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On minimizing maximum transient energy growth

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Abstract

The problem of minimizing the maximum transient energy growth is considered. This problem has importance in some fluid flow control problems and other classes of non-linear systems. Conditions for the existence of static controllers that restrict the maximum transient energy growth to unity are established. An explicit parametrization of all linear controllers ensuring monotonic decrease of the transient energy is derived. It is shown that by means of a Q-parametrization, the problem of minimizing the maximum transient energy growth can be posed as a convex optimization problem that can be solved by means of a Ritz approximation of the free parameter. By considering the transient energy growth at an appropriate sequence of discrete time points, the minimal maximum transient energy growth problem can be posed as a semidefinite problem. The theoretical developments are demonstrated on two numerical problems.

1 Introduction

In stable linear systems, it is possible for the system state trajectory, following some initial perturbation, to grow to large values before decreasing again and converging to the origin despite the eigenvalues having very negative real parts. This behaviour is highly undesirable, particularly for certain non-linear systems, where linear eigenvalue analysis at an equilibrium point indicates very good stability, but the transient behaviour of the trajectories means that even very small initial perturbations in the state variables can cause the states to leave the domain of attraction resulting in instability.

This phenomenon is known to occur in fluid dynamic systems. For example, laminar flow can become turbulent even for Reynolds numbers for which linear stability analysis predicts stable eigenvalues. The reason for this was only realized by fluid dynamicists fairly recently (Trefethen et al., 1993). In the fluid dynamics community, the distance from the equilibrium is usually measured by the energy of the perturbations, that being the square of their Euclidian distance from the origin, and the maximum transient energy growth following some energy-bounded initial state perturbation is of interest in many fluid systems (e.g. Reddy and Henningson, 1993; Schmid and Henningson, 2001). For fluid control systems, a useful control objective is the minimization of the maximum transient energy of the flow perturbations (Bewley and Liu, 1998). Reducing the maximum transient energy is also important for a class of partially linear cascade systems initially investigated by Sussmann and Kokotovic (1991), see Sepulchre, Jankovic and Kokotovic (1997) for a review.

The problem of constraining transient trajectory norms has been considered elsewhere (recent results have been reported in Hinrichsen and Pritchard, 2000; Pritchard, 2000; Hinrichsen, Plischke and Wirth, 2002; Plischke and Wirth, 2004; Wirth, 2004). A Linear Matrix Inequality (LMI) approach to minimization of maximum transient energy growth has been proposed by Boyd et al. (1994).

This paper is organized as follows. In Section 2, following definitions of transient energy and maximum transient energy growth, some conditions for unity maximum transient energy growth are established. An upper bound on the maximum transient energy growth is given, and methods for evaluating the maximum transient energy growth and the upper bound are proposed. Section 3 considers the problem of determining static gain controllers that minimize the maximum transient energy growth. An explicit parametrization of all linear controllers ensuring monotonic decrease of the transient energy is derived. For systems where such controllers do not exist, a state feedback static controller that minimizes an upper bound may be determined. In Section 4, dynamic feedback controllers are considered. It is shown that the problem of determining a controller to minimize maximum transient energy growth may be solved by convex optimization over the free parameter in a Q-parametrization of the problem. Furthermore, by considering the response at a finite set of time points, an approximation of the problem can posed as a semidefinite program that can be solved using standard methods. In Section 5, the theory is illustrated with some numerical examples.

Notation

M^T denotes the transpose of a matrix M
M^* denotes the complex conjugate transpose of a matrix M
$M^\dagger$ denotes the Moore-Penrose inverse of the matrix M
$M^\perp$ denotes the left null space of a matrix M, that is $M^\perp = U_2^T$ where $\begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = M$

is the singular value decomposition of M
$\|M\| := \max\left\{\sqrt{\lambda_i} : \lambda_i \text{ are the eigenvalues of } M^T M\right\}$ denotes the spectral norm of a real matrix M
$\| M \|_F$ denotes the Frobenius norm of a matrix M
$\|x\| := \sqrt{x^T x}$ denotes the Euclidian 2-norm of a vector x
$\text{vec}(M)$ denotes the vector formed by stacking the columns of matrix M
$\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote, respectively, the largest and smallest eigenvalues of the matrix M
I_n represents the identity matrix of dimension $n \times n$

2 Maximum transient energy growth

Consider the stable linear time-invariant system described by

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$ which has the solution

$$x(t) = \Phi(t)x_0, \tag{2}$$

where $\Phi(t)$ is the state transition matrix given by $\Phi(t) = e^{At} = \sum_{i=0}^{\infty} A^i t^i / i!$.

Definition 1. *The transient energy, $\mathcal{E}(t)$, is defined as*

$$\mathcal{E}(t) := \max_{\|x(0)\|=1} \|x(t)\|^2 . \tag{3}$$

In practice, the transient energy, $\mathcal{E}(t)$, usually requires appropriate weights on the states, that is

$$\mathcal{E}(t) = \max_{\|Wx(0)\|=1} \|Wx(t)\|^2 , \tag{4}$$

where $W > 0$ is a constant weighting matrix. For the remaining results in this paper to be applicable to such cases, a simple change of variables $\tilde{x} = Wx$ should be performed.

Definition 2. *The maximum transient energy growth, $\widehat{\mathcal{E}}$, is defined as*

$$\widehat{\mathcal{E}} := \max_{t\geq 0} \mathcal{E}(t). \tag{5}$$

Lemma 1. *The maximum transient energy growth, $\widehat{\mathcal{E}}$, of the system described by* (1) *is lower bounded by unity.*

Proof. At $t = 0$ then $x = x(0)$ and $\|x(0)\| = 1$. Hence

$$\max_{t\geq 0} \max_{\|x(0)\|=1} \|x(t)\|^2 \geq 1. \tag{6}$$

□

2.1 Evaluating maximum transient energy growth

The following lemma gives the conditions on the state matrix, A, for there to be no transient energy growth.

Lemma 2. *The maximum transient energy growth, $\widehat{\mathcal{E}}$, of the system described by* (1) *is unity if and only if $A + A^T \leq 0$.*

Proof.

Sufficiency: if $\mathrm{d}\left\|x(t)\right\|^2/\mathrm{d}t \leq 0$ for all $t \geq 0$ and all $\{x(t) : \|x(0)\| = 1\}$, then $\max\{\|x(t)\| : t \geq 0\}$ $= \|x(0)\| = 1$. Differentiating $\|x(t)\|^2$ gives

$$\frac{\mathrm{d}x^T(t)x(t)}{\mathrm{d}t} = x^T(t)(A + A^T)x(t) \tag{7}$$

which, if $A + A^T \leq 0$, is non-positive for all $x(t)$, $t \geq 0$.

