REPORT NO. 133
June, 1960.

THE COLLEGE OF AERONAUTICS

CRANFIELD

Pressure Fluctuations in an Incompressible

Turbulent Boundary Layer

- by -

G. M. Lilley, M. Sc., D. I. C.,
A. F. R. Ae. S., A. M. I. Mech. E.

CORRIGENDA

Summary	- Line 8		Replace 4.6 C_f by 3.1 C_f
Page 3	- Equation (10)		Replace X_2 by x_2
Page 5	- Equation (19)		Replace $e^{i\kappa y_2}$ by $e^{i\kappa_2 y_2}$
Page 6	- Equation (25)		the variable of integration is κ_2
Page 7	- Equation (29)		Replace $e^{-0.31y}$ by $e^{-0.31\,y/\delta_1}$
	Line 17		Replace $\beta = 0.31$ by $\beta\delta_1 = 0.31$
Page 8	- Line 17		Insert 'wave number' before spectrum function
Page 9	- Lines 9 to 11		Delete and replace by,

*Since $$\int_0^\infty \frac{\bar{k}^2 e^{-\bar{k}^2}\, d\bar{k}}{\bar{k} + 0.62} = 0.27367$$

we see that

$$\frac{\sqrt{\bar{p}^2}}{\frac{1}{2}\rho_0 u_e^2} = 3.1\, C_f \qquad \ldots\ldots\ldots(37)$$

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* The author wishes to thank Dr. N. Curle for pointing out the error in the text, and showing that the integral can be evaluated from the tabulated values of a similar integral by Goodwin, E. T. and Staton, J., Q. J. M. A. M. Vol. 1, 1948, p. 319.

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SUMMARY

In a recent paper by Lilley and Hodgson an approximate analysis is given of the pressure fluctuations on a rigid wall under a turbulent boundary layer. One of the approximate results given in that paper was that $\sqrt{\overline{p^2}}\,/\tfrac{1}{2}\rho_o u_e^2 \approx 5\,C_f$, although strictly the analysis gave $\sqrt{\overline{p^2}}\,/\tfrac{1}{2}\rho_o u_e^2 \sim C_f^{5/4}\sqrt{\frac{u_e\delta}{\nu_o}}$.

The present paper presents a more exact analysis by using the method of generalised Fourier transforms. The final result is that $\sqrt{\overline{p^2}}\,/\tfrac{1}{2}\rho_o u_e^2 \approx 4.6\,C_f$ and is independent of the boundary layer thickness, except in so far as this is a function of the wall shear stress.

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NOTATION

B	- constant in Coles' 'Law of the wake'
C_f	$= \tau_W / \frac{1}{2} \rho_o u_e^2$ local skin friction coefficient
f	longitudinal velocity correlation coefficient
K	von Karman constant
l_2	transverse scale of turbulence
$\underset{\sim}{k}$	wave number vector in the plane (x_1, x_3)
P	pressure covariance
p	fluctuating pressure
$\underset{\sim}{r}$	separation vector
R_{22}	velocity covariance
s	Laplace transform operator
t	time
$\underset{\sim}{x}$	co-ordinates
$\underset{\sim}{X}$	co-ordinates in plane (x_1, x_3)
$\bar{u}$	mean velocity
u_e	velocity outside boundary layer
u	turbulent velocity component
u_τ	$= \sqrt{\tau_W / \rho_o}$ shear velocity
dZ	Fourier coefficient of velocity
β	mean shear parameter
δ	boundary layer thickness
δ_1	displacement thickness
$\underset{\sim}{\kappa}$	three-dimensional wave number vector
ν_o	kinematic viscosity
σ	inverse turbulent scale
ρ	density

Notation (Continued)

τ	mean shear
τ_o	mean shear parameter
τ_w	wall shear stress
$d\tilde{\omega}$	Fourier coefficient of pressure
π	pressure spectrum function
Φ	energy spectrum function

Other symbols, not listed above, are defined where they appear in the text.

