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Water exclusion from tunnel cavities in the saturated capillary fringe

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Abstract The problem of water flow around a tunnel cavity located in the saturated capillary fringe on top of a very permeable, freely-draining substratum is considered for the critical non-leakage condition when there is uniform vertical downward flow through the upper surface of the saturated region. In this critical condition the soil-water pressure is equal to zero everywhere on the cavity wall that is also a streamline. The conditions at the upper fringe boundary are that the soil-water pressure is equal to the air-entry value of the soil and the flux through this surface is the uniform infiltration rate. The cavity surface and the fringe boundary which is elevated above the cavity position, are found through conformal mapping and the use of integral representations of non-standard mixed boundary-value problems. They are calculated for a range of infiltration rates and compared with those obtained by assuming the upper fringe boundary to be horizontal. The exact analysis given here gives larger tunnel cavities than those given by the approximate treatment of the problem. The results have application in the design of underground repositories against entry of seepage water, the construction of protective capillary barriers and in the design of interceptor drainage systems.

Keywords : Tunnel cavity; Water exclusion; Capillary fringe; Conformal mapping; Mixed boundary value problems

1. Introduction

The analysis of water flow around cavities in unsaturated soils shows the effect on soil-water behaviour resulting from man-made excavations. It also facilitates the understanding of the role of macropores in unsaturated flow. Philip et al. (1989a) initiated analytical studies of the perturbation of vertical unsaturated soil-water flow due to the presence of cavities, using the so-called quasi-linear model that assumes an exponential dependence of hydraulic conductivity on soil-water pressure, in an investigation of the flow around circular cylindrical cavities. These were developed further in a series of papers (Philip, 1989a, b; 1990; Philip et al., 1989a, b; Knight et al., 1989) concerned

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with the soil-water behaviour in the presence of two- and three- dimensional cavities of various shapes.

Water will flow into a cavity when the soil-water pressure at any point on its wall becomes greater than atmospheric. Philip's analysis elucidated conditions under which an air-filled cavity (that has the same effect when not leaking water as an impermeable stone) excludes descending seepage water (the subcritical regime) or admits water (the supercritical regime). In the critical regime some part of the cavity boundary is just less than atmospheric pressure so that any further increase leads to water seeping into the cavity. There is a critical cavity shape where the soil-water pressure everywhere on the cavity wall is atmospheric and the cavity wall is also a streamline. For an infinite flow field and a soil having an exponential hydraulic conductivity function as assumed in Philip's analysis, the cavity shape is parabolic-cylindrical for two dimensional tunnel cavities and paraboloidal for three-dimensional cavities (Philip et al., 1989b). The analysis has been further developed by Fujii and Kacimov (1998) and Kacimov (2000).

A different physical situation occurs when cavities are located in a saturated capillary fringe above a water table when there is infiltration from the unsaturated soil above, seeping downwards to a water table. Youngs (2002) calculated the critical shape of tunnel cavities for water exclusion in this situation. His analysis notes that the hydraulic conductivity in this tension-saturated region is the same as that of the saturated soil under positive pressure and that the saturated region is bounded at its upper surface by the locus of points where the soil-water pressure is equal to the air-entry value of the soil, as used in many problems discussed by Polubarinova-Kochina (1977). Youngs' critical shape approximated to the parabolic form found by Philip et al. (1989b) for an exponential dependent hydraulic conductivity function that does not give a tension-saturated capillary fringe above a water table.

Since the hydraulic conductivity is uniform throughout the tension-saturated flow region, the flow pattern in this region is obtained by solving Laplace's equation so that in the case of two-dimensional flows conformal mapping techniques can be used. This was used by Kacimov and Nicolaev (1992) in their investigations on subcritical flows around impermeable holes and was also used by Youngs (2002). However, Youngs' solution was an approximation to the real situation for uniform precipitation in that his upper boundary was assumed horizontal, equivalent to assuming some variable accretion over the upper surface. When there is uniform accretion, the upper fringe surface is elevated in the vicinity above the cavity and its shape is part of the mathematical solution. We note further that the fringe height and hence the extent of the tension-saturated region increase with the precipitation rate. In this paper a solution is given for this situation, using conformal mapping and boundary-value problem theory techniques to obtain both the critical tunnel shape and the fringe boundary. In particular, we calculate the width and height of the tunnel as functions of the infiltration intensity and the air-entry pressure.

2. Analytical Solution

We consider a homogeneous, isotropic soil, which is underlain by a horizontal highly permeable layer into which all percolated water drains and above which is located a tunnel cavity D_1CD_2, as illustrated in Fig.1. The origin of coordinates (x, y) is at point O in the centre of the tunnel. Along E_1D_1 and E_2D_2 at the base of the soil region next to the highly permeable layer the soil-water pressure is atmospheric so that the pressure head p is zero and hence the hydraulic head $h = 0$. The tunnel D_1CD_2 protrudes into the capillary fringe. The upper surface of the fringe A_1BA_2 is a surface of constant pressure head where $p = P$ (P negative). Additionally, along A_1BA_2 water enters uniformly from the unsaturated zone above the fringe with a constant rate $q = q'k$ where $0 < q' < 1$ and k is the hydraulic conductivity of the saturated soil, so that each segment of the fringe boundary receives an amount of water proportional to the horizontal projection of the segment. We assume that the soil-water pressure on the walls of the tunnel is maintained at a negligibly small negative pressure $-\epsilon$ relative to atmospheric pressure so that no water leaks into the tunnel. We also assume that the shape of the tunnel is critical for the particular value of q so that the streamline along BC bifurcates at C and CD_1 and CD_2 are streamlines.

We introduce a complex position coordinate $z = x + \mathrm{i}\,y$ and designate the flow domain as G_z. We also introduce the complex potential $w = \phi + \mathrm{i}\,\psi$ where $\phi = -kh$ is the seepage velocity potential and ψ the stream function. The specific discharge is $\vec{V} = \nabla\phi$. The Zhukovskii function $\theta = w - ikz = \phi + ky + i(\psi - kx) = \theta_1 + i\theta_2$.

The boundary conditions in our problem are

$$\begin{array}{ll} \phi = 0, \quad \text{along} \quad E_1D_1 \quad \text{and} \quad E_2D_2 \\ \phi = -ky, \ \psi = 0 \quad \text{along} \quad D_1CD_2 \\ \phi = -kP - ky, \ \psi = qx \quad \text{along} \quad A_1BA_2 \end{array} \tag{1}$$

In the w-plane the image of G_z is a strip G_w but with a cut D_1CD_2 and a curved unknown boundary A_1A_2, which is shown schematically in Fig.2a. In the θ-plane the image of G_z is a strip G_θ (Fig.2b).

