_{i+\frac{1}{2}}^n$ the intercell flux at interface $i+\frac{1}{2}$, at time level n. All explicit conservative schemes can be written in this form; they differ in their choice of $\mathbf{F}_{i+\frac{1}{2}}$. The class of schemes of interest here, the so-called 'Godunov-type' schemes have in common that $\mathbf{F}_{i+\frac{1}{2}}$ is computed from either the exact or an approximate solution of the Riemann problem $\mathbf{RP}(\mathbf{U}_i, \mathbf{U}_{i+1})$ at the interface $i+\frac{1}{2}$. For this reason, we now describe the Riemann problem for a general system of N hyperbolic equations.

3.2 *The Riemann Problem*

The Riemann problem $\mathbf{RP}(\mathbf{U}_L, \mathbf{U}_R)$ for a system of N one-dimensional hyperbolic conservation laws is an initial value problem on the computational domain $-\infty < x < +\infty$. At time $t = 0$ this domain contains contains the initial data:

$$\mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L & \text{if } x < 0 \\ \mathbf{U}_R & \text{if } x > 0 \end{cases} \tag{7}$$

The solution $\mathbf{U} = \mathbf{U}(\zeta)$ to this problem depends only on the similarity variable $\zeta = \frac{x}{t}$, and consists of $N+1$ constant states separated by N nonlinear waves.

For the Isothermal Gas Equations there are three constant states in the solution of $\mathbf{RP}(\mathbf{U}_L, \mathbf{U}_R)$: the original left and right states $\mathbf{U}_L$ and $\mathbf{U}_R$ and a constant state between the two waves, known as the star-state $\mathbf{U}^*$. Each of the two waves can be one of two possibilities — either a shock wave, or a rarefaction wave. Thus the pair of waves in the solution will be one of four possible combinations. One such combination is depicted in Fig. 2.

3.3 *Godunov's scheme*

Godunov's scheme [4] is an explicit, conservative scheme of the form (6). The intercell flux $\mathbf{F}_{i+\frac{1}{2}}$ is found from the exact solution of the Riemann problem $\mathbf{RP}(\mathbf{U}_i, \mathbf{U}_{i+1})$ by taking:

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{U}(\zeta = 0)) \tag{8}$$

i.e taking the (constant) value of the solution on the t-axis. This will be one of the $N+1$ constant states in the solution of the Riemann problem, or, in the case of sonic flow (when a rarefaction straddles the t-axis) the solution inside the rarefaction on the axis.

Note that the intercell flux (8) can be written as

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_R + \mathbf{F}_L) - \frac{1}{2}(\Delta\mathbf{F}^+ - \Delta\mathbf{F}^-) \tag{9}$$

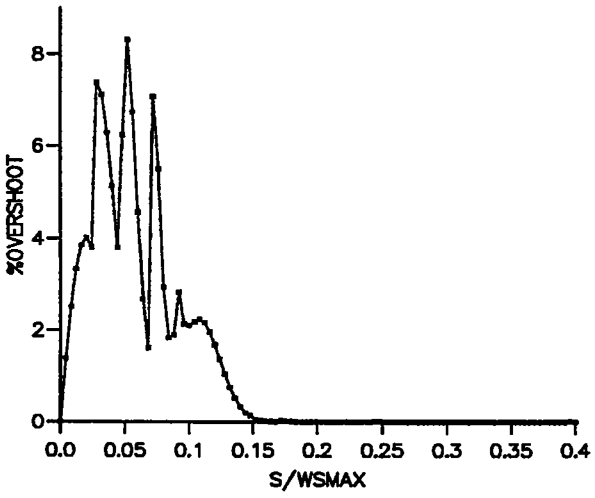

Figure 10 *%Overshoots for Godonuv's scheme for shocks of various speeds.*

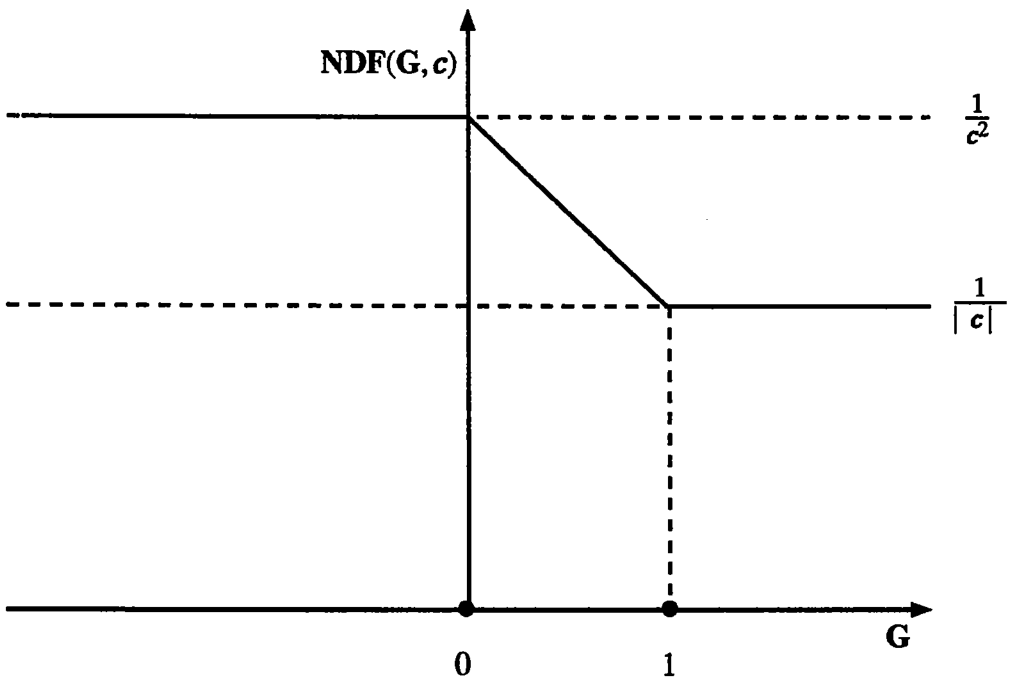

Figure 11 *Our first-order NDF for the WAF dissipation mechanism.*

where:

$$\begin{aligned}\Delta \mathbf{F}^{+} &= \mathbf{F}_R - \mathbf{F}(\mathbf{U}(\zeta = 0)) \\ \Delta \mathbf{F}^{-} &= -\mathbf{F}_L + \mathbf{F}(\mathbf{U}(\zeta = 0))\end{aligned} \tag{10}$$

so that the flux difference across the cell interface $\Delta \mathbf{F} = \mathbf{F}_R - \mathbf{F}_L = \Delta \mathbf{F}^{+} + \Delta \mathbf{F}^{-}$. It is the form (9) of Godunov's scheme that gives rise to the term 'flux-difference splitting schemes', another name often used when refering to Godunov-type schemes. The other Godunov-type schemes, such as Roe's and Osher's schemes, can be written in this form.

Godunov's method is, by definition, upwind scheme: the t-axis where the intercell flux is evaluated, is always upwind of every wave in the solution of the Riemann problem. It is a first order method, and, in the vast majority of cases is diffuse enough to be monotone in the presence of shocks, and needs no modifications to ensure that TVD criteria are satisfied. As will be seen later, its' monoticity breaks down near slowly moving shocks, and, as already mentioned, it is our aim to investigate a way to reduce this problem.