Necessity: if $A + A^T \nleq 0$, then there exists an x such that $x^T(A + A^T)x > 0$, hence there exists an x_0 such that $x^T(A + A^T)x > 0$ at $t = 0$. Thus $\|x(t)\|^2 > 1$ for some $t > 0$, so it is necessary that $A + A^T \leq 0$. □

If $A + A^T > 0$, then $\widehat{\mathcal{E}}$ can be evaluated by means of a line search over time of the spectral norm of $\Phi(t)$, the procedure suggested by Bewley and Liu (1998), since it is well known that

$$\max_{\|x\|=1} \|Mx\| = \|M\| . \tag{8}$$

The search procedure can be improved by establishing the necessary conditions for the maximum point of $\|\Phi(t)\|$. Clearly, at a turning point, the state derivative vector, $\dot{x}$, should be orthogonal to the state vector, x. Thus a local search can be made over t to obtain the inner product of $\dot{x}$ and x to be zero, that is $x^T Ax = 0$. Note that for second-order systems, the turning points can be determined analytically without recourse to a search over t.

Lemma 3. *The transient energy, $\mathcal{E}(t)$, of a system described by* (1) *is monotonically decreasing if and only if $A + A^T < 0$.*

Proof. The proof follows the same lines as Lemma 2. □

Remark 1. *A system satisfying Lemma 2 is not necessarily asymptotically stable. Lemma 3 provides a useful sufficient condition for unity transient energy growth and asymptotic stability, since it is well-known that asymptotic stability is ensured if there exists a $P > 0$ such that $PA + A^TP < 0$. If $P = I$, then from Lemma 2, there is also unity maximum transient energy growth.*

2.2 Normal systems

As discussed in the introduction, large transient energy growth is related to the non-normal nature of the system. Here we show that systems with a state matrix, A, that is normal have unity maximum transient energy growth.

Definition 3. *A matrix M is normal if $M^TM = MM^T$.*

The following lemma from (Maciejowski, 1989, p. 91) is well-known.

Lemma 4. *Suppose that a matrix M is normal and if $M = E\Lambda E^{-1}$ where $\Lambda = \mathrm{diag}\{\lambda_i\}$ is the spectral matrix of eigenvalues and E is the modal matrix of right eigenvectors, then $E^* = E^{-1}$ (that is, the eigenvectors of a normal matrix form an orthonormal set).*

The following result shows that systems with a normal state matrix, A, always have unity transient energy growth.

Lemma 5. *The maximum transient energy growth, $\widehat{\mathcal{E}}$, of the system described by (1) is unity if A is normal.*

Proof. From Lemma 4

$$
\begin{aligned}
A + A^* &= E\Lambda E^* + (E\Lambda E^*)^* && (9)\\
&= E\left(\Lambda + \Lambda^*\right)E^*. && (10)
\end{aligned}
$$

Hence $\{\lambda_i(A + A^*)\} = \{\lambda_i(A) + \lambda_i(A)^*\}$, and since the system is stable, $\lambda_i(A + A^*) < 0$ for all i. Thus, since A is real, then $A + A^T < 0$. From Lemma 2, if $A + A^T \leq 0$, then $\widehat{\mathcal{E}} = 1$. □

2.3 A bounding ellipsoid

An upper bound on the maximum transient energy growth can be obtained by means of a Lyapunov function that describes an ellipsoid that bounds the trajectory (known as a positive invariant set (Blanchini, 1999)). This results in the well-known Lyapunov inequality.

Lemma 6. *$\widehat{\mathcal{E}}_u \geq \widehat{\mathcal{E}}$ is an upper bound on the maximum transient energy growth, $\widehat{\mathcal{E}}$, for a system described by (1), where*

$$
\widehat{\mathcal{E}}_u := \lambda_{\max}(P)\lambda_{\max}(P^{-1}) \qquad (11)
$$

where $P = P^T > 0$ satisfies

$$
PA + A^T P < 0. \qquad (12)
$$

Proof. The function $L(x) = x^T P x$ is a Lyapunov function if $\mathrm{d}L/\mathrm{d}t < 0$, that is if P satisfies $PA + A^T P < 0$. If $L(x)$ is a Lyapunov function, then if $x(0)$ is in the ellipsoidal set $\{\xi^T P \xi \leq 1\}$, then $x(t)$ will remain in set $\{\xi^T P \xi \leq 1\}$ for all $t \geq 0$. Since $\lambda_{\min}(P)\left\|\xi\right\|^2 \leq \xi^T P \xi \leq \lambda_{\max}(P)\left\|\xi\right\|^2$ (Chen, 1970, Theorem 8.18); the identity $\lambda_{\min}(P) = 1/\lambda_{\max}(P^{-1})$ gives (11). □

The upper bound can be minimised by solving the following LMI generalized eigenvalue problem (GEVP) (Boyd et al., 1994, p. 65):

$$
\begin{aligned}
&\min \gamma\\
&\text{subject to} && (13)\\
&\quad I \leq P \leq \gamma I, \quad PA + A^T P < 0
\end{aligned}
$$

where $P > 0$ is real and symmetric. The inequality $I \leq P \leq \gamma I$ ensures that $\gamma \geq \lambda_{\max}(P) \geq \lambda_{\min}(P) \geq 1$, thus $\lambda_{\max}(P)/\lambda_{\min}(P) \leq \gamma$ and so $\widehat{\mathcal{E}} \leq \widehat{\mathcal{E}}_u \leq \gamma$.

3 Optimal static gain feedback controllers

Now consider the linear time-invariant plant

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{14}$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times \ell}$, $u(t) \in \mathbb{R}^\ell$, $C \in \mathbb{R}^{m\times n}$, $y(t) \in \mathbb{R}^m$. Furthermore, it is assumed that $B^T B > 0$, that is B has full column rank, and $CC^T > 0$, that is C has full row rank, (i.e. all actuators and sensors are independent).

3.1 Monotonically decreasing transient energy

In this section, conditions are given for all static output feedback controllers that ensure that the transient energy is monotonically decreasing, and hence have unity maximum transient energy growth.

Theorem 1. *For the system of* (14)*, the following are equivalent:*

1. *There exists a control $u = Ky$ where K is a constant matrix such the the transient energy is monotonically decreasing.*
2. *The following two conditions hold*

$$B^\perp \left(A + A^T\right) B^{\perp T} < 0 \text{ or } BB^T > 0, \tag{15}$$

$$C^{T\perp} \left(A + A^T\right) C^{T\perp T} < 0 \text{ or } C^T C > 0. \tag{16}$$

Furthermore, if the above statements hold, all controller matrices K that ensure that the transient energy is monotonically decreasing are given by

$$K = -R^{-1}B^T\Psi C^T(C\Psi C^T)^{-1} + S^{1/2}L(C\Psi C^T)^{-1/2} \tag{17}$$

where

$$S := R^{-1} - R^{-1}B^T \left[\Psi - \Psi C^T(C\Psi C^T)^{-1}C\Psi\right] BR^{-1} \tag{18}$$

where L is an arbitrary matrix such that $\|L\| < 1$ and $R > 0$ is an arbitrary matrix such that

$$\Psi := \left(BR^{-1}B^T - A - A^T\right)^{-1} > 0. \tag{19}$$

Proof. For the closed loop system transient energy to satisfy Lemma 3 and so be monotonically decreasing, it is required that

$$(A + BKC) + (A + BKC)^T < 0. \tag{20}$$

The remainder follows directly by application of Theorem 2.3.12 of Skelton, Iwasaki and Grigoriadis (1998, p. 29), with the condition that B is full column rank and C has full row rank. □