We map conformally G_θ onto the upper half-plane of an auxiliary plane $\zeta = \xi + i\eta$ (Fig.2c) by the function:

$$\theta = \frac{\mathrm{i}\,kP}{\pi} \log \frac{a+\zeta}{a-\zeta} \tag{2}$$

where the branch of the logarithm for $-a < \zeta < a$ is fixed in the upper half-plane. In (2) the parameter a is determined from the given maximum horizontal width x_m of the tunnel (point D_1) as

$$\frac{x_m}{|P|} = \frac{1}{\pi} \log \frac{a+1}{a-1} \tag{3}$$

In order to solve the flow problem we have to find the second characteristic analytic function. Youngs (2002) evaluated this function by conformal mapping of G_w onto G_θ that is impossible in our case because A_1BA_2 in Fig.2a is unknown. Riezenkampf (see Polubarinova-Kochina, 1977, p.138-145) and Strack (1989, p.555) introduced an auxiliary function in which the real part is constant along the flow domain boundary. In our case this function does not possess this property along D_1CD_2 and hence will not help. Another option is to use the hodograph $\vec{V}$. The hodograph domain G_V is shown in Fig.2d where the cut $E_1A_1M_1BM_2A_2E_2$ is along the circumference of a radius k but shifted a distance $-q$ along the v axis, and points M_1 and M_2 correspond to the inflexion points on the fringe boundary. As we can see from Fig.2d the hodograph plane is a circular sextagon. It would be a formidable task to map this sextagon onto ζ even by the Polubarinova-Kochina method of linear differential equations (Polubarinova-Kochina, 1977, p.240-290) as we did in another problem concerning the design of drains and soil channels (Kacimov and Obnosov, 2002).

Instead of conformal mapping of the second domain as in the hodograph method, we use a different (simpler, as we believe) method based on the theory of boundary-value problems (Gakhov, 1977, p.472-474) employed recently in other free-boundary problems in subsurface mechanics (Ilyinsky and Kacimov, 1992). Thus, we formulate the so-called mixed boundary value problem for an analytic function $w^*(\zeta) = \mathrm{i}\, w = -\psi + \mathrm{i}\, \phi$. This satisfies the following boundary conditions along the real axis L of our auxiliary variable

$$\begin{gathered} \phi = 0\ ,\ \ -a < \xi < -1 \text{ and } 1 < \xi < a, \\ \psi = 0\ ,\ \ -1 < \xi < -1; \quad \psi = g(\xi)\ ,\ |\xi| > a \end{gathered} \tag{4}$$

The function $g(\xi)$ in (4) is found from the following arguments. According to (2) at $|\xi| > a$

$$\psi - kx = -\frac{k|P|}{\pi} \log \left| \frac{a+\xi}{a-\xi} \right| \tag{5}$$

On the other hand, from the uniform infiltration condition $x = \psi/q$. We substitute this expression into (5) that yields along the fringe boundary

$$g(\xi) = \frac{k|P|q'}{(1-q')\pi} \log \left| \frac{a+\xi}{a-\xi} \right| \tag{6}$$

As (4) shows, at the transition points $\pm a, \pm 1$ the boundary condition type changes from Re w^* to Im w^*; that is, we arrive at a mixed boundary value problem.

The simplest boundary-value problem for an analytic function $w^*(\zeta)$ is the Schwartz one: given the real part $\phi(\tau)$ of w (τ is an arc coordinate of L) along the boundary L of the domain D in which the solution is sought, to determine w in the whole domain. When D is the upper half plane, the solution to this Schwartz problem is given by a singular (Cauchy-type) integral (Polubarinova-Kochina,1977, p. 202-211). A mathematically similar solution for a harmonic function in a unit circle is given by the so-called Poisson formula (Nehari, 1975, p.17). In mixed boundary-value problems the contour L is divided into segments along which either the real or the imaginary part of w is specified. The boundary functions and hence the whole function w may exhibit different properties when they approach these transition points, and w may be finite or infinite.

Volterra (1883) pioneered the solution of a mixed problem with two transition points in a half-plane, so that L in his problem was divided into three segments with Re(w) prescribed along one segment and Im(w) along the other two. Signorini (1916) derived a formula for $w^*(\zeta)$ with an arbitrary number of points where $\psi(\xi)$ condition changes to $\phi(\xi)$. Volterra and Signorini posited that $w^*(\zeta)$ is finite at all these points. In our problem, $w^*(\zeta) \to \infty$ as $\zeta \to \pm a$. Obnosov (1981) generalized the Signorini formula to the case when $w^*(\zeta)$ has integrable singularities at arbitrary transition points. This is the most general class including all known solutions to the mixed boundary problem found by Volterra, Signorini, Keldysh and Sedov (1937) (see Gakhov, 1977, p.472-474).

All approaches to solve the mixed problem are based on the reduction of this problem to the Schwartz problem (factually, to the Cauchy integral formula, see Nehari, 1975, p.94) by a proper choice of extra multipliers (R_0 and R_0^+ in (7) below) that must also guarantee that the function searched belongs to the class predetermined. (Detailed explanations are given in Polubarinova-Kochina (1977, p.208-209)).

Thus, from (4) the analytic function w^* in the upper half-plane is represented by (Obnosov, 1981)

$$w^*(\zeta) = \frac{R_0(\zeta)}{\pi \mathrm{i}} \int_M \frac{-g(\tau, a)\mathrm{d}\tau}{(\tau - \zeta) R_0^+(\tau)} + \mathrm{i}\, c_0 R_0(\zeta) \tag{7}$$

where $M = (-\infty, -a) \cup (a, \infty)$, c_0 is a real constant to be determined later and the function $R_0(\zeta)$ is given by

$$R_0(\zeta) = \sqrt{\frac{\zeta^2 - 1}{\zeta^2 - a^2}} \tag{8}$$

and $R_0^+(\tau)$ is the limit value of the chosen branch on the real axis, selected so that it is positive in the half-plane $\zeta > 0$ at $\zeta = \xi > a$; . From this branch fixation we can conclude that $R_0^+(\tau) > 0$ if $|\tau| > a$ and $|\tau| < 1$, and $\mathrm{Im}[\mathrm{sign}(\tau) R_0^+(\tau)] < 0$, $\mathrm{Re}[R_0^+(\tau)] = 0$ if $1 < |\tau| < a$. We evaluate the limit value of the integral (7) if $\zeta \to \xi$, $|\xi| > |a|$ via the Sokhotskij-Plemel formula (Henrichi, 1986, page 100). Recall that this formula gives the limit of the Cauchy type integrals in (7) when $\zeta \to \xi$. In other words, when approaching L the Cauchy type integrals (Nehari, 1975) call for special treatment. We note that the principal value of the integral should be taken (Gakhov, 1977) so that a singularity at $\tau = \zeta$ is isolated.

