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Independence of helicopter rotor derivatives under non-uniformity of induced velocity distribution at low forward speed

-by-

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SUMMARY

A radial parabolic induced velocity distribution agreeing closely with flight measurements has been used for the hovering case. To this has been added a second induced velocity distribution, varying linearly from the front to the rear of the rotor disc, to allow for the effect of forward speed. The magnitude of this second induced velocity term depends on the advance ratio μ.

Values of the force coefficients C_H and C_{YS}, the flapping coefficients a_o, a_1 and b_1, and the rotor derivatives x_q, z_q, y_P, x_u, z_u, x_w, z_w and y_v have been calculated for a typical case for the low forward speed region ($\mu = 0 - 0.14$) for both uniform and non-uniform induced velocity and the results compared. Additional values of the flapping coefficients have been calculated for the speed range $\mu = 0.14 - 0.24$ and the results compared with flight measurements and with values based on the Mangler induced velocity distribution. Good agreement has been obtained.

The values obtained for the rotor derivatives show that the effect of non-uniform induced velocity is almost negligible except in the case of z_q which is a very small derivative.

MEP

CONTENTS

LIST OF SYMBOLS

a	Blade section lift curve slope
a_o	Blade coning angle
a_1	First harmonic longitudinal flapping coefficient
A	Parameter in expression for λ_T (5-16)
A'	Parameter in expression for λ_U (5-16A)
A_o	Blade collective pitch angle
A_1	Coefficient of $-\cos\psi$ in expression for cyclic feathering
b	Number of blades
b_1	First harmonic lateral flapping coefficient
B	Parameter in expression for λ_T (5-16)
B'	Parameter in expression for λ_U (5-16A)
B_1	Coefficient of $-\sin\psi$ in expression for cyclic feathering
c	Blade chord
C	Arbitrary constant
C_L	Blade section lift coefficient
C_H	H force coefficient $= H / \pi R^2 \rho (\Omega R)^2$
C_T	Thrust coefficient $= T / \pi R^2 \rho (\Omega R)^2$
C_{YS}	Lateral force coefficient $= Y_S / \pi R^2 \rho (\Omega R)^2$
D	Drag force on blade
F	Aerodynamic force on blade
H	Drag force in plane of rotor disc
i	Incidence of rotor disc
I_1	Blade moment of inertia about flapping hinge
L	Lift force on blade
M_A	Moment of aerodynamic forces about flapping hinge

/M_D ...

List of Symbols (Contd.)

M_D	Moment of dynamic forces about flapping hinge
P	Rate of roll (positive OY → OZ)
q	Rate of pitch (positive OZ → OX)
r	Radial distance along blade from hub
R	Blade radius
T	Rotor thrust force
u,v,w	Disturbance velocities along OX, OY, OZ respectively
U	Resultant air velocity relative to blade element ($\approx U_T$)
U_P	Air velocity component perpendicular to blade and to the rotor cone
U_T	Air velocity component perpendicular to blade and tangential to the rotor cone
V	Velocity of forward flight
V'	Resultant air velocity relative to rotor disc (5-11)
X,Y,Z	Forces along OX, OY, OZ respectively
X_u, Y_v, Z_w	$\frac{\partial X}{\partial u}$, $\frac{\partial Y}{\partial v}$, $\frac{\partial Z}{\partial w}$ respectively
x	Fractional distance along blade, $x = r/R$
x_u etc.	Non-dimensional form of derivative, $x_u = X_u/\rho(\Omega R)(\pi R^2)$ etc.
x_q etc.	Non-dimensional form of derivative, $x_q = X_q/\rho(\Omega R)(\pi R^2)R$ etc.
a	Angle of attack of blade element, $a = \theta - \phi$
β	Instantaneous blade flapping angle, measured from no-feathering plane
γ	Lock's inertia number, $\gamma = \frac{\rho a c R^4}{I_1}$
δ	Blade section drag coefficient
θ	Instantaneous blade pitch angle measured from the tip-path plane
λ	Inflow factor

/λ_o ...

List of Symbols (Contd)

λ_o, λ_1 etc.	$\frac{\nu_o}{\Omega R}$, $\frac{\nu_1}{\Omega R}$ etc.
λ_w	$\lambda_w = \frac{w}{\Omega R}$
μ	Advance Ratio, $\mu = \frac{V \cos i}{\Omega R}$
ν	Induced velocity through the rotor disc at any point (r, ψ)
ν_m	Mean induced velocity, $\nu_m = \frac{1}{\pi R^2} \int_o^R r dr \int_o^{2\pi} \nu d\psi$
ν_T	Value of the induced velocity at $r = R$ in hovering, and $r = R$, $\psi = \pm \frac{\pi}{2}$ in forward flight
ν_u	Uniform induced velocity
ν_o	Radial induced velocity distribution, $\frac{\nu_o}{\nu_T} = - x^2 + 2x$
ν_1	Parameter in expression $\nu_1 x \cos \psi$ for the induced velocity distribution due to forward speed
ρ	Air density
σ	Solidity factor, $\sigma = \frac{bc}{\pi R}$
ϕ	$\phi \approx \frac{U_P}{U_T}$
ψ	Blade azimuth angle measured from the downwind position in direction of rotation
Ω	Angular velocity of rotor

/1. ...

1. Introduction

Although it is well known that the induced velocity distribution through a rotor is far from uniform, little has been written concerning the effect of this non-uniformity on blade flapping coefficients and rotor derivatives.

Glauert (1) suggested a triangular distribution of induced velocity from the front to the rear of the rotor disc. This distribution gave values of the lateral flapping coefficient b_1 agreeing more closely with experimental measurements than values predicted using a uniform distribution.

Martin (7) used the induced velocity distribution calculated by Mangler (10), who treated the rotor disc as a circular wing, to obtain values of b_1 which compared favourably with flight measurements by Myers (13). By considering the effect of this greater lateral tilt on the force coefficients he concluded that there would be a significant effect on the rotor derivatives.

The Mangler induced velocity distribution was calculated on the assumption that the perturbation velocities due to the rotor disc were small compared with the free-stream velocity. It is, therefore, not applicable at low forward speeds, (below $\mu = 0.1$ say).

To investigate the effect of non-uniform induced velocity at low forward speeds a parabolic radial distribution has been chosen which agrees well with flight measurements by Brotherhood (9) on a hovering helicopter. To this has been added a distribution varying linearly from the front to the rear of the rotor disc and depending in magnitude on the advance ratio μ. Values have been calculated for the flapping coefficients, the force coefficients and the rotor derivatives. These have been compared with values obtained assuming a uniform distribution of induced velocity over the rotor disc, and with the results obtained by Martin (7) and the flight measurements by Myers (13).

2. Notation

The British system of notation has been adopted i.e. all forces and moments are referred to axes attached to the tip-path plane. The angle of incidence of the rotor disc is taken as being positive when the disc is tilted forward with respect to the direction of flight. The system of axes is shown in Fig. 1.

/The expression ...

The expression for the cyclic feathering of the blades with respect to the tip-path plane is

$$\theta = A_o - A_1 \cos\psi - B_1 \sin\psi \quad \ldots\ldots\ldots\ldots(2\text{-}1)$$

where ψ is the azimuth angle in the plane of the disc and is measured from the downstream direction in the direction of rotation of the blades.

The expression for the blade flapping angle with respect to the no-feathering plane is

$$\beta = a_o - a_1 \cos\psi - b_1 \sin\psi + \text{terms in higher harmonics} \quad \ldots\ldots\ldots\ldots(2\text{-}2)$$

It has been shown by Lock (2) and others that, for the flapping and feathering systems to be equivalent, the first harmonic flapping coefficients are related to the cyclic feathering coefficients by the following expressions.

$$\left.\begin{array}{l} a_1 = B_1 \\ b_1 = -A_1 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(2\text{-}3)$$

3. The Flow Relative to the Rotor Disc

For the rotor with forward velocity V, the component of V in a plane parallel to the tip-path plane is given by

$$\mu \Omega R = V \cos i \quad \ldots\ldots\ldots\ldots(3\text{-}1)$$

where $\mu = V \cos i / \Omega R$ $\quad \ldots\ldots\ldots\ldots(3\text{-}2)$

is known as the 'advance ratio'.

The velocity perpendicular to the tip-path plane is

$$\lambda \Omega R = V \sin i + \nu \quad \ldots\ldots\ldots\ldots(3\text{-}3)$$

where ν is the induced flow through the rotor disc, and λ is the 'inflow factor'.

/4. ...

4. The Flow Relative to a Blade Element

For purposes of estimating derivatives the rotor is assumed to have a pitching velocity q and a rolling velocity p. Using the expression for cyclic feathering given by (2-1) the following expressions are obtained for the velocity components relative to a blade element at radius $r = xR$.

(i) The velocity component perpendicular to the blade in a plane parallel to the tip-path plane.

$$U_T = (x + \mu \sin \psi)\, \Omega R \qquad \text{.............(4-1)}$$

(ii) The velocity component perpendicular to the blade and to the cone surface

$$U_P = (a_o \mu \cos \psi + \lambda - \frac{p}{\Omega} x \sin \psi - \frac{q}{\Omega} x \cos \psi)\Omega R \qquad \text{.............(4-2)}$$

where a_o is the base angle of the rotor cone.

(iii) The spanwise velocity along the blade is

$$(\mu \cos \psi - \lambda a_o)\Omega R$$

The effect of this spanwise velocity is not considered in the subsequent analysis since the dominant term $\mu \cos \psi$ will be small at low forward speeds.

The angle of incidence of the rotor blade element is

$$\alpha = \theta - \phi \qquad \text{.............(4-3)}$$

where $\phi = \tan^{-1} \frac{U_P}{U_T} \approx \frac{U_P}{U_T}$ since $U_P \ll U_T$

Hence

$$\alpha = A_o - A_1 \cos \psi - B_1 \sin \psi - \frac{a_o \mu \cos\psi + \lambda - \frac{p}{\Omega} x \sin\psi - \frac{q}{\Omega} x \cos \psi}{x + \mu \sin \psi} \qquad \text{.............(4-4)}$$

/5. ...

5. The Induced Velocity

5.1. The Induced Velocity in Hovering

Measurements by Brotherhood (9) show that the induced velocity in hovering is far from uniform over the rotor disc. His experimental values agree well with values calculated from propeller strip theory.

It was found that the induced velocity distribution, as measured by Brotherhood, could be approximated very closely (see Fig. 4) by the following simple expression.

$$\frac{\nu_o}{\nu_T} = - x^2 + 2x \qquad \text{.............(5-1)}$$

where ν_T is the value of the induced velocity at the edge of the rotor disc and $x = r/R$. This expression represents a parabolic distribution varying from zero at the centre of the disc to a maximum value at the edge of the disc.

The following integrals are now evaluated for later reference. Note that $\lambda_T = \nu_T/\Omega R$, $\lambda_o = \nu_o/\Omega R$

$$\left.\begin{aligned} \int_o^1 \lambda_o \, dx &= \frac{2}{3}\lambda_T \\ \int_o^1 \lambda_o x \, dx &= \frac{5}{12}\lambda_T \\ \int_o^1 \lambda_o x^2 dx &= \frac{3}{10}\lambda_T \\ \int_o^1 \lambda_o x^3 dx &= \frac{7}{30}\lambda_T \end{aligned}\right\} \qquad \text{............(5-2)}$$

5.2. The Induced Velocity at Moderate Forward Speeds ($\mu > 0.14$)

Following Glauert (1) it was decided to superimpose an induced velocity distribution, varying linearly from the front to the rear of the rotor disc, on the induced velocity distribution in hovering, to account for the effect of forward speed. This linear induced velocity distribution is given

/in non- ...

in non-dimensional form by

$$\lambda_1 x \cos\psi = \left(\frac{\nu_1}{\Omega R}\right) x \cos\psi \qquad \text{.............(5-3)}$$

This represents an induced velocity varying linearly from a value $-\lambda_1$ at the front of the disc to $+\lambda_1$ at the rear of the disc.

The choice of the value for λ_1 is arbitrary. Glauert (1) suggested letting it have the same value as λ_o which, in his paper, represented an induced velocity uniform over the whole of the disc. It was decided to let $\lambda_1 \doteqdot \lambda_T$, for $\mu > 0.14$ and later calculations of the flapping coefficients a_o, a_1, and b_1 showed good agreement with experimental values given in Ref. 13, and also with values calculated by Martin (7) using the Mangler induced velocity distribution (see Figs. 7-9).

The effect of the angle of incidence of the tip-path plane on the induced velocity distribution has been ignored since the incidence is small in practice ('Gyrodyne condition').

5.3. The Induced Velocity at Low Forward Speeds

At zero forward speed λ_1 is zero and at moderate and high forward speeds the choice of $\lambda_1 = \lambda_T$ appears to give good agreement with flight measurements for the flapping coefficients. To cover the low forward speed range it was decided to assume an exponential increase, from $\lambda_1 = 0$ to $\lambda_1 = \lambda_T$, given by

$$\lambda_1 = \lambda_T\,(1 - e^{-c\mu}) \qquad \text{.............(5-4)}$$

and to choose c such that $\lambda_1 = 0.9\,\lambda_T$ for $\mu = 0.10$. This gives $c = 23$ and

$$\lambda_1 = \lambda_T\,(1 - e^{-23\mu}) \qquad \text{.............(5-5)}$$

Again this expression for λ_1 is somewhat arbitrary but gives the proper end conditions (i.e. $\lambda_1 = 0$ for $\mu = 0$; $\lambda_1 \doteqdot \lambda_T$ for $\mu > 0.14$).

/5.4. ...

5.4. The Variation of λ_T with μ

For hovering, the value of λ_T may be determined from momentum theory.

The thrust T is given by

$$T = \int_0^R \rho 2\pi r \, dr. \, 2\nu^2$$

Putting $x = r/R$ and substituting for ν from (5-1)

$$T = 4\pi R^2 \rho \nu_T^2 \int_0^1 (x^5 - 4x^4 + 4x^3) \, dx$$

whence $T = \frac{44}{30} \pi R^2 \rho \nu_T^2$

Now the thrust coefficient $C_T = \dfrac{T}{\rho \pi R^2 (\Omega R)^2}$

therefore

$$\lambda_T = \sqrt{\frac{30}{44} C_T} = 0.826 \sqrt{C_T} \quad \text{............(5-6)}$$

The corresponding expression for uniform induced velocity is

$$\lambda_U = 0.707 \sqrt{C_T} \quad \text{............(5-6A)}$$

where λ_U is the non-dimensional form of the uniform induced velocity.