3.4 *The Weighted Average Flux method*

The Weighted Average Flux (WAF) method (Toro, [5,6]) is a second order extension of Godunov's scheme. In this case the intercell flux for (6) is defined, for a set of N hyperbolic conservation laws on a uniform grid in one dimension, as:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{\delta x} \int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \mathbf{F}(\overline{\mathbf{U}}(\frac{x}{\frac{1}{2}\delta t}))dx \tag{11}$$

where $\overline{\mathbf{U}}(\frac{x}{\frac{1}{2}\delta t})$ is the solution of $\mathbf{RP}(\mathbf{U}_i, \mathbf{U}_{i+1})$ at time $t = \frac{\delta t}{2}$. In practice, in the majority of cases the solution to the Riemann problem can be approximated as a set of $N+1$ constant states separated by N shock waves, whence (11) reduces to a weighted average of the $N+1$ fluxes on the constant states (hence the name of the scheme):

$$\mathbf{F}_{i+\frac{1}{2}} = \sum_{k=1}^{N+1} \mathbf{W}_k \mathbf{F}_{i+\frac{1}{2}}^{(k)} \tag{12}$$

The weights $\mathbf{W}_k$ are defined in terms of the local Courant numbers $c_k = \frac{a_k \delta t}{\delta t}$, where a_k is the speed of wave k, as:

$$\mathbf{W}_k = \frac{1}{2}(c_k - c_{k-1}) \tag{13}$$

with $c_0 = -1$ and $c_{N+1} = +1$, and the fluxes $\mathbf{F}_{i+\frac{1}{2}}^{k}$ are the star fluxes between waves $k-1$ and k. See Fig. 3. Note that the special case of sonic flow must again be treated specially: since weight k is found from integrating between waves $k-1$ and k, if these waves have speeds of opposite sign, (where the speed of a rarefaction is taken to be the speed of the *head* of the wave) and either wave is a sonic rarefaction we replace the constant state $\mathbf{F}_{i+\frac{1}{2}}^{k}$ by the state on the t-axis inside the rarefaction.

The intercell flux (12) leads to a second-order scheme (the Lax-Wendroff scheme for the linear advection equation) and thus fails to satisfy Total Variation Diminishing (TVD) criteria (see Harten, [8]) in regions of high gradients. A TVD WAF flux (12) is:

$$\mathbf{F}_{i+\frac{1}{2}}^{WAF} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} \mathbf{A}_k c_k \Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} \tag{14}$$

where:

$$\Delta \mathbf{F}_{i+\frac{1}{2}}^{(k)} = \mathbf{F}_{i+\frac{1}{2}}^{(k+1)} - \mathbf{F}_{i+\frac{1}{2}}^{(k)} \tag{15}$$

is the flux jump across wave k, and $\mathbf{A}$ is a numerical dissipation function. Note that if $\mathbf{A}_k = 1\ \forall k$ then (14) reduces to (12), and if $\mathbf{A}_k = \frac{1}{|c_k|}\ \forall k$ then (14) reduces to the Godunov flux. If the WAF

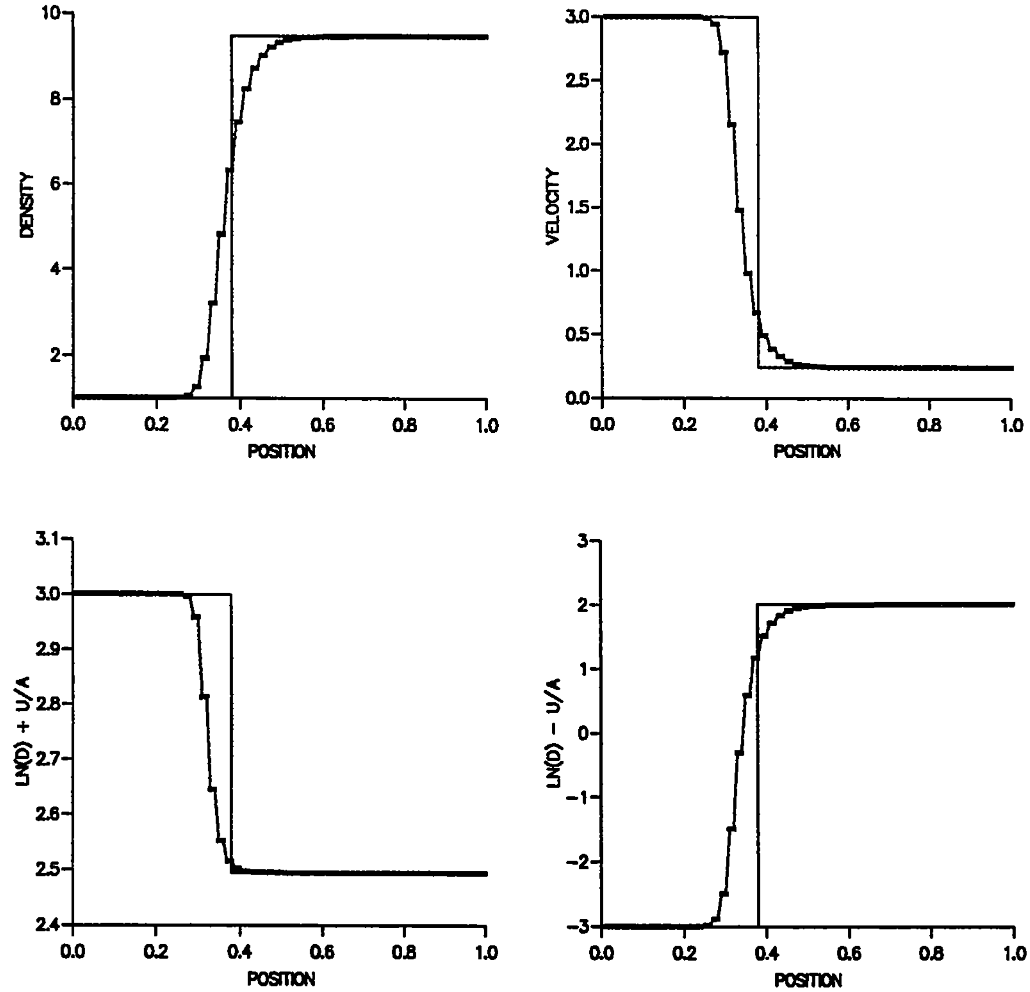

Figure 9 *Solution to the model problem, with the Isothermal Gas Equations, using the Lax-Friedrichs scheme. The solid line represents the exact solution; the symbols show the numerical solution.*

scheme is applied to the linear advection equation, some algebra shows that the numerical dissipation function $\mathbf{A}_k = \mathbf{A}_k(r_i)$ must be a function of the *flow parameter* r defined as:

$$r_i = \frac{\Delta \mathbf{q}^k_{upw}}{\Delta \mathbf{q}^k_{loc}} \tag{16}$$

where $\Delta \mathbf{q}^k_{loc}$ is the jump in some variable across the wave k at the interface $i+\frac{1}{2}$, and $\Delta \mathbf{q}^k_{upw}$ is the corresponding jump at the nearest upwind interface. The conserved variable $\mathbf{q}$ often used is the density. TVD regions in the $\mathbf{A} - r$ plane can be derived inside which the function $\mathbf{A}$ gives a TVD scheme. If these TVD regions are then (empirically) used in the case of nonlinear systems, such as the Isothermal Gas Equations or Euler equations, they are generally sucessful. For full details, see, for example, Toro [5]. Here we simply define the numerical dissipation function we use in this paper, known as "Minaaa":

$$\begin{array}{ll} \mathbf{A}(r,c) = 1 & r \in [1,\infty) \\ \mathbf{A}(r,c) = (1-(1-\mid c \mid)r)/\mid c \mid & r \in [0,1] \\ \mathbf{A}(r,c) = 1/\mid c \mid & r \in (-\infty, 0] \end{array} \tag{17}$$

At this stage, we feel it necessary to make an important point on the term 'numerical dissipation function'(NDF). These functions can be thought of as amplifying the upwind contribution in the intercell flux (12), and are therefore sometimes known as "amplifiers". They are related to limiter functions, and, inparticular, are related to the flux-limiters $\mathbf{B}_k$ (Sweby, [10]) via:

$$\mathbf{A} = (1-\mid c \mid)\mathbf{B} \tag{18}$$

NOTE: Whenever we refer to Godunov's scheme, or the WAF scheme here, we imply an exact Riemann solver is to be used.

