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CRANFIELD

The aerodynamic derivatives of an aerofoil oscillating in an infinite staggered cascade

- by -

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SUMMARY

Thin aerofoil theory is used to obtain, in integral form the aerodynamic derivatives of an aerofoil oscillating in an infinite cascade. The theory allows for arbitrary stagger angle and phase difference between adjacent blades of the cascade. The expressions obtained reduce, for zero stagger and for in-phase and antiphase oscillations, to known results.

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LIST OF SYMBOLS

a	elastic axis position measured from midchord
c	chord length
F_1, F_2, F_3	functions of η and μ defined by equations (34)
G_1, G_2, G_3	integrals involving the F-functions defined by equations (42)
G_4	a function of G_3 and H_3 defined in equation (51)
G_5	a function of G_3, H_3, I_3 and J_3 defined in equation (57)
H_1, H_2, H_3	integrals involving the G-functions defined by equations (48)
h	bending displacement of the aerofoils
I_1, I_2, I_3	integrals involving the F-functions defined by equations (54)
J_1, J_2, J_3	integrals involving the G-functions defined by equations (55)
k	reduced frequency $\left(\frac{\omega c}{2U}\right)$
L	lift per unit span
M	moment about the elastic axis
p	$\frac{ik}{2\lambda} + m$
$\underset{\sim}{q}$	$u - iv$
s	spacing between adjacent blades
U	undisturbed free stream velocity
u	local velocity resolved in free stream direction
v	local velocity resolved normal to free stream direction
x, ξ	co-ordinates in stream direction; origin at mid-chord
θ	angular displacement of aerofoil
β	stagger angle
γ	vorticity distribution
Γ	total circulation
λ	$\frac{\pi c}{2s} e^{i\beta}$
ρ	density
ϕ	velocity potential
ϵ	$1/\tanh \lambda$
η	$\epsilon \tanh \lambda\xi$
μ	$\epsilon \tanh \lambda x$

1. Introduction

The flexure-torsion flutter of aerofoils in unstaggered cascade has been the subject of theoretical studies by Lilley (1) and Mendelson and Carroll (2). These authors use thin aerofoil theory to derive the lift and moment equations for an aerofoil moving in phase or in antiphase with its neighbour. Lilley includes structural stiffness terms and determines the conditions for flutter to occur. Sisto (3) finds a general expression for the vorticity at any point on the oscillating aerofoil in the form of an integral equation which is solved approximately for the case of zero stagger angle. The numerical results for the derivatives agree with the exact calculations of Mendelson and Carroll and the approximate values found by Lilley.

Legendre (4), using a conformal transformation method, has considered the general case of flutter in a cascade with stagger. This is an extension of the work of Timman (5) for zero stagger. Expressions are given for the velocity potential and circulation from which the pressure distribution can be calculated. Eichelbrenner (6) gives details of calculations based on Legendre's method for one gap/chord ratio and one stagger angle. He simplifies Legendre's integral expressions by the extended use of theta and zeta functions.

The present paper uses thin aerofoil theory to extend the work of Mendelson and Carroll to include arbitrary stagger angle and phase difference between adjacent blades. An integral equation relating local velocity and vorticity is solved and the aerodynamic derivatives are found in integral form.

2. The Lift and Moment Equations

Consider an infinite cascade of oscillating aerofoils of unit semi-chord at zero incidence, set at a stagger angle β and having a gap s (Fig. 1). The uniform velocity far upstream of the cascade is U. We shall assume that the oscillations are of small amplitude so that velocity perturbations are small compared with the free stream velocity.

The equations of motion for the perturbed motion reduce to

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

Defining a perturbation velocity potential ϕ such that

$$u = \frac{\partial \phi}{\partial x} \quad ; \quad v = \frac{\partial \phi}{\partial y} \tag{3}$$

equations (1) and (2) become respectively

$$\frac{\partial^2\phi}{\partial x\partial t} + U^2 \frac{\partial^2\phi}{\partial x^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (4)$$

$$\frac{\partial^2\phi}{\partial y\partial t} + U \frac{\partial^2\phi}{\partial x\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (5)$$

Adding (4) and (5) we have, if the operator $d = \frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy$,

$$dp = -\rho\, d\left[\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x}\right] \qquad (6)$$

The difference in velocity above (u_u) and below (u_l) an aerofoil is

$$u_u - u_l = \left(\frac{\partial\phi}{\partial x}\right)_u - \left(\frac{\partial\phi}{\partial x}\right)_l \qquad (7)$$

and, from thin aerofoil theory, this velocity difference can be represented by a distribution of vorticity along the chordline of the aerofoil and its wake,

$$u_u - u_l = \gamma(x,t) \qquad (8)$$

Thus substituting (7) and (8) in (6)

$$\Delta p = p_u - p_l = -\rho\left(U\gamma + \frac{\partial}{\partial t}\int_{-1}^{x}\gamma(x,t)\,dx\right) \qquad (9)$$

and since there can be no pressure difference across the wake

$$U\,\gamma_w(x,t) + \frac{\partial}{\partial t}\int_{1}^{x}\gamma_w(x,t)\,dx + \frac{\partial}{\partial t}\int_{-1}^{1}\gamma(x,t)\,dx = 0 \qquad (10)$$

where $\gamma_w(x,t)$ is the vorticity in the wake.