For moderate and high forward speeds Glauert (1) has developed the following formula for the thrust, by treating the rotor disc as a circular wing of span 2R, and having elliptical loading.

$$T = (\pi R^2 \rho V') \, \nu_m \quad \text{............(5-7)}$$

where V' is the resultant velocity at the rotor disc given by

$$V' = \left[(V \sin i + \nu_m)^2 + (V \cos i)^2 \right]^{\frac{1}{2}} \quad \text{.....(5-8)}$$

and ν_m is the mean induced velocity given by

$$\nu_m = \frac{1}{\pi R^2} \int_0^R r\,dr \int_0^{2\pi} \nu \,d\psi \qquad \text{.............(5-9)}$$

Substituting $\nu = \nu_T\,(-x^2 + 2x) + \Omega R\lambda_1 x \cos\psi$ into (5-7) gives

$$\left.\begin{aligned} \nu_m &= 5/6\ \nu_T \\ \text{or}\quad \lambda_m &= \frac{\nu_m}{\Omega R} = 5/6\ \lambda_T \end{aligned}\right\} \qquad \text{.............(5-10)}$$

(5-8) can be written as

$$V' = \Omega R(\lambda^2 + \mu^2)^{\frac{1}{2}} \qquad \text{.............(5-11)}$$

and by substituting (5-10), (5-11) and the expression for the thrust coefficient in (5-7) the expression for λ_T becomes

$$\lambda_T = \frac{6}{5}\ \lambda_m = \frac{3}{5}\ \frac{C_T}{\left[(\mu i + \lambda_m)^2 + \mu^2\right]^{\frac{1}{2}}} \qquad \text{........(5-12)}$$

This leads to a quartic equation for λ_T which cannot be solved in general terms. However for high forward speeds and low angles of incidence, i.e. $\mu^2 >> (\mu i + \lambda_m)^2$, λ_T is given by the simplified expression

$$\lambda_T = 0.6\ \frac{C_T}{\mu} \qquad \text{.............(5-13)}$$

The corresponding expression for uniform induced velocity is

$$\lambda_U = \frac{1}{2}\ \frac{C_T}{\mu} \qquad \text{.............(5-13A)}$$

Due to the difficulty in solving (5-12) for λ_T and also to the doubtful validity of this expression at low forward speeds it was decided to use an empirical expression for λ_T of the form

$$\lambda_T = \frac{A}{B + \mu} \qquad \text{.............(5-14)}$$

and to choose A and B to satisfy the following conditions:

$$\begin{aligned} \lambda_T &= 0.826\ \sqrt{C_T} \quad \text{for } \mu = 0 \\ \lambda_T &= 0.6\ \ C_T/\mu \quad \text{for } \mu = 0.25 \end{aligned} \qquad \text{........(5-15)}$$

A and B are then given by

$$\left.\begin{aligned} A &= \frac{0.6\ C_T}{1 - 2.9\sqrt{C_T}} \\ B &= \frac{0.727\sqrt{C_T}}{1 - 2.9\sqrt{C_T}} \end{aligned}\right\} \qquad \text{.............(5-16)}$$

Similarly for uniform induced velocity

$$\lambda_U = \frac{A'}{B' + \mu} \qquad \text{.............(5-14A)}$$

and

$$\left.\begin{aligned} \lambda_U &= 0.707\sqrt{C_T} \quad \text{for } \mu = 0 \\ \lambda_U &= 0.5\ C_T/\mu \quad \text{for } \mu = 0.25 \end{aligned}\right\} \qquad \text{.............(5-15A)}$$

giving

$$\left.\begin{aligned} A' &= \frac{0.5\ C_T}{1 - 2.83\sqrt{C_T}} \\ B' &= \frac{0.707\sqrt{C_T}}{1 - 2.83\sqrt{C_T}} \end{aligned}\right. \qquad \text{.............(5-16A)}$$

Curves of λ_T and λ_U against μ for a thrust coefficient $C_T = .0055$ are presented in Fig. 5.

5.5. The Derivatives of λ_T, λ_1 and λ_U

5.5.1. The Derivatives of λ_T

$$\lambda_T = \lambda_T(\mu, C_T) \quad \text{where} \quad C_T = C_T(\mu)$$

therefore

$$\frac{d\lambda_T}{d\mu} = \frac{\partial\lambda_T}{\partial\mu} + \frac{\partial\lambda_T}{\partial C_T} \cdot \frac{\partial C_T}{\partial\mu} \qquad \text{.............(5-17)}$$

/where ...

where $\frac{\partial \lambda_T}{\partial \mu}$ and $\frac{\partial \lambda_T}{\partial C_T}$ are obtained by differentiating (5-14), giving

$$\frac{\partial \lambda_T}{\partial \mu} = - \frac{A}{(B+\mu)^2}$$

and

$$\frac{\partial \lambda_T}{\partial C_T} = \frac{1}{(1-2.9\sqrt{C_T})^2 (B+\mu)} \left[0.6(1-1.45\sqrt{C_T}) - \frac{0.364\,\lambda_T}{\sqrt{C_T}} \right]$$

5.5.2. The Derivatives of λ_U

The corresponding expressions for uniform induced velocity are

$$\frac{d\lambda_U}{d\mu} = \frac{\partial \lambda_U}{\partial \mu} + \frac{\partial \lambda_U}{\partial C_T} \cdot \frac{\partial C_T}{\partial \mu} \qquad \text{.............(5-17A)}$$

where

$$\frac{\partial \lambda_U}{\partial \mu} = - \frac{A'}{(B'+\mu)^2} \qquad \text{.............(5-18A)}$$

and

$$\frac{\partial \lambda_U}{\partial C_T} = \frac{1}{(1-2.83\sqrt{C_T})^2 (B'+\mu)} \left[0.50(1-1.42\sqrt{C_T}) - \frac{0.355\,\lambda_U}{\sqrt{C_T}} \right]$$

.............(5-19A)

5.5.3. The Derivatives of λ_1

$$\lambda_1 = (1 - e^{-23\mu})\lambda_T = \lambda_1(\lambda_T, \mu) \quad \text{where} \quad \lambda_T = \lambda_T(\mu)$$

therefore

$$\frac{d\lambda_1}{d\mu} = \frac{\partial \lambda_1}{\partial \lambda_T} \frac{d\lambda_T}{d\mu} + \frac{\partial \lambda_1}{\partial \mu}$$

giving

$$\frac{d\lambda_1}{d\mu} = (1 - e^{-23\mu}) \frac{d\lambda_T}{d\mu} + 23\, e^{-23\mu} \lambda_T \qquad \text{.....(5-20)}$$

Also

$$\frac{\partial \lambda_1}{\partial C_T} = (1 - e^{-23\mu}) \frac{\partial \lambda_T}{\partial C_T} \qquad \text{.............(5-21)}$$

6. The Thrust Coefficient

The thrust T is given by the double integral

$$T = \frac{b}{2\pi} \int_0^1 dx \int_0^{2\pi} \frac{dT}{dx} \, d\psi \quad \ldots\ldots\ldots\ldots(6\text{-}1)$$

The resultant force on a blade element of area $cR\,dx$ is

$$dF = \frac{1}{2} \rho \, a \, c \, \Omega^2 R^3 \left(\frac{U}{\Omega R}\right)^2 \alpha \, dx \quad \ldots\ldots\ldots\ldots(6\text{-}2)$$

where U is the resultant of U_T and U_P and $U \approx U_T$ since $U_T >> U_P$.

Also the resultant force F is very nearly perpendicular to the tip-path plane so that

$$dT \approx dF = \frac{1}{2} \rho \, a \, c \, \Omega^2 R^3 \left(\frac{U_T}{\Omega R}\right)^2 \alpha \, dx \quad \ldots\ldots(6\text{-}3)$$

By substituting (6-3), (4-4) and (4-2) in (6-1) the expression for the thrust coefficient becomes (see Appendix I)

$$C_T = \frac{a\sigma}{2} \left[\frac{A_o}{3} \left(1 + \frac{3}{2}\mu^2\right) - \frac{\mu i}{2} - \frac{5}{12}\lambda_T - \frac{\mu B_1}{2} + \frac{\mu P}{4\Omega} \right] \quad \ldots\ldots\ldots\ldots(6\text{-}4)$$

and for uniform induced velocity

$$C_T = \frac{a\sigma}{2} \left[\frac{A_o}{3} \left(1 + \frac{3}{2}\mu^2\right) - \frac{\mu i}{2} - \frac{\lambda_U}{2} - \frac{\mu B_1}{2} + \frac{\mu P}{4\Omega} \right] \quad (6\text{-}4A)$$

7. The Feathering Coefficients

For equilibrium of the rotor disc the cyclic feathering must be such that the aerodynamic moment produced on a blade balances the dynamic moment about the flapping hinge given by

$$M_D = I_1 \Omega^2 a_o - 2q\Omega I_1 \sin\psi + 2p\Omega I_1 \cos\psi \quad \ldots.(7\text{-}1)$$

where I_1 is the blade moment of inertia about the flapping hinge.

The aerodynamic moment about the flapping hinge is

/given by ...

given by

$$M_A = \int_o^1 x \frac{dF}{dx} \; dx \qquad \ldots\ldots\ldots\ldots(7\text{-}2)$$

Substituting for $\frac{dF}{dx}$ from (6-3) the following expression is obtained for M_A (see Appendix II)

$$M_A = \frac{1}{2} \rho a c \Omega^2 R^4 \Big[\left(-\frac{3}{10} \lambda_T - \frac{\mu i}{3} + \frac{A_o}{4} + \frac{\mu P}{6\Omega} - \frac{\mu B_1}{3} + \frac{\mu^2 A_o}{4} \right)$$
$$+ \sin \psi \left(\frac{P}{4\Omega} - \frac{\mu^2 i}{2} - \frac{5}{12} \mu \lambda_T - \frac{B_1}{4} + \frac{2}{3} \mu A_o - \frac{3}{8} \mu^2 B_1 \right)$$
$$+ \cos \psi \left(-\frac{\mu a_o}{3} - \frac{\lambda_1}{4} + \frac{P}{4\Omega} - \frac{A_1}{4} - \frac{1}{8} \mu^2 A_1 \right)$$
$$+ \text{terms in higher harmonics} \Big] \qquad \ldots\ldots\ldots\ldots(7\text{-}3)$$

Comparing (7-1) with (7-3)

$$a_o = \frac{\gamma}{2} \left[\frac{A_o}{4} (1+\mu^2) - \frac{3}{10} \lambda_T - \frac{\mu i}{3} - \frac{\mu B_1}{3} + \frac{\mu P}{6\Omega} \right] \quad (7\text{-}4)$$

$$A_1 = -\frac{4}{1+\frac{1}{2}\mu^2} \left[\frac{\mu a_o}{3} + \frac{\lambda_1}{4} - \frac{q}{4} + \frac{4P}{\gamma\Omega} \right] \ldots\ldots\ldots(7\text{-}5)$$

$$B_1 = \frac{4}{1+\frac{3}{2}\mu^2} \left[\frac{2}{3} \mu A_o - \frac{\mu^2 i}{2} - \frac{5}{12} \mu \lambda_T + \frac{P}{4\Omega} + \frac{4q}{\gamma\Omega} \right] \ldots\ldots\ldots\ldots(7\text{-}6)$$

where $\gamma = \frac{\rho a c R^4}{I_1}$ is known as Lock's inertia number.

The corresponding expressions for uniform induced velocity are

$$a_o = \frac{\gamma}{2} \left[\frac{A_o}{4} (1+\mu^2) - \frac{\lambda_U}{3} - \frac{\mu i}{3} - \frac{\mu B_1}{3} + \frac{\mu P}{6\Omega} \right] \quad \ldots(7\text{-}4A)$$

$$A_1 = -\frac{4}{1+\frac{1}{2}\mu^2} \left[\frac{\mu a_o}{3} - \frac{q}{4\Omega} + \frac{4P}{\gamma\Omega} \right] \quad \ldots\ldots\ldots\ldots(7\text{-}5A)$$

$$B_1 = \frac{4}{1+\frac{3}{2}\mu^2} \left[\frac{2}{3} \mu A_o - \frac{\mu^2 i}{2} - \frac{\mu}{2} \lambda_U + \frac{P}{4\Omega} + \frac{4q}{\gamma\Omega} \right] \quad (7\text{-}6A)$$

/8. ...

8. The H Force Coefficient

The H force is the drag force in the tip-path plane. From Fig. 2

$$dH = (dD \cos \phi + dL \sin \phi)\sin \psi - (dL \cos \phi - dD \sin \phi)\sin a_o \cos \psi \qquad \text{............(8-1)}$$

Now a_o and ϕ are both small angles so that

$$dH = dD \sin \psi + \phi \, dL \sin \psi - a_o \, dL \cos \psi \qquad \text{..(8-2)}$$

The term $a_o \phi \, dD \cos \psi$ is neglected since a_o and ϕ are both small and dD is small compared with dL.

Now $$dL = \tfrac{1}{2} \rho c \, C_L U^2 \, dr \qquad \text{............(8-3)}$$

where C_L is the local blade lift coefficient $= a(\theta - \phi)$

and $$dD = \tfrac{1}{2} \rho c \delta U^2 \, dr \qquad \text{............(8-4)}$$

where δ = blade section profile drag coefficient, assumed constant.

Substituting (8-3) and (8-4) in (8-2) and putting $U = U_T$ and $\phi = U_P/U_T$

$$dH = \tfrac{1}{2} \rho c \, U_T^2 \left\{ \left[\delta + a \frac{U_P}{U_T} \left(\theta - \frac{U_P}{U_T} \right) \right] \sin\psi - a_o a \left(\theta - \frac{U_P}{U_T} \right) \cos\psi \right\} R \, dx \qquad \text{............(8-5)}$$

Martin (7) neglected terms involving ϕ^2 but retained such terms as $\phi\theta$, $a_o\phi$ and $a_o\theta$. Since ϕ, θ and a_o are all of the same order this simplification was not considered to be justifiable, and the terms involving ϕ^2 have been retained.