3.5 *Other methods*

In the course of this paper we shall also need to refer to other numerical schemes.

Osher's scheme [7] is a first-order flux-difference splitting scheme for systems of hyperbolic conservation laws. His approach to the Riemann problem is to consider the state space rather than physical space. The flux-difference between left and right states may be written:

$$\mathbf{F}_R - \mathbf{F}_L = \int_{\mathbf{U}_L}^{\mathbf{U}_R} \frac{\delta \mathbf{F}}{\delta \mathbf{U}} \mathrm{d}\mathbf{U} \tag{19}$$

where the integral is evaluated along an arbitrary path Γ in the state space. By choosing a path that consists of a K-simple wave, followed by a $K-1$ simple wave, and so on down to a 1-simple wave, a path uniquely determined by the left and right states, Osher derives the first-order intercell flux

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\sum_{k=1}^{N} \int_{\Gamma^{(k)}} \overline{\mathbf{r}}^{(k)} \mid \overline{\lambda}^{(k)} \mid \alpha^{(k)} \tag{20}$$

where $\overline{\lambda}^{(k)}$, $\overline{\mathbf{r}}^{(k)}$ and $\alpha^{(k)}$ are the kth eigenvalue, right eigenvector and innerproduct of the left eigenvector with $\Delta \mathbf{U}$ respectivly.

The *Lax-Friedrichs scheme* is another scheme of the form (6), with intercell flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\frac{\delta x}{\delta t}(\mathbf{U}_{i+1} - \mathbf{U}_i) \tag{21}$$

This is a first-order scheme, one that is far more diffusive than Godunov's scheme. It is not, directly, a Godunov-type scheme: however it will be of use later.

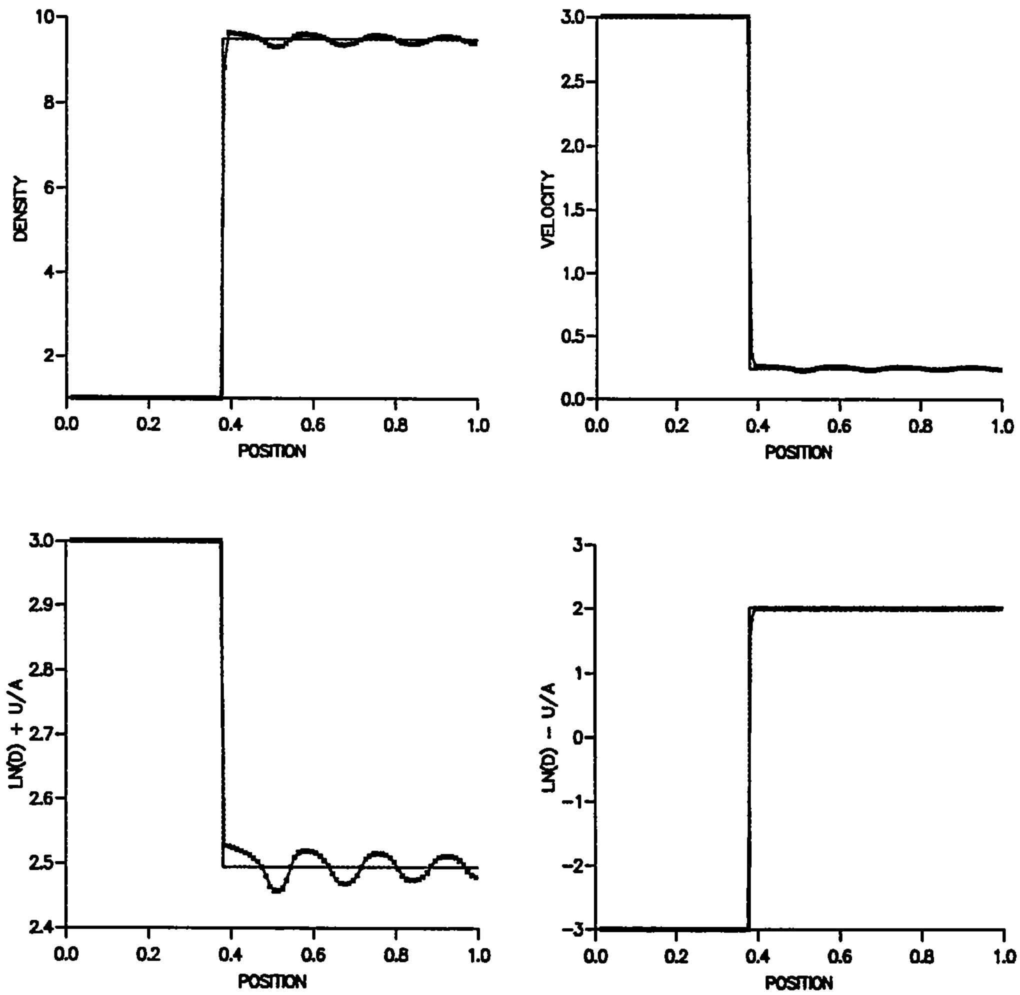

Figure 8 *Solution to the model problem, with the Isothermal Gas Equations, using the WAF scheme with Minaaa. The solid line represents the exact solution; the symbols show the numerical solution.*

4 Summary of Roberts' paper

The problem of resolving slowly moving shock waves via Godunov-type schemes is studied in detail in a paper by Roberts [1]. He examines the shock capturing schemes of Godunov, Roe and Osher when applied to a one-dimensional model problem consisting of a nearly stationary shock. Both scalar and systems of non-linear equations are examined, and it is found that while the solutions obtained when solving the scalar equations are well behaved, those resulting from the systems of equations display numerical noise behind the shock wave in the form of low-frequency oscillations. The aim of this section is to describe the model problem, reproduce some of Roberts' results and briefly review his explanation for the causes of the problem.

4.1 *The model problem*

The model problem is illustrated in Fig. 4. The one-dimensional domain divided into 100 computational cells of width δx, and an initial profile consisting of two constant states separated by an exact shock jump is set. The states are chosen to give a ratio of shock speed to maximum eigenvalue of the system of about 1:50. For the Isothermal Gas Equations (the system for which results will be displayed here) suitable pairs of states can be readily obtained from the Rankine Hugoniot condition $\Delta \mathbf{F} = S\Delta \mathbf{U}$; we take:

$$\begin{array}{ll} \rho_L = 1.0 & \rho_R = 9.4864 \\ u_L = 3.0 & u_R = 0.2446 \end{array} \quad a = 1.0 \tag{22}$$

giving a shock speed of $S = -0.08$. We will also demonstrate the situation for a scalar equation, namely the invicid Burgers equation: here suitable initial conditions are found to be:

$$u_L = 0.96 \quad u_R = -1.0 \tag{23}$$

giving a shock speed of $S = -0.02$. The results are taken after 2000 timesteps. The oscillations produced are independent of Courant number CFL; we use CFL=0.8 in all our calculations.