Remark 2. *A matrix R that satisfies* (19) *can be obtained by $R = I/\rho$. For the case where $BB^T > 0$ (i.e. B is full rank n), ρ is obtained simply by rearranging $BRB^T - A - A^T > 0$ giving the inequality*

$$\rho > \lambda_{\max}(B^{-1}(A + A^T)(B^T)^{-1}). \tag{21}$$

For the case where $B^{\perp}(A + A^T)B^{\perp T} < 0$, ρ is obtained by an application of Theorem 2.3.10, of Skelton, Iwasaki and Grigoriadis (1998, p. 26), this being an extension to Finsler's Theorem.

Finsler's Theorem is presented below in an appropriate form for use with Theorem 1.

Theorem 2 (Finsler's Theorem). *Given $\Gamma = (A + A^T)$, the following statements are equivalent:*

1. *There exists a scalar ρ such that*

$$\rho BB^T - \Gamma > 0. \tag{22}$$

2. *The following condition holds*

$$P := B^{\perp}\Gamma B^{\perp T} < 0. \tag{23}$$

If the above statements hold, then all scalars ρ satisfying (22) *are given by*

$$\rho > \rho_{\min} := \lambda_{\max}\left\{B^{\dagger}(\Gamma - \Gamma B^{\perp T}P^{-1}B^{\perp}\Gamma)B^{\dagger T}\right\}. \tag{24}$$

3.2 Minimal upper-bound on maximum transient energy growth

An LMI (Boyd et al., 1994, p. 100) can be formulated to obtain a controller that minimizes the upper bound $\widehat{\mathcal{E}}_u$. Expanding (12) for $u = Kx$ gives

$$PA + A^TP + PBK + K^TB^TP < 0. \tag{25}$$

By the change of variable, $Q = P^{-1}$ and $Y = KQ$ the LMI

$$AQ + QA^T + BY + Y^TB^T < 0. \tag{26}$$

is obtained. Now since $\lambda_{\max}(P)\lambda_{\max}(P^{-1}) = \lambda_{\max}(Q)\lambda_{\max}(Q^{-1})$, we can obtain a controller that minimizes the upper bound on the maximum transient energy growth by solving the following LMI generalized eigenvalue problem (GEVP):

$$\begin{aligned} &\min \gamma \\ &\text{subject to} \\ &\quad I \leq Q \leq \gamma I, \quad AQ + QA^T + BY + Y^TB^T < 0 \end{aligned} \tag{27}$$

and the upper-bound minimizing controller is $K = YQ^{-1}$.

4 Optimal dynamic feedback controllers

Consider the linear time-invariant plant

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{28}$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times \ell}$, $u(t) \in \mathbb{R}^\ell$, $C \in \mathbb{R}^{m\times n}$, $y(t) \in \mathbb{R}^m$ with controller

$$\begin{aligned} \dot{x_k} &= A_k x_k + B_k y, \quad x_k(0) = x_{k0}, \\ u &= C_k x_k + D_k y, \end{aligned} \tag{29}$$

with $A_k \in \mathbb{R}^{n_k\times n_k}$, $x_k(t) \in \mathbb{R}^n_k$, $B_k \in \mathbb{R}^{n_k\times m}$, $C \in \mathbb{R}^{n_k\times \ell}$, $D \in \mathbb{R}^{m\times \ell}$. The closed loop system is given by

$$\dot{x}_c = A_c x_c, \quad x_c(0) = x_{c0} \tag{30}$$

where

$$Ac := \begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix}, \quad x_c := \begin{bmatrix} x \\ x_k \end{bmatrix}. \tag{31}$$

4.1 Unity transient energy growth

Lemma 7. *A necessary condition for unity transient energy growth, $\widehat{\mathcal{E}} = 1$, of the plant* (28) *with a stabilizing feedback controller* (29) *is that* $(A + BD_kC) + (A + BD_kC)^T \leq 0$.

Proof. From Definition 1, the transient energy of the plant (28) is given by

$$\mathcal{E}(t) := \max_{\|x(0)\|=1} \|x(t)\|^2 . \tag{32}$$

Let us replace $\mathcal{E}(t)$ by a modified energy function $\mathcal{E}_\epsilon(t)$ where

$$\mathcal{E}_\epsilon(t) := \max_{\|W_\epsilon^{-1} x_c(0)\|=1} \|W_\epsilon x_c(t)\|^2 . \tag{33}$$

where $W_\epsilon := \mathrm{diag}(I_n, \epsilon I_{n_k})$ and $\epsilon \in \mathbb{R}_+$. Clearly as $\epsilon \to 0$, $\mathcal{E}_\epsilon \to \mathcal{E}$. Applying Lemma 2 to (31), $\max_t\{\mathcal{E}_\epsilon(t)\} = 1$ if and only if $W_\epsilon(A_c + A_c^T)W_\epsilon \leq 0$, that is

$$\begin{bmatrix} A_D + A_D^T & (BC_k + (BC_k)^T)\epsilon \\ (B_kC + C^T B_k^T)\epsilon & (A_k + A_k^T)\epsilon^2 \end{bmatrix} \leq 0, \tag{34}$$

where $A_D = A + BD_kC$. It is known (e.g. Horn and Johnson, 1985, p. 397) that all the principal submatrices of a negative semidefinite matrix are negative semidefinite. Hence $(A + BD_kC) + (A + BD_kC)^T \leq 0$ is a necessary condition for (34) to hold and for $\widehat{\mathcal{E}} = 1$. □

Remark 3. *From the above lemma, it is clear that if no static controller that achieves unity transient energy growth exists, then no dynamic controller exists either.*

4.2 Minimal transient energy growth by convex optimization

The operation

$$\max_{t\geq 0} \|\Phi(t)\| \tag{35}$$

represents a norm on the matrix function $\Phi(t)$. By means of a Q-parametrization, control system performance indices that are norms can be minimized by exploiting the convex properties of norms (Boyd and Barratt, 1991). For simplicity, here we just consider the case for an open loop stable system. Details on a parametrization for the unstable case are given in Boyd and Barratt (1991).

Assuming that the system given by (28) is stable, a convex realization of the closed loop system is given by

$$H(s) = U_1(s) + U_2(s)Q(s)U_3(s) \tag{36}$$

where $U_1(s) = (sI - A)^{-1}$, $U_2(s) = (sI - A)^{-1}B$ and $U_3(s) = C(sI - A)^{-1}$, and $Q(s)$ is the free parameter transfer function matrix with dimension $\ell \times m$ and is stable and proper. It is clear that $\Phi(t) = \mathcal{L}^{-1}[H(s)] = 1/2\pi \int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega$.