The H force is given by the double integral

$$H = \frac{b}{2\pi} \int_0^1 dx \int_0^{2\pi} \frac{dH}{dx} \, d\psi \qquad \text{............(8-6)}$$

Substituting (8-5) in (8-6) the following expression

/for the ...

for the H force coefficient $C_H = \dfrac{H}{\rho\pi R(\Omega R)^2}$ is obtained (see Appendix III)

$$C_H = \frac{a\sigma}{2}\left[\frac{\delta\mu}{2a} + \frac{\mu A_o}{2}\left(\mu i + \frac{2}{3}\lambda_T\right) - \frac{B_1}{4}\left(\mu i + \frac{5}{6}\lambda_T\right) + \frac{A_1 a_o}{6} + \frac{\mu a_o^2}{4} + \frac{a_o\lambda_1}{6}\right.$$

$$\left. - \frac{\mu A_1\lambda_1}{16} + \frac{P}{2\Omega}\left(\mu i + \frac{5}{6}\lambda_T - \frac{A_o}{3} + \frac{3}{8}\mu B_1\right)\right] \quad \text{...(8-7)}$$

The corresponding expression for uniform induced velocity is

$$C_H = \frac{a\sigma}{2a}\left[\frac{\delta\mu}{2} + \frac{\mu A_o}{2}(\mu i + \lambda_U) - \frac{B_1}{4}(\mu i + \lambda_U) + \frac{A_1 a_o}{6} + \frac{\mu a_o^2}{4}\right.$$

$$\left. + \frac{P}{2\Omega}\left(\mu i + \lambda_U - \frac{A_o}{3} - \frac{3}{8}\mu B_1\right)\right] \quad \text{............(8-7A)}$$

9. The Side Force Coefficient C_{YS}

From Fig. 2

$$dY_S = -\left(a_o\, dL \sin\psi + \phi dL \cos\psi + dD \cos\psi\right) \quad \text{............(9-1)}$$

Substituting as in expression for H force

$$dY_S = -\tfrac{1}{2}\rho c U_T^2 \left\{\left[a_o a\left(\theta - \frac{U_P}{U_T}\right)\right]\sin\psi + \left[a\frac{U_P}{U_T}\left(\theta - \frac{U_P}{U_T}\right) + \delta\right]\cos\psi\right\} R dx$$

$$\text{............(9-2)}$$

Performing the double integration as before gives (see Appendix IV)

$$C_{YS} = \frac{a\sigma}{2}\left\{\frac{a_o}{2}\left[3\mu^2 i + 2\mu\lambda_T + B_1\left(\frac{1}{3} + \mu^2\right) - \frac{3}{2}\mu A_o\right] + \frac{\lambda_1}{2}\left(\mu i + \frac{5}{6}\lambda_T\right)\right.$$

$$+ \frac{A_1}{4}\left(\mu i + \frac{5}{6}\lambda_T\right) - \frac{\lambda_1 A_o}{6} - \frac{\mu B_1\lambda_1}{16} - \frac{P}{2\Omega}\left(\frac{a_o}{3} + \frac{\mu A_1}{8}\right)$$

$$\left. - \frac{q}{2\Omega}\left(\mu i + \frac{5}{6}\lambda_T - \frac{A_o}{3} + \frac{\mu B_1}{8}\right)\right\} \quad \text{...(9-3)}$$

The corresponding expression for uniform induced

/velocity is ...

velocity is

$$C_{YS} = \frac{a\sigma}{2}\left\{ \frac{a_o}{2}\left[3\mu^2 i + 3\mu\lambda_U + B_1\left(\frac{1}{3} + \mu^2\right) - \frac{3}{2}\mu A_o\right] + \frac{A_1}{4}(\mu i + \lambda_U)\right.$$

$$\left. - \frac{P}{2\Omega}\left(\frac{a_o}{3} + \frac{\mu A_1}{8}\right) - \frac{q}{2\Omega}\left(\mu i + \lambda_U - \frac{A_o}{3} + \frac{\mu B_1}{8}\right)\right\} \quad \ldots\ldots\ldots(9\text{-}3A)$$

10. The Rotor Stability Derivatives

The rotor derivatives of importance, for the case of zero flapping hinge offset, are.-

(i) The force-angular velocity derivatives

x_q , z_q , y_P

(ii) The force-velocity derivatives

x_u, z_u, y_v, x_w and z_w

Russell (6) and others have shown the basic equations for estimating rotor derivatives to be

$$\Delta X = - T\Delta a_1 - \Delta H \quad \ldots\ldots\ldots\ldots(10\text{-}1)$$

$$\Delta Y = T\Delta b_1 + \Delta Y_S \quad \ldots\ldots\ldots\ldots(10\text{-}2)$$

$$\Delta Z = H\Delta a_1 - \Delta T \quad \ldots\ldots\ldots\ldots(10\text{-}3)$$

These relations follow immediately from Fig. 3.

For the case of controls fixed a change in longitudinal flapping Δa_1 results in a change of incidence of the disc $\Delta i = -\Delta a_1$ i.e. $\frac{\partial i}{\partial a_1} = -1 = \frac{\partial i}{\partial B_1}$.

In estimating the rotor derivatives the change in induced velocity in the disturbed motion was taken into account. This was done by assuming equations (5-17) and (5-18) to apply in the disturbed state. This assumption seems reasonable provided the disturbed motion takes place slowly.

/In the ...

In the expressions for the derivatives, in the following sections, A_1 is replaced by $-b_1$ and B_1 by a_1, from (2-3). Equations with the suffix 'A' refer to the uniform induced velocity case.

11. The Force-Angular Velocity Derivatives x_q, y_P and z_q

The force-angular velocity derivatives follow from equations (10-1) and (10-3)

$$x_q = \Omega \left(- C_T \frac{\partial a_1}{\partial q} - \frac{\partial C_H}{\partial q}\right) \quad \text{.............(11-1)}$$

$$y_P = \Omega \left(C_T \frac{\partial b_1}{\partial p} + \frac{\partial C_{YS}}{\partial p}\right) \quad \text{.............(11-2)}$$

$$z_q = \Omega \left(C_H \frac{\partial a_1}{\partial q} - \frac{\partial C_T}{\partial q}\right) \quad \text{.............(11-3)}$$

C_T and C_H are obtained from equations (6-4) and (8-7) respectively. The expressions for the partial derivatives are

$$\frac{\partial C_T}{\partial P} = \frac{\mu \, a\sigma}{8\,\Omega\left[1 + \frac{5}{24}\, a\sigma \frac{\partial \lambda_T}{\partial C_T}\right]} \quad \text{.............(11-4)}$$

$$\frac{\partial C_T}{\partial P} = \frac{\mu \, a\sigma}{8\,\Omega\left[1 + \frac{1}{4}\, a\sigma \frac{\partial \lambda_U}{\partial C_T}\right]} \quad \text{.............(11-4A)}$$

$$\frac{\partial C_T}{\partial q} = 0 \quad \text{.....(11-5),(11-5A)}$$

$$\frac{\partial a_o}{\partial p} = \frac{\gamma\mu}{12\,\Omega} - \frac{3\gamma}{20} \frac{\partial \lambda_T}{\partial C_T} \cdot \frac{\partial C_T}{\partial p} \quad \text{.............(11-6)}$$

$$\frac{\partial a_o}{\partial p} = \frac{\gamma\mu}{12\,\Omega} - \frac{\gamma}{6} \frac{\partial \lambda_U}{\partial C_T} \cdot \frac{\partial C_T}{\partial p} \quad \text{.............(11-6A)}$$

$$\frac{\partial a_o}{\partial q} = 0 \qquad \text{.....(11-7),(11-7A)}$$

$$\frac{\partial a_1}{\partial p} = \frac{1}{1 - \frac{1}{2}\mu^2}\left[\frac{1}{\Omega} - \frac{5}{3}\mu\frac{\partial\lambda_T}{\partial C_T} \cdot \frac{\partial C_T}{\partial p}\right] \qquad \text{.............(11-8)}$$

$$\frac{\partial a_1}{\partial p} = \frac{1}{1 - \frac{1}{2}\mu^2}\left[\frac{1}{\Omega} - 2\mu\frac{\partial\lambda_U}{\partial C_T} \cdot \frac{\partial C_T}{\partial p}\right] \qquad \text{..........(11-8A)}$$

$$\frac{\partial a_1}{\partial q} = \frac{16}{\gamma\Omega\,(1-\frac{1}{2}\mu^2)} \qquad \text{.....(11-9),(11-9A)}$$

$$\frac{\partial b_1}{\partial p} = \frac{4}{1 + \frac{1}{2}\mu^2}\left[\frac{\mu}{3}\;\frac{\partial a_o}{\partial P} + \frac{1}{4}\frac{\partial\lambda_1}{\partial C_T} \cdot \frac{\partial C_T}{\partial p} + \frac{4}{\gamma\Omega}\right] \qquad \text{............(11-10)}$$

$$\frac{\partial b_1}{\partial p} = \frac{4}{1 + \frac{1}{2}\mu^2}\left[\frac{\mu}{3}\;\frac{\partial a_o}{\partial p} + \frac{4}{\gamma\Omega}\right] \qquad \text{............(11-10A)}$$

$$\frac{\partial b_1}{\partial q} = -\frac{1}{\Omega\,(1 + \frac{1}{2}\mu^2)} \qquad \text{....(11-11),(11-11A)}$$

$\partial\lambda_T/\partial C_T$ etc. are obtained from equations (5-18) and (5-19).

$$\frac{\partial C_H}{\partial q} = \frac{a\sigma}{2}\left[\frac{\partial a_1}{\partial q}\left(\frac{\mu a_1}{4} - \frac{\mu^2 A_o}{2} - \frac{1}{4}\mu i - \frac{5}{24}\lambda_T\right) + \frac{\partial b_1}{\partial q}\left(\frac{\mu\lambda_1}{16} - \frac{a_o}{6}\right)\right] \qquad \text{............(11-12)}$$

$$\frac{\partial C_H}{\partial q} = \frac{a\sigma}{2}\left[\frac{\partial a_1}{\partial q}\left(\frac{\mu a_1}{4} - \frac{\mu^2 A_o}{2} - \frac{1}{4}\mu i - \frac{1}{4}\lambda_U\right) - \frac{a_o}{6}\;\frac{\partial b_1}{\partial q}\right] \qquad \text{..(11-12A)}$$

$$\frac{\partial C_{YS}}{\partial p} = \frac{a\sigma}{2}\left\{\frac{\partial a_o}{\partial p}\left[\frac{3}{2}\mu^2 i + \mu\lambda_T + \frac{a_1}{2}\left(\frac{1}{3} + \mu^2\right) - \frac{3}{4}\mu A_o\right] + \frac{\partial a_1}{\partial p}\left[\frac{a_o}{2}\left(\frac{1}{3} - 2\mu^2\right) - \frac{7}{16}\mu\lambda_1 + \frac{\mu b_1}{4}\right]\right.$$

$$- \frac{\partial b_1}{\partial p}\left(\frac{1}{4}\mu i + \frac{5}{24}\lambda_T\right) + \frac{\partial\lambda_T}{\partial P}\left(\mu a_o + \frac{5}{12}\lambda_1 - \frac{5}{24}b_1\right) +$$

$$\frac{\partial\lambda_1}{\partial p}\left(\frac{\mu i}{2} + \frac{5}{12}\lambda_T - \frac{A_o}{6} + \frac{\mu a_1}{4}\right) - \frac{1}{2\Omega}\left(\frac{a_o}{3} - \frac{\mu b_1}{8}\right) \qquad \text{......(11-13)}$$

$$\frac{\partial C_{YS}}{\partial p} = \frac{a\sigma}{2}\left\{\frac{\partial a_o}{\partial p}\left[\frac{3}{2}\mu^2 i + \frac{3}{2}\mu\lambda_U + \frac{a_1}{2}\left(\frac{1}{3}+\mu^2\right) - \frac{3}{4}\mu a_o\right] + \frac{\partial a_1}{\partial p}\frac{a_o}{2}\left(\frac{1}{3} - 2\mu^2\right)\right.$$
$$\left. - \frac{\partial b_1}{\partial p}\left(\frac{\mu i}{4} + \frac{\lambda_U}{4}\right) + \frac{\partial \lambda_U}{\partial p}\left(\frac{3}{2}\mu a_o - \frac{b_1}{4}\right) - \frac{1}{2\Omega}\left(\frac{a_o}{3} - \frac{\mu b_1}{8}\right)\right\} \qquad \text{.............(11-13A)}$$

12. The force-Velocity Derivatives x_u and z_u

From equations (10-1) and (10-3)

$$x_u = -C_T\frac{\partial a_1}{\partial \mu} - \frac{\partial C_H}{\partial \mu} \qquad \text{.............(12-1)}$$

$$z_u = C_H\frac{\partial a_1}{\partial \mu} - \frac{\partial C_T}{\partial \mu} \qquad \text{.............(12-2)}$$