4.2 *Numerical results*

Roberts produced results on the model problem for the invicid Burgers equation, the Isothermal Gas Equations, and the Euler equations, using the three schemes mentioned above. We only reproduce a few of his results here, nonetheless sufficient to demonstrate the phenonema: we concentrate on the Burgers equation and the Isothermal Gas Equations, using Godunov's and Osher's schemes. We omit results from Roe's scheme, since they are virtually identical to those from Godunov's scheme. All results, both in this section and later, are displayed against the exact solution to the problem.

Fig. 5 shows the result obtained from applying Godunov's scheme to Burgers' equation. As can be seen, the solution is perfectly monotone, as one would expect of a solution from this scheme. This illustrates that when applying flux-difference splitting schemes to scalar equations, the slow shock oscillations do not appear.

Not so in the case of the Isothermal Gas Equations. We show profiles of both density and velocity, as well as the profiles of the two Riemann invariants ($\ln \rho \pm \frac{u}{a}$), two quantities most clearly demonstrate the point. Fig. 6 displays these profiles for the solution to the model problem as computed by Godunov's scheme. The 'right invariant' ($\ln \rho - \frac{u}{a}$) is well behaved, showing a monotone profile. However, the 'left invariant' ($ln\rho + \frac{u}{a}$) (the one corresponding to the shock wave) displays a low frequency oscillation originating at the shock and being convected downstream. Its' wavelength is $\approx 15\delta x$.

Compare this with the solution of the same problem given by Osher's scheme in Fig. 7: as with Godunov's scheme, the right invariant has a perfectly monotone solution, but the left invariant shows a significant difference — oscillations are still present, but their amplitude is an order of magnitude smaller than those of Godunov, and are effectivly negligible.

Roberts also points out that the usual TVD procedures do not damp these oscillations, but actually makes the phenomena worse. Fig. 8 shows the result of using WAF with the well established

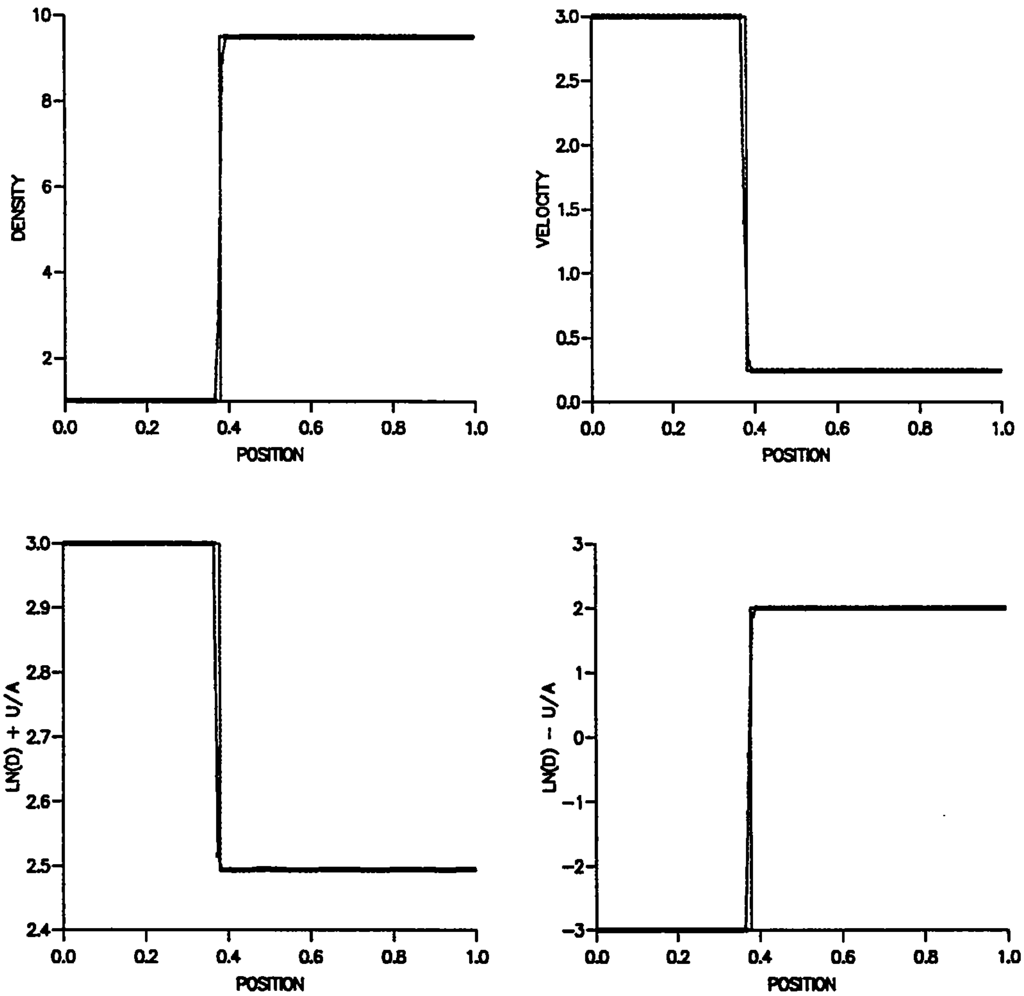

Figure 7 *Solution to the model problem, with the Isothermal Gas Equations, using Osher's scheme. The solid line represents the exact solution; the symbols show the numerical solution.*

NDF Minaaa — a TVD combination that is second order over most of the flow. The oscillations are clearly larger.

4.3 *Brief description of the causes*

The oscillations behind the slowly-moving shocks were originally explained by Woodward and Collela [2,3]. They argue that, since dissipation in Godunov's scheme is proportional to the eigenvalue (wavespeed), near slow shocks there is insufficient dissipation to ensure the correct entropy production at the shock. They conclude that more dissipation must be added and develop several artifical viscosity mechanisms to do so.

The problem with this argument is that it does not explain why solutions to scalar equations do not display the error. Roberts, however, develops an explanation that does take this into account: he explains the production of noise in terms of the internal structure of the shock. He shows that to avoid noise being generated and transported downstream, any internal shock zones must move smoothly along the Hugoniot curve between the two states separated by the shock. Since, in the case of scalar equations, this condition is automatically satisfied by any states that may arise, solutions are monotone. Roberts goes on to show that no shock with only a one zone transition may satisfy his conditions for a monotone solution, illustrating that Woodward and Collela's arguement is essentially correct.

The arguement also explains why Osher's scheme behaves far better near slow shocks: Osher's intercell flux is evaluated in the state space along a path that lies close to the Hugoniot curve, preventing internal states of the shock from deviating far from this curve, and so the amplitude of the noise in the downstream running waves remains small.

It should now be obvious why the usual mechanisms for enforcing TVD conditions (such as the amplifiers for WAF do not damp the oscillations, but make them worse. They are designed to make second-order schemes TVD: they simply detect regions of high gradient and increase numerical diffusion there, up to a limit of the "diffuseness" of the typical first-order schemes, such as Godunov's. However, such first-order schemes display such noise themselves, so the diffusion added by the TVD mechanisms is bound not to be sufficient to damp the oscillations.