If $\Gamma(t)$ is the total circulation about the aerofoil

$$\Gamma(t) = \int_{-1}^{1}\gamma(x,t)\,dx$$

and (10) becomes

$$U\,\gamma_w(x,t) + \frac{\partial}{\partial t}\int_{1}^{x}\gamma_w(x,t)\,dx + \frac{\partial\Gamma(t)}{\partial t} = 0 \qquad (11)$$

If the oscillation is simple harmonic all quantities have a time variation proportional to $e^{i\omega t}$ and we can express our equations in terms of the reduced frequency k defined by $k = \frac{\omega c}{2U}$ (where $\frac{c}{2}$ is the semi-chord and is taken as unity). Equation (11) becomes

$$\gamma_W(x) + ik \int_1^x \gamma_W(x)\, dx + ik\Gamma = 0 \tag{12}$$

where $\gamma_W(x)$ and Γ are now the amplitudes of the wake vorticity and circulation respectively and are thus complex quantities independent of time.

Equation (12) can be solved for the vorticity in the wake in terms of the circulation round the wing in the form

$$\gamma_W(x) = -ik\Gamma\, e^{ik(1-x)} \tag{13}$$

and equation (9) becomes

$$\Delta p = -\rho U \left[\gamma(x) + ik \int_{-1}^{x} \gamma(x)\, dx \right] \tag{14}$$

Integrating (14), the lift on the aerofoil is

$$L = -\rho U \left[\int_{-1}^{1} \gamma(x)\, dx + ik \int_{-1}^{1} \int_{-1}^{x} \gamma(\xi)\, d\xi\, dx \right] \tag{15}$$

and, if the moment is measured about the elastic axis $x = a$,

$$M = -\rho U \left[\int_{-1}^{1} (x-a)\, \gamma(x) dx + ik \int_{-1}^{1} (x-a) \int_{-1}^{x} \gamma(\xi) d\xi\, dx \right] \tag{16}$$

3. The vorticity distribution in the cascade

For the infinite cascade the induced complex velocity $dq_n = du_n - i\,d\,v_n$ at any point x on the reference blade (zeroth blade) due to the vorticity $\gamma_n(\xi)$ at the point $z_n = \xi + i\,n\,s\,e^{-i\beta}$ on the n^{th} aerofoil and its wake is given by

$$d\,q_n(x) = -\frac{i\,\gamma_n(\xi)\,d\xi}{2\pi(x - z_n)}$$

or writing $\lambda = \dfrac{\pi\,c\,e^{i\beta}}{2s}$ (with $\frac{c}{2} = 1$) where λ is complex.

[It is real for $\beta = 0$ and imaginary for $\beta = \frac{\pi}{2}$.]

$$d\,q_n(x) = \frac{\lambda\,\gamma_n(\xi)\,d\xi}{2\pi^2\left[n - \frac{i\lambda(\xi - x)}{\pi}\right]} \tag{17}$$

The complex velocity induced by the complete cascade is

$$q(x) = \frac{\lambda}{2\pi^2}\int_{-1}^{\infty}\sum_{\infty}^{\infty}\frac{\gamma_n(\xi)\,d\xi}{n - \frac{i\lambda}{\pi}(\xi - x)} \tag{18}$$

If the phase difference between two adjacent blades is $\delta = 2\pi m$ $(0 \leqslant m < 1)$ then the phase difference δ_n between the n^{th} blade and the reference (n=0) blade is

$$\delta_n = 2\pi\,m\,n$$

and

$$\gamma_n(\xi) = \gamma_o(\xi)e^{2i\pi mn} \tag{19}$$

where $\gamma_o(\xi) \equiv \gamma(\xi)$ is the vorticity distribution on the reference blade.

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CORRIGENDUM

Professor Sisto has pointed out that equation (19)

$$\gamma_n(\xi) = \gamma_0(\xi)e^{2i\pi mn}$$

is valid only if m = 0 or $\frac{1}{2}$, and that arbitrary phase difference can be included only if different complex operators are used for harmonic time dependence and for the complex velocities.

The appropriate sentences of the summary, introduction and conclusion should be amended to:

"the theory allows for arbitrary stagger angle and for phase differences of 0 and π between adjacent blades"

Thus from (18) and (19)

$$\underset{\sim}{q}(x) = \frac{\lambda}{2\pi^2} \int_{-1}^{\infty} \sum_{\infty}^{\infty} \frac{\gamma(\xi) e^{2i\pi mn}}{n - \frac{i\lambda}{\pi}(\xi - x)} \, d\xi \tag{20}$$

But

$$\sum_{-\infty}^{\infty} \frac{e^{2i\pi mn}}{n - \frac{i\lambda}{\pi}(\xi - x)} = \frac{i\pi . e^{\lambda(1-2m)(\xi-x)}}{\sinh \lambda(\xi - x)} \qquad \text{(Ref.7)}$$

and thus (20) becomes

$$\underset{\sim}{q}(x) = \frac{i\lambda}{2\pi} \int_{-1}^{\infty} \frac{\gamma(\xi) e^{\lambda(1-2m)(\xi-x)}}{\sinh \lambda(\xi - x)} \, d\xi \tag{21}$$

We can now use (13) to express $q(x)$ in terms of the local vorticity and total circulation on the aerofoil. We obtain

$$\begin{aligned} \underset{\sim}{q}(x) = \; & \frac{\lambda k \Gamma}{2\pi} \int_{1}^{\infty} \frac{e^{ik(1-\xi)} \, e^{\lambda(1-2m)(\xi-x)}}{\sinh \lambda(\xi - x)} \, d\xi \\ & + \frac{i\lambda}{2\pi} \int_{-1}^{1} \frac{\gamma(\xi) \, e^{\lambda(1-2m)(\xi-x)}}{\sinh \lambda(\xi - x)} \, d\xi \end{aligned} \tag{22}$$

as the integral equation which must be solved for the local vorticity $\gamma(x)$ on the aerofoil in terms of the perturbation velocity $\underset{\sim}{q}(x)$.