The expressions for the partial derivatives are

$$\frac{\partial C_T}{\partial \mu} = \frac{a\sigma}{2\left[1 + \frac{5a\sigma}{24}\frac{\partial\lambda_T}{\partial C_T}\right]}\left(\mu A_o - \frac{i}{2} - \frac{5}{12}\frac{\partial\lambda_T}{\partial \mu}\right) \qquad \text{.........(12-3)}$$

$$\frac{\partial C_T}{\partial \mu} = \frac{a\sigma}{2\left[1 + \frac{1}{4}a\sigma\frac{\partial\lambda_U}{\partial C_T}\right]}\left(\mu A_o - \frac{i}{2} - \frac{1}{2}\frac{\partial\lambda_U}{\partial \mu}\right) \qquad \text{.........(12-3A)}$$

$$\frac{\partial a_o}{\partial \mu} = \frac{\gamma}{2}\left[\frac{\mu A_o}{2} - \frac{3}{10}\frac{d\lambda_T}{d\mu} - \frac{i}{3} - \frac{a_1}{3}\right] \qquad \text{.............(12-4)}$$

$$\frac{\partial a_o}{\partial \mu} = \frac{\gamma}{2}\left[\frac{\mu A_o}{2} - \frac{1}{3}\frac{d\lambda_U}{d\mu} - \frac{i}{3} - \frac{a_1}{3}\right] \qquad \text{.............(12-4A)}$$

$$\frac{\partial a_1}{\partial \mu} = \frac{4}{1 - \frac{1}{2}\mu^2}\left[\frac{2}{3}A_o - \mu i - \frac{5}{12}\lambda_T - \frac{5}{12}\mu\frac{d\lambda_T}{d\mu} - \frac{3}{4}\mu a_1\right] \qquad \text{....(12-5)}$$

$$\frac{\partial a_1}{\partial \mu} = \frac{4}{1 - \frac{1}{2}\mu^2}\left[\frac{2}{3}A_o - \mu i - \frac{1}{2}\lambda_U - \frac{1}{2}\mu\frac{d\lambda_U}{d\mu} - \frac{3}{4}\mu a_1\right] \qquad \text{....(12-5A)}$$

$$\frac{\partial b_1}{\partial \mu} = \frac{4}{1 + \frac{1}{2}\mu^2}\left[\frac{a_o}{3} + \frac{\mu}{3}\frac{\partial a_o}{\partial \mu} + \frac{1}{4}\frac{d\lambda_1}{d\mu} + \mu b_1\right] \qquad \text{.............(12-6)}$$

$$\frac{\partial b_1}{\partial \mu} = \frac{4}{1+\frac{1}{2}\mu^2}\left[\frac{a_o}{3} + \frac{\mu}{3}\frac{\partial a_o}{\partial \mu} + \mu b_1\right] \qquad \text{............(12-6A)}$$

$$\frac{\partial C_H}{\partial \mu} = \frac{a\sigma}{2}\left\{\frac{\delta}{2a} + A_o\left(\mu i + \frac{1}{3}\lambda_T\right) - \frac{a_1 i}{4} + \frac{a_o^2}{4} + \frac{b_1\lambda_1}{16} + \right.$$
$$\frac{\partial a_o}{\partial \mu}\left(\frac{\mu a_o}{2} + \frac{\lambda_1}{6} - \frac{b_1}{6}\right) + \frac{\partial a_1}{\partial \mu}\left(\frac{\mu a_1}{4} - \frac{\mu^2 A_o}{2} - \frac{\mu i}{4} - \frac{5}{24}\lambda_T\right)$$
$$\left. + \frac{\partial b_1}{\partial \mu}\left(\frac{\mu\lambda_1}{16} - \frac{a_o}{6}\right) + \frac{d\lambda_T}{d\mu}\left(\frac{\mu A_o}{3} - \frac{5a_1}{24}\right) + \frac{d\lambda_1}{d\mu}\left(\frac{a_o}{6} + \frac{\mu b_1}{16}\right)\right\} \qquad \text{............(12-7)}$$

$$\frac{\partial C_H}{\partial \mu} = \frac{a\sigma}{2}\left\{\frac{\delta}{2a} + A_o\left(\mu i + \frac{\lambda_U}{2}\right) + \frac{a_o^2}{4} + \frac{a_1 i}{4} + \frac{\partial a_o}{\partial \mu}\left(\frac{\mu a_o}{2} - \frac{b_1}{6}\right) + \right.$$
$$\frac{\partial a_1}{\partial \mu}\left(\frac{\mu a_1}{4} - \frac{\mu^2 A_o}{2} - \frac{\mu i}{4} - \frac{\lambda_U}{4}\right)$$
$$\left. - \frac{a_o}{6}\frac{\partial b_1}{\partial \mu} + \frac{d\lambda_U}{d\mu}\left(\frac{\mu A_o}{2} - \frac{a_1}{4}\right)\right\} \qquad \text{............(12-7A)}$$

13. The Force-Velocity Derivatives x_w and z_w

From equations (10-1) and (10-3)

$$x_w = \Omega R\left\{-C_T\frac{\partial a_1}{\partial w} - \frac{\partial C_H}{\partial w}\right\} \qquad \text{............(13-1)}$$

$$z_w = \Omega R\left\{C_H\frac{\partial a_1}{\partial w} - \frac{\partial C_T}{\partial w}\right\} \qquad \text{............(13-2)}$$

The effect of a disturbance velocity w in the positive z direction is to cause a uniform flow w through the rotor disc in the negative z direction. The inflow through the disc then becomes

$$\lambda\,\Omega R = \Omega R\,(\mu i + \lambda_o + \lambda_1 x\cos\psi) - w$$

or non-dimensionally

$$\lambda = \mu i + \lambda_o + \lambda_1 x\cos\psi - \lambda_w \qquad \text{............(13-3)}$$

/where ...

where $\lambda_W = \frac{W}{\Omega R}$ ....(13-4)

It follows, therefore, that

$$\frac{\partial}{\partial w} = \frac{1}{\Omega R}\frac{\partial}{\partial \lambda_W} = -\frac{1}{\Omega R}\frac{\partial}{\partial(\mu i)} \qquad \text{..............(13-5)}$$

Equations (13-1) and (13-2) may then be written as

$$x_W = -C_T\frac{\partial a_1}{\partial \lambda_W} - \frac{\partial C_H}{\partial \lambda_W} \qquad \text{..............(13-6)}$$

$$z_W = C_H\frac{\partial a_1}{\partial \lambda_W} - \frac{\partial C_T}{\partial \lambda_W} \qquad \text{..............(13-7)}$$

where

$$\frac{\partial}{\partial \lambda_W} = -\frac{\partial}{\partial(\mu i)} \qquad \text{..............(13-8)}$$

The relevant partial derivatives are

$$\frac{\partial C_T}{\partial \lambda_W} = \frac{a\sigma}{4}\frac{1}{\left[1 + \frac{5a\sigma}{24}\frac{\partial \lambda_T}{\partial C_T}\right]} \qquad \text{..............(13-9)}$$

$$\frac{\partial C_T}{\partial \lambda_W} = \frac{a\sigma}{4}\frac{1}{\left[1 + \frac{a\sigma}{4}\frac{\partial \lambda_U}{\partial C_T}\right]} \qquad \text{..............(13-9A)}$$

$$\frac{\partial a_o}{\partial \lambda_W} = \frac{\gamma}{2}\left[\frac{1}{3} - \frac{3}{10}\frac{\partial \lambda_T}{\partial C_T}\cdot\frac{\partial C_T}{\partial \lambda_W}\right] \qquad \text{..............(13-10)}$$

$$\frac{\partial a_o}{\partial \lambda_W} = \frac{\gamma}{2}\left[\frac{1}{3} - \frac{1}{3}\frac{\partial \lambda_U}{\partial C_T}\cdot\frac{\partial C_T}{\partial \lambda_W}\right] \qquad \text{..............(13-10A)}$$

$$\frac{\partial a_1}{\partial \lambda_W} = \frac{4}{1-\frac{1}{2}\mu^2}\left[\frac{\mu}{2} - \frac{5}{12}\mu\frac{\partial \lambda_T}{\partial C_T}\cdot\frac{\partial C_T}{\partial \lambda_W}\right] \qquad \text{..............(13-11)}$$

$$\frac{\partial a_1}{\partial \lambda_W} = \frac{4}{1-\frac{1}{2}\mu^2}\left[\frac{\mu}{2} - \frac{1}{2}\mu\frac{\partial \lambda_U}{\partial C_T}\cdot\frac{\partial C_T}{\partial \lambda_W}\right] \qquad \text{..............(13-11A)}$$

$$\frac{\partial b_1}{\partial \lambda_W} = \frac{4}{1+\frac{1}{2}\mu^2}\left[\frac{\mu}{3}\frac{\partial a_o}{\partial \lambda_W} + \frac{1}{4}\frac{\partial \lambda_1}{\partial C_T}\cdot\frac{\partial C_T}{\partial \lambda_W}\right] \qquad \text{.............(13-12)}$$

$$\frac{\partial b_1}{\partial \lambda_W} = \frac{4}{1+\frac{1}{2}\mu^2}\left[\frac{\mu}{3}\frac{\partial a_o}{\partial \lambda_W}\right] \qquad \text{.............(13-12A)}$$

$$\frac{\partial C_H}{\partial \lambda_W} = \frac{a\sigma}{2}\left\{\frac{a_1}{4} - \frac{\mu A_o}{2} + \frac{\partial a_o}{\partial \lambda_W}\left(\frac{\lambda_1}{6} + \frac{\mu a_o}{2} - \frac{b_1}{6}\right) + \right.$$

$$\frac{\partial a_1}{\partial \lambda_W}\left(\frac{\mu a_1}{4} - \frac{\mu^2 A_o}{2} - \frac{\mu i}{4} - \frac{5}{24}\lambda_T\right) + \frac{\partial b_1}{\partial \lambda_W}\left(\frac{\mu \lambda_1}{16} - \frac{a_o}{6}\right)$$

$$\left. + \frac{\partial \lambda_T}{\partial \lambda_W}\left(\frac{\mu A_o}{3} - \frac{5}{24}a_1\right) + \frac{\partial \lambda_1}{\partial \lambda_W}\left(\frac{a_o}{6} + \frac{\mu b_1}{16}\right)\right\} \qquad \text{........(13-13)}$$

$$\frac{\partial C_H}{\partial \lambda_W} = \frac{a\sigma}{2}\left\{\frac{a_1}{4} - \frac{\mu A_o}{2} + \frac{\partial a_o}{\partial \lambda_W}\left(\frac{\mu a_o}{2} - \frac{b_1}{6}\right) + \right.$$

$$\left.\frac{\partial a_1}{\partial \lambda_W}\left(\frac{\mu a_1}{4} - \frac{\mu^2 A_o}{2} - \frac{\mu i}{4} - \frac{1}{4}\lambda_U\right) - \frac{a_o}{6}\frac{\partial b_1}{\partial \lambda_W} + \frac{\partial \lambda_U}{\partial \lambda_W}\left(\frac{\mu A_o}{2} - \frac{a_1}{4}\right)\right\} \qquad \text{.............(13-13A)}$$

14. The Force-Velocity Derivative y_v

A velocity v in the positive y direction causes the H force vector to rotate through an angle $v/V \cos i$ giving a component $-Hv/V \cos i$ in the y direction. In addition there is a change in the lateral tilt of the rotor disc $\Delta b_1 = -a_1 \frac{v}{V \cos i}$ giving rise to a force $-Ta_1 \frac{v}{V \cos i}$ in the y direction.

Therefore

$$\frac{\Delta Y}{v} = -\frac{1}{V \cos i}(H + Ta_1) \qquad \text{.............(14-1)}$$

whence

$$y_v = -\frac{1}{\mu}(C_H + C_T a_1) \qquad \text{.............(14-2)}$$

This expression is not applicable for the hovering condition where $\mu = 0$, but by symmetry in hovering

$$y_v = x_u \qquad \text{.............(14-3)}$$

/15. ...

15. Calculation of Force Coefficients, Flapping Coefficients and Rotor Derivatives for a Typical Case

Values of force coefficients, flapping coefficients and rotor derivatives have been calculated for a typical case using values given in ref. 13. The details of the configuration are given in Appendix V.

Values have been worked out for both uniform and non-uniform induced velocity distribution. The results of the flapping coefficients at moderate forward speeds are compared with results calculated by Martin (7) using the Mangler induced velocity distribution, and with flight measurements given in ref. 13.

The results of the calculations are presented as follows.-

Fig. 7. a_o vs μ ($\mu = 0.14 - 0.24$)
' 8. a_1 vs μ (' ' ')
' 9. b_1 vs μ (' ' ')
' 10. a_o, a_1, b_1 vs μ ($\mu = 0 - 0.14$)
' 11. C_H, C_{YS} vs μ (' ' ')
' 12. x_q vs μ ($\mu = 0 - 0.14$)
' 13. y_p vs μ (' ' ')
' 14. z_q vs μ (' ' ')
' 15. z_u vs μ (' ' ')
' 16. x_u vs μ (' ' ')
' 17. z_w vs μ (' ' ')
' 18. x_w vs μ (' ' ')
' 19. y_v vs μ (' ' ')

16. Discussion

Referring to Figs. 7 - 9 it can be seen that the flapping coefficients, as calculated from the induced velocity distribution adopted, give good agreement with the flight measurements of ref. 13 and Martin's results (7), based on the Mangler induced velocity distribution. In particular the values of the lateral tilt of the disc, b_1, compare favourably, whereas those for the uniform induced velocity distribution considerably underestimate the actual case.

The values of a_1, the longitudinal flapping coefficient are underestimated by all three theoretical induced velocity distributions. This is due to the fact that no account is taken of lateral asymmetry of the flow through the rotor disc. Certainly such asymmetry must exist since the effect of cyclic blade feathering (and/or flapping) is to produce a different lift distribution over the retreating blade than over the advancing blade. However at low forward speeds this difference will be small and its effect on the induced velocity distribution can probably be ignored. At higher forward speeds it could possibly be taken into account by introducing a term λ_2 x sin ψ into the expression for the induced velocity, where λ_2 would be a function of the advance ratio μ. It would probably be difficult to find an expression for $\lambda_2(\mu)$ analytically, but an empirical expression based on experimental results might well be used.

It is doubtful if the expression adopted for the induced velocity actually represents in any detail the true flow distribution through the rotor disc, except at or very near the hovering state. What it does represent is the overall trend of an increase in induced velocity from the front to the rear of the disc, which has been observed. This appears to be sufficient for the estimation of flapping coefficients and hence also of rotor derivatives. The Mangler induced velocity distribution, on the other hand, probably gives a much truer picture of the details of the flow through the rotor. Measurements by Fail and Eyre (11) and by Falabella and Meyer (12) appear to confirm that the prediction of upflow over a region of the forward part of the disc is correct. However the Mangler distribution involves somewhat complicated expressions and it would appear that the much simpler representation of the flow used here is sufficient for the purpose of estimating rotor derivatives.