1. Introduction

In a recent paper[1] a brief review is given of the theoretical and experimental research on the pressure fluctuations in incompressible turbulent shear flows. On the theoretical side the methods used by Kraichnan[2] are discussed, although the analysis in that paper depended upon a slightly different model of the turbulent flow than that used by Kraichnan. In order to obtain numerical results a certain integral had to be evaluated approximately and the accuracy of the resulting expression for $\overline{p^2}$ as a function of the skin friction coefficient cannot therefore be easily established. The present paper employs a different method of approach and thereby avoids this difficult integral. It is shown that the final results show good agreement with the earlier approximate results.

2. Analysis

The equation for the fluctuating pressure in an incompressible turbulent shear flow is[2]

$$\nabla^2 p(\underset{\sim}{x}, t) = -2 \rho_0 \tau \frac{\partial u_2(\underset{\sim}{x}, t)}{\partial x_1} \tag{1}$$

where $\underset{\sim}{x} \equiv (x_1, x_2, x_3)$, ρ_0 is the constant density, τ is the mean shear $\frac{\partial \bar{u}_1}{\partial x_2}$ with $\bar{u}_1$ a function of x_2 only, and u_2 is the turbulent velocity component in the direction x_2.

Let us consider the special case where the shear flow is the boundary layer flowing over the surface at $x_2 = 0$ with co-ordinates (x_1, x_3) in the plane of the surface. Then, if $\underset{\sim}{X} \equiv (x_1, x_3)$ in the plane parallel to the surface and distance x_2 from it we can write[3] the three-dimensional Fourier-Stieltjes transforms of $p(\underset{\sim}{x}, t)$ and $u_2(\underset{\sim}{x}, t)$ respectively as

$$p(x_2; \underset{\sim}{X}, t) = \int e^{i(\underset{\sim}{k} \cdot \underset{\sim}{X} + \omega t)} \, d\tilde{\omega}(x_2; \underset{\sim}{k}, \omega) \tag{2}$$

and

$$u_2(x_2; \underset{\sim}{X}, t) = \int e^{i(\underset{\sim}{k} \cdot \underset{\sim}{X} + \omega t)} \, dZ_2(x_2; \underset{\sim}{k}, \omega) \tag{3}$$

where $\underset{\sim}{k}$ is the wave number in the plane and ω is the frequency.

If we substitute for p and u_2 in equation (1) in terms of the Fourier coefficients $d\tilde{\omega}$ and dZ_2 defined in (2) and (3) we obtain

$$\frac{d^2}{dx_2^2}(d\tilde{\omega}) - k^2(d\tilde{\omega}) = -i\,2\,\rho_0\,\tau(x_2)\,k_1\,(dZ_2) \qquad (4)$$

where $k^2 = k_1^2 + k_3^2$.

The boundary conditions for p are $p = 0$ as $x_2 \to \infty$ and $\frac{\partial p}{\partial x_2} = 0$ as $x_2 \to 0$. Thus by means of the Laplace transform method, writing $\lim_{x_2 \to 0}(d\tilde{\omega}) = (d\tilde{\omega})_0$ and $\lim_{x_2 \to 0}\frac{d(d\tilde{\omega})}{dx_2} = (d\tilde{\omega})_1 = 0$ and $\overline{d\tilde{\omega}} = \int_0^\infty (d\tilde{\omega})\,e^{-sx_2}\,dx_2$, we find

$$(s^2 - k^2)\,\overline{(d\tilde{\omega})} = s(d\tilde{\omega}_0) - \int_0^\infty 2i\,\rho_0\,k_1\tau(dZ_2)e^{-sx_2}\,dx_2 \qquad (5)$$

or

$$\overline{(d\tilde{\omega})} = \frac{(d\tilde{\omega})_0}{2(s-k)} + \frac{(d\tilde{\omega})_0}{2(s+k)} - \frac{i\,\rho_0\,k_1}{k(s-k)}\int_0^\infty \tau(dZ_2)e^{-sx_2}dx_2 \qquad (6)$$
$$+ \frac{i\,\rho_0\,k_1}{k(s+k)}\int_0^\infty \tau(dZ_2)e^{-sx_2}dx_2$$

which on interpreting gives

$$d\tilde{\omega} = \frac{(d\tilde{\omega})_0}{2}\left(e^{kx_2} + e^{-kx_2}\right) - \frac{i\,\rho_0\,k_1}{k}\int_0^{x_2} e^{k(x_2 - x_2')}\,\tau(dZ_2)dx_2'$$
$$+ \frac{i\,\rho_0\,k_1}{k}\int_0^{x_2} e^{-k(x_2 - x_2')}\,\tau(dZ_2)\,dx_2' \qquad (7)$$