If we put

$$\begin{aligned} \mathrm{tank}\; \lambda x &= \mu/\epsilon \\ \mathrm{tank}\; \lambda\xi &= \eta/\epsilon \\ \mathrm{tank}\; \lambda &= 1/\epsilon \end{aligned} \tag{23}$$

where μ, η, ϵ are complex,

then (22) becomes

$$\begin{aligned} \underset{\sim}{q}(\mu) - \frac{k\Gamma}{2\pi} e^{ik} \frac{(\epsilon+\mu)^m}{(\epsilon-\mu)^{m-1}} \int_{C_2} \frac{(\epsilon-\eta)^{m-1+\frac{ik}{2\lambda}}}{(\epsilon+\eta)^{m+ik/2\lambda}} \frac{d\eta}{\mu-\eta} \\ = \frac{i}{2\pi} \cdot \frac{(\epsilon+\mu)^m}{(\epsilon-\mu)^{m-1}} \int_{C_1} \frac{\gamma(\eta)(\epsilon-\eta)^{m-1}}{(\epsilon+\eta)^m} \frac{d\eta}{\mu-\eta} \end{aligned} \tag{24}$$

where the contour C_1 is the path of $\eta = \frac{\tanh \lambda\xi}{\tanh \lambda}$ in the range $-1 \leq \xi \leq 1$ and C_2 is the path of η in the range $1 \leq \xi \leq \infty$. Fig. 2 shows the contour C_1 for the case $\lambda = e^{i\pi/4}$.

Equation (24) can be simplified by writing

$$Q(\mu) = \underset{\sim}{q}(\mu) \cdot \frac{(\epsilon - \mu)^{m-1}}{(\epsilon + \mu)^{m}} - \frac{k\,\Gamma e^{ik}}{2\pi} \int_{C_2} \frac{(\epsilon - \eta)^{m-1+\frac{ik}{2\lambda}}}{(\epsilon+\eta)^{m+\frac{ik}{2\lambda}}} \frac{d\eta}{\mu - \eta}$$

and the integral equation reduces to

$$Q(\mu) = \frac{i}{2\pi} \int_{C_1} \frac{\gamma(\eta)\,(\epsilon - \eta)^{m}}{(\epsilon+\eta)^{m-1}(\epsilon^2 - \eta^2)} \frac{d\eta}{\mu - \eta} \tag{25}$$

$\eta = \epsilon$ corresponds to the point $\xi = \infty$ and hence does not lie on the contour C_1. Thus the kernel of equation 25

$$K(\eta) = \frac{\gamma(\eta)\,(\epsilon - \eta)^{m}}{2(\epsilon+\eta)^{m-1}(\epsilon^2 - \eta^2)} \tag{26}$$

has no singularities on C_1.

Thus we have to invert the equation

$$Q(\mu) = \frac{1}{i\pi} \int_{C_1} K(\eta) \frac{d\eta}{\eta - \mu} \tag{27}$$

which is a special case of the general Cauchy equation

$$a\,\phi(t) + \frac{b}{\pi i} \int_{C} \frac{\phi(s)}{s-t}\,ds = f(t) \tag{28}$$

with $a = 0$ and $b = 1$ and C an unclosed continuous contour.

The solution of this equation is (Ref. 8)

$$\phi(t) = \frac{a}{a^2 - b^2} f(t) - \frac{b}{(a^2 - b^2)\pi i} \left(\frac{t - \alpha}{t - \beta}\right)^{\bar{p}} \int_{C} \left(\frac{s-\beta}{s-\alpha}\right)^{\bar{p}} f(s)\,\frac{ds}{s-t}. \quad \ldots\ldots \tag{29}$$

with $$\bar{p} = \frac{1}{2\pi i} \ln \frac{a+b}{a-b}$$

and α, β the first and last points of the contour C.

In the special case considered here

$$\bar{p} = \frac{1}{2\pi i} \ln(-1) = \frac{1}{2}$$

$$\alpha = \eta_{\xi=-1} = -1$$

and $$\beta = \eta_{\xi=1} = 1$$

thus the solution of equation (27) is

$$K(\eta) = \frac{1}{\pi i}\left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{2}} \int_{C_1}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}} Q(\mu)\,\frac{d\mu}{\mu-\eta} \tag{30}$$

or, substituting back the actual vorticity and velocity distributions

$$\frac{\gamma(\eta)}{\epsilon^2-\eta^2} = \frac{2}{\pi i}\frac{(\epsilon+\eta)^{m-1}}{(\epsilon-\eta)^m}\left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{2}} \int_{C_1}\left\{g(\mu)\cdot\frac{(\epsilon-\mu)^{m-1}}{(\epsilon+\mu)^m}\right.$$
$$\left. -\frac{k\Gamma e^{ik}}{2\pi}\int_{C_2}\frac{(\epsilon-\eta_1)^{m-1+\frac{ik}{2\lambda}}}{(\epsilon+\eta_1)^{m+ik/2\lambda}}\frac{d\eta_1}{\mu-\eta_1}\right\}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}}\frac{d\mu}{\mu-\eta} \tag{31}$$

