A New Kinetic Approach to the Shock Wave Problem

It is indeed curious that such plausible expansion techniques as the Chapman-Enskog sequence for solving the Boltzmann equation, lead to higher order approximations which, on the basis of experimental evidence, fail to yield correct predictions in situations where they are expected to be valid. The present paper explores the possibility of a solution based on an integral iteration technique, as distinct from an expansion technique, and its application to the normal shock problem.

In reference (1) an attempt was made to iterate on the B.E. starting from the N.S. distribution function in order to determine shock wave structure; in the analysis, however, a wrong assumption was made and some incorrect conclusions were drawn. In the present investigation the method is revised and generalised, and the convergence problem is studied.

In the formulation of the method the hard sphere molecule assumption is made and the following form of B.E. is used as a basis for the proposed iteration scheme,

$$\frac{Df}{Dt} + f\ L(f) = G(f)$$

where L and G are the familiar loss and gain terms. The scheme is initiated by using a local Maxwellian to approximate these two terms and the B.G.K. model results as the first model equation whose approximate solution is known (2). This is similar to the N.S. distribution function which, by a similar argument, can also be obtained by departing from an approximate solution of the linearized B.E. Next $f_{N.S.}$, which may rightly be considered as the initial distribution function, is used to determine a second model equation; corrected expressions for the gain and loss terms are obtained, having additional terms depending on the original N.S. flow parameters

$$\frac{\mu}{p}\frac{d\bar{u}_\alpha}{dx_\alpha}, \quad \frac{\mu}{p}\frac{d\sqrt{2RT}}{dx_\alpha}$$

and their products, and on exponential and error-type functions of the peculiar velocity and its components. The derivation is made in 3-D.

The suitability of the initial distribution function $f_{N.S.}$ can be verified by studying the convergence of the scheme. On the assumption that squares of differences between two consecutive iterations of the distribution function are negligible compared to the difference itself, a similar set of conditions for convergence as that of Willis (3) is obtained and the scheme is shown to converge at least at and in the neightbourhood of the free molecular flow and continuum flow régimes.

In the second part of the paper the integral of the second model equation is formally applied to the steady plane shock wave problem and three integral equations for the macroscopic variables n, $\bar{u}$ and T are obtained. To carry out the integration w.r.t. the molecular velocity components perpendicular to the flow direction, the loss term appearing in the exponential of these three moments is approximated by a function dependent on x only (but not on the peculiar velocity) and whose form, different for each moment, is determined by satisfying the conservation requirement. The final form of the three moment equations is a generalisation of but reduces to the B.G.K. model results. The computation of shock profiles based on N.S. solutions, though lengthy, should present no difficulties.

As an illustration of the type of results obtained, the contributions of the gain term of the second model equation which appears in the integral equations of the flow variables n, $\bar{u}$ and T are G_1, $\xi_x G_1$ and $\xi_x^2 G_1 + G_2$ respectively, where

$$G_i = A_i(U) + \frac{\theta}{t_{xx}} B_i(U) + \frac{\theta}{t_x} C_i(U) + \left(\frac{\theta}{t_{xx}}\right)^2 D_i(U) + \left(\frac{\theta}{t_x}\right)^2 E_i(U) + \frac{\theta^2}{t_x t_{xx}} F_i(U)$$

$$i = 1,\ 2$$

and

$$\frac{\theta}{t_{xx}} = 2\,\frac{\mu}{p}\frac{d\bar{u}}{dx} \quad ; \quad \frac{\theta}{t_x} = 3\,\frac{\mu}{p}\frac{d\sqrt{2RT}}{dx} ,$$

ξ_x being the molecular speed in the flow direction, $\bar{u}$ the local macroscopic speed and U the corresponding dimensionless peculiar speed component. A plot of the functions A_i ...., F_i is shown in fig.1 and 2, where the curves $C_i(U)$ and $F_i(U)$ change sign as U becomes negative.

For the B.G.K. model

$$G_1 = G_2 = \exp\,(-\,U^2)$$

which approximate the curves $A_1(U)$ and $A_2(U)$.

According to the analysis of Liepmann et al (2) regarding $\frac{\theta}{t_{xx}}$ and $\frac{\theta}{t_x}$, the additional terms in G_i will yield a significant contribution, as the

Mach number increases, where the N.S. equations cease to be valid, i.e. particularly on the low density side of the shock ahead of the point of maximum stress. Liepmann et al show also that on the assumption of constant total enthalpy across the shock (correct to within less than 3%)

$$\frac{\theta}{t_x} \sim - M \frac{\theta}{t_{xx}}$$

where M is the Mach number.

Hence, although $\frac{\theta}{t_{xx}}$ remains of order unity even for an infinite shock strength, $\frac{\theta}{t_x}$ will become increasingly large upstream as the free stream Mach number increases.

In conlusion, although the plane shock wave problem is characterised by a single parameter, the Mach number, significant local rarefied effects occur as the shock strength increases. These effects, characterised by the parameters $\frac{\theta}{t_x}$ and $\frac{\theta}{t_{xx}}$ which arise naturally in the equations and which may be regarded as local Knudsen numbers, or Reynolds numbers, are intimately related, and a consequence, of the Mach number. The present investigation is an attempt to account for such rarefied effects and to shed light on the validity and limitations of the B.G.K. model.

The method should be equally applicable to other fluid flow problems of linear or nonlinear nature.