Fig. 10 shows the values of the flapping coefficients over the low forward speed range. a_1 is the same for both uniform and non-uniform induced velocity. b_1 is much greater

/for the ...

for the non-uniform induced velocity distribution because of the term $\lambda_1/4$ which takes account of the longitudinal asymmetry of flow through the rotor disc. A_o is slightly smaller for the non-uniform induced velocity case indicating that the resultant aerodynamic force acts closer to the blade root than for the uniform induced velocity case.

Fig. 11 shows the variation of the drag force coefficient C_H and the side force coefficient C_{YS} with μ for the two cases. It is interesting to note that C_H is somewhat smaller for the case of non-uniform induced velocity than for the case of uniform induced velocity. This is due to the term $A_1 a_o/6$ being greater in magnitude than the additional terms involving λ_1. C_{YS} is negative for both cases but is considerably greater in magnitude for non-uniform induced velocity. This is due to the larger values of $A_1 - = b_1$ and also to the terms involving λ_1.

The force-angluar velocity derivatives are shown in Figs. 12-14. The derivative x_q is the same for both cases in as much as $C_T \; \partial a_1/\partial q$ is the same and the contribution from $\partial C_H/\partial q$ is small and very nearly the same. y_P is also unaffected by non-uniform induced velocity since $C_T \; \partial b_1/\partial P$ and $\partial C_{YS}/\partial P$ are virtually identical for the two cases. The derivative z_q is slightly different for uniform and non-uniform induced velocity. It is proportional to C_H since $\partial a_1/\partial q$ is the same for both cases and $\partial C_T/\partial q = 0$. This derivative is exceedingly small and would probably be ignored in most stability calculations.

With regard to the force-velocity derivatives it can be seen from Figs. 15 and 17 that z_u and z_w are virtually the same for uniform and non-uniform induced velocity. The expressions for $\partial C_T/\partial \mu$ and $\partial C_T/\partial \lambda_w$ are very nearly the same for the two cases and the $C_H \; \partial a_1/\partial \mu$ and $C_H \; \partial a_1/\partial \lambda_w$ contributions to these 'z' derivatives are negligible.

The derivatives x_u and x_w are also virtually identical for uniform and non-uniform induced velocity. The $C_T \; \partial a_1/\partial \mu$ and $C_T \; \partial a_1/\partial \lambda_w$ terms are dominant in the

/expressions for ...

expressions for these 'x' derivatives so that the small changes in $\partial C_H/\partial\mu$ and $\partial C_H/\partial\lambda_w$ for the two cases are relatively unimportant.

The derivative y_v is also very nearly the same for both uniform and non-uniform induced velocity. The dominant term in the expression for y_v is $C_T a_1$ which is identical for the two cases. The small differences in C_H have little effect.

Summarising it can be said that the only derivative appreciably affected by non-uniform induced velocity is z_q which is very small and relatively unimportant.

It appears that, at low forward speeds, non-uniform induced velocity has no significant effect on rotor derivatives. At higher forward speeds it is possible that its effect might be more significant. Certainly if a lateral asymmetry of flow through the rotor disc were taken into account the values of a_1 and its derivatives would be different for uniform and non-uniform induced velocity. This would affect all derivatives to some extent and particularly x_q, x_u, x_w and y_v. For a highly loaded rotor at high forward speeds it would be expected that C_H would be larger relative to C_T than for the case of the lightly loaded rotor at low forward speeds considered here. This would mean that the $C_H\,\partial a_1/\partial\mu$ and $C_H\,\partial a_1/\partial\lambda_w$ contribution to z_u and z_w would be significant and the effect of non-uniform induced velocity might be important. There is some doubt about this last statement, however, for at high forward speeds and high disc loadings, the main contributions to C_H would probably come from the μA_o and μa_o^2 terms with the result that C_H would be very nearly the same for both uniform and non-uniform induced velocity.

17. Conclusions

1) An important effect of non-uniform induced velocity is to increase considerably the magnitude of the lateral flapping coefficient b_1.

2) The value of C_H is somewhat less for the case of non-uniform than for uniform induced velocity and the value of C_{YS} considerably greater.

3) The effect of non-uniform induced velocity on rotor derivatives at low forward speeds is almost negligible except in the case of z_q which is a very small derivative.

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APPENDIX I

Derivation of Thrust Coefficient for Non-uniform Induced Velocity

$$T = \frac{b}{2\pi} \int_0^1 dx \int_0^{2\pi} \frac{dT}{dx} \cdot d\psi$$

From (4-1) and (6-3)

$$C_T = \frac{T}{\rho\pi R^2 (\Omega R)^2} = \frac{a\sigma}{4\pi} \int_0^1 dx \int_0^{2\pi} (x + \mu \sin\psi)^2 \alpha \, d\psi$$

Substituting for α from (4-4)

Integrand

$$= (x+\mu \sin\psi)^2 \left[A_o - A_1 \cos\psi - B_1 \sin\psi - \frac{a_o \mu\cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x \cos\psi}{x + \mu \sin\psi} \right]$$

$$= -(x+\mu \sin\psi)(a_o\mu \cos\psi + \mu i + \lambda_o + \lambda_1 x \cos\psi - \frac{P}{\Omega} x \sin\psi - \frac{q}{\Omega} x \cos\psi)$$

$$+ (x^2 + 2\mu x \sin\psi + \mu^2 \sin^2\psi)(A_o - A_1 \cos\psi - B_1 \sin\psi)$$

$$= \sin\psi\left(\frac{P}{\Omega} x^2 - \mu^2 i - \mu\lambda_o - B_1 x^2 + 2\mu A_o x\right) + \cos\psi\left(-\mu a_o x - \lambda_1 x^2 + \frac{q}{\Omega} x^2 - A_1 x^2\right)$$

$$+ \sin\psi\cos\psi\left(-\mu^2 a_o - \mu\lambda_1 x + \mu \frac{q}{\Omega} x - 2\mu A_1 x\right) + \sin^2\psi\left(\mu \frac{P}{\Omega} x - 2\mu B_1 x + \mu^2 A_o\right)$$

$$- \mu^2 B_1 \sin^3\psi - \mu^2 A_1 \sin^2\psi \cos\psi - \lambda_o x - \mu i x + A_o x^2$$

$$\therefore \quad C_T = \frac{a\sigma}{2} \int_0^1 \left\{ -\lambda_o x - \mu i x + A_o x^2 + \frac{\mu P x}{2\Omega} - \mu B_1 x + \frac{\mu^2 A_o}{2} \right\} dx$$

$$C_T = \frac{a\sigma}{2} \left[\frac{A_o}{3}\left(1 + \frac{3}{2}\mu^2\right) - \frac{\mu i}{2} - \frac{5}{12}\lambda_T - \frac{\mu B_1}{2} + \frac{\mu P}{4\Omega} \right]$$

/Appendix II ...

APPENDIX II

Derivation of Expression for M_A

$$M_A = \int_0^1 x \frac{dF}{dx} \, dx$$

From (4-1) and (6-3)

$$M_A = \frac{1}{2} \rho a c R^4 \int_0^1 x(x+\mu \sin \psi)^2 \alpha \, dx$$

Substituting for α from (4-4)

Integrand

$$= - \lambda_o x^2 - \mu i x^2 + A_o x^2 + \sin\psi \left(\frac{P}{\Omega} x^3 - \mu^2 i x - \mu\lambda_o x - B_1 x^3 + 2\mu A_o x^2\right)$$

$$+ \cos\psi \left(-\mu a_o x^2 - \lambda_1 x^3 + \frac{q}{\Omega} x^3 - A_1 x^3\right) + \sin\psi \cos\psi \left(-\mu^2 a_o x - \mu\lambda_1 x^2 + \mu \frac{q}{\Omega} x^2 - 2\mu A_1 x^2\right)$$

$$+ \sin^2\psi \left(\mu \frac{P}{\Omega} x^2 - 2\mu B_1 x^2 + \mu^2 A_o x\right) - \mu^2 B_1 x \sin^3\psi - \mu^2 A_1 x \sin^2\psi \cos\psi$$

Now

$$\sin^2\psi = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\psi$$

$$\sin^3\psi = \frac{3}{4}\sin\psi - \frac{1}{4}\sin 3\psi$$

$$\sin^2\psi \, . \cos\psi = \frac{1}{4}\cos\psi - \frac{1}{4}\cos 3\psi$$

so integrand

$$= - \lambda_o x^2 - \mu i x^2 + A_o x^3 + \frac{\mu P}{2\Omega} x^2 - \mu B_1 x^2 + \mu^2 \frac{A_o}{2} x$$

$$+ \sin\psi \left(\frac{P}{\Omega} x^3 - \mu^2 i x - \mu\lambda_o x - B_1 x^3 + 2\mu A_o x^2 - \frac{3}{4}\mu^2 B_1 x\right)$$

$$+ \cos\psi \left(-\mu a_o x^2 - \lambda_1 x^3 + \frac{q}{\Omega} x^3 - A_1 x^3 - \frac{1}{4}\mu^2 A_1 x\right)$$

+ terms in higher harmonics

Therefore

$$M_A = \tfrac{1}{2}\rho a c \Omega^2 R^4 \left[\left(-\frac{3}{10}\lambda_T - \frac{\mu i}{3} + \frac{A_o}{4} + \frac{\mu P}{6\Omega} - \frac{\mu B_1}{3} + \frac{\mu^2 A_o}{4}\right)\right.$$

$$\left. + \sin\psi \left(\frac{P}{4\Omega} - \frac{\mu^2 i}{2} - \frac{5}{12}\mu\lambda_T - \frac{B_1}{4} + \frac{2}{3}\mu A_o - \frac{3}{8}\mu^2 B_1\right)\right.$$

$$+ \cos\psi \left(-\frac{\mu a_o}{3} - \frac{\lambda_1}{4} + \frac{P}{4\Omega} - \frac{A_1}{4} - \frac{1}{8}\mu^2 A_1 \right)$$

$$\left. + \text{ terms in higher harmonics.} \right]$$

APPENDIX III

Derivation of C_H for Non-Uniform Induced Velocity

$$H = \frac{b}{2\pi} \int_o^1 dx \int_o^{2\pi} \frac{dH}{dx} \, d\psi$$

From (4-1), (4-4) and (8-5)

$$C_H = \frac{\sigma a}{4\pi} \int_o^1 dx \int_o^{2\pi} (x + \mu \sin\psi)^2$$

$$\left\{ \left[\frac{\delta}{a} + \left(\frac{\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x \cos\psi - \frac{P}{\Omega} x \sin\psi - \frac{q}{\Omega} x \cos\psi}{x + \mu \sin\psi} \right) \right. \right.$$

$$\left. \times \left(A_o - A_1 \cos\psi - B_1 \sin\psi - \frac{\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x \cos\psi}{x + \mu \sin\psi} \right) \right] \sin\psi$$

$$- a_o \left(A_o - A_1 \cos\psi - B_1 \sin\psi - \frac{\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x\cos\psi}{x + \mu \sin\psi} \right) \cos\psi \, d\psi$$

$$\text{Integrand} = \left[-(\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x\cos\psi)^2 \right] \sin\psi$$

$$+ (x + \mu\sin\psi) \left\{ (\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x\cos\psi) \left[(A_o - A_1\cos\psi - B_1 \sin\psi) \times \sin\psi \right] \right.$$

$$\left. + a_o (\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x\cos\psi) \cos\psi \right\}$$

$$+ (x^2 + 2\mu x \sin\psi + \mu^2 \sin^2\psi) \left[\frac{\delta}{a} \sin\psi - a_o (A_o - A_1 \cos\psi - B_1 \sin\psi) \cos\psi \right]$$

$$= \left[-(\mu^2 a_o \cos^2\psi + \mu^2 i^2 + \lambda_o^2 + \lambda_1^2 x^2 \cos^2\psi + \frac{P^2}{\Omega^2} x^2 \sin^2\psi + \frac{q^2}{\Omega^2} x^2 \cos^2\psi + 2\mu^2 a_o i \cos\psi \right.$$

$$+ 2\mu a_o \lambda_o \cos\psi$$

$$+2\lambda_1\mu a_o x\cos^2\psi - 2\mu a_o \frac{P}{\Omega} x\sin\psi\cos\psi - 2\mu a_o \frac{q}{\Omega} x\cos^2\psi + 2\mu i\lambda_o + 2\mu i\lambda_1 x\cos\psi - 2\mu i \frac{P}{\Omega} x \sin\psi$$

$$-2\mu i \frac{q}{\Omega} x\cos\psi + 2\lambda_o\lambda_1 x\cos\psi - 2\lambda_o \frac{P}{\Omega} x\sin\psi - 2\lambda_o \frac{q}{\Omega} x\cos\psi - 2\lambda_1 \frac{P}{\Omega} x^2\sin\psi\cos\psi - 2\lambda_1 \frac{q}{\Omega} x^2\cos^2\psi + \frac{Pq}{\Omega^2} x^2 \sin\psi\cos\psi \Big] \sin\psi$$

$$+(x+\mu\sin\psi)\Big\{(\mu i+\lambda_o)(A_o\sin\psi - A_1\sin\psi\cos\psi - B_1\sin^2\psi) + (\lambda_1 x + \mu a_o - \frac{q}{\Omega} x)(A_o\sin\psi\cos\psi - A_1\sin\psi\cos^2\psi + B_1\sin^2\psi\cos\psi)$$

$$-\frac{P}{\Omega}x(A_o\sin^2\psi - A_1\sin^2\psi\cos - B_1\sin^3\psi) + \mu^2 a_o\cos^2\psi + \mu i a_o\cos\psi + a_o\lambda_o\cos\psi + a_o\lambda_1 x\cos^2\psi - \frac{a_o P}{\Omega} x\sin\psi\cos\psi - \frac{a_o q}{\Omega} x\cos^2\psi$$

$$+(x^2 + 2\mu x\sin\psi + \mu^2\sin^2\psi)(\frac{\delta}{a}\sin\psi - a_o A_o\cos\psi + a_o A_1\cos^2\psi + a_o B_1\sin\psi\cos\psi)$$

$$= \sin\psi(-\mu^2 i^2 - \lambda_o^2 - 2\mu i\lambda_o + \mu i A_o x + \lambda_o A_o x + \frac{\delta}{a} x^2) + \cos\psi(a_o\mu i x + a_o\lambda_o x - a_o A_o x^2)$$