But $d\tilde{\omega} \to 0$ as $x_2 \to \infty$ so that

$$(d\tilde{\omega})_0 = \frac{2\,i\,\rho_0\,k_1}{k}\int_0^\infty e^{-kx_2'}\,\tau(x_2')\,dZ_2(x_2')\,dx_2' \qquad (8)$$

Hence from (7) and (8) after some rearrangement

$$d\tilde{\omega}(x_2) = \frac{i\,\rho_0\,k_1}{k}\int_0^\infty e^{-k(x_2+x_2')}\,\tau(dZ_2)\,dx_2' + \frac{i\,\rho_0\,k_1}{k}\int_0^\infty e^{-k|x_2-x_2'|}\,\tau(dZ_2)\,dx_2' \qquad (9)$$

a result previously derived by Kraichnan.

The pressure spectrum function $\pi(x_2;\,\underset{\sim}{k},\omega)$ in the plane x_2 is therefore related to p by[4]

$$\begin{aligned}\pi(x_2;\,\underset{\sim}{k},\omega) &= \frac{1}{8\pi^3}\iint \overline{p(x_2;\,\underset{\sim}{X},\,t)\,p(x_2;\,\underset{\sim}{X}+\underset{\sim}{r},\,t+t')} \\ &\qquad .\,e^{-i(\underset{\sim}{k}\,.\,\underset{\sim}{r}+\omega t')}\,d\underset{\sim}{r}\,dt' \\ &= \frac{\overline{d\tilde{\omega}(x_2;\,\underset{\sim}{k},\omega)\,d\tilde{\omega}^*(x_2;\,\underset{\sim}{k},\omega)}}{dk_1\,dk_3\,d\omega}\end{aligned} \qquad (10)$$

and similarly the energy spectrum function is

$$\begin{aligned}\Phi_{22}(x_2,\,x_2';\,\underset{\sim}{k},\omega) &= \frac{1}{8\pi^3}\iint \overline{u_2(x_2;\,\underset{\sim}{X},\,t)\,u_2(x_2';\,\underset{\sim}{X}+r,\,t+t')} \\ &\qquad .\,e^{-i(\underset{\sim}{k}\,.\,\underset{\sim}{r}+\omega t')}\,d\underset{\sim}{r}\,dt' \\ &= \frac{\overline{dZ_2(x_2;\,\underset{\sim}{k},\omega)\,dZ_2^*(x_2';\,\underset{\sim}{k},\omega)}}{dk_1\,dk_2\,d\omega}\end{aligned} \qquad (11)$$

Hence on the surface $x_2 = 0$ the three-dimensional pressure spectrum function is

$$\begin{aligned}\pi(0\,;\,\underset{\sim}{k},\omega) &= \frac{4\,\rho_0^2\,k_1^2}{k^2}\int_0^\infty\!\!\int_0^\infty e^{-k(x_2'+x_2'')}\,\tau(x_2')\,\tau(x_2'') \\ &\qquad .\,\Phi_{22}(x_2',\,x_2'';\,\underset{\sim}{k},\omega)\,dx_2'\,dx_2''\end{aligned} \qquad (12)$$

and the two-dimensional pressure spectrum function is (if t is the time delay)

$$\bar{\pi}(0\ ;\ \underset{\sim}{k}\ ,\ t) = \frac{4\ \rho_0^2\ k_1^2}{k^2} \int_0^\infty e^{-2kx_2'} \tau(x_2')\ dx_2' \int_{-x_2'}^\infty e^{-ky_2} \tau(x_2' + y_2)$$

$$\bar{\Phi}_{22}(x_2', y_2; \underset{\sim}{k}\ ,\ t)\ dy_2 \tag{13}$$

We note in passing that the velocity covariance

$$\overline{u_2(x_2; \underset{\sim}{X}\ ,\ t)\ u_2(x_2'; \underset{\sim}{X} + \underset{\sim}{r}\ ,\ t + t')} \equiv R_{22}(x_2, x_2'; \underset{\sim}{r}\ ,\ t')$$

$$= \iint \Phi_{22}(x_2, x_2'; \underset{\sim}{k}\ ,\omega)\ e^{i(\underset{\sim}{k}\ .\ \underset{\sim}{r} + \omega t')}\ d\underset{\sim}{k}\ d\omega$$

$$= \iint e^{i(\underset{\sim}{k}\ .\ \underset{\sim}{r} + \omega t')}\ \overline{dZ_2(x_2; \underset{\sim}{k}\ ,\omega)\ dZ_2^*(x_2'; \underset{\sim}{k}\ ,\omega)} \tag{14}$$