A parametric equation for the shape of the tunnel wall follows from (7) at $\zeta \to \xi$, $-1 < \xi < 1$ as

$$\begin{aligned} x(\xi) &= -\theta_2(\xi)/k = \frac{|P|}{\pi} \log \left| \frac{a + \xi}{a - \xi} \right| \\ y(\xi) &= -\phi/k = -\sqrt{\frac{1 - \xi^2}{a^2 - \xi^2}} \left(\frac{1}{\pi} \int_M \sqrt{\frac{\tau^2 - a^2}{\tau^2 - 1}} \frac{g(\tau, a)/k \mathrm{d}\tau}{\tau - \xi} + c_0 \right) \end{aligned} \tag{9}$$

and that for the fringe at $\zeta \to \xi \in M$, (i.e., $|\xi| > a$) as

$$\begin{aligned} x(\xi) &= \psi(\xi)/q = \frac{|P|}{\pi(1 - q)} \log \left| \frac{a + \xi}{a - \xi} \right| \\ y(\xi) &= |P| - \phi/k = |P| - \sqrt{\frac{1 - \xi^2}{a^2 - \xi^2}} \left(\frac{1}{\pi} \int_M \sqrt{\frac{\tau^2 - a^2}{\tau^2 - 1}} \frac{g(\tau, a)/k \mathrm{d}\tau}{\tau - \xi} + c_0 \right) \end{aligned} \tag{10}$$

In the limit $\xi \to +a$ where $x \to {}^{\pm}\infty$ we have

$$y(a) = H = |P|/(1 - q') \tag{11}$$

At $\xi \to a$ the limit value $\phi(a)$ must be finite and hence

$$c_0 = -\frac{1}{\pi} \int_M \sqrt{\frac{\tau^2 - a^2}{\tau^2 - 1}} \frac{g(\tau, a)/k \mathrm{d}\tau}{\tau - a}$$

Thus, in (10)

$$y(\xi) = |P| - \sqrt{\frac{(\xi^2-1)(\xi-a)}{\xi+a}}\frac{1}{\pi}\int_M \sqrt{\frac{\tau+a}{(\tau^2-1)(\tau-a)}}\frac{g(\tau,a)/k\mathrm{d}\tau}{\tau-\xi} \tag{12}$$

3. Computed Tunnel Shapes

Figure 3 shows the computed critical tunnel shape and fringe surface for $x_m/|P| = 0.5$ when $q' = 0.75$, 0.5 and 0.25, obtained from (9) for the tunnel and from (10), after some algebraic manipulation of (12) to avoid computational difficulties, for the fringe (see Appendix). They were found by fixing the hydrological parameters q and P and specifying the tunnel half-width $x_m = x_{D_1}$. The height of the tunnel and the height of the capillary fringe both increase as the accretion rate q increases.

In the approximate analysis of Youngs (2002) A_1BA_2 was assumed horizontal at an elevation H above E_1E_2, which requires a non-uniform distribution of infiltration across the fringe, the distribution of which was found *a posteriori* in the solution with q' a minimum above the apex of the tunnel. His tunnel cavity is described by

$$\frac{y}{|P|} = \frac{1}{\pi}(1-\frac{H}{|P|})\operatorname{arccos}\left(\frac{(1+t_L)\cosh\pi x/|P| + t_L - 1}{2}\right) \tag{13}$$

where

$$t_L = \frac{3-\cosh\pi x_m/|P|}{1+\cosh\pi x_m/|P|}$$

The tunnel cavity shapes given by (13) are compared with those given by (9) for $x_m/|P| = 0.5$ for $q' = 0.75$, 0.5 and 0.25 in Fig.3, assuming H to be the fringe height at $x \to {}^{\pm}\infty$ given by $H = |P|/(1-q')$. It is seen that (13) gives a slightly smaller cavity than that given by the present analysis because of the different boundary conditions at the top of the fringe used in the two cases.

Figure 4 shows the critical tunnel shapes for $q' = 0.75$, 0.5 and 0.25 when the tunnel height is fixed at $L/|P| = 1.0$. Note that by fixing L calls for a different procedure in computing the shapes than that used when x_m is fixed. In the latter a was found from (3) and then used in (9) and (10). When L is fixed, a was obtained by solving the second of the equations in (9) at $\xi = 0$ when $y(0) = 0$. It is seen that as q' increases the tunnel becomes narrower and the fringe surface becomes higher and flatter. Figure 5 shows the effect on the tunnel shape for different tunnel heights for the same accretion rate. The tunnel size increases overall while the fringe boundary rises in the vicinity above it, becoming more humped.

In Fig.6 the tunnel height $L/|P|$ calculated from (9)is plotted against $x_m/|P|$ for $q' = 0.75$, 0.5 and 0.25. Also shown in Fig.6 is the relationship given by Youngs' (2002) approximate analysis. Figure 7 shows the calculated fringe height above the tunnel $y_0/|P|$ at $x = 0$ for the same values of q'.

Youngs (2002) suggested the approximate relationship

$$\frac{x_m}{L} = \frac{1-q'}{q'} \tag{14}$$

to describe the width to height ratio of the tunnel cavity with the infiltration rate. This is shown in Fig.8 in which results from the present analysis are compared. It is seen that (14) gives a good fit to the computed results, especially at large values of q' for which it was suggested.

4. Discussion

The analysis given here leads to the shape of tunnel cavities in the critical condition for water exclusion when they are located in the saturated capillary fringe on top of a freely draining very permeable stratum and there is uniform flow across the fringe boundary. It uses conformal mapping and integral representations of non-standard mixed boundary-value problems to find a solution for conditions presented at the fringe boundary, namely, for the mixed boundary condition of a surface at constant soil-water pressure head and of uniform flux. The shapes of the cavity and the fringe surface, that is elevated in the vicinity above the cavity, are found in the solution for a given ratio of infiltration rate to soil hydraulic conductivity. The cavity sizes obtained in the analysis differ only slightly from those given by Youngs' (2002) approximate analysis in which the fringe surface is assumed to be horizontal. An interesting extension of Youngs' (2002) analysis shows that under certain conditions a cavity can span a water table and yet remain dry.