Now $$\Gamma = \int_{-1}^{1}\gamma(\xi)d\xi$$

thus, using the substitution of equation (23), the circulation on the reference blade can be expressed as

$$\Gamma = \frac{\epsilon}{\lambda}\int_C \frac{\gamma(\eta)}{\epsilon^2-\eta^2}\,d\eta \tag{32}$$

Thus, from (31), putting $p = m+ik/2\lambda$

$$\Gamma = \frac{2\epsilon}{\pi i\lambda} \int_{C_1} \frac{(\epsilon+\eta)^{m-1}}{(\epsilon-\eta)^m} \left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{2}} \left[\int_{C_1} \left\{ \underset{\sim}{q}(\mu) \frac{(\epsilon-\mu)^{m-1}}{(\epsilon+\mu)^m} - \frac{k\Gamma e^{ik}}{2\pi} \cdot \int_{C_2} \frac{(\epsilon-\eta_1)^{p-1}}{(\epsilon+\eta_1)^p} \frac{d\eta_1}{\mu-\eta_1} \right\} \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}} \frac{d\mu}{\mu-\eta} \right] d\eta$$

which can be manipulated to give the circulation in terms of the velocity over the blade and the cascade geometry in the form

$$\Gamma = \frac{\frac{2\epsilon}{\pi\lambda} \int_{C_1} \frac{(\epsilon+\eta)^{m-1}}{(\epsilon-\eta)^m} \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}} \left[\int_{C_1} \underset{\sim}{q}(\mu) \frac{(\epsilon-\mu)^{m-1}}{(\epsilon+\mu)^m} \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}} \frac{d\mu}{\mu-\eta} \right] d\eta}{1 + \frac{\epsilon k e^{ik}}{\pi^2\lambda} \int_{C_1} \frac{(\epsilon+\eta)^{m-1}}{(\epsilon-\eta)^m} \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}} \int_{C_1} \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}} \left[\int_{C_2} \frac{(\epsilon-\eta_1)^{p-1}}{(\epsilon+\eta_1)^p} \frac{d\eta_1}{\mu-\eta_1} \right] \frac{d\mu}{\mu-\eta} d\eta} \qquad (33)$$

If now we write $F_1(\eta) = \frac{(\epsilon+\eta)^{m-1}}{(\epsilon-\eta)^m} \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}}$

$$F_2(\mu,\eta) = \frac{(\epsilon-\mu)^{m-1}}{(\epsilon+\mu)^m} \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}} \cdot \frac{1}{\mu-\eta} \qquad (34)$$

$$F_3(\eta) = \int_{C_1} \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}} \left[\int_{C_2} \frac{(\epsilon-\eta_1)^{p-1}}{(\epsilon+\eta_1)^p} \frac{d\eta_1}{\mu-\eta_1} \right] \frac{d\eta}{\mu-\eta}$$

then (33) becomes

$$\Gamma = \frac{\frac{2\epsilon}{\pi\lambda} \int_{C_1} F_1(\eta) \int_{C_1} \underset{\sim}{q}(\mu) F_2(\mu,\eta)\, d\mu\, d\eta}{1 + \frac{\epsilon k e^{ik}}{\pi^2\lambda} \int_{C_1} F_1(\eta) F_3(\eta)\, d\eta} \qquad (35)$$

and substituting into (31) we have the vorticity distribution on the reference blade given in terms of the velocity distribution $\underset{\sim}{q}(\mu)$ by

$$\frac{\gamma(\eta)}{\epsilon^2 - \eta^2} = \frac{2}{\pi} F_1(\eta) \left[\int_{C_1} \underset{\sim}{q}(\mu) F_2(\mu,\eta) - \frac{k\epsilon e^{ik}}{\pi^2\lambda} F_3(\eta) \left\{ \frac{\int_{C_1} \tilde{F}_1(\eta) \int_{C_1} \underset{\sim}{q}(\mu) F_2(\mu,\eta)\, d\mu\, d\eta}{1 + \frac{\epsilon k e^{ik}}{\pi^2\lambda} \int_{C_1} F_1(\eta) F_3(\eta)\, d\eta} \right\} \right.$$

..... (36)

4. The velocity of flow over the reference blade

The vertical displacement of a point x on the aerofoil at time t is

$$y\,(x,t) = h + (x-a)\,\theta \tag{37}$$

where h and θ are functions of t and a is the position of the elastic axis measured from mid chord.

We assume that the induced perturbation velocity in the stream direction is small enough, compared with the free stream velocity, to be neglected. The velocity normal to the surface must be zero (relative to the surface) at all points of the surface.

Thus

$$\begin{aligned} u(x,t) &= U \\ v(x,t) &= U\,\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \end{aligned} \tag{38}$$

or

$$v(\mu,t) = \dot{h} + U\theta - a\dot{\theta} + \frac{\dot{\theta}}{2\lambda}\log\frac{\epsilon+\mu}{\epsilon-\mu}$$

and

$$\underset{\sim}{q}(\mu,\,t) = U - i\left[\dot{h} + U\theta - a\dot{\theta} + \frac{\dot{\theta}}{2\lambda}\log\frac{\epsilon+\mu}{\epsilon-\mu}\right] \tag{39}$$