Also the two-dimensional energy spectrum function is

$$\bar{\Phi}_{22}(x_2', y_2; \underset{\sim}{k}\ ,\ t) = \int_{-\infty}^\infty \Phi_{22}(x_2', y_2\ ;\ \underset{\sim}{k}\ ,\omega)\ e^{i\omega t}\ d\omega \tag{15}$$

and so the more conventional three-dimensional energy spectrum function[5] is

$$\bar{\bar{\Phi}}_{22}(x_2'; \underset{\sim}{k}\ ,\ t) = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{\Phi}_{22}(x_2'; y_2; \underset{\sim}{k}\ ,\ t)\ e^{-iy_2 k_2}\ dy_2$$

$$= \frac{1}{8\pi^3} \int \overline{u_2(\underset{\sim}{x}', t')u_2(\underset{\sim}{x}' + \underset{\sim}{r}\ ,\ t' + t)}\ e^{-i(\underset{\sim}{\kappa}\ .\ \underset{\sim}{r})} d\underset{\sim}{r} \qquad \ldots\ldots (16)$$

or

$$\bar{\Phi}_{22}(x_2'; y_2; \underset{\sim}{k}\ ,\ t) = \int_{-\infty}^\infty \bar{\bar{\Phi}}_{22}(x_2'; \underset{\sim}{\kappa}\ ,\ t)\ e^{iy_2 k_2}\ dk_2 \tag{17}$$

where $\underset{\sim}{\kappa} \equiv (\kappa_1, \kappa_2, \kappa_3)$ and $k_1 \equiv \kappa_1$; $k_2 \equiv \kappa_2$; $k_3 \equiv \kappa_3$.

From (13) and (17)

$$\bar{\bar{\pi}}(0; \underset{\sim}{k}, t) = \frac{4\rho_0^2 k_1^2}{k^2} \int_0^\infty e^{-2k x_2'} \tau(x_2') \, dx_2' \cdot \int_{-\infty}^{\infty} \bar{\bar{\Phi}}_{22}(x_2'; \underset{\sim}{\kappa}, t) d\kappa_2$$

$$\cdot \int_{-x_2'}^{\infty} e^{-ky_2} e^{i\kappa_2 y_2} \tau(x_2' + y_2) \, dy_2 \qquad \ldots\ldots(18)$$

and for zero time delay the spectrum function of the surface pressure fluctuations is

$$\bar{\pi}(0; \underset{\sim}{k}) = \frac{4\rho_0^2 k_1^2}{k^2} \int_0^\infty e^{-2k x_2'} \tau(x_2') \, dx_2'$$

$$\cdot \int_{-\infty}^{\infty} \bar{\bar{\Phi}}_{22}(x_2'; \underset{\sim}{\kappa}) \, d\kappa_2 \qquad (19)$$

$$\cdot \int_{-x_2'}^{\infty} e^{-ky_2} e^{i\kappa y_2} \tau(x_2' + y_2) \, dy_2$$

which can be evaluated when the turbulent energy spectrum function is given as well as the distribution of the mean velocity $\bar{u}_1$ as a function of x_2.

We see, following Kraichnan, that (19) is simplified when the mean shear, τ, is expressed as

$$\tau(x_2) = \tau_0 \, e^{-\beta x_2} \qquad (20)$$

where τ_0 and β are constant.