The situation considered is the exclusion of water from tunnel cavities of critical shape where the tunnel wall is a streamline. A wider tunnel of the same height with the given flow rate would leak water while a narrower one might, but not necessarily, continue to exclude water seepage. When there is water entry, the boundary conditions on the tunnel walls are mixed with a seepage surface that is unknown *a priori* and is part of the solution. The problem is thus mathematically more difficult. Knight et al. (1989) and Philip et al. (1989a, b) show that the flow pattern is governed primarily by the apical curvature of a cavity. A blunter apex than that for the critical situation would leak water whereas a sharper one would divert the streamline from the stagnation point at the apex away from the tunnel wall.

The engineering applications of this study are in the design of underground repositories, including the construction of protective capillary barriers and in the design of interceptor drainage systems. It might also give an insight into the emergence and growth of stalactites in caves as well as providing a basic understanding of the role of macropores in soil-water flow. The work is an example of the use of analytical tools to provide solutions to practical problems.

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Appendix

In order to compute values of $y(\xi)$ from eq.(12) we make the transformations $\tau = a/s$ and $t = a/\xi$, to produce

$$\frac{y(a/t)}{|P|} = 1 - \frac{q'}{\pi^2} \frac{\sqrt{(1-t)(a^2-t^2)}}{(1-q')\sqrt{1+t}} \int_{-1}^{1} \sqrt{\frac{1+s}{(a^2-s^2)(1-s)}} \log\left(\frac{1+s}{1-s}\right) \frac{\mathrm{d}s}{t-s}, \quad -1 < t < 1 \tag{A1}$$

Because of symmetry, $y(-a/t) \equiv y(a/t)$, and after algebraic manipulation eq.(A1) results in

$$\frac{y(a/t)}{|P|} = 1 + \frac{q'}{\pi^2} \frac{\sqrt{(a^2-t^2)(1-t^2)}}{(1-q')} \int_{-1}^{1} \frac{\log[(1+s)/(1-s)]}{\sqrt{(a^2-s^2)(1-s^2)}} \frac{\mathrm{d}s}{s-t}, \quad 0 < t < 1 \tag{A2}$$

for the right hand branch of the fringe boundary.

The integral in (A2) has three singularities. At $s = \pm 1$ it has regular singularities so that we deal with common improper integrals. The singularity at $s = t$ is more subtle. At these internal points the improper integral does not exist in the common sense and we take the principal value of the integral (Polubarinova-Kochina, 1977,p. 204). Thus, we isolate the singular points and write

$$\int_{-1}^{1} = I_1 + I_2 + I_3 + I_4 \tag{A3}$$

in which

$$I_1 = \int_{-1}^{-1+\epsilon_1}, \; I_2 = \int_{-1+\epsilon_1}^{t-\epsilon_2}, \; I_3 = \int_{t+\epsilon_2}^{1-\epsilon_1}, \; I_4 = \int_{1-\epsilon_1}^{1} \tag{A4}$$

where ϵ_1 ϵ_2 are small constants. Integrals I_2 and I_3 in (A4) are found numerically by computer algebra routines (Wolfram, 1991). Integrals I_1 and I_3 can be evaluated asymptotically. Thus

$$I_4 \asymp \frac{1}{\sqrt{2(a^2-1)(1-t)}} \int_{1-\epsilon_1}^{1} \frac{\log 2 - \log(1-s)}{\sqrt{1-s}} ds \tag{A5}$$

or upon integration

$$I_4 \asymp \frac{2\sqrt{\epsilon_1}[2 + \log 2 - \log \epsilon_1]}{\sqrt{2(a^2-1)(1-t)}} \tag{A6}$$

Thus, for the two end singularities we get

$$I_1 + I_4 \asymp \frac{\sqrt{2\epsilon_1}}{\sqrt{(a^2-1)}} \left(\frac{2 + \log 2 - \log \epsilon_1}{1-t} + \frac{\log 2 - 2 + \log \epsilon_1}{1+t} \right) \tag{A7}$$

The desingularised integrals are put into (A2) and values of $y(a/t)/|P|$ computed. Practical computations showed that sufficiently small values of ϵ_1 and ϵ_2 were required in order for the fringe height to converge to the known value of $y = |P|/(1-q')$ as $x \to {}^{\pm}\infty$.

Figure Captions

Fig.1 Uniform precipitation flowing around a critical tunnel cavity located in the saturated capillary fringe on a water-bearing stratum at atmospheric pressure: the physical domain G_z.

Fig.2. Flow in the capillary fringe: (a) the complex potential domain G_w; (b) the Zhukovskii function domain G_θ; (c) the auxiliary half-plane $\zeta = \xi + i\eta$; and (d) the hodograph domain G_V.

Fig.3. The critical tunnel cavity and fringe surface calculated for $x_m/|P| = 0.5$ and for $q' = 0.25$, 0.5 and 0.75: the line of the cavity is eq.(9) and the dotted line is eq.(13).

Fig.4. The critical tunnel cavity and fringe surface calculated for $q' = 0.5$ and $L/|P| = 0.5$, 1.0 and 2.0.

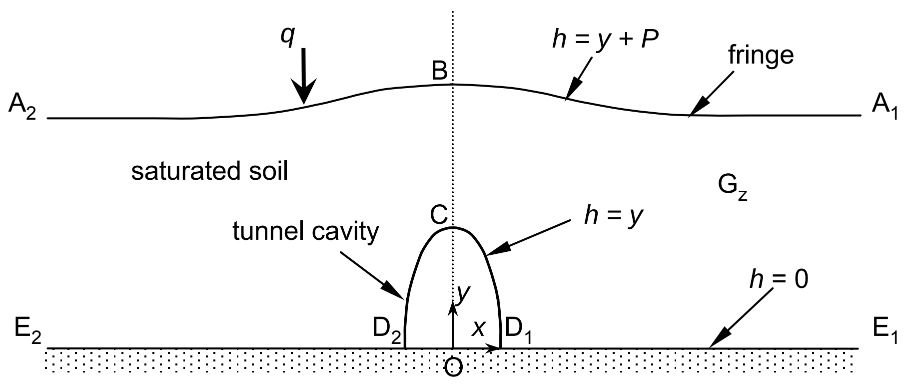
Fig.5. The critical tunnel cavity and fringe surface calculated for $L/|P| = 1.0$ and for $q' = 0.25$, 0.5 and 0.75.