The full details of the development of slow shock oscillations are not of importance here: all we need know is that one way of eliminating the problem lies in adding artificial viscosity to a scheme. For full details, see Roberts' paper [1].

5 Our Adaptation of Godunov's scheme.

As stated in the introduction, the aim of this paper is to describe a modification to Godunov's scheme that, if not eliminates, atleast vastly reduces the problem of low-frequency noise behind slowly-moving shock waves. It is in this section that we present our ideas.

5.1 *The Artificial Viscosity Mechanism*

Our first observation in studying this slow shock phenonema was that the Lax- Friedrichs scheme produces a perfectly monotone solution when applied to Roberts' model problem (see Fig. 9). However, as can be seen from the figure, this scheme has two major disadvantages: firstly the shock is *very* diffused, and secondly the scheme produces an undesirable 'pairing' of points. However, the fact that the solution is monotone is important — it indicates that if a scheme is diffusive enough, the oscillations will be damped.

The idea is to construct a scheme that is identical to Godunov's scheme over the vast majority of the computational domain, but 'moves' towards the Lax-Friedrichs scheme near slow-shocks. The trick is to find a mechanism that makes this possible.

One answer lies in the WAF scheme and in the mechanism that makes WAF a TVD scheme. Recall from section 3 that the WAF scheme is made TVD via numerical dissipation functions $\mathbf{A}_k = \mathbf{A}_k(r_k, c_k)$ where the c_k are local Courant numbers, and the r_k are the corresponding flow parameters. When

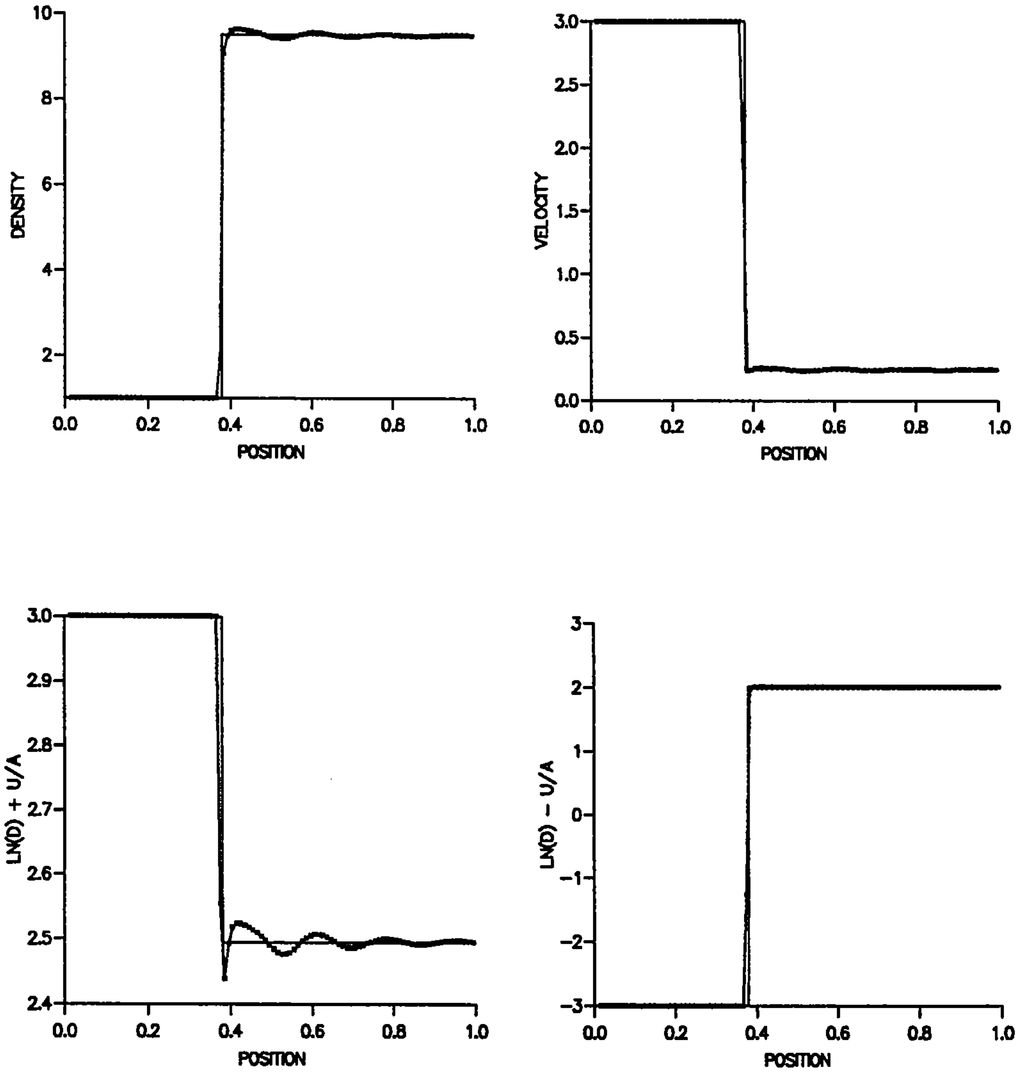

Figure 6 *Solution of the model problem, with the Isothermal Gas Equations, using Godunov's scheme. The solid line represents the exact solution; the symbols show the numerical solution.*

$\mathbf{A}_k = \frac{1}{|c_k|}\ \forall k$ this flux reduces to the Godunov intercell flux. Substituting $\mathbf{A}_k = \frac{1}{c_k^2}\ \forall k$ into (14), placing the resulting flux into the conservative formula, and using the definition $c_k = \frac{a_k \delta t}{\delta x}$ gives the scheme:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\frac{\delta t}{\delta x}\sum_{k=1}^{N}\frac{1}{a_k}\Delta\mathbf{F}^{(k)}_{i+\frac{1}{2}} \tag{24}$$

If we assume that across all N waves the Rankine-Hugoniot condition:

$$\Delta\mathbf{F} = S\Delta\mathbf{U} \tag{25}$$

holds, substituting this into (24) and rearranging gives the scheme:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\frac{\delta x}{\delta t}(\mathbf{U}_{i+1} - \mathbf{U}_i) \tag{26}$$

which is the Lax-Friedrichs intercell flux (21). This relationship between WAF and Lax-Friedrichs led us to apply WAF to the Isothermal Gas Equations, using a constant numerical dissipation function of $\mathbf{A}_k = \frac{1}{c_k^2}\ \forall k$. The result of this experiment was, as expected, a solution identical to that of the Lax-Friedrichs scheme. Thus (for a first order scheme) we set $\mathbf{A} = \frac{1}{|c|}$ over most of the flow field, and, near slow shocks, "move" towards the more diffuse pseudo Lax-Friedrichs scheme at $\mathbf{A} = \frac{1}{c^2}$.