The problem is then posed as follows

$$\widehat{\mathcal{E}}_{\min} = \min_{\text{stable } Q} \max_{t\geq 0} \|\Phi(t)\| \tag{37}$$

Approximations of the set of all stable, proper $Q(s)$ can be parameterized by means of a Ritz approximation (Boyd and Barratt, 1991; Linnemann, 1999). The final optimal controller is given by $K_{\text{opt}} = (I + Q_{\text{opt}}G)^{-1}Q_{\text{opt}}$.

4.2.1 A Ritz approximation of $Q(s)$

A state-space basis for the Ritz approximation proposed by Linnemann (1999) is used in this paper to parametrize the free parameter, $Q(s)$ and is described below. Let $\{\lambda_i\}_{i=1}^{\infty} = \{\lambda_1, \lambda_2, \ldots\}$ be a sequence of real or complex (in conjugate pairs) numbers such that

$$\text{i) } \operatorname{Re}(\lambda_i) > 0 \text{ for all } i, \tag{38}$$

$$\text{ii) } \sum_{i=1}^{\infty} \frac{\operatorname{Re}(\lambda_i)}{|\lambda_i|^2} = \infty, \tag{39}$$

and any λ_i may be repeated. Then there exists a sequence of functions that provides an orthonormal basis for the space $\mathcal{L}_2$. Thus we can approximate a function in $\mathcal{L}_2$ by a truncated sequence $\{\lambda_i\}_{i=1}^{q} := \Lambda_q$ to an arbitrary accuracy. A state-space realization of the orthonormal basis is provided in Linnemann (1999) and is summarized here for convenience. For each complex pair in the sequence Λ_q, let

$$F_j = \begin{bmatrix} -\operatorname{Re}(\lambda_j) & \operatorname{Re}(\lambda_j) - |\lambda_j| \\ \operatorname{Re}(\lambda_j) + |\lambda_j| & -\operatorname{Re}(\lambda_j), \end{bmatrix} \tag{40}$$

$$h_j = \sqrt{2\operatorname{Re}(\lambda_j)}\begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{41}$$

and for each real value in in the sequence Λ_q, let

$$F_j = -\lambda_j, \tag{42}$$
$$h_j = \sqrt{2\lambda_j}; \tag{43}$$

and let

$$F_q = \begin{bmatrix} F_1 & 0 & \cdots & \cdots & 0 \\ -h_2^T h_1 & F_2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ -h_{k-1}^T h_1 & \ddots & \ddots & F_{k-1} & 0 \\ -h_k^T h_1 & -h_k^T h_2 & \cdots & -h_k^T h_{k-1} & F_k \end{bmatrix}, \qquad h_q = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_k^T \end{bmatrix}, \tag{44}$$

where k is the sum of the number of complex pairs and the number of real values in Λ_q. Then, a state space realization is given by

$$A_q = \operatorname{diag}(F_q, \ldots, F_q) \in \mathbb{R}^{qm \times qm} \tag{45}$$
$$B_q = \operatorname{diag}(h_q, \ldots, h_q) \in \mathbb{R}^{qm \times m} \tag{46}$$

and $C_q \in \mathbb{R}^{\ell \times qm}$ and $D_q \in \mathbb{R}^{\ell \times m}$ are free parameters. Let $\tilde{Q} = [C_q, D_q]$, then it is clear that $H(s, \tilde{Q})$ is an affine function of $\tilde{Q}$, and hence so is $\Phi(t, \tilde{Q})$.

Given a particular Λ_q, the problem given by (37) is hence convex in $\tilde{Q}$, and can be solved using, for example, the ellipsoidal algorithm (Boyd and Barratt, 1991). Some example problems have been solved by Whidborne, Mckernan and Steer (2005). However, the search over time for the peak of the maximum transient energy growth, described in Section 2.1 is computationally intensive, and so the method is not very efficient. However, the problem can be solved approximately by adopting the method described below.

4.2.2 An LMI approximate solution

The problem given by (37) can be solved approximately by choosing an appropriate sequence of points in time, $\{t_i\}_{i=1}^N = \{t_1, t_2, \ldots, t_N\}$, $t_i > 0$, and minimizing the maximum energy growth over all t_i. This translates simply into an LMI, since it is well-known that, for some real matrix M,

$$\|M\| < \gamma \Leftrightarrow \begin{bmatrix} \gamma I & M \\ M^T & \gamma I \end{bmatrix} > 0. \tag{47}$$

Thus if at each time point t_i, the following LMI is satisfied,

$$\begin{bmatrix} \gamma I & \Phi^T(t_i) \\ \Phi^T(t_i) & \gamma I \end{bmatrix} > 0, \tag{48}$$

where $\Phi(t_i)$ is the state transition matrix at time $t = t_i$. Then, providing the sequence $\{t_i\}_{i=1}^N$ is carefully chosen, $\widehat{\mathcal{E}} \lesssim \gamma^2$. Hence the problem given by (37) can be approximated by the semidefinite program (SDP)

$$\begin{aligned} &\min \gamma \\ &\text{such that} \\ &\begin{bmatrix} \gamma I & \Phi^T(t_i, \tilde{q}) \\ \Phi^T(t_i, \tilde{q}) & \gamma I \end{bmatrix} > 0, \quad i = 1, \ldots, N \end{aligned} \tag{49}$$

where $\tilde{q} = \operatorname{vec}(\tilde{Q})$ and $\Phi(t_i, \tilde{q})$ depends affinely on the decision vector $\tilde{q}$.

4.3 Minimal $\mathcal{L}_\infty$-transient peaking

The transient energy growth given by Definition 2 is a measure of the "peaking" of the state variables. It could be regarded as a "worst-case" peaking measure. An alternative measure of the peaking given by the $\mathcal{L}_\infty$-norm of the transient response has been used to investigate bounded peaking for LQR optimal control (Francis and Glover, 1978). Here we will dub this the $\mathcal{L}_\infty$-*transient peaking* measure.

Definition 4. *The $\mathcal{L}_\infty$-transient peaking measure, $\widehat{\mathcal{L}}$, is defined as*

$$\widehat{\mathcal{L}} := \max_{t\geq 0} \| \Phi(t) \|_F. \tag{50}$$

From standard results on norms, we have that, for all t, $\|\Phi(t)\| \leq \| \Phi(t) \|_F \leq \sqrt{n}\, \|\Phi(t)\|$. Clearly at $t = 0$, since $\Phi(0) = I$, then $\| \Phi(0) \|_F = \|\Phi(0)\|$.