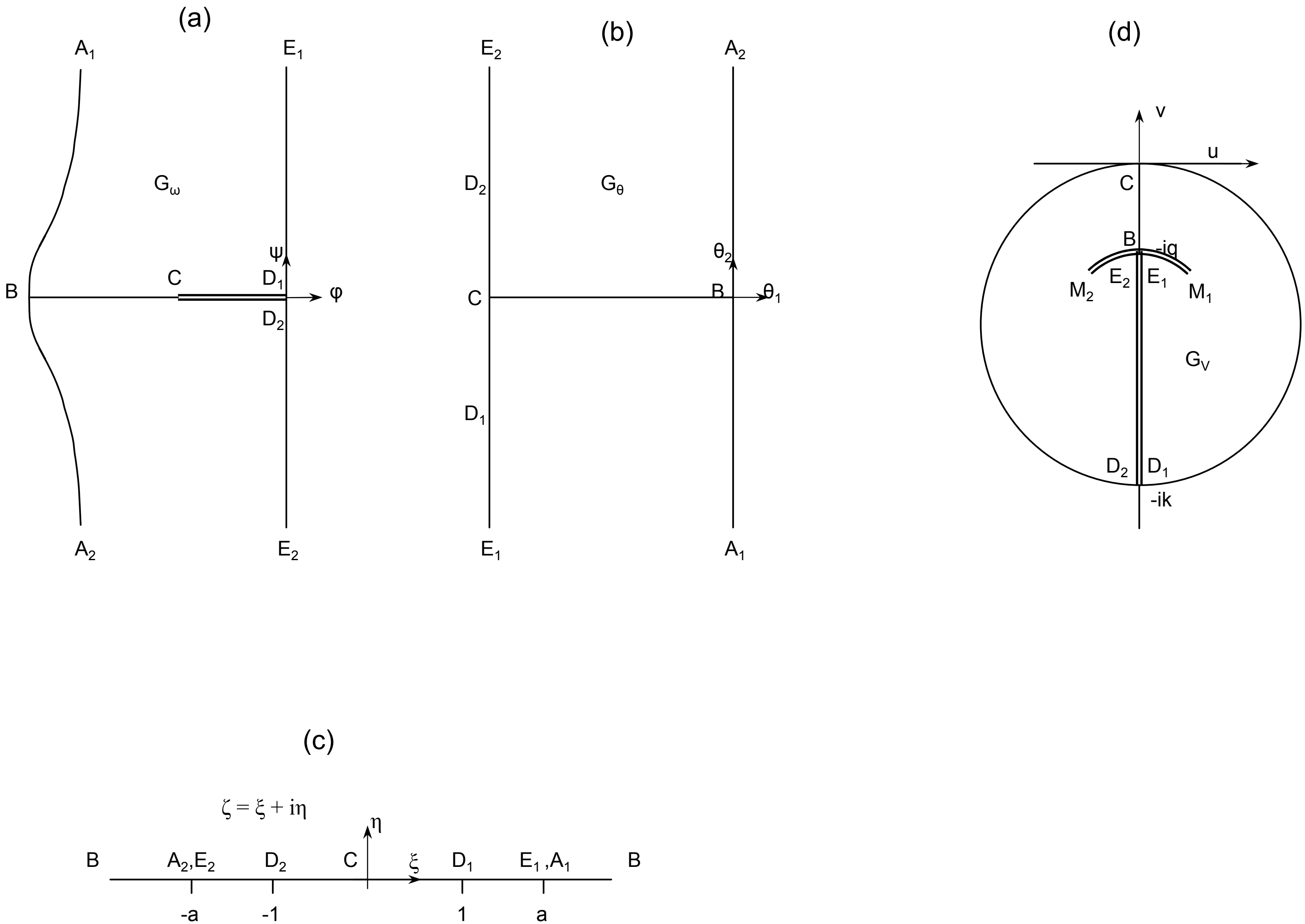
Fig.6. The tunnel height $L/|P|$ plotted as a function of tunnel width $x_m/|P|$ for $q' = 0.25$, 0.5 and 0.75: the lines are the relationships given by eq.(9) for $x = 0$, the dotted lines are those given by eq.(13).

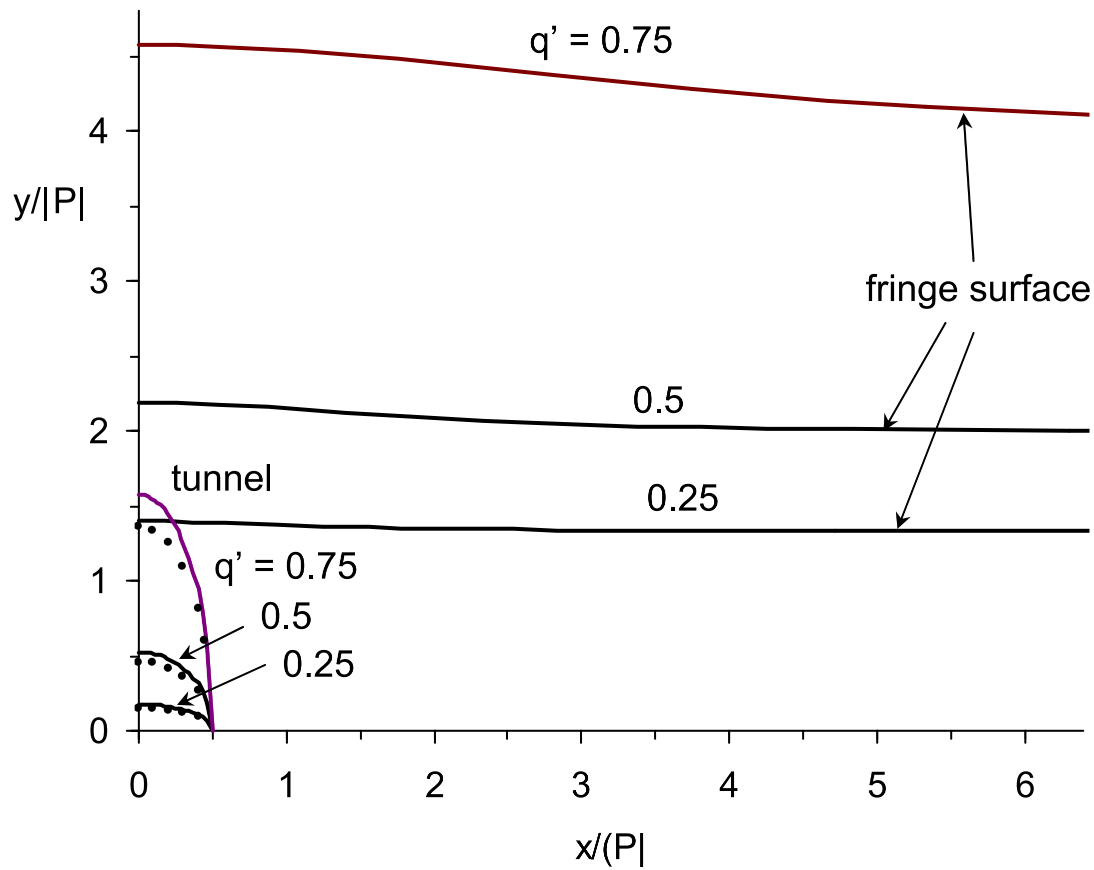
Fig.7. The fringe height $y_0/|P|$ above the tunnel plotted as a function of tunnel width $x_m/|P|$ for $q' = 0.25$, 0.5 and 0.75, given by eq.(10) for $x = 0$.