3. A simple relation for $\overline{\overline{\Phi}}_{22}$

Let $\underset{\sim}{r} \equiv (r_1, r_2, r_3)$ then if $f(r)$ is the conventional longitudinal velocity correlation coefficient in isotropic turbulence a possible form for R_{22} is given by

$$R_{22}(x_2'; \underset{\sim}{r}) = \overline{u_2^2}(x_2') \left[f + \frac{r_1^2 + r_3^2}{2r} f' \right] \tag{21}$$

with

$$\overline{\overline{\Phi}}_{22}(x_2'; \kappa) = \frac{\overline{u_2^2}(x_2')}{8\pi^3} \int R_{22}(x_2'; \underset{\sim}{r}) e^{-\underset{\sim}{k} \cdot \underset{\sim}{r}} \, d\underset{\sim}{r} \tag{22}$$

Hence if $\quad f(r) = \exp(-\sigma^2 r^2)$ $\hfill (23)$

then

$$\overline{\overline{\Phi}}_{22}(x_2'; \underset{\sim}{\kappa}) = \frac{\overline{u_2^2}(x_2') \, k^2 e^{-\kappa^2/4\sigma^2}}{32\pi^{3/2} \sigma^5} \tag{24}$$

and

$$\int_{-\infty}^{\infty} \overline{\overline{\Phi}}_{22}(x_2'; \underset{\sim}{\kappa}) e^{i y_2 \kappa_2} d = \frac{\overline{u_2^2}(x_2') \, k^2 \, e^{-k^2/4\sigma^2} e^{-\sigma^2 y_2^2}}{16\pi\sigma^4} \tag{25}$$

Thus with $\tau(x_2)$ given by (20) and $\overline{\overline{\Phi}}_{22}$ by (24) we find from (19) that

$$\overline{\pi}(0; \underset{\sim}{k}) = \frac{\rho_0^2 \tau_0^2 k_1^2 e^{-k^2/4\sigma^2}}{4} \int_0^{\infty} e^{-2(k+\beta)x_2'} \overline{u_2^2}(x_2') \, dx_2' \, . \int_{-x_2'}^{\infty} e^{-(k+\beta)y_2} e^{-\sigma^2 y_2^2} dy_2 \tag{26}$$

4. The mean velocity distribution

In the case of a boundary layer in zero pressure gradient the mean velocity distribution in both the inner and outer regions is given by Coles[6] as (replacing $\bar{u}_1$ by $\bar{u}$ and x_2 by y)

$$\bar{u}/u_\tau = \frac{1}{K} \ln\left(\frac{u_\tau y}{\nu}\right) + B + \frac{\pi}{K}\left(1 + \sin\frac{\pi}{2}\ \left(\frac{2y}{\delta} - 1\right)\right) \qquad (27)$$

where the last term is Coles 'Law of the wake', and K is the von Karman constant. If we choose the values of K and π to be respectively, 0.4 and 0.55, then

$$\frac{u_e}{u_\tau}\ \frac{d\ \bar{u}/u_e}{d\ y/\delta} = \frac{2.5}{y/\delta} + 4.33 \cos\frac{\pi}{2}\ \left(\frac{2y}{\delta} - 1\right) \qquad (28)$$

where u_e and u_τ are respectively the external velocity and the shear velocity, $\sqrt{\tau_w/\rho_o}$, and the mean shear, τ, as defined above, is given by $d\bar{u}/dy$.

If therefore we define τ as given by (20), we find that a reasonable fit with (28) is obtained if

$$\tau = \frac{3.7\ u_\tau}{\delta_1}\ e^{-0.31\ y} \qquad (29)$$

provided y is outside the laminar sub-layer, so that $\tau_o = \frac{3.7\ u_\tau}{\delta_1}$ and $\beta = 0.31$.

5. The surface pressure spectrum function

In order to have a value of the velocity covariance R_{22} which can be compared with experiment we will modify slightly the value of R_{22} as given by (21). The modification is to replace $e^{-\sigma^2 y_2^2}$ by e^{-y_2/l_2} where $l_2(x_2')$ is the scale of the energy containing eddies in the direction normal to the surface. Thus we will put

$$R_{22}(x_2';\ y_2;\ r) = \overline{u_2^2}\ (x_2')\ e^{-y_2/l_2}\ e^{-\sigma^2 r^2}\ (1 - \sigma^2\ r^2) \qquad (30)$$

where <u>here</u> $r \equiv \sqrt{r_1^2 + r_3^2}$.