In a similar manner as for the maximum transient energy growth, the problem of minimizing the transient peaking by output feedback for the system given by (28) can be posed as a convex optimization problem by means of the Q-parametrization. Similarly the problem can be approximated by an SDP, since it is well-known that

$$\| M \|_F < \gamma \Leftrightarrow \begin{bmatrix} \gamma I & \mathrm{vec}(M) \\ \mathrm{vec}(M)^T & \gamma \end{bmatrix} > 0. \tag{51}$$

Then, given a carefully chosen sequence $\{t_i\}_{i=1}^N$, the problem of minimizing $\widehat{\mathcal{L}}$ can hence be approximated by the SDP

$$\begin{aligned} &\min \gamma \\ &\text{such that} \\ &\begin{bmatrix} \gamma I & \mathrm{vec}(\Phi^T(t_i,\tilde{q})) \\ \mathrm{vec}(\Phi(t_i,\tilde{q}))^T & \gamma \end{bmatrix} > 0, \quad i = 1,\dots,N. \end{aligned} \tag{52}$$

5 Examples

5.1 Example 1

The following example is adapted from (Trefethen et al., 1993). The linear system is given by

$$\dot{x} = \begin{bmatrix} -1/a & 1 \\ 0 & -2/a \end{bmatrix} x \tag{53}$$

where $a = 40$. The maximum transient energy growth for the open-loop system is calculated as $\widehat{\mathcal{E}} = 100.313$ which occurs at time

$$t = \ln\left(\frac{4(1+a^2)}{5a^2 - 4 - 3a\sqrt{a^2-8}}\right) a = 27.65. \tag{54}$$

The transient energy, $\mathcal{E}(t)$, is shown in Figure 1. Solving the GEVP of (13), an upper bound on $\widehat{\mathcal{E}}$ is obtained as $\widehat{\mathcal{E}}_u = 188.472$. This shows that there can be quite a degree of conservatism in $\widehat{\mathcal{E}}_u$. The angle between the right eigenvectors is given by $\tan^{-1}(1/a) = 1.43°$, which shows the non-normal nature of the system.

Case a

Now consider the linear system

$$\dot{x} = \begin{bmatrix} -1/a & 1 \\ 0 & -2/a \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{55}$$

where $a = 40$, with feedback controller $u = Kx$.

If the feedback control matrix is taken as $K = kI_2$, then from Lemma 2, the closed loop system has unity transient energy growth if and only if $A + A^T + 2kI_2 \leq 0$, that is if $k \leq -\lambda_{\max}(A + A^T)/2$. Taking $k = -\lambda_{\max}(A + A^T)/2 = -(\sqrt{a^2+1} - 3)/2a$ gives the closed loop transient energy, $\mathcal{E}(t)$, shown in Figure 2. With this controller, the angle between the right eigenvectors does not change from the open-loop case, which emphasizes that non-normality is not a necessary condition for low transient energy growth. This is explained at more length in Mckernan, Whidborne and Papadakis (2005).

The index, $\phi = \| E \|_F \| F \|_F$, where E and F are the modal matrices of the right and left eigenvectors respectively is often used as a measure of the eigenvalue sensitivities (Liu, Yang and Whidborne, 2002, p. 210). This index is also a measure of the normality of the system matrix. The eigenvalue sensitivity index for the open loop eigenvalues is calculated as $\phi = 80.025$. With ϕ as the objective function, and with $\rho = 1$, Theorem 1 combined with non-linear programming was used to search through $\{L : \|L\| < 1\}$ to find a controller that ensures asymptotic stability, with unity transient energy growth and minimal ϕ. The result was a value of L given by

$$L = \begin{bmatrix} 0.03646 & -0.54872 \\ -0.55090 & 0.06246 \end{bmatrix} \tag{56}$$

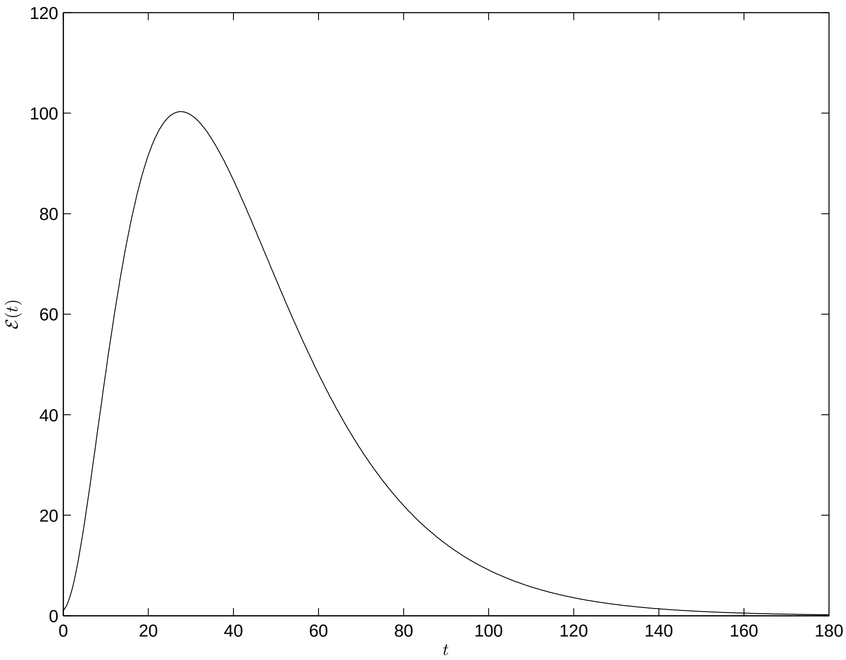

Figure 1: Open-loop transient energy for Example 1

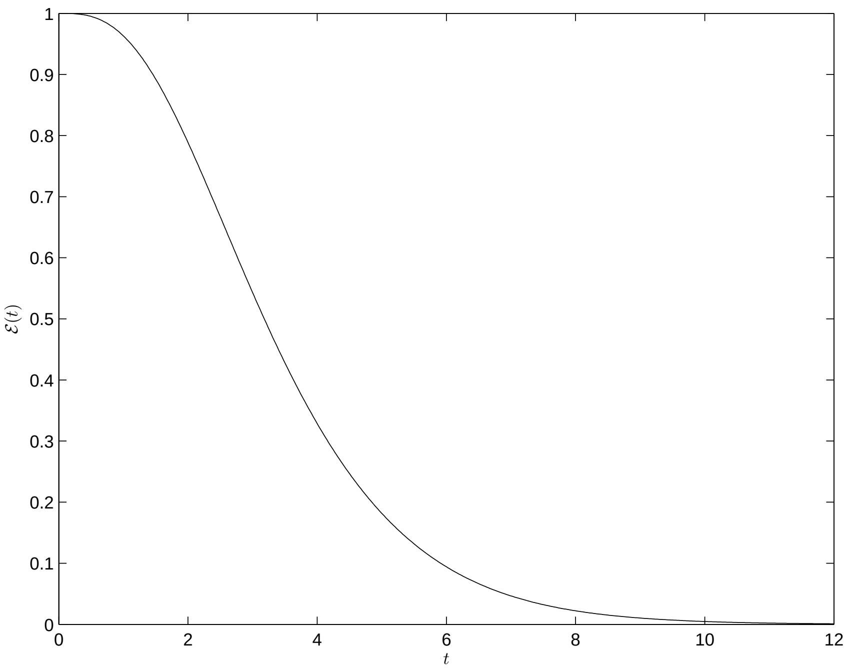

Figure 2: Closed-loop transient energy with minimizing diagonal controller for Example 1 - Case a

which gives a controller

$$K = \begin{bmatrix} -0.64913 & -0.49950 \\ -0.50050 & -0.62413 \end{bmatrix} \tag{57}$$

and a normal closed loop state matrix with minimal sensitivity index $\phi = 2$. The closed loop transient energy, $\mathcal{E}(t)$, is shown in Figure 3.