Fig.8. The width-height ratio x_m/L of critical tunnel cavities plotted as a function of q', the precipitation rate q expressed as a fraction of the hydraulic conductivity k: the line is eq.(14); symbols are values calculated from eq.(9), circles $x_m/|P| = 0.25$, triangles $x_m/|P| = 0.5$ and squares $x_m/|P| = 1.0$.

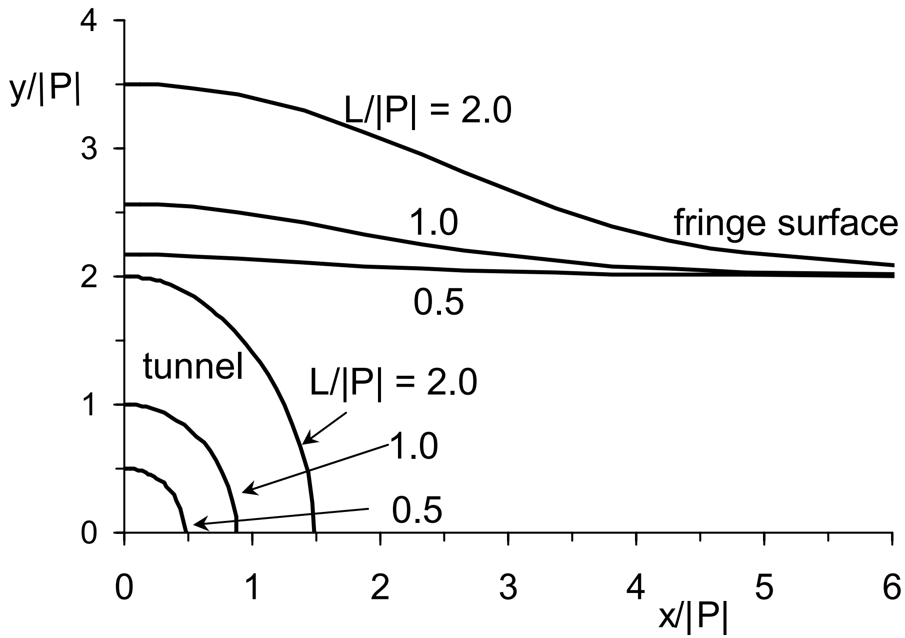
Fig.9. Water exclusion from a tunnel cavity spanning a water table maintained by artesian pressure in a lower very permeable substratum.


q
B
h = y + P
fringe
A2
A1
saturated soil
Gz
C
h = y
tunnel cavity
h = 0
y
E2
D2
x
D1
E1
O

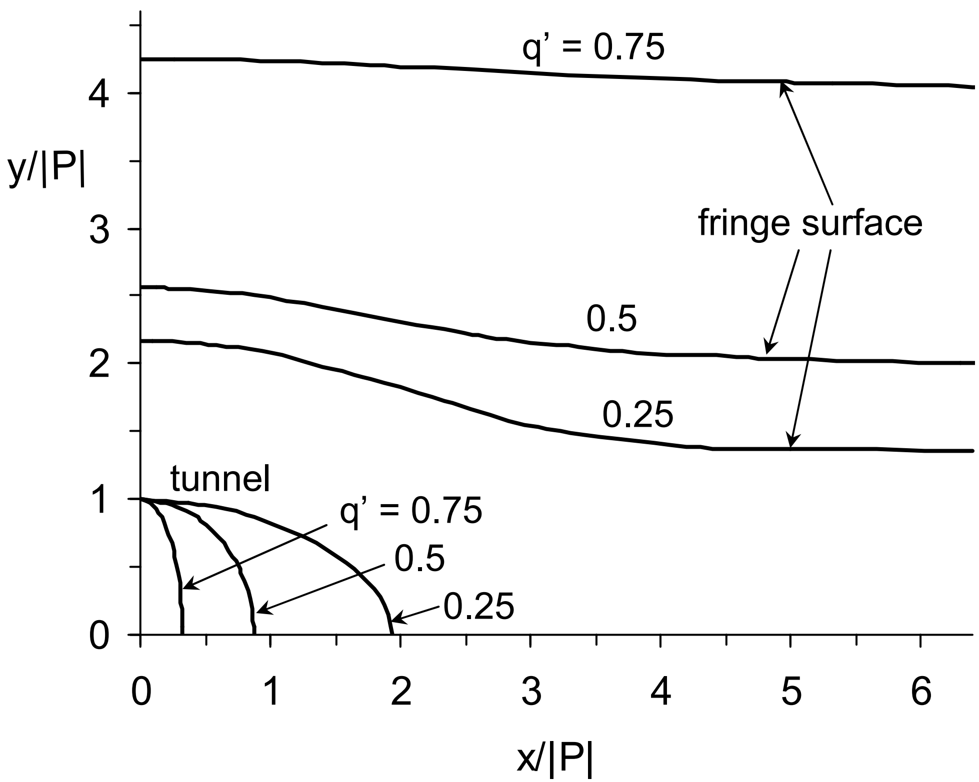

(a)
A$_1$
E$_1$
G$_\omega$
ψ
B
C
D$_1$
φ
D$_2$
A$_2$
E$_2$
(b)
E$_2$
A$_2$
D$_2$
G$_\theta$
θ$_2$
C
B
θ$_1$
D$_1$
E$_1$
A$_1$
(d)
v
u
C
B
-iq
E$_2$
E$_1$
M$_2$
M$_1$
G$_V$
D$_2$
D$_1$
-ik
(c)
$\zeta = \xi + i\eta$
η
B
A$_2$,E$_2$
D$_2$
C
ξ
D$_1$
E$_1$,A$_1$
B
-a
-1
1
a