Now q, h and θ all have a time variation proportional to $e^{i\omega t}$. Therefore, considering the amplitude of the time dependent terms, we have

$$\underset{\sim}{q}(\mu) = \omega\left[h - \theta\left\{a + \frac{i}{k} - \frac{1}{2\lambda}\log\frac{\epsilon+\mu}{\epsilon-\mu}\right\}\right] \tag{40}$$

and substituting for $q(\mu)$ in (36) from (40) we obtain the vorticity distribution on the reference blade in terms of the blade motion in the form

$$\frac{\gamma(\eta)}{\epsilon^2-\eta^2} = \frac{2\omega}{\pi}\,F_1(\eta)\left[\left(h-\theta\left[a+{}^{i}/k\right]\right)\int_{C_1}F_2(\mu,\eta)\,d\mu + \frac{\theta}{2\lambda}\int_{C_1}F_2(\mu,\eta)\,\log\frac{\epsilon+\mu}{\epsilon-\mu}\,d\mu - \frac{k\epsilon\,e^{ik}}{\pi^2\lambda}\,F_3(\eta)\left\{\frac{\left(h-\theta\left[a+{}^{i}/k\right]\right)G_1 + \frac{\theta}{2\lambda}G_2}{1+\frac{k\epsilon\,e^{ik}}{\pi^2\lambda}\,G_3}\right\}\right] \tag{41}$$

where

$$G_1 = \int_{C_1} F_1(\eta) \int_{C_1} F_2(\mu,\eta)\, d\mu d\eta$$

$$G_2 = \int_{C_1} F_1(\eta) \int_{C_1} F_2(\mu,\eta) \log \frac{\epsilon+\mu}{\epsilon-\mu}\, d\mu d\eta \qquad (42)$$

$$G_3 = \int_{C_1} F_1(\eta)\, F_3(\eta)\, d\eta$$

5. The Aerodynamic Derivatives

5.1. The lift derivatives

From (15) the lift per unit span is given by

$$L = -\rho U \left[\int_{-1}^{1} \gamma(x)dx + ik \int_{-1}^{1} \int_{-1}^{x} \gamma(\xi)\, d\xi\, dx \right]$$

or using the transformations defined in (23)

$$L = -\rho U \left[\frac{\epsilon}{\lambda} \int_{C_1} \frac{\gamma(\eta)}{\epsilon^2-\eta^2}\, d\eta + \frac{ik\epsilon^2}{\lambda^2} \int_{C_1} \frac{1}{\epsilon^2-\eta^2} \int_{C_1'(\eta)} \frac{\gamma(\xi)}{\epsilon^2-\xi^2}\, d\xi\, d\eta \right] \qquad \ldots\ldots (43)$$

where $C_1'(\eta)$ is the part of the contour C between -1 and η

From (41), using the G functions defined by (42)

$$\int_{C_1} \frac{\gamma(\eta)}{\epsilon^2-\eta^2} d\eta = \frac{2\omega}{\pi} \left[\left\{ h - \theta(a+{}^i/k) \right\} G_1 + \frac{\theta}{2\lambda} G_2 - \frac{k\, e^{ik\epsilon}}{\pi^2\lambda} G_3 \left\{ \frac{(h-\theta a-\theta^i/k)G_1 + \frac{\theta}{2\lambda} G_2}{1 + \frac{\epsilon\, ke^{ik}}{\pi^2\lambda} G_3} \right\} \right] \qquad \ldots\ldots (44)$$

Similarly

$$\int_{C_1'} \frac{\gamma(\xi)}{\epsilon^2-\eta^2} d\xi = \frac{2\omega}{\pi} \left[\left\{ h - \theta(a+{}^i/k) \right\} G_1' + \frac{\theta}{2\lambda} G_2' - \frac{k\epsilon\, e^{ik}}{\pi^2\lambda} G_3' \left\{ \frac{(h-\theta a - \frac{i\theta}{k})G_1 + \frac{\theta}{2\lambda}G_2}{1+ \frac{\epsilon\, ke^{ik}}{\pi^2\lambda} G_3} \right\} \right] \qquad \ldots\ldots (45)$$

where
$$G_1'(\eta) = \int_{C_1'(\eta)} F_1(\xi) \int_{C_1} F_2(\mu,\xi)\,d\mu d\xi$$
$$G_2'(\eta) = \int_{C_1'(\eta)} F_1(\xi) \int_{C_1} F_2(\mu,\xi)\,d\mu d\xi \tag{46}$$
$$G_3'(\eta) = \int_{C_1'(\eta)} F_1(\xi)\, F_3(\xi)\, d\xi$$

and furthermore

$$\int_{C_1} \frac{1}{\epsilon^2 - \eta^2} \int_{C_1'(\eta)} \frac{\gamma(\xi)}{\epsilon^2 - \xi^2}\, d\xi d\eta = \frac{2\omega}{\pi} \left[\left\{ h - \theta(a + {}^{i}/k) \right\} H_1 + \frac{\theta}{2\lambda} H_2 - \frac{k\epsilon e^{ik}}{\pi^2\lambda} H_3 \left\{ \frac{(h - \theta a - \frac{i\theta}{k})\, G_1 + \frac{\theta}{2\lambda} G_2}{1 + \frac{\epsilon\, ke^{ik}}{\pi^2\lambda} G_3} \right\} \right] \tag{47}$$

where
$$H_1 = \int_{C_1} \frac{G_1'(\eta)}{\epsilon^2 - \eta^2}\, d\eta \; ; \; H_2 = \int_{C_1} \frac{G_2'(\eta)}{\epsilon^2 - \eta^2}\, d\eta \; ; \; H_3 = \int_{C_1} \frac{G_3'(\eta)}{\epsilon^2 - \eta^2} \tag{48}$$