$$+ \sin\psi\cos\psi(-2\mu^2 a_o i - 2\mu a_o\lambda_o - 2\mu i\lambda_1 x + 2\mu i \frac{q}{\Omega} x - 2\lambda_o\lambda_1 x + 2\lambda_o \frac{q}{\Omega} x - \mu i A_1 x - \lambda_o A_1 x$$

$$+ A_o\lambda_1 x^2 + A_o \frac{q}{\Omega} x^2 + \mu a_o A_o x + a_o \frac{P}{\Omega} x^2 + \mu^2 i a_o + \mu\lambda_o a_o + a_o B_1 x^2 - 2\mu a_o A_o x)$$

$$+\sin^2\psi(2\mu i \frac{P}{\Omega} x + 2\lambda_o \frac{P}{\Omega} x - B_1\mu i x - B_1\lambda_o x - \frac{P}{\Omega} A_o x^2 + A_o\mu^2 i + A_o\mu\lambda_o + 2\frac{\delta}{a}\mu x)$$

$$+\cos^2\psi(\mu a_o^2 x + a_o\lambda_1 x^2 - a_o \frac{q}{\Omega} x^2 + a_o A_1 x^2) + \sin^2\psi\cos\psi(2\mu a_o \frac{P}{\Omega} x + 2\lambda_1 \frac{P}{\Omega} x - \frac{Pq}{\Omega^2} x^2$$

$$-\lambda_1 B_1 x^2 + B_1 \frac{q}{\Omega} x^2 - \mu a_o B_1 x + A_1 \frac{P}{\Omega} x^2 - \mu^2 A_1 i - \mu A_1\lambda_o + \mu A_o\lambda_1 x - \mu A_o \frac{q}{\Omega} x + \mu^2 a_o A_o$$

$$-\mu a_o \frac{P}{\Omega} x + 2\mu a_o B_1 x - \mu^2 a_o A_o) + \sin\psi\cos^2\psi(-\mu^2 a_o^2 - \lambda_1^2 x^2 - \frac{q^2}{\Omega^2} x^2 - 2\lambda_1\mu a_o x + 2\mu a_o \frac{q}{\Omega} x$$

$$+2\lambda_1 \frac{q}{\Omega} x^2 - \lambda_1 A_1 x^2 + A_1 \frac{q}{\Omega} x^2 - \mu a_o A_1 x + \mu^2 a_o^2 + \mu a_c\lambda_1 x - \mu a_o \frac{q}{\Omega} x + 2\mu a_o A_1 x)$$

$$+\sin^2\psi\cos^2\psi(-\mu\lambda_1 A_1 x + \mu A_1 \frac{q}{\Omega} x - \mu^2 a_o A_1 + \mu^2 a_o A_1) + \sin^3\psi\cos\psi\,(-\mu B_1\lambda_1 x + \mu B_1 \frac{q}{\Omega} x$$

$$+\mu A_1 \frac{P}{\Omega} x - \mu^2 a_o B_1 + \mu^2 a_o B_1) + \sin^3\psi(-\frac{P^2}{\Omega^2} x^2 + B_1 \frac{P}{\Omega} x^2 - \mu^2 B_1 i - \mu\lambda_o B_1 - \mu A_o \frac{P}{\Omega} x + \mu^2 \frac{\delta}{a})$$

$$+\mu B_1 \frac{P}{\Omega} x \sin^4\psi$$

Now

$$\int_0^{2\pi} (\cos\theta, \sin\theta, \sin\theta\cos\theta, \sin^2\theta\cos\theta, \cos^2\theta\sin\theta, \sin^3\theta, \sin^3\theta\cos\theta)\, d\theta = 0$$

$$\int_0^{2\pi} (\sin^2\theta, \cos^2\theta) d\theta = \pi, \quad \int_0^{2\pi} \sin^2\theta . \cos^2\theta \, d\theta = \frac{\pi}{4}, \quad \int_0^{2\pi} \sin^4\theta \, d\theta = \frac{3\pi}{4}$$

Therefore

$$C_H = \frac{a\sigma}{2} \int_0^1 \left\{ \mu i \frac{P}{\Omega} x + \lambda_o \frac{P}{\Omega} x - \frac{B_1}{2} \mu i x - \frac{B_1}{2} \lambda_o x - \frac{P}{2\Omega} A_o x^2 + \frac{A_o \mu^2 i}{2} + \frac{\mu A_o \lambda_o}{2} \right.$$
$$+ \frac{\delta}{a} \mu x + \frac{\mu a_o^2 x}{2} + \frac{a_o \lambda_1 x^2}{2} - \frac{a_o q x^2}{2\,\Omega} + \frac{a_o}{2} A_1 x^2 - \frac{\mu\lambda_1 A_1 x}{8} + \mu A_1 \frac{q}{8\Omega} x$$
$$\left. + \frac{3}{8} \mu B_1 \frac{P}{\Omega} x \right\} dx$$

Finally

$$C_H = \frac{a\sigma}{2} \left[\frac{\delta\mu}{2a} + \frac{\mu A_o}{2} (\mu i + \frac{2}{3} \lambda_T) - \frac{B_1}{4} (\mu i + \frac{5}{6} \lambda_T) + \frac{A_1 a_o}{6} + \frac{\mu a_o^2}{4} \right.$$
$$\left. + \frac{a_o \lambda_1}{6} - \frac{\mu A_1 \lambda_1}{16} + \frac{P}{2\Omega} (\mu i + \frac{5}{6} \lambda_T - \frac{A_o}{3} + \frac{3}{8} \mu B_1) \right].$$

/Appendix IV ...

APPENDIX IV

Derivation of C_{YS} for Non-Uniform Induced Velocity

$$Y_S = \frac{b}{2\pi} \int_0^1 dx \int_0^{2\pi} \frac{dY_S}{dx} \cdot d\psi$$

From (4-1), (4-4) and (9-2)

$$C_{YS} = -\frac{a\sigma}{4\pi} \int_0^1 dx \int_0^{2\pi} (x + \mu \sin\psi)^2 \Bigg\{ a_o \Bigg(A_o - A_1 \cos\psi - B_1 \sin\psi - \frac{\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x\cos\psi}{x + \mu \sin\psi} \Bigg) \sin\psi$$

$$+ \Bigg[\frac{\delta}{a} + \Bigg(\frac{\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - q/\Omega\, x\cos\psi}{x + \mu \sin\psi} \Bigg)$$

$$\Bigg(A_o - A_1 \cos\psi - B_1 \sin\psi - \frac{\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x\cos\psi}{x + \mu \sin\psi} \Bigg) \Bigg] \cos\psi \Bigg\} d\psi$$

$$\text{Integrand} = -(\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x \cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x \cos\psi)^2 \cos\psi$$

$$+ (x + \mu\sin\psi) \Big[-a_o (\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x\sin\psi - \frac{q}{\Omega} x \cos\quad) \sin\psi$$

$$+ (\mu a_o \cos\psi + \mu i + \lambda_o + \lambda_1 x\cos\psi - \frac{P}{\Omega} x \sin\psi - \frac{q}{\Omega} x\cos\psi)(A_o - A_1 \cos\psi - B_1 \sin\psi)\cos\psi \Big]$$

$$+ (x^2 + 2\mu x\sin\psi + \mu^2 \sin^2\psi) \Big[a_o (A_o - A_1 \cos\psi - B_1 \sin\psi) \sin\psi + \frac{\delta}{a} \cos\psi \Big]$$

$$= \Big[-(\mu^2 a_o^2 \cos^2\psi + \mu^2 i^2 + \lambda_o^2 + \lambda_1^2 x^2 \cos^2\psi + \frac{P^2}{\Omega} x^2 \sin^2\psi + \frac{q^2}{\Omega^2} x^2 \cos^2\psi + 2\mu^2 a_o i\cos\psi + 2\mu a_o \lambda_o \cos\psi$$

$$+ 2\lambda_1 \mu a_o x\cos^2\psi - 2\mu a_o \frac{P}{\Omega} x\sin\psi \cos\psi - 2\mu a_o \frac{q}{\Omega} x\cos^2\psi + 2\mu i\lambda_o + 2\mu i\lambda_1 x\cos\psi - 2\mu i \frac{P}{\Omega} x\sin\psi$$

$$- 2\mu i \frac{q}{\Omega} x\cos\psi + 2\lambda_o \lambda_1 x\cos\psi - 2\lambda_o \frac{P}{\Omega} x\sin\psi - 2\lambda_o \frac{q}{\Omega} x\cos\psi - 2\lambda_1 \frac{P}{\Omega} x^2 \sin\psi \cos\psi$$

$$- 2\lambda_1 \frac{q}{\Omega} x^2 \cos^2\psi + \frac{Pq}{\Omega^2} \sin\psi \cos\psi \Big] \cos\psi$$

$$+(x+\mu\sin\psi)\Big\{ -a_o\mu\sin\psi\cos\psi - a_o\mu i\sin\psi - a_o\lambda_o\sin\psi - a_o\lambda_1 x\cos\psi\sin\psi$$

$$+a_o\frac{P}{\Omega}x\sin^2\psi + a_o\frac{q}{\Omega}x\sin\psi\cos\psi$$

$$+(\mu i+\lambda_o)(A_o\cos\psi - A_1\cos^2\psi - B_1\sin\psi\cos\psi) + (\mu a_o+\lambda_1 x - \frac{q}{\Omega}x)(A_o\cos^2\psi - A_1\sin\psi\cos^2\psi$$

$$-B_1\sin^2\psi\cos\ )$$

$$-\frac{P}{\Omega}x(A_o\sin\psi\cos\psi - A_1\sin\psi\cos^2\psi - B_1\sin^2\psi\cos\psi) + (x^2+2\mu x\sin\psi+\mu^2\sin^2\psi)$$

$$(a_oA_o\sin\psi - a_oA_1\sin\psi\cos\psi - a_oB_1\sin^2\psi + \frac{\delta}{a}\cos\psi)$$

$$= \sin\psi(-a_o\mu ix - a_o\lambda_o x + x^2a_oA_o) + \cos\psi(-\mu^2i^2 - \lambda_o^2 - 2\mu i\lambda_o + \mu iA_ox + \lambda_oA_ox + x^2\frac{\delta}{a})$$

$$+\sin\psi\cos\psi\,(2\mu i\frac{P}{\Omega}x$$

$$+2\lambda_o\frac{P}{\Omega}x - \mu a_ox - a_o\lambda_1x^2 + a_o\frac{q}{\Omega}x^2 - \mu iB_1x - \lambda_oB_1x - A_o\frac{P}{\Omega}x^2 + \mu^2iA_o + \mu\lambda_oA_o - a_oA_1x^2 + 2\mu\frac{\delta}{a}x^2)$$

$$+\sin^2\psi\,(a_o\frac{P}{\Omega}x^2 - \mu^2a_oi - \mu a_o\lambda_o - a_oB_1x^2 + 2\mu a_oA_ox) + \cos^2\psi\,(-2\mu^2a_oi - 2\mu a_o\lambda_o - 2\mu i\lambda_1x$$

$$+2\mu i\frac{q}{\Omega}x - 2\lambda_o\lambda_1x + 2\lambda_o\frac{q}{\Omega}x - \mu iA_1x - \lambda_oA_1x + \mu a_oA_ox + \lambda_1a_ox^2 - A_o\frac{q}{\Omega}x^2)$$

$$+\sin^2\psi\cos\psi\,(-\frac{P^2}{\Omega^2}x^2 + B_1\frac{P}{\Omega}x^2 - 2\mu a_oA_1x + \mu^2\frac{\delta}{a} - \mu^2a_o - \mu a_o\lambda\,x + \mu a_o\frac{q}{\Omega}x - \mu^2iB_1$$

$$-\mu\lambda_oB_1 - \mu A_o\frac{P}{\Omega}x\ )$$

$$+\cos^2\psi\sin\psi\,(2\mu a_o\frac{P}{\Omega}x + 2\lambda_1\frac{P}{\Omega}x^2 - \frac{Pq}{\Omega^2}x^2 - \mu a_oB_1x - \lambda_1B_1x^2 + B_1\frac{q}{\Omega}x^2 + A_1\frac{P}{\Omega}x^2$$

$$-\mu^2iA_1 - \mu\lambda_oA_1$$

$$+\mu^2a_oA_o + \mu\lambda_1A_ox - \mu A_o\frac{q}{\Omega}x) + \sin^3\psi\,(\mu a_o\frac{P}{\Omega}x - 2\mu a_oB_1x + \mu^2a_oA_o)$$

$$+\cos^3\psi\,(-\mu^2a_o - \lambda_1^2x^2 - \frac{q^2}{\Omega^2}x^2 - 2\lambda_1\mu a_ox + 2\mu a_o\frac{q}{\Omega}x + 2\lambda_1\frac{q}{\Omega}x^2 - \mu A_1a_ox - \lambda_1A_1x^2$$

$$+A_1\frac{q}{\Omega}x^2$$

$$+\sin^2\psi\cos^2\psi\left(-\mu^2 a_o B_1-\mu\lambda_1 B_1 x+\mu B_1\frac{q}{\Omega}x+\mu A_1\frac{P}{\Omega}x\right)+\sin\psi\cos\psi\left(\mu B_1\frac{P}{\Omega}x-\mu^2 a_o A_1\right)$$

$$+\sin\psi\cos^3\psi\left(-\mu^2 a_o A_1-\mu\lambda_1 A_1 x+\mu A_1\frac{q}{\Omega}x\right)-\mu^2 a_o B_1\sin^4\psi$$

Now $\int_o^{2\pi}(\cos\theta,\sin\theta,\sin\theta\cos\theta,\sin^2\theta\cos\theta,\cos^3\theta\sin\theta,\sin^3\theta,\cos^3\theta,$
$\sin^3\theta\cos\theta,\cos^3\theta\sin\theta)\,d\theta = 0$

$$\int_o^{2\pi}(\sin^2\theta,\cos^2\theta)d\theta=\pi,\quad \int_o^{2\pi}\sin^2\theta\cos^2\theta\,d\theta=\frac{\pi}{4},\quad \int_o^{2\pi}\sin^4\theta\,d\theta=\frac{3\pi}{4}$$

Therefore

$$C_{YS}=-\frac{a\sigma}{4}\int_o^1\left\{\frac{a_o P}{\Omega}x^2-\mu^2 a_o i-\mu a_o\lambda_o-a_o B_1 x+2\mu a_o A_o x-2\mu^2 a_o i-2\mu a_o\lambda_o\right.$$

$$-2\mu i\lambda_1 x+2\mu i\frac{q}{\Omega}x-2\lambda_o\lambda_1 x+2\lambda_o\frac{q}{\Omega}x-\mu i A_1 x-\lambda_o A_1 x+\mu a_o A_o x+\lambda_1 A_o x^2$$

$$\left.-A_o\frac{q}{\Omega}x^2-\mu^2 a_o\frac{B_1}{4}-\mu\lambda_1 B_1\frac{x}{4}+\mu\frac{B_1}{4}\frac{q}{\Omega}x+\frac{\mu A_1 Px}{4\ \Omega}-\frac{3}{4}\mu^2 a_o B_1\right\}dx$$

Finally

$$C_{YS}=\frac{a\sigma}{2}\left\{\frac{a_o}{2}\left[3\mu^2 i+2\mu\lambda_T+B_1\left(\frac{1}{3}+\mu^2\right)-\frac{3}{2}\mu A_o\right]+\frac{\lambda_1}{2}\left(\mu i+\frac{5}{6}\lambda_T\right)\right.$$

$$+\frac{A_1}{4}\left(\mu i+\frac{5}{6}\lambda_T\right)-\frac{\lambda_1 A_o}{6}+\frac{\mu B_1\lambda_1}{16}-\frac{P}{2\Omega}\left(\frac{a_o}{3}+\frac{\mu A_1}{8}\right)-\frac{q}{2\Omega}$$

$$\left.\left(\mu i+\frac{5}{6}\lambda_T-\frac{A_o}{3}+\frac{\mu B_1}{8}\right)\right\}.$$

/Appendix V ...