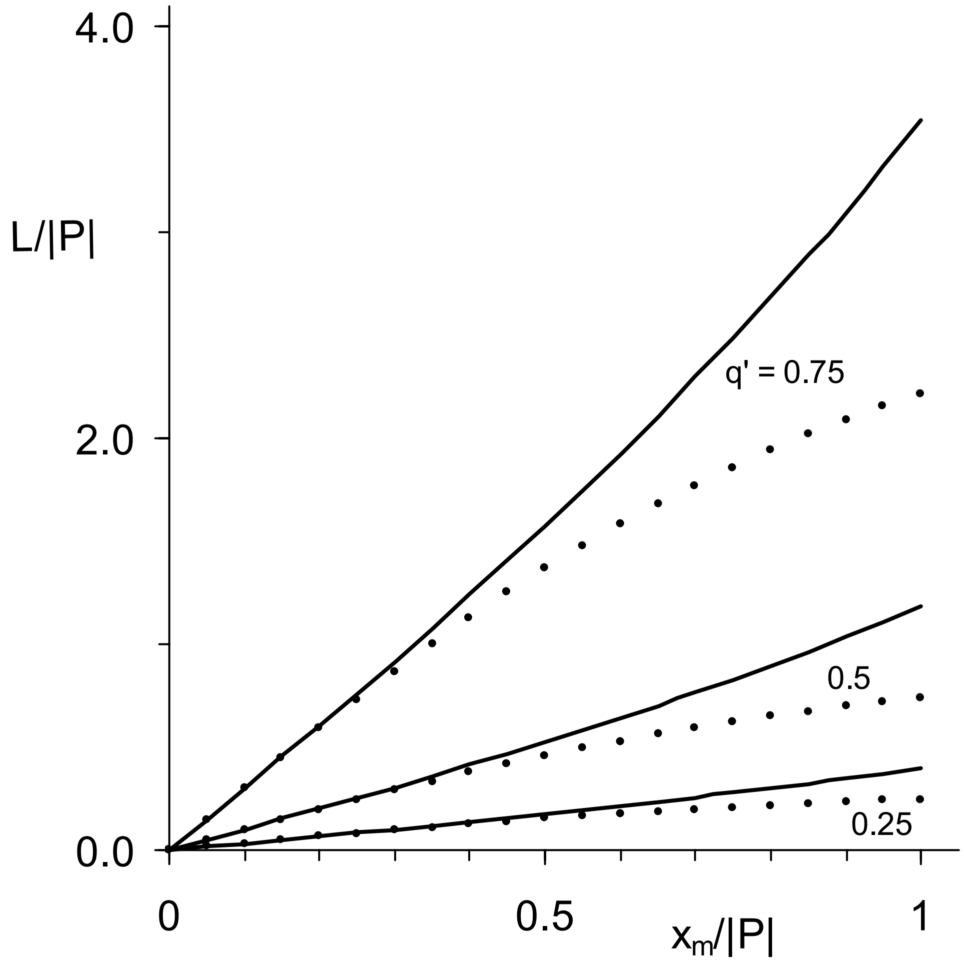
q' = 0.75
4
y/|P|
3
fringe surface
0.5
2
tunnel
0.25
q' = 0.75
1
0.5
0.25
0
0
1
2
3
4
5
6
x/(P|

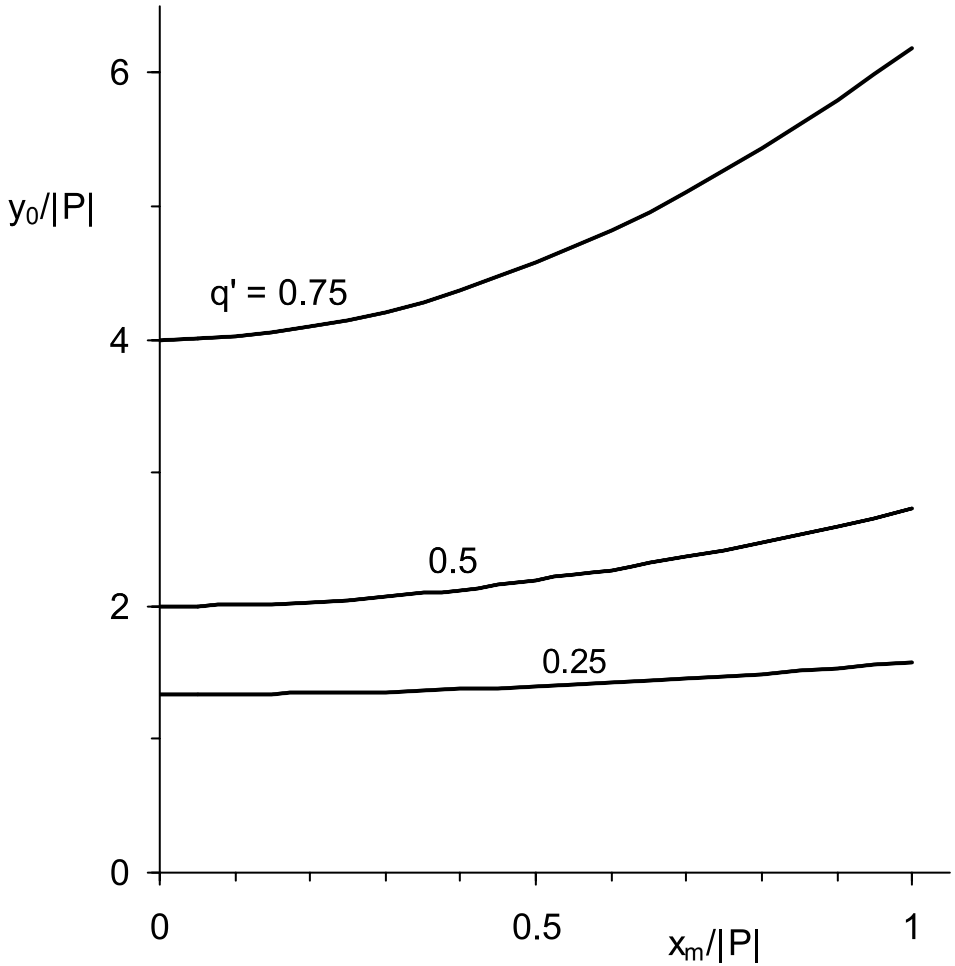


y/|P|
4
3
2
1
0
L/|P| = 2.0
1.0
0.5
fringe surface
tunnel
L/|P| = 2.0
1.0
0.5
0
1
2
3
4
5
6
x/|P|