REFERENCES

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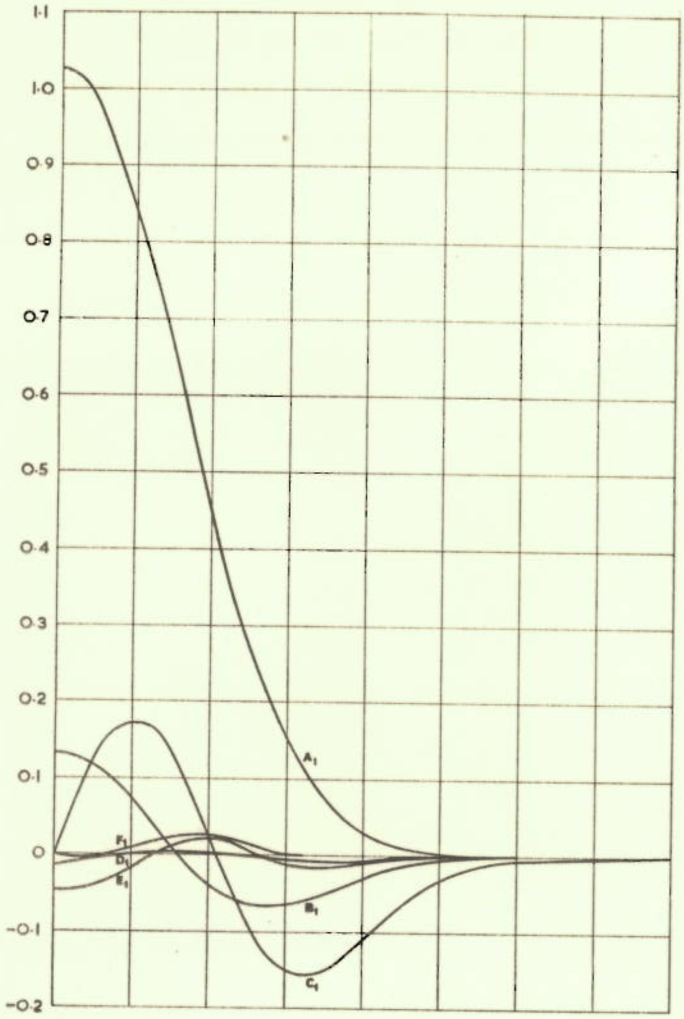

Fig 1 Gain term contribution — G_1

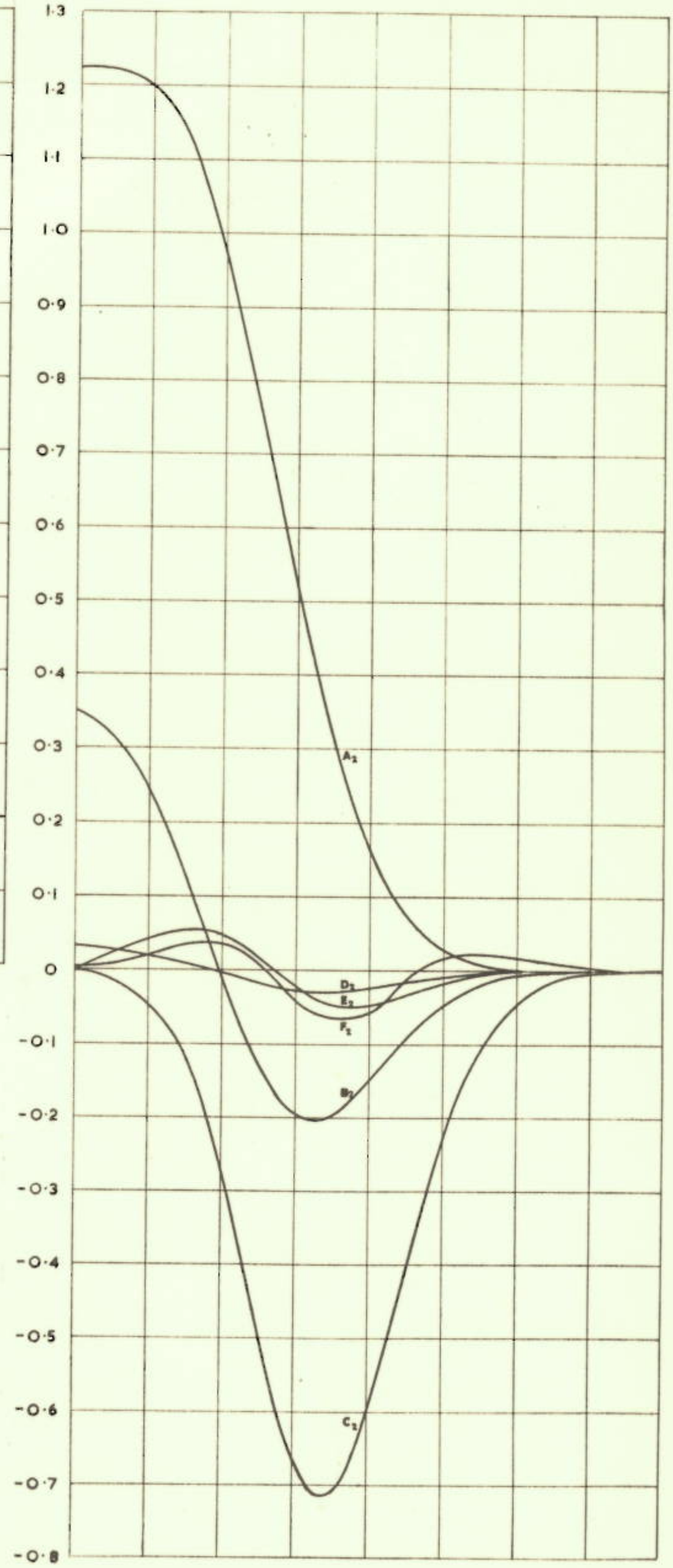

Fig 2 Gain term contribution — G_2