If $k \equiv \sqrt{k_1^2 + k_3^2}$ then the two-dimensional energy spectrum function is

$$\bar{\Phi}_{22}\,(x_2';\,y_2;\,k) \;=\; \frac{\overline{u_2^2}\,(x_2')e^{-y_2/l_2}\;k^2\;e^{-k^2/4\,\sigma^2}}{16\,\pi\,\sigma^4} \qquad (31)$$

which is similar to (25) except for the inclusion of e^{-y_2/l_2} and the exclusion of $e^{-y_2^2\,\sigma^2}$.

The surface pressure spectrum function is now (after integration with respect to y_2)

$$\bar{\pi}\,(0;\,\underset{\sim}{k}) \;=\; \frac{\rho_o^2\;\tau_o^2\;k_1^2\;e^{-k^2/4\sigma^2}}{4}\int_o^\infty \frac{\overline{u_2^2}(x_2')e^{-(k\,+\,\beta\,-\,1/l_2)x_2'}\,dx_2'}{\left[k+\beta+\dfrac{1}{l_2(x_2')}\right]} \;\ldots\ldots \qquad (32)$$

and when $\overline{u_2^2}$ and l_2 are constants,

$$\bar{\pi}\;(0;\,\underset{\sim}{k}) \;=\; \frac{\rho_o^2\;\tau_o^2\;k_1^2\;e^{-k^2/4\sigma^2}\;\overline{u_2^2}}{4\pi\,\sigma^4\;\;(k+\beta+1/l_2)(k+\beta-1/l_2)} \qquad (33)$$

If we now insert the values

$$\frac{\delta_1\tau_o}{u_e} \;=\; 3.7\;\frac{u_\tau}{u_e}\;\;;\;\; \frac{\sqrt{\overline{u_2^2}}}{u_\tau} \;=\; 0.8$$

$$\sigma\,\delta_1 \;=\; 1/2 \qquad;\quad \delta_{1}/l_2 \;=\; 0.31 \quad;\;\; \beta\delta_1 = \;0.31$$

where the first is found following (29), the second from the results of Laufer[7], the third and fourth from the results of Grant[8] and the last from (29) also, (δ_1 is the displacement thickness) then the surface pressure spectrum function becomes

$$\frac{\bar{\pi}(0;\,\underset{\sim}{k})}{(\tfrac{1}{2}\rho_o\;u_e^2\,\delta_1)^2} \;=\; \frac{44.7\;(^{u}\tau/u_e)^4\;\;(k_1\delta_1)^2\;\;e^{-(k\;\delta_1)^2}}{(k\;\;\delta_1)^2\;\;+\;\;0.62\;(k\;\;\delta_1)} \qquad (34)$$

where $(u_\tau/u_e)^2 \;=\; C_{f/2}$ and C_f is the local skin friction coefficient.

6. The mean square value of the fluctuating pressure

If we write

$$P(o) = \overline{p^2} \quad \text{and } \bar{k} = k\,\delta_1$$

then on integration of (34) in the plane over all wave numbers we find that

$$P \,/ (\tfrac{1}{2}\,\rho_o\,u_e^2\,)^2 = 35\,C_f^2 \int_0^\infty \frac{\bar{k}^2\, e^{-\bar{k}^2}\, d\bar{k}}{(\bar{k} + 0.62)} \tag{35}$$

or,

$$\frac{\sqrt{\overline{p^2}}}{\frac{1}{2}\,\rho_o\,u_e^2} = 5.9\;C_f \sqrt{\int_0^\infty \frac{\bar{k}^2\, e^{-\bar{k}^2}\, d\bar{k}}{(\bar{k} + 0.62)}} \tag{36}$$

Since

$$0.72 > \int_0^\infty \frac{\bar{k}^2\, e^{-\bar{k}^2}\, d\bar{k}}{\bar{k} + 0.62} > 0.5$$

we see that

$$\frac{\sqrt{\overline{p^2}}}{\frac{1}{2}\,\rho_o\,u_e^2} \doteq 4.6\;C_f \tag{37}$$

in agreement with the results given in Ref. 1. This suggests that the pressure fluctuations under a turbulent boundary layer are proportional to the external dynamic pressure, and the skin friction coefficient and are independent of the boundary layer thickness except in so far as the skin friction coefficient is a function of boundary layer thickness. Equation (37) is based on the assumptions (a) that equilibrium conditions prevail in the turbulent boundary layer, and (b) that the external velocity to the layer is constant.

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