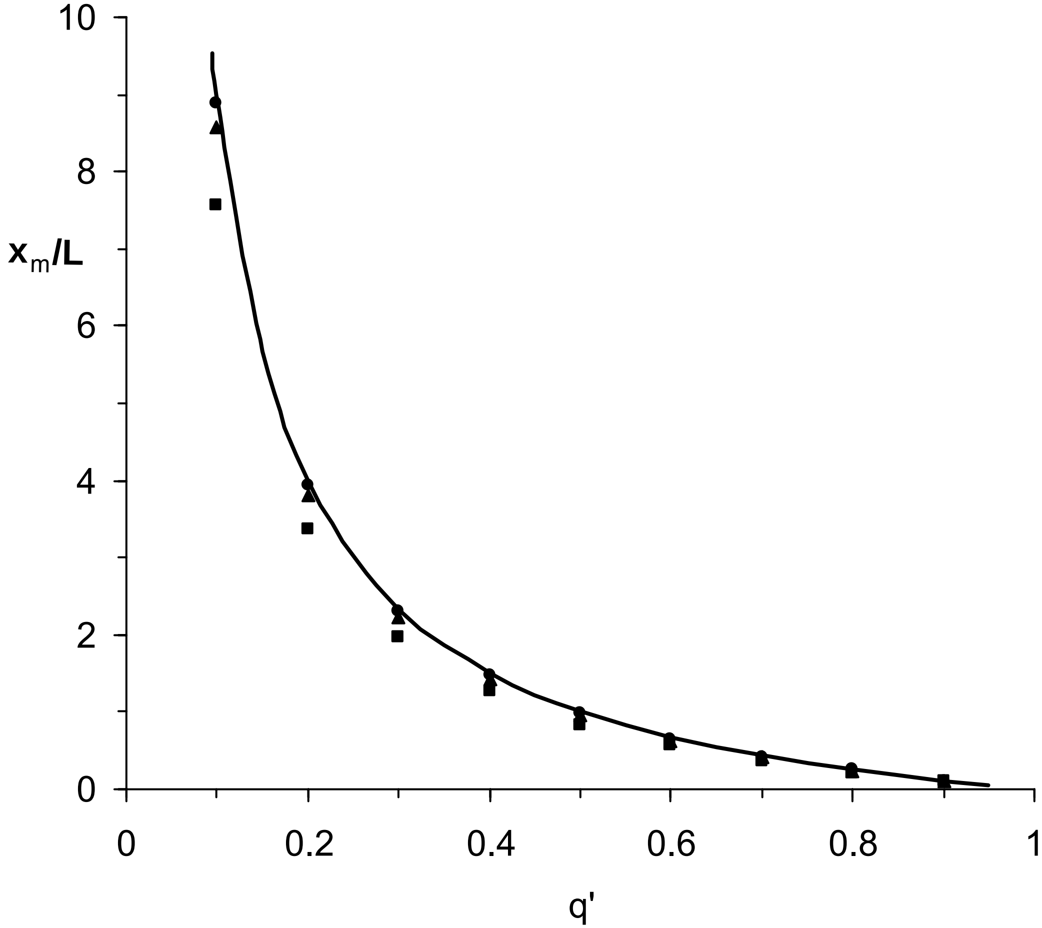


q' = 0.75
y/|P|
fringe surface
0.5
0.25
tunnel
q' = 0.75
0.5
0.25
x/|P|
0
1
2
3
4
5
6

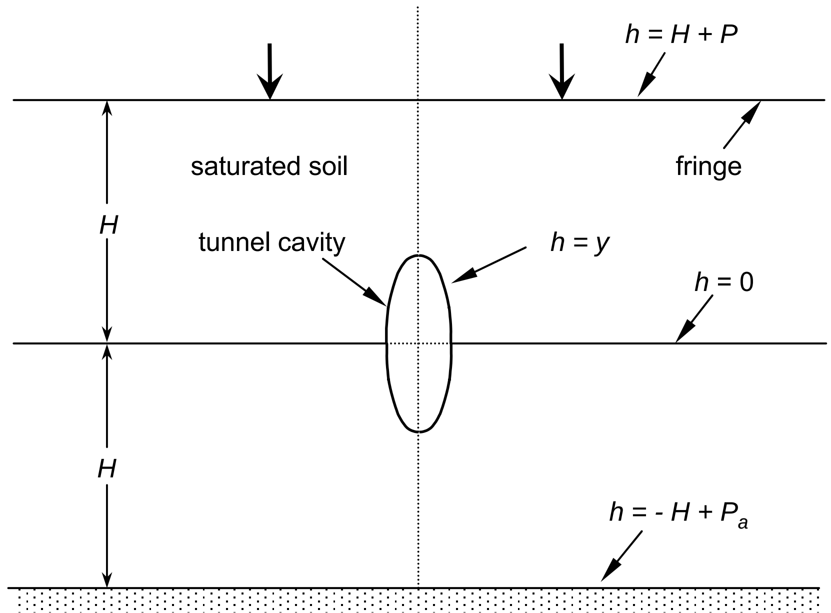

4.0
2.0
0.0
L/|P|
q' = 0.75
0.5
0.25
0
0.5
1
x_m/|P|



6
4
2
0
y0/|P|
q' = 0.75
0.5
0.25
0
0.5
1
xm/|P|



10
8
6
4
2
0
x_m/L
0
0.2
0.4
0.6
0.8
1
q'



$h = H + P$
saturated soil
fringe
H
tunnel cavity
$h = y$
$h = 0$
H
$h = -H + P_a$