Substituting from (44) and (47) into (43)

$$\frac{L}{-\pi\rho\omega^2} = \frac{2\,\epsilon}{\pi^2 \lambda k} \left[\left(h - \theta\, a - \frac{i\theta}{k} \right) \left\{ G_1 + \frac{ik\epsilon}{\lambda} H_1 - \frac{k\epsilon e^{ik}}{\pi^2\lambda} G_1 \left(\frac{G_3 + \frac{ik\epsilon}{\lambda} H_3}{1 + \frac{\epsilon ke^{ik}}{\pi^2\lambda} G_3} \right) \right\} + \frac{\theta}{2\lambda} \left\{ G_2 + \frac{ik\epsilon}{\lambda} H_2 - \frac{k\epsilon e^{ik}}{\pi^2\lambda} G_2 \left(\frac{G_3 + \frac{ik\epsilon}{\lambda} H_3}{1 + \frac{\epsilon ke^{ik}}{\pi^2\lambda} G_3} \right) \right\} \right] \tag{49}$$

The two dimensional lift derivatives are found by collecting the coefficients of $-\frac{1}{k^2}$, $\frac{i}{k}$ and 1 in (49). We write z for h to conform with the usual notation for such derivatives.

Thus, remembering that the chord length is 2

$$l_z = L_{z}/\rho U^2 = 0 \; ; \; l_{\dot z} = L_{\dot z}/2\rho U = \frac{i\epsilon}{\pi^2\lambda} G_1 \tag{50}$$

$$l_{\ddot{z}} = L_{\ddot{z}}/4\rho = \frac{\epsilon^2}{2\pi^2\lambda^2}\left[i\,H_1 - \frac{e^{ik}}{\pi^2}\,G_1G_4\right] \quad \text{where } G_4 = \frac{G_3 + \frac{ik\epsilon}{\lambda}H_3}{1+\frac{\epsilon k e^{ik}}{\pi^2\lambda}G_3} \quad \ldots\ldots \quad (51)$$

and

$$l_\theta = L_\theta/2\rho U^2 = \frac{i\,\epsilon}{\pi^2\lambda}G_1 \quad ; \quad l_{\dot{\theta}} = L_{\dot{\theta}}/4\rho U = \frac{i\,\epsilon}{4\pi^2\lambda}\left[2aG_1 - \frac{2\epsilon}{\lambda}\left(H_1 - \frac{ie^{ik}}{\pi^2}G_1G_4\right) - \frac{G_2}{\lambda}\right]$$

$$l_{\ddot{\theta}} = L_{\ddot{\theta}}/8\rho = \frac{\epsilon^2}{8\pi^2\lambda^2}\left[\frac{iH_2}{\lambda} - 2ia\,H_1 + \frac{e^{ik}}{\pi^2}\left(2aG_1 - \frac{G_2}{\lambda}\right)G_4\right] \qquad (52)$$

5.2. The moment derivatives

From (16) the moment is given by

$$M = -\rho U\left[\int_{-1}^{1}(x-a)\gamma(x)\,dx + ik\int_{-1}^{1}(x-a)\int_{-1}^{x}\gamma(\xi)\,d\xi\,dx\right]$$

or, using the transformations of (23)

$$M = -\rho U\left[\frac{\epsilon}{\lambda}\int_{C_1}\left[\log\frac{\epsilon+\eta}{\epsilon-\eta} - a\right]\frac{\gamma(\eta)}{\epsilon^2-\eta^2}\,d\eta + \frac{ik\epsilon^2}{\lambda^2}\int_{C_1}\frac{\log\frac{\epsilon+\eta}{\epsilon-\eta} - a}{\epsilon^2-\eta^2}\int_{C_1'}\frac{\gamma(\xi)}{\epsilon^2-\xi^2}\,d\xi\,d\eta\right] \quad \ldots\ldots \quad (53)$$

Substituting for the vorticity from (44) and (45) and using the functions I_1, I_2, I_3 and J_1, J_2, J_3 defined by

$$I_1 = \int_{C_1} F_1(\eta)\,\log\frac{\epsilon+\eta}{\epsilon-\eta}\int_{C_1} F_2(\mu,\eta)\,d\mu\,d\eta$$

$$I_2 = \int_{C_1} F_1(\eta)\,\log\frac{\epsilon+\eta}{\epsilon-\eta}\int_{C_1} F_2(\mu,\eta)\,\log\frac{\epsilon+\mu}{\epsilon-\mu}\,d\mu\,d\eta \qquad (54)$$

$$I_3 = \int_{C_1} F_1(\eta)\,F_3(\eta)\,\log\frac{\epsilon+\eta}{\epsilon-\eta}\,d\eta$$

$$J_n = \int_{C_1} \frac{G_n'(\eta)\log\frac{\epsilon+\eta}{\epsilon-\eta}}{\epsilon^2-\eta^2}\,d\eta \qquad n = 1,2,3 \tag{55}$$

the moment equation (53) becomes

$$\frac{M}{-\pi\rho\omega^2} = \frac{2\epsilon}{\pi^2\lambda k}\left[\left(h-\theta a-\frac{i\theta}{k}\right)\left\{I_1 - aG_1 + \frac{k\epsilon}{\lambda}\left[i(J_1 - aH_1) - \frac{e^{ik}}{\pi^2}G_1G_5\right]\right\}\right.$$
$$\left. + \frac{\theta}{2\lambda}\left\{I_2 - aG_2 + \frac{k\epsilon}{\lambda}\left[i(J_2 - aH_2) - \frac{e^{ik}}{\pi^2}G_2G_5\right]\right\}\right] \tag{56}$$

where
$$G_5 = \frac{I_3 - aG_3 + \frac{ik\epsilon}{\lambda}\left[J_3 - aH_3\right]}{1 + \frac{\epsilon k e^{ik}}{\pi^2\lambda}G_3} \tag{57}$$