APPENDIX V

Data Used in Computing Force Coefficients, Flapping Angles and Rotor Derivatives

The values of the rotor derivatives etc. were calculated for a typical single rotor helicopter, the Sikorsky HNS-1 (Army YR-AB). The required data was obtained from Ref. 13. The values of A_o and i used were the actual measured flight values given in this reference. These values were extrapolated over the low forward speed range (Fig. 6). This procedure is considered satisfactory since it is only necessary to have these values of the right order for purposes of showing the effect of non-uniform induced velocity.

Other relevant data is listed below.

$C_T = 0.0055$

$\Omega = 225$ R.P.M.

$\gamma = 12.1$

$\sigma = 0.06 = \frac{bc}{\pi R}$ at 0.75 R

Blade aerofoil section N.A.C.A. 0012 $\begin{cases} a = 5.73/\text{rad.} \\ \delta = 0.006 \end{cases}$

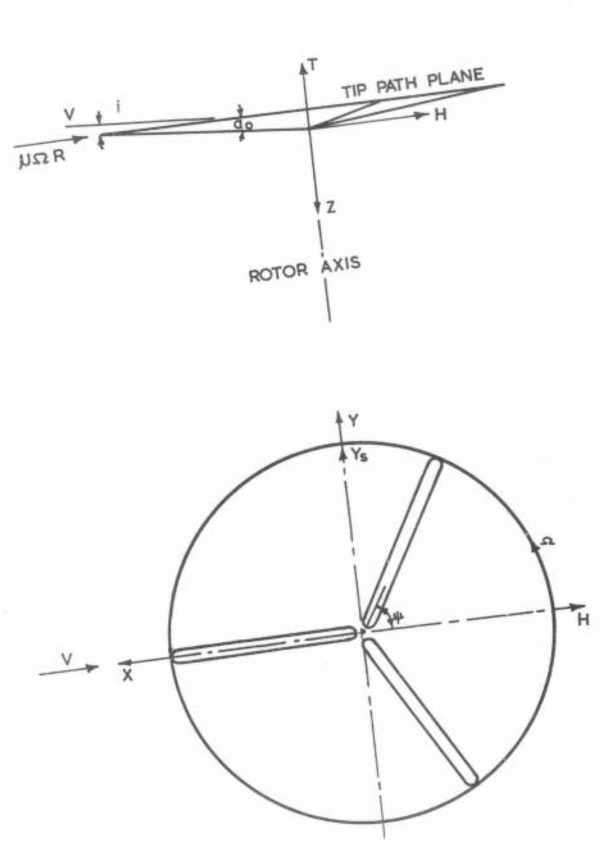