An alternative approach is to start with the Lax Friedrichs scheme and subtract diffusion near smooth flow and fast shocks to improve accuracy. This can be done by again considering the WAF flux in the form:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\frac{\delta t}{\delta x}\sum_{k=1}^{N} a_k \mathbf{F}^{(k)} \tag{27}$$

Using thr Rankine-Hugoniot condition (25), and substituting $c_k = \frac{a_k \delta t}{\delta x}$ gives:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\frac{\delta x}{\delta t}\sum_{k=1}^{N} c_k^2 \Delta\mathbf{U}^{(k)} \tag{28}$$

Note that this flux still gives a scheme that reduces to the Lax-Wendroff scheme on the Linear Advection Equation, and is thus still second order. Replacing c_k^2 in (28) by the function $\mathbf{A}_k$ (of one or more parameters, to be specified later) we obtain a general intercell flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\frac{\delta x}{\delta t}\sum_{k=1}^{N} \mathbf{A}_k \Delta\mathbf{U}^{(k)} \tag{29}$$

Note that if $\mathbf{A}_k = 1\ \forall k$, (29) reduces to the Lax-Friedrichs flux, and $\mathbf{A}_k = | c_k |\ \forall k$ gives a first-order scheme analgous to Godunov's scheme. Again the hope is that a first-order scheme that is monotone at slow shocks can be obtained by using this $\mathbf{A}_k$ as a numerical dissipation function: in this case, we require $\mathbf{A}_k = | c_k |$ in regions of smooth flow and near fast shocks, and move $\mathbf{A}_k$ "up" towards $\mathbf{A}_k = 1$ near slow shocks.

So we have two mechanisms by which we can diffuse our solution. Numerical experiments have shown that they give similar results, so results we will present later will only be from the first of these: the WAF scheme and its' TVD mechanism.

The question still remains of exactly *when* to diffuse the solution. What is needed is a parameter, **G**, that will detect both the existence and location of a shock in the flowfield, and whether this shock is slow or not. The NDF **A** can then be made a function of **G** as well as the flow parameters and local Courant numbers.

5.2 *When is a shock slow?*

One answer to the this question lies in the theory of characteristics. Our inspiration for this came from a comment in Roberts' paper:

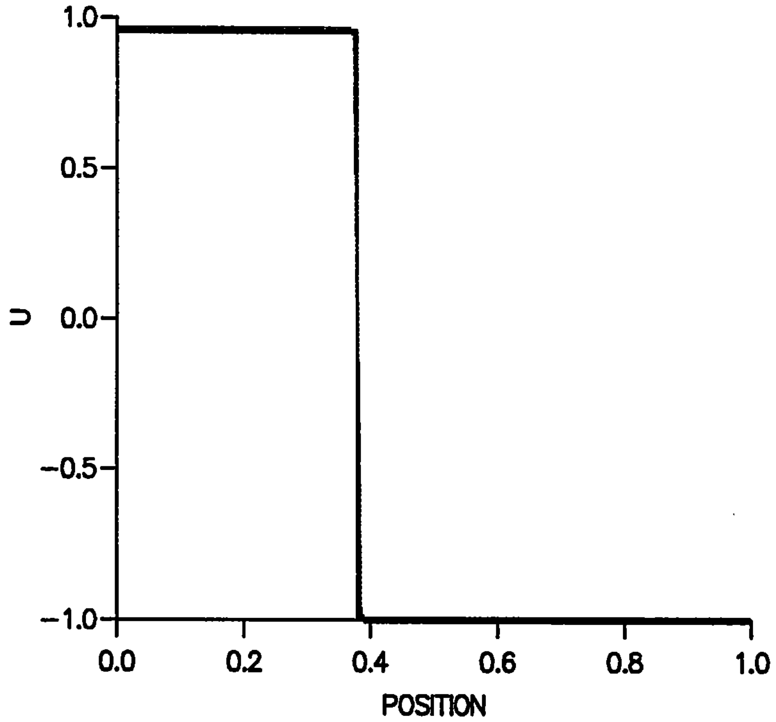

Figure 5 *Godunov's scheme on the invicid Burger's equation. The solid line represnts the exact solution; the symbols show the numerical solution.*

'... a "fast" shock, where the eigenvalues *do not change sign* across the shock wave.'

This lead us to define, for our purposes, a slow shock as one across which the eigevalues do change sign, a definition that can be exploited in a way that would tell us when to add numerical dissipation (even if not by how much). In terms of solving Riemann problems and applying the WAF method to obtain numerical solutions, eigenvalues correspond to wavespeeds; since these wavespeeds are computed in the process of applying WAF anyway, they are readily available at no extra cost.

Before this property can be used it is important to make clear what we mean by the phrase 'eigenvalues change sign' — the question of which eigenvalues is of obvious importance.

Consider a non-linear system of N one-dimensional hyperbolic equations. Suppose we have an exact shock separating two constant states U_L and U_R. For each state, we have N eigenvalues. In characteristic theory, N types of shock are defined for the equations in terms of these eigenvalues. For $k \in [1, N]$ a k-shock is defined as:

$$\begin{array}{c} \lambda_k(\mathbf{U}_R) < S < \lambda_{k+1}(\mathbf{U}_R) \\ \lambda_{k-1}(\mathbf{U}_L) < S < \lambda_k(\mathbf{U}_L) \end{array} \tag{30}$$

where S is the shock speed (see, e.g Smoller [9] for further details). Thus it is natural to compare λ_k's when concerned with a k-shocks.

A numerical experiment was performed to check these ideas. Taking Roberts' model problem, and reversing the shock (so that $S > 0$) a whole range pairs of states separated by 1-shocks for the Isothermal Gas Equations can be deduced from the Rankine-Hugoniot conditions. Keeping $\rho_L = 1.0$ and $u_L = 3.0$ and varying the right state along the Hugoniot curve (the locus of states connected to U_L by the Rankine-Hugoniot conditions) and insisting that the shock remains a 1-shock gives a set of shocks with $\frac{S}{WSMAX}$ in the interval $[0, 0.5]$, where WSMAX is the maximum wavespeed in the domain. We took 100 of these, with $\frac{S}{WSMAX}$ spaced at regular intervals in the range $[0.0, 0.4]$, ran the Godunov scheme for 1000 timesteps on each and recorded the maximum deviation of the first Riemann invariant from the exact solution to the problem as a percentage of exact shock height. The results are shown in Fig. 10.

Note how the solutions do not oscillate for $\frac{S}{WSMAX} \geq 0.15$. Why 0.15? It can be shown, via the Rankine Hugoniot condition (25) that for this set of shocks, the point where the characteristics either side of the shock change from being of opposite sign to being of same sign occurs when $\frac{S}{WSMAX} = 0.1464466$ — approximately the point where oscillations disappear.

5.3 *The Slow-Shock Detection Parameter*

When calculating Godunov's intercell flux $\mathbf{F}_{i+\frac{1}{2}}$ from the solution of $\mathbf{RP}(\mathbf{U}_i, \mathbf{U}_{i+1})$ we are left with $N+1$ constant states upon each of which we can calculate N characteristics, separated by N waves — each either a fast or slow shock, or some other wave type (such as rarefaction wave, or a contact). For each wave we require a parameter $\mathbf{G}$, dependent on the characteristics either side, that indicates the type of wave it is.