Collecting the coefficients of $-\frac{1}{k^2}$, $\frac{i}{k}$, 1 we obtain the moment derivatives

$$m_z = M_z/2\rho U^2 = 0 \quad , \quad m_{\dot z} = M_{\dot z}/4\rho U = -\frac{i\,\epsilon}{2\pi^2\lambda}(I_1 - aG_1)$$
$$m_{\ddot z} = M_{\ddot z}/8\rho = \frac{\epsilon^2}{4\pi^2\lambda^2}\left[i(J_1 - aH_1) - \frac{e^{ik}}{\pi^2}G_1G_5\right] \tag{58}$$

and

$$m_\theta = M_\theta/4\rho U^2 = -\frac{i\,\epsilon}{2\pi^2\lambda}(I_1 - aG_1)$$
$$m_{\dot\theta} = M_{\dot\theta}/8\rho U = \frac{\epsilon}{4\pi^2\lambda}\left[i\left\{a(I_1 - aG_1) - \frac{1}{2\lambda}(I_2 - G_2)\right\} - \frac{\epsilon}{\lambda}\left\{i(J_1 - aH_1) - \frac{e^{ik}}{\pi^2}G_1G_5\right\}\right]$$
$$m_{\ddot\theta} = M_{\ddot\theta}/16\rho = \frac{\epsilon^2}{8\pi^2\lambda^2}\left[i\left\{\frac{1}{2\lambda}(J_2 - aH_2) - a(J_1 - aH_1)\right.\right.$$
$$\left.\left. + \frac{e^{ik}}{\pi^2}G_5\left(aG_1 - \frac{G_2}{2\lambda}\right)\right\}\right] \tag{59}$$

5.3. Comparison with previous results

The basic equations of this paper (15, 16 and 24) are in agreement, for the special case of zero stagger and antiphase oscillation, with those of Lilley (Ref. 1, eqns. 2.10, 2.11 and 2.27) and the solution of the integral equation also agrees. Lilley expresses the aerodynamic derivatives directly in elliptic functions and further comparison of the two papers is not possible except that the present author also finds that $l_z = m_z = 0$ and $l_{\dot z} = l_\theta$ and $m_{\dot z} = m_\theta$.

Mendelson and Carroll (Ref. 2) present their results for the unstaggered cascade oscillating in phase or in antiphase in the form of functions L_h, L_α, M_h, M_α, which show the dependence of the lift L and moment M on flexural displacement h and angular displacement α, such that

$$L = \pi\rho\omega^2 \left\{ L_h h + \left[L_\alpha - (\tfrac{1}{2} + a)L_h\right] \alpha \right\}$$

$$M = \pi\rho\omega^2 \left\{ \left[M_h - (\tfrac{1}{2} + a)L_h\right] h + \left[M_\alpha - (\tfrac{1}{2} + a)(L_\alpha + M_h) + (\tfrac{1}{2}+a)^2 L_h\right]\alpha \right\}$$

(Ref. 2 eqn.B.37)

In corresponding form the results of the present paper for the special cases are

$$L_h = -\frac{2\epsilon}{\pi^2\lambda}\left\{ \frac{G_1}{k} + \frac{i\epsilon H_1}{\lambda} - \frac{\epsilon e^{ik}}{\pi^2\lambda} G_1 G_4 \right\}$$

$$L_\alpha = \tfrac{1}{2} L_h + \frac{2i\epsilon}{\pi^2\lambda k}\left(\frac{G_1}{k} + \frac{i\epsilon H_1}{\lambda} - \frac{\epsilon e^{ik}}{\pi^2\lambda} G_1 G_4\right) - \frac{\epsilon^2}{\pi^2\lambda}\left(\frac{G_2}{k} + \frac{i\epsilon H_2}{\lambda} - \frac{\epsilon e^{ik}}{\pi^2\lambda} G_2 G_4\right)$$

$$M_h = -\frac{2\epsilon}{\pi^2\lambda}\left\{\frac{I_1 - aG_1}{k} + \frac{\epsilon}{\lambda}\left[i\,(J_1 - aH_1) - \frac{e^{ik}}{\pi^2} G_1 G_5\right]\right\} \qquad (60)$$

$$M_\alpha = \tfrac{1}{2}(L_\alpha + M_h - \tfrac{1}{2}L_h) + \frac{1}{k^2} L_h - \frac{2\epsilon}{\pi^2\lambda}\left\{\frac{aI_1}{k} + \frac{i\varepsilon}{k} J_1 - \frac{\epsilon e^{ik}}{\pi^2\lambda} G_1 (G_5 - kG_4) \right\}$$

$$+ \frac{i}{k^2}\left\{I_1 - a\,G_1 + \frac{k\varepsilon}{\lambda}\left[i(J_1 - aH_1) - \frac{e^{ik}}{\pi^2} G_1 G_5\right]\right\}$$

$$+ \frac{1}{2\lambda}\left\{I_2 - a\,G_2 + \frac{k\varepsilon}{\lambda}\left[i(J_2 - a\,H_2) - \frac{e^{ik}}{\pi^2} G_2 G_5\right]\right\}$$

with $m = 0$ or $\frac{1}{2}$, λ real and the integrals along C_1 becoming integrals along the η-axis between -1 and 1, the integrals along C_2 becoming integrals along the η-axis between 1 and ϵ and $C_1'(\eta)$ becoming that part of the η-axis between -1 and $\log \frac{\epsilon+\eta}{\epsilon-\eta}$

If we substitute for the G, H, I and J integrals in (60) we obtain results which show substantial agreement with equations B.38, 39, 40 and 41 of reference 2. However Mendelson and Carroll have been able to simplify the integrals

$$\int_{-1}^{1} \int_{-1}^{x} \gamma(\xi)\, d\xi\, dx \quad \text{and} \quad \int_{-1}^{1} (x - a) \int_{-1}^{x} \gamma(\xi)\, d\xi\, dx$$

further than the present author and hence the H and J integrals of this paper are more complicated than the corresponding integrals of reference 2.