Case b

Now consider the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1/a & 1 \\ 0 & -2/a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{58}$$

where $a = 40$, with feedback controller $u = ky$.

For a simple second order system such as this example, the conditions for unity transient energy growth can be obtained in a straightforward manner. The closed-loop system matrix is

$$A + BkC = \begin{bmatrix} -1/a & 1 \\ k & -2/a \end{bmatrix}. \tag{59}$$

From Lemma 2, the maximum transient energy growth is unity if and only if $(A+BkC)+(A+BkC)^T \leq 0$ giving $(1+k)^2 \leq 8/a^2$, that is $(-1-2\sqrt{2}/a) \leq k \leq (-1+2\sqrt{2}/a)$. Figure 4 shows the maximum transient energy growth as a function of the controller gain k. This confirms the

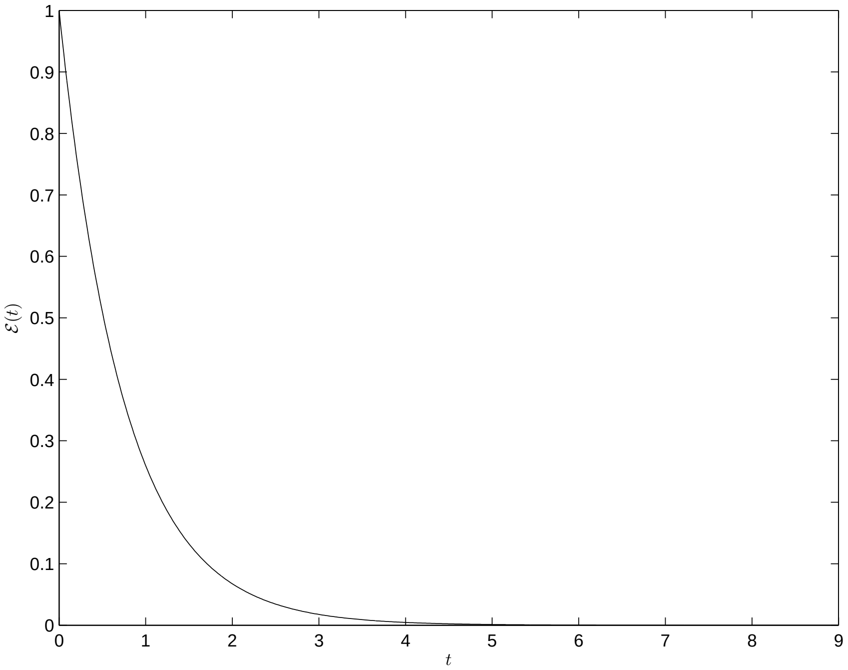

Figure 3: Closed-loop transient energy with a sensitivity index minimizing controller for Example 1 - Case a

bounds on k for unity maximum transient energy growth as well as showing the convex nature of the problem. Choosing $K = -1$ gives the closed loop transient energy, $\mathcal{E}(t)$, shown in Figure 5.

5.2 Example 2

The following linear system is taken from Plischke and Wirth (2004)

$$\dot{x} = Ax + Bu \tag{60}$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -625 \\ 0 & -1 & -30 & 400 & 0 & 0 & 250 \\ -2 & 0 & -1 & 0 & 0 & 0 & 30 \\ 5 & -1 & 5 & -1 & 0 & 0 & 200 \\ 11 & 1 & 25 & -10 & -1 & 1 & -200 \\ 200 & 0 & 0 & -150 & -100 & -1 & -1000 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{61}$$

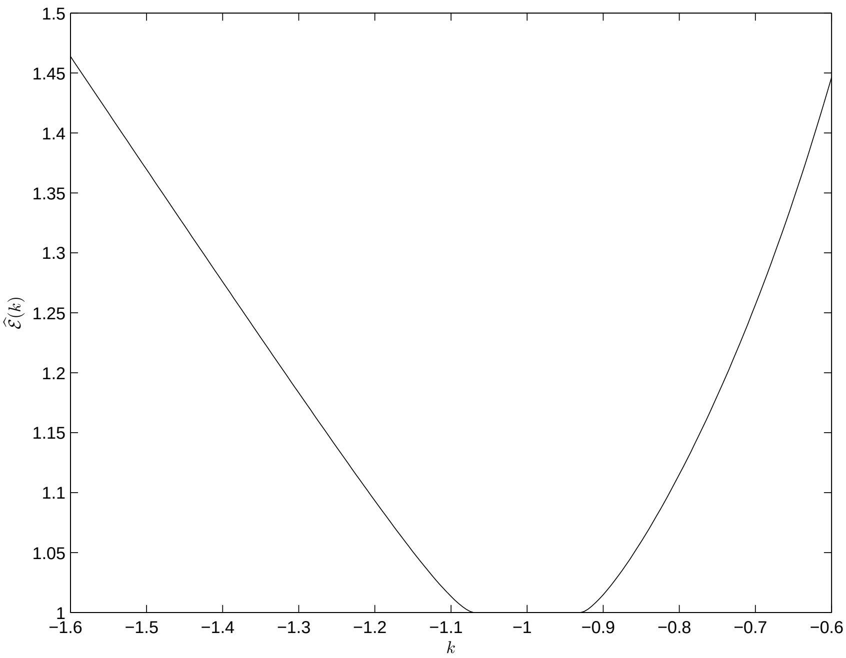

Figure 4: Maximum transient energy growth as a function of controller gain k for Example 1 - Case b