Consider the case of a k-shock defined as in (30). Define $\lambda_L = \lambda_k(\mathbf{U}_L)$ and $\lambda_R = \lambda_k(\mathbf{U}_R)$, substituting into (30), and rearranging gives the relation:

$$\lambda_R < S < \lambda_L \tag{31}$$

Suppose $S < 0$: from (31), $\lambda_R < 0$, so dividing (31) through by λ_R gives:

$$1 > \frac{S}{\lambda_R} > \frac{\lambda_L}{\lambda_R} \tag{32}$$

On the other hand, if $S > 0$, from (31) $\lambda_L > 0$, so dividing (31) through by λ_L gives:

$$\frac{\lambda_R}{\lambda_L} < \frac{S}{\lambda_L} < 1 \tag{33}$$

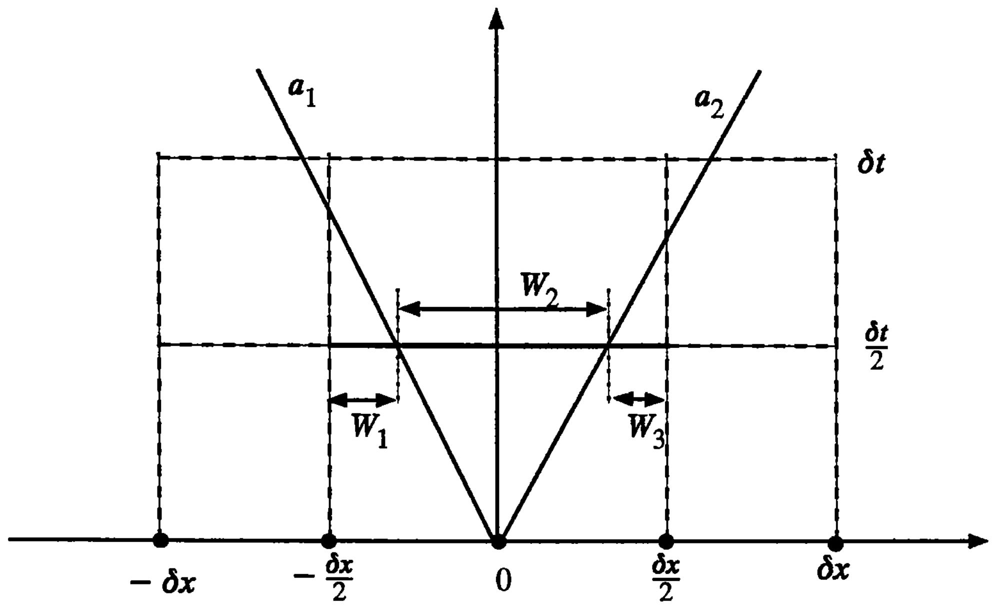

Figure 3 *Weights for the WAF scheme.*

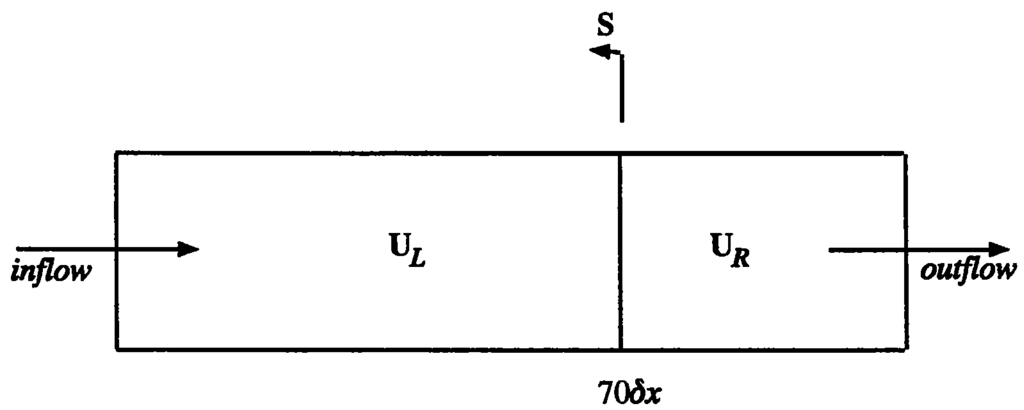

Figure 4 *The model problem.*

If we define λ_U to be the characterstic *upwind* of the shock and λ_D to be the characteristic *downwind* of the shock, we find that both (32) and (33) reduce to:

$$\frac{\lambda_D}{\lambda_U} < \frac{S}{\lambda_U} < 1 \tag{34}$$

Hence we can define a '*shock parameter*' $\mathbf{G}$ as:

$$\mathbf{G} = \frac{\lambda_D}{\lambda_U} \tag{35}$$

where $\mathbf{G} \in (-\infty, 1]$.

The parameter $\mathbf{G}$ can be used as our shock-speed detection parameter. Notice that when $\mathbf{G} > 0$, we have a "fast" shock, and when $\mathbf{G} < 0$ we have a "slow" shock: hence we should add diffusion when $\mathbf{G} < 0$, but not when $\mathbf{G} > 0$.

Note that, as it stands, $\mathbf{G}$ can only be defined *once we know we have a shock.* However this does not pose a problem, atleast not when using the exact Riemann solver: in the coding of Godunov's scheme (or WAF), once a Riemann problem has been solved at a cell interface, logic necessarily exists for each wave to determine whether it is a shock or a rarefaction, depending on the relative values of some variable (e.g density for the Isothermal Gas Equations) either side of the wave. The question may also arise of what to do about $\mathbf{G}$ in the case of a rarefaction. We need assign some value where diffusion will not be added — else we shall introduce the undesirable feature of degrading the resolution of rarefactions. It is sufficient to set $\mathbf{G} = 1$ in this case, since at this point, $\mathbf{G}$ reflects parallel characteristics, i.e shocks of zero strengths where no diffusion will be added in any case.

5.4 *Limits on the numerical dissipation functions*

Before we can design our NDFs $\mathbf{A} = \mathbf{A}(c, \mathbf{G})$ we need to know of any limits on $\mathbf{A}$. We shall discuss these in terms of WAF amplifiers. It has been shown (Billett [11]) that for constant coefficient schemes the hyperbolic linear advection equation, Harten's TVD conditions (Harten, [8]) restrict us to the region between the Lax-Friedrichs scheme and the first-order upwind scheme (equivalent to Godunov), and that stability considerations restrict us to the larger region between the Lax-Friedrichs scheme and the second order Lax-Wendroff scheme. Thus, when using WAF amplifiers to add dissipation to Godunov's scheme at $\mathbf{A} = \frac{1}{|c|}$, it would be wise not to venture further than the Lax-Friedrichs scheme at $\mathbf{A} = \frac{1}{c^2}$. Fortunately, as shown above, Lax-Friedrichs is monotone for slow-shocks, so it is not necessary to go further. When trying to improve accuracy, stability conditions are satisfied as far as the second-order $\mathbf{A} = 1$ (Lax-Wendroff on the linear equation) so this does not restrict us; however to remain TVD we must restrict ourselves to the regions, dependent on the flow-parameter r (Toro [6]).

5.5 *A First-order NDF*

After performing some numerical experiments we realized that our original aim of eliminating slow-shock oscillations without adding dissipation to any fast shocks could not be achieved with the present scheme. The reason appears to be this: within any shock structure, there are very few local slow-shocks, and these are present in the centre of the structure. Any amount of dissipation (within stability limits) added at these points turns out to be insufficient to damp the oscillations, since it only widens the shock structure by about a point.

We have therefore been forced to limit our ambitions for the present, and accept the necessity of adding dissipation to fast shocks. The NDF used is illustrated in Fig. 11: for $\mathbf{G} = 1$ (smooth flow) we keep the scheme at Godunov with $\mathbf{A} = \frac{1}{|c|}$; for negative $\mathbf{G}$ (a slow shock) we take the Lax-Friedrichs flux at $\mathbf{A} = \frac{1}{c^2}$, and in the remaining region of fast shocks, where $\mathbf{G} \in [0, 1]$, we linearly interpolate between the Lax-Friedrichs and Godunov amplifiers.