6. Conclusion

Thin aerofoil theory can be used to find the aerodynamic derivatives of an aerofoil oscillating in an infinite cascade. The theory takes account of stagger angle and phase difference between adjacent blades of the cascade. The derivatives are expressed in terms of complex integrals (except for the degenerate case of zero stagger and antiphase oscillation when the integrals are real) which have to be evaluated along the aerofoil and its wake.

7. References

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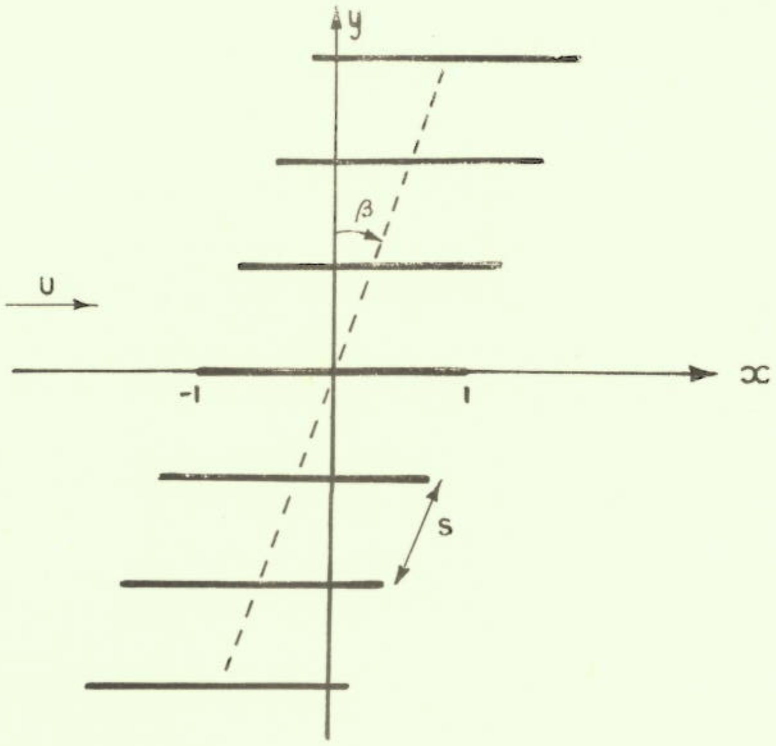

FIG. 1. CASCADE GEOMETRY

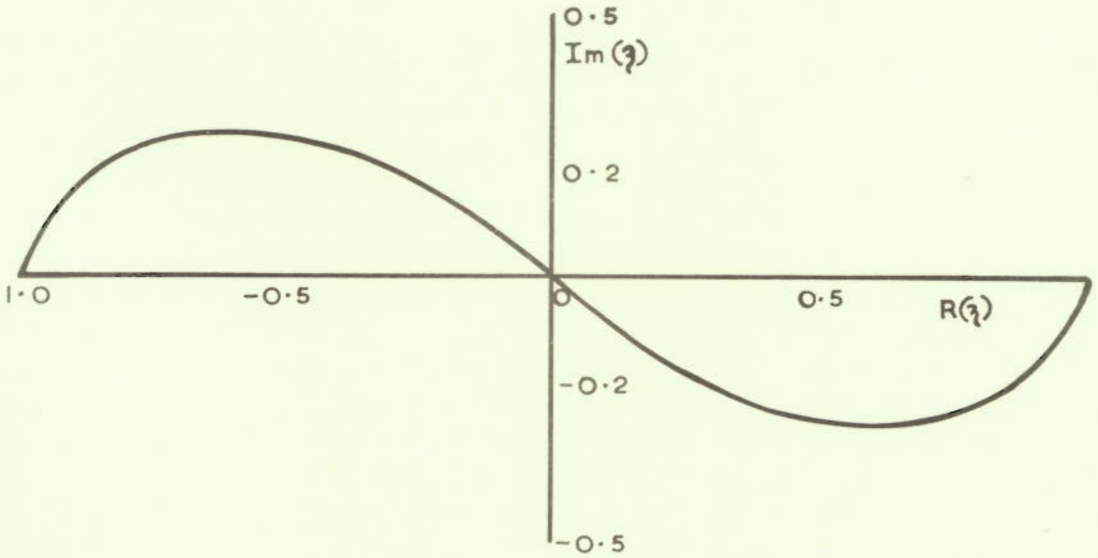

FIG. 2. THE TRANSFORMED AEROFOIL

$$\eta = \frac{\tanh \lambda \xi}{\tanh \lambda};\ -1 \le \xi \le 1;\ \left[\lambda = \frac{\pi}{s} e^{i\beta};\ \beta = 45^{\circ}\ s = \pi\right]$$