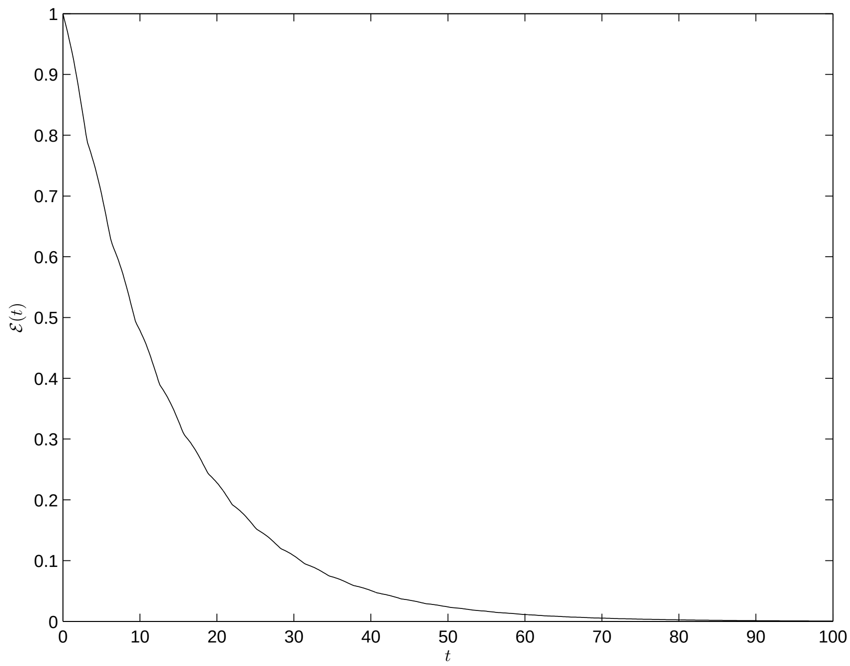

Figure 5: Closed-loop transient energy for Example 1 - Case b

and

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{62}$$

The maximum transient energy growth for the open-loop system is calculated as $\widehat{\mathcal{E}} = 358148$ by means of a line search over time of the spectral norm of $\Phi(t)$. The transient energy $\mathcal{E}(t)$ is shown in Figure 6. Solving the GEVP of (13), an upper bound on $\widehat{\mathcal{E}}$ is obtained as $\widehat{\mathcal{E}}_u = 439709$.

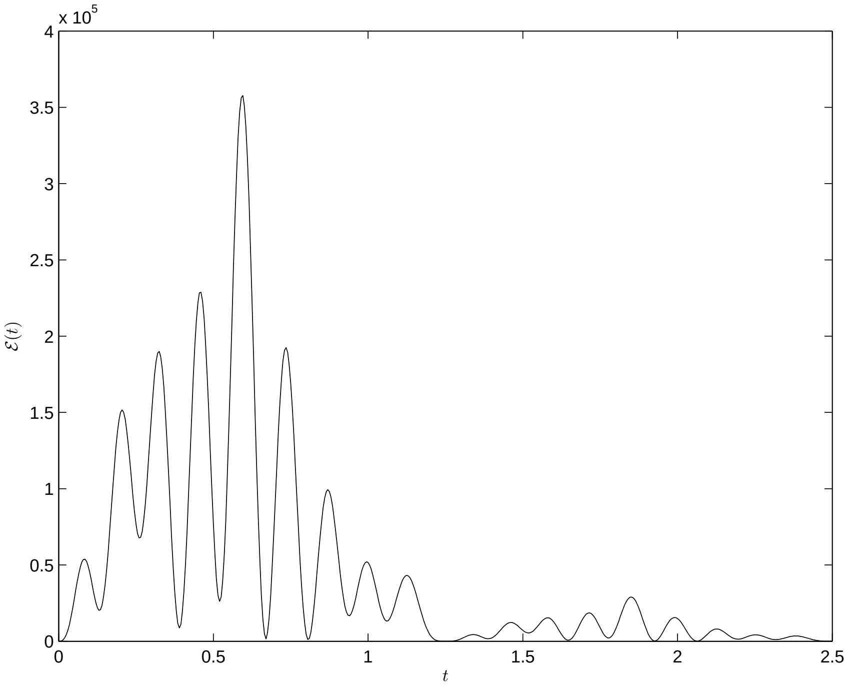

Figure 6: Open-loop transient energy for Example 2

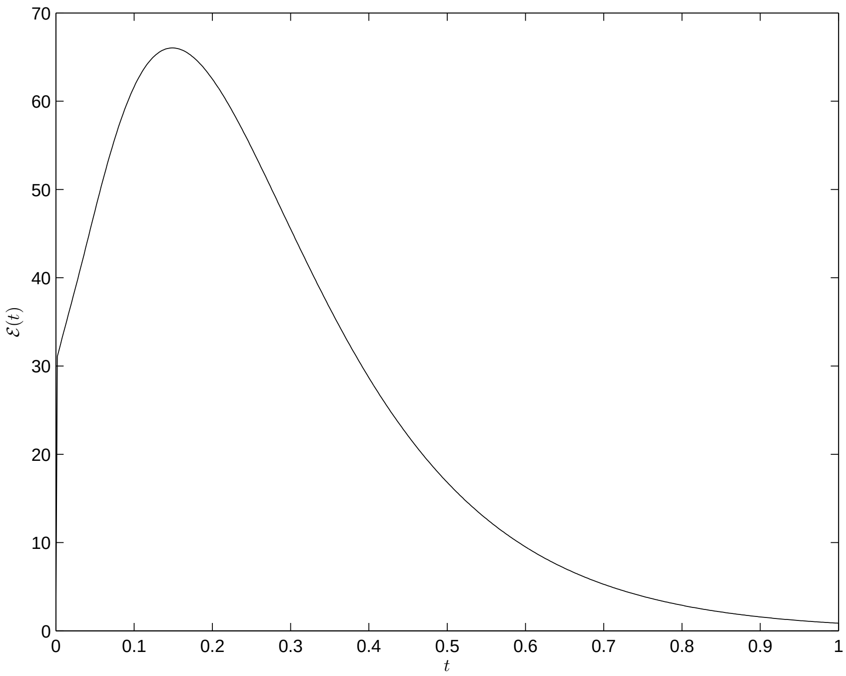

Figure 7: Transient energy with an upper-bound minimizing controller for Example 2

No monotonically decreasing transient energy controller was found to exist. The GEVP of (27) is solved to obtain a controller that minimizes the upper-bound on the maximum transient energy growth. The minimal upper bound is $\widehat{\mathcal{E}}_u = 178.55$ with maximum transient energy growth of $\widehat{\mathcal{E}} = 66.05$. The transient energy of the closed-loop system, $\mathcal{E}(t)$, is shown in Figure 7.

The dynamic controller problem is now considered. A sequence of SDPs given by (49) were solved using Ritz approximations formed from eigenvalues located at -50 as described in Section 4.2.1. The Ritz approximations were parametrized by Λ_q, $q = 0, \ldots, 9$ where $\Lambda_0 = \{\,\}$, $\Lambda_1 = \{50\}$, $\Lambda_2 = \{50, 50\}$, etc. The time sequence $\{t_i\}_{i=1}^N$ was kept constant for the whole sequence of problems, and was chosen by trial and error. Table 1 shows the solutions to the SDPs along with the actual maximum transient energy growth resulting from the controllers obtained from solving (49). It can be seen that the solution converges with increasing q. For $q \geq 9$, the chosen time sequence $\{t_i\}_{i=1}^N$ is no longer dense enough to give a good approximation to the $\widehat{\mathcal{E}}$ minimization problem. The transient energy of the closed-loop system, $\mathcal{E}(t)$, for the controller for $q = 8$ is shown in Figure 8.