This NDF was much more sucessful at damping the oscillations behind slow shocks: for Roberts' problem, the amplitude of the oscillations were an order of maginitude smaller when measured as a

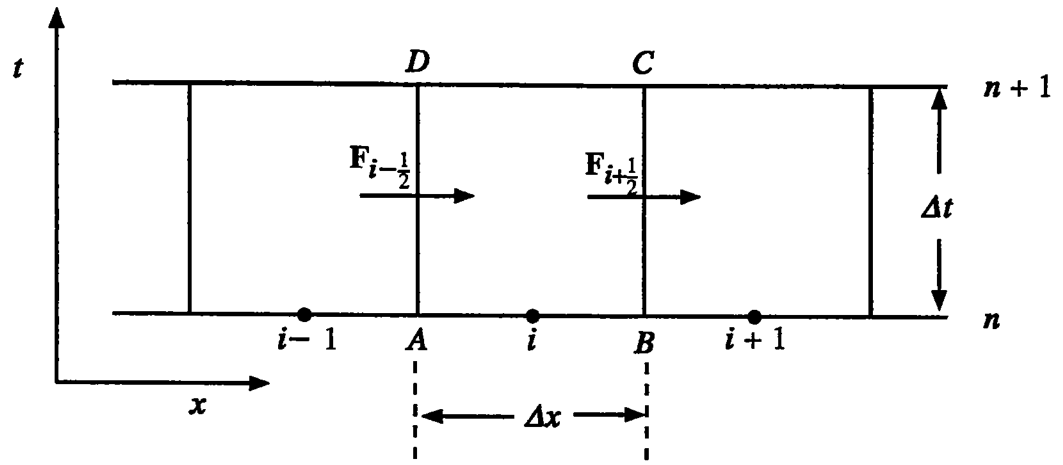

Figure 1 *The computational domain.*

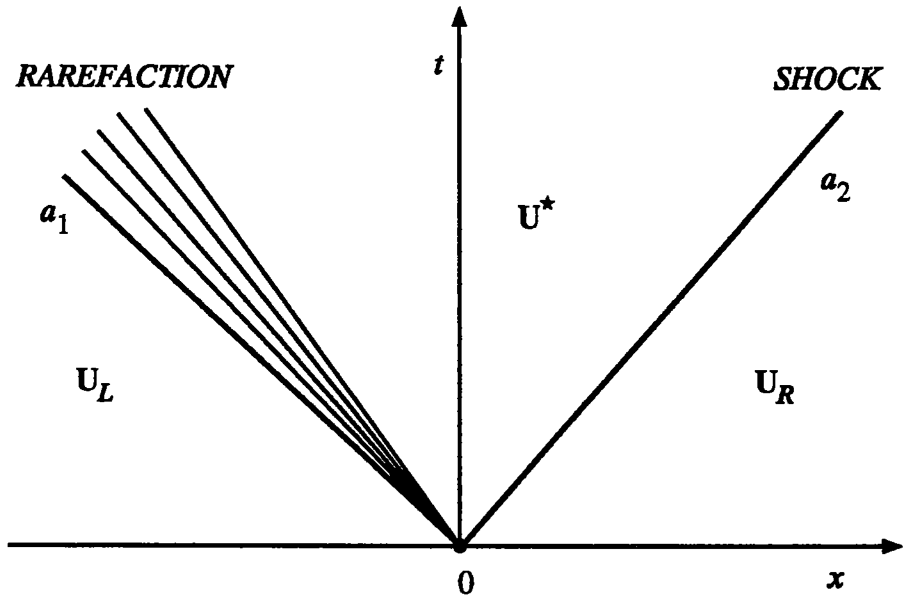

Figure 2 *Solution of the Riemann problem for a system of two hyperbolic conservation laws.*

percentage of shock height, and not significantly larger in amplitude than the oscillations produced by Osher's scheme. See Fig. 12. However, we were disapointed at the width of the shock needed to attain this: there are about 9 points in the shock, as opposed to the 4 or 5 we had hoped for. This is, however, still half the width of the Lax-Friedrichs shock shown earlier, so it is by no means a failure. Note also that the unattractive pairing of points characteristic of the Lax-Friedrichs scheme appears to some extent in the centre of the shock. This was to be expected.

5.6 *A Second-order Extension.*

The NDF discussed above reduces to Godunov's scheme in smooth parts of the flow-field, and is hence only first order there. We shall now describe how to attain second order accuracy in these regions, while still heavily dissipating slow shocks.

The method is simple: all is required is a simple translation of any conventional NDF, such as Minaaa, defined above. Suppose $\mathbf{A}_1 = \mathbf{A}_1(r, c)$ is such an NDF, and denote by $\mathbf{NDF} = \mathbf{NDF}(\mathbf{G}, c)$ the dissipation function defined in the last subsection. We can then define a numerical dissipation function as:

$$\mathbf{A}(r, \mathbf{G}, c) = \mathbf{A}_1(r, c) + \mathbf{NDF}(\mathbf{G}, c) - \frac{1}{|c|} \tag{36}$$

Note that this function reduces to the NDF $\mathbf{A}_1$ whenever our our $\mathbf{NDF}$ returns the Godunov scheme, and will thus give accuracy as good as that of $\mathbf{A}_1$ in this region. At very fast shock, when our $\mathbf{NDF}$ does not stray far from Godunov's scheme, we shall remain close to this accuracy. It is only near slower shocks that this changes: in this case, $\mathbf{NDF}$ is equal to, or close to, a value of $\frac{1}{c^2}$, and $\mathbf{A}_1$ will be close to $\frac{1}{|c|}$, since this is required for it to remain in the TVD regions near shocks, and hence we are left with a $\mathbf{A} \approx \frac{1}{c^2}$, giving the Lax-Friedrichs scheme. This should keep the slow shock monotone.

Results are shown for the above function applied to two problems. For both problems, NDF used as $\mathbf{A}_1$ was Minaaa (as defined earlier).

The first is Roberts' model problem, shown in Fig. 13. Note that the oscillations are larger in amplitude than those given by the the first-order $\mathbf{NDF}$. This is caused by the shock being slightly narrower. The reduction in shock width occurs because once the slow shock has been widened by the dissipation added, the flow parameter in the centre of the shock structure moves closer to unity, thus reducing the amount of dissipation added.

The second problem given to this NDF was designed to test its performance on smooth flow and fast shocks. It is analgous to Sod's problem for the Euler equations; the initial conditions consist of a discontinuity separating two constant states:

$$\begin{matrix} \rho_L = 1.0 & \rho_R = 0.125 \\ u_L = 0.0 & u_R = 0.0 \end{matrix} \quad a = 1 \tag{37}$$

The solution consists of a rarefaction travelling to the left, with a sonic point near its tail, and a fast shock travelling to the right. Results, shown in Fig. 14, were good: the rarefaction is captured as accurately as the usual Minaaa in the WAF scheme, and the shock is barely any wider that normal.

6 Summary

The aim of this paper was to present a modification of Godunov's scheme that uses the WAF dissipation mechanism to tackle the problem of loss of monotonicity behind slowly moving shock waves. The scheme has both first- and second-order versions. The second-order version consists of a simple modification of existing second order amplifiers. We have demonstrated it to be second order in smooth regions of flow, almost second order near fast shocks, and to resolve slow shocks with ≈ 9 points, with oscillations an order of magnitude smaller than those produced by the original Godunov scheme.

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