Table 1: Solutions to sequence of approximant SDPs

q	0	1	2	3	4	5	6	7	8	9
γ^2_{min}	508.39	148.30	65.25	57.37	54.32	53.04	52.27	51.82	51.54	51.29
$\widehat{\mathcal{E}}$	511.34	140.00	65.80	57.75	54.50	53.64	52.44	51.91	51.72	56.65

For the final example, it is assumed that only the sixth state variable can be measured. Eigenvalues located at -50 are again chosen for the Ritz approximation, and for $\Lambda_q = \Lambda_9$, the maximum transient energy gain is $\widehat{\mathcal{E}} = 11919.1$, and the transient energy, $\mathcal{E}(t)$, of the resulting closed-loop system is shown in Figure 9.

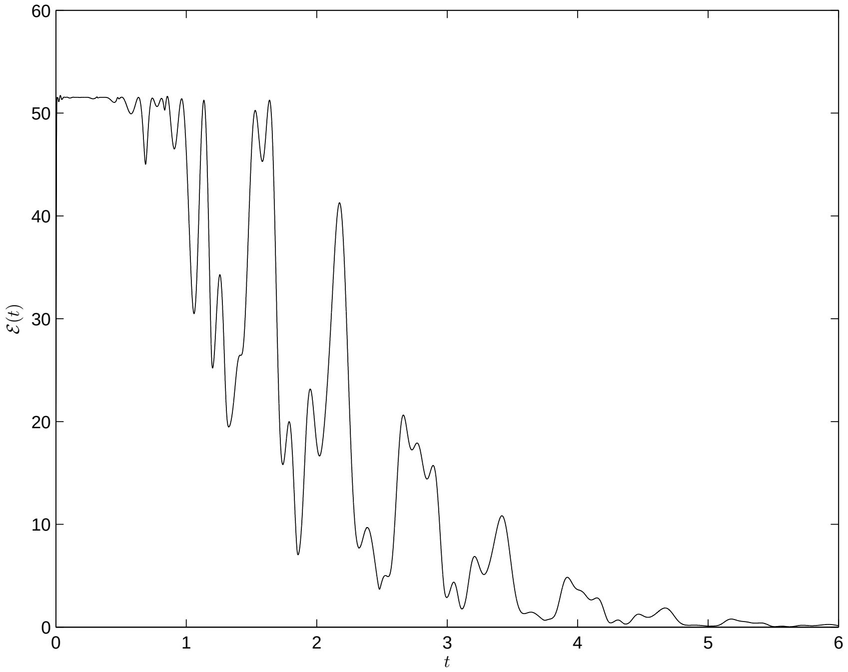

Figure 8: Transient energy with an approximately minimizing state feedback controller with $q = 8$ for Example 2

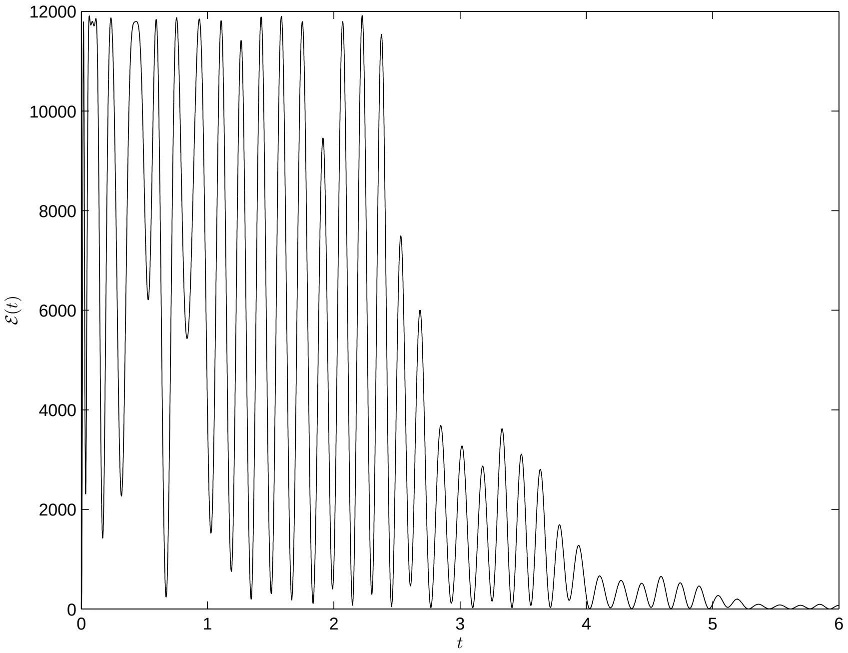

Figure 9: Transient energy with an approximately minimizing output feedback controller with $q = 9$ for Example 2

6 Discussion and Conclusions

Methods to calculate the maximum transient energy growth of linear systems are given. A characterization of all constant gain controllers that ensure the transient energy is monotonically decreasing (and so provide asymptotic stability with unity maximum transient energy growth) are provided. The upper bound problem LMI's were suggested in Boyd et al. (1994), but some similar results have also appeared in Hinrichsen, Plischke and Wirth (2002) and Plischke and Wirth (2004). These have been extended to consider the robust problem in Wirth (2004). Other upper bounds have been proposed in Hinrichsen and Pritchard (2000). An LMI that also constrains the control effort energy has been proposed in Mckernan, Whidborne and Papadakis (2005).

It is also shown in this paper that if no constant gain controller that restricts the maximum transient energy growth to unity exists, then no dynamic controller exists either. It is shown that by a Q-parametrization, the problem of minimizing the maximum transient energy growth is convex in the free parameter Q. Hence, by means of a Ritz approximation, sub-optimal controllers that minimize the maximum transient energy growth can also be obtained by convex programming. Solving this problem is computationally very demanding, so an SDP that can solve the problem to an arbitrary accuracy is proposed.

The methods are illustrated by two numerical examples. The controllers resulting from the Q-parametrization are high order, and although they lead to low maximum transient energy growth, it is clear that these controllers do not provide good control system designs. The intention is not necessarily to design controllers that meet all desired closed-loop requirements, but to provide designers with a means of determining the minimum of the maximum transient energy gain so that the specifications for the controller design can be sensibly set. Alternatively, additional performance criteria can be included in the SDP to improve the design. Note that the results of Hinrichsen, Plischke and Wirth (2002) and Plischke and Wirth (2004) allow for the inclusion of a decay rate constraint, i.e. $\|e\|^{At} < Me^{\alpha t}$. The inclusion of such constraints within the suggested approach is straightforward.

One difficulty with the Q-parametrization approach is the choice of the eigenvalues for the Ritz approximant. This is discussed in more detail in Linnemann (1999). The Q-parametrization allows for an observer structure that includes state and estimator gain matrices (Boyd, Barratt and Norman, 1990). This is required if the plant is not open-loop stable. However, the sub-optimal state feedback controller from Section 3.2 could be used in the observer structure, and it is envisaged that this would improve convergence of the Ritz approximant. Determining an appropriate sub-optimal estimator gain matrix remains for future work.

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