FIG. I. ROTOR DISC AXES

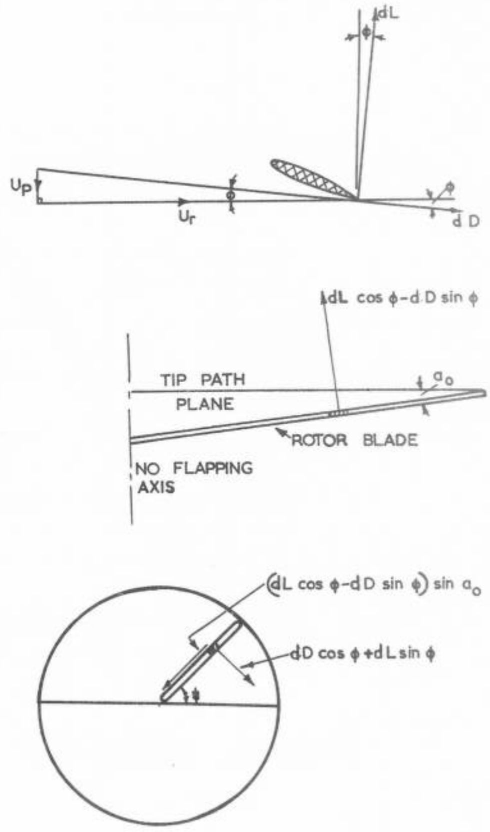

FIG. 2. ROTOR DISC FORCES

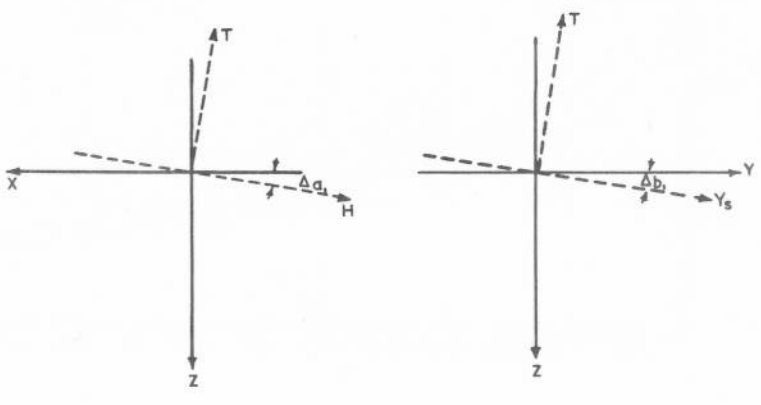

FIG 3 EFFECT OF CHANGES IN LONGITUDINAL AND LATERAL FLAPPING

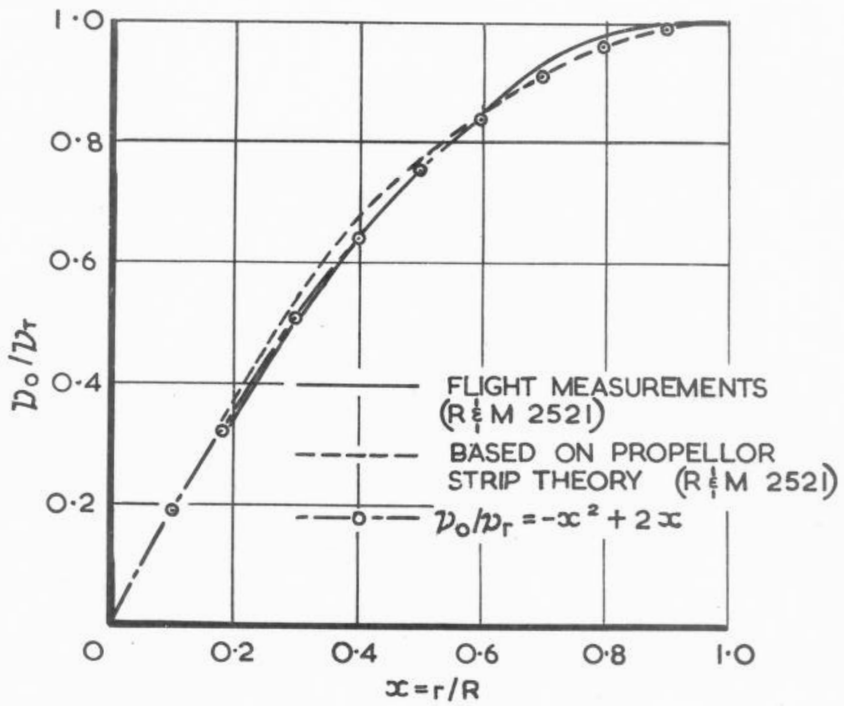

FIG. 4.. INDUCED VELOCITY IN HOVERING

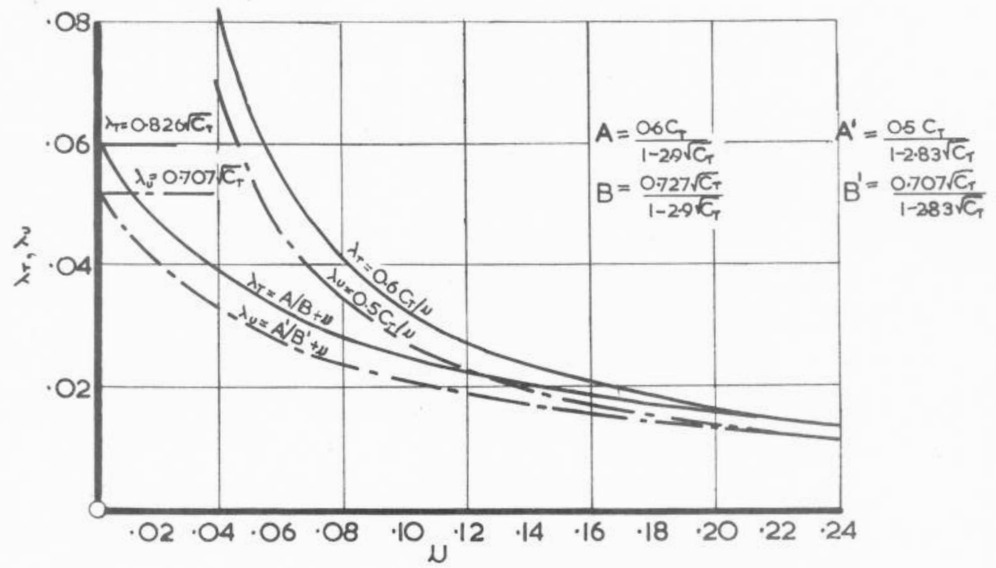

FIG. 5. λ_T, λ_μ vs μ

$C_T = ·0055$

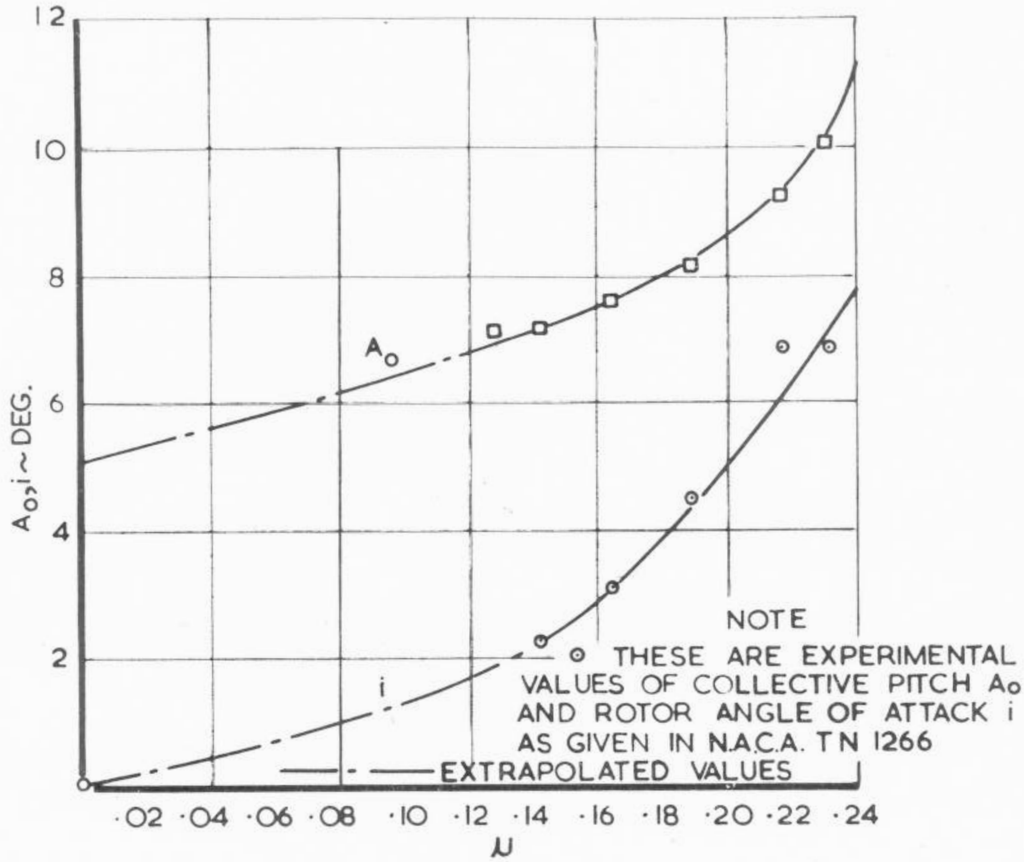

FIG 6 A_O,i VS μ
γ = 12·1
Ω = 225 R.P.M.
C_T=0055

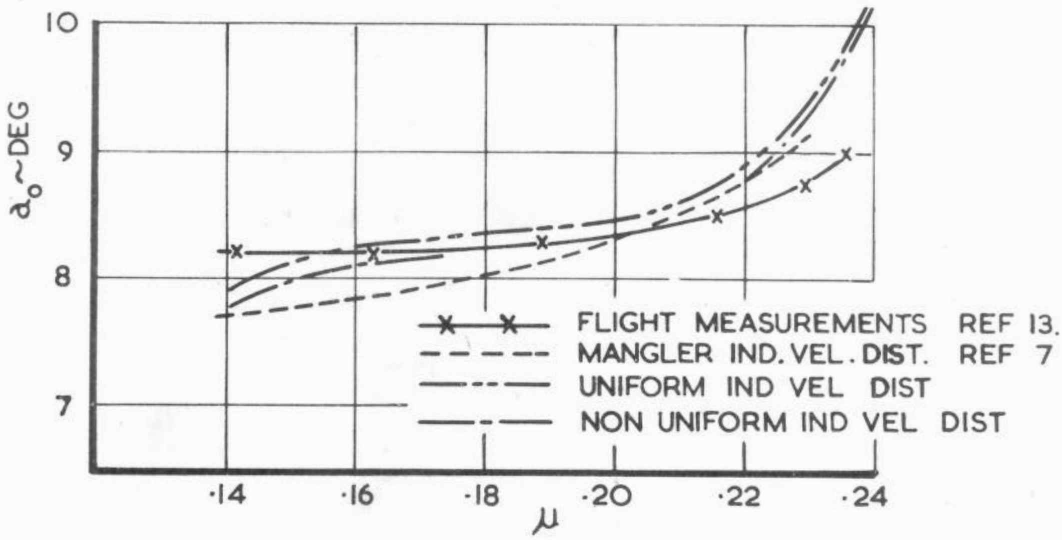

FIG 7 a_o VS μ

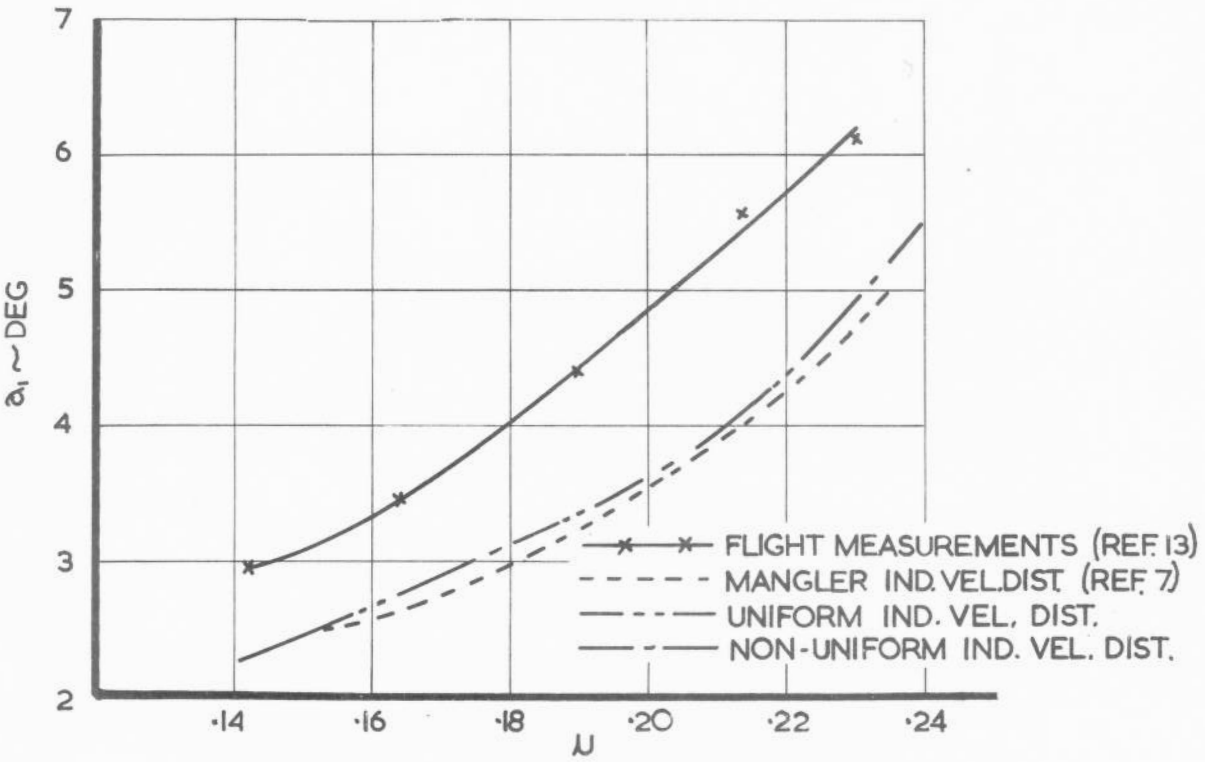

FIG 8 a_1 vs μ

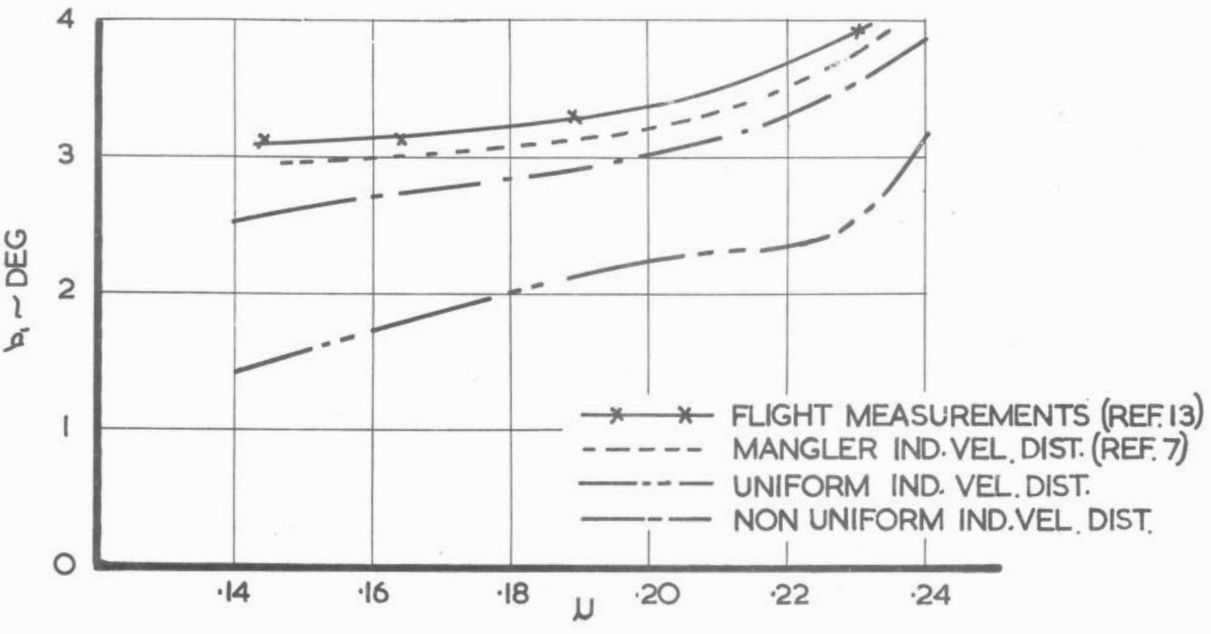

FIG 9 b_1 vs μ

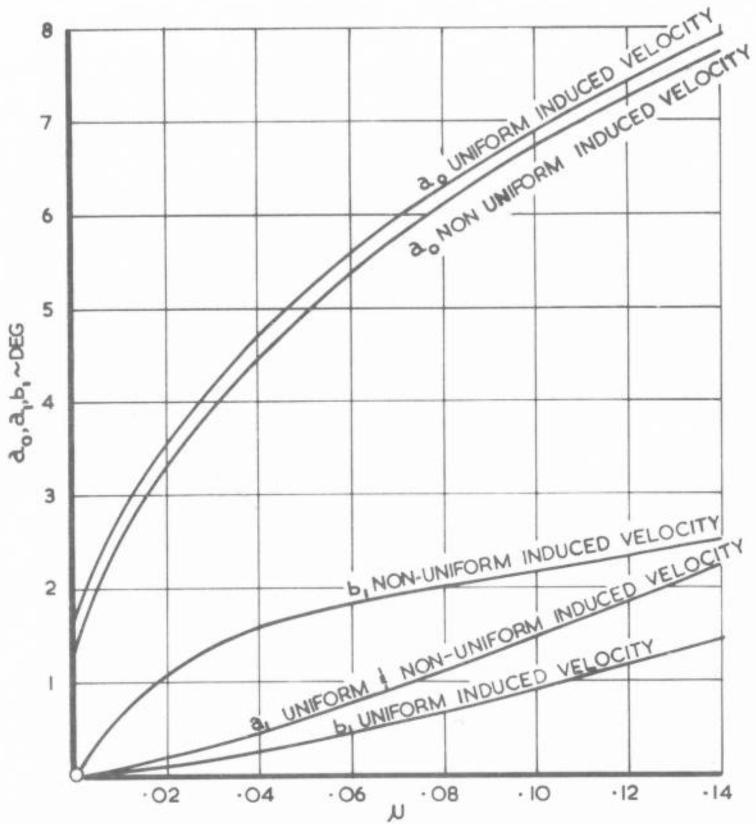

FIG. 10. FLAPPING COEFFICIENTS

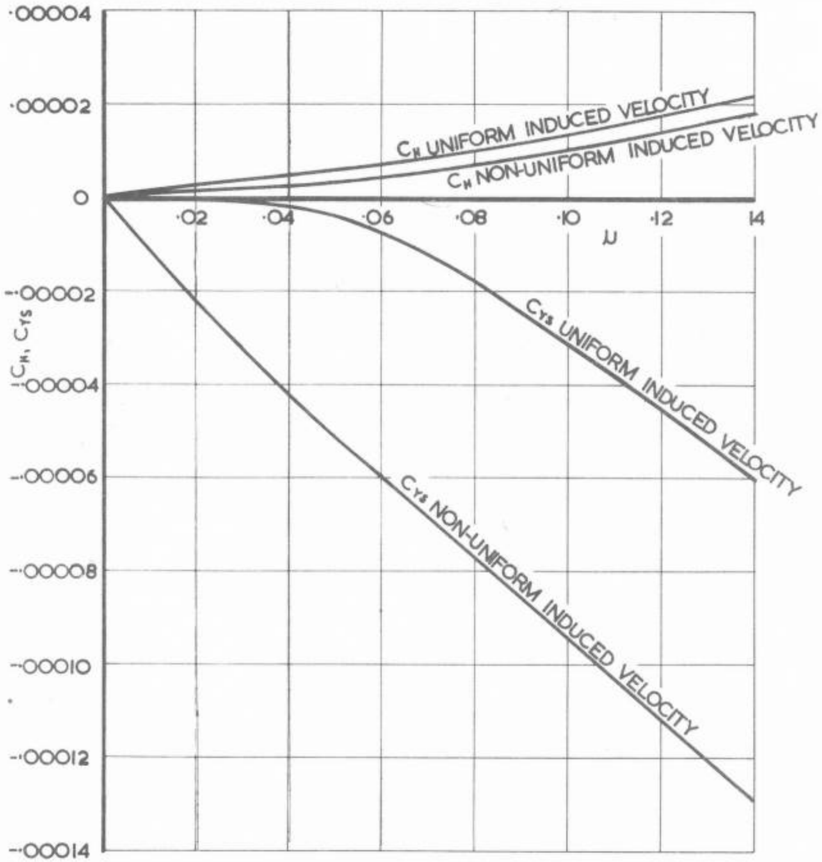

FIG 11 C_H, C_{YS} VS μ

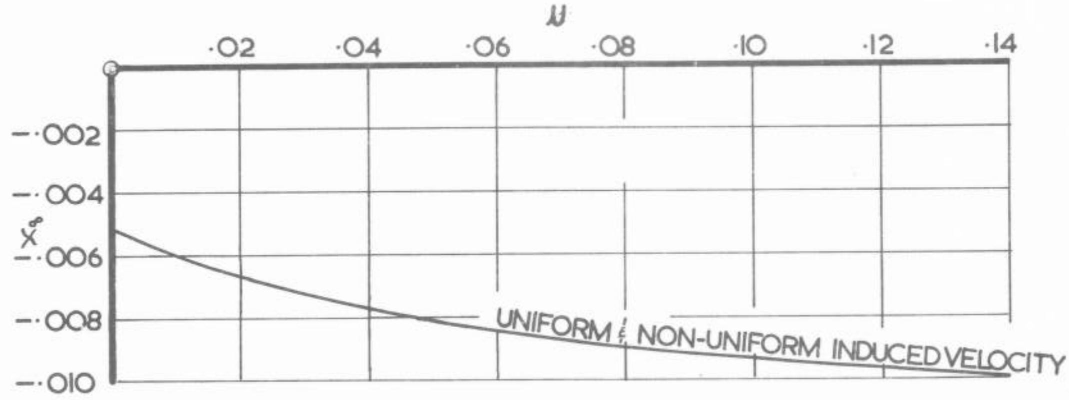

FIG 12 X_q vs μ

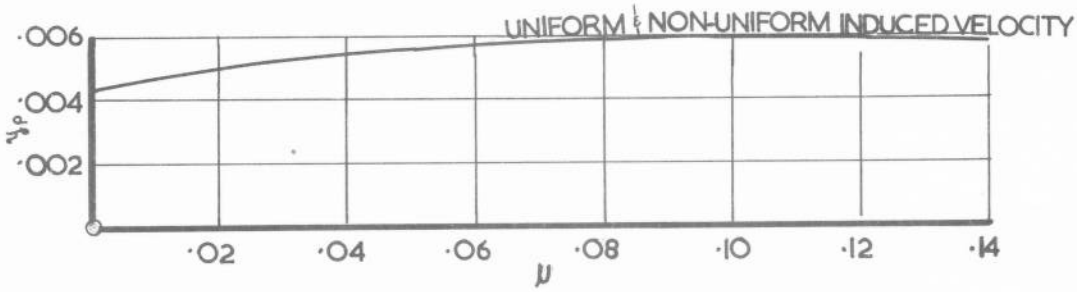

FIG 13 y_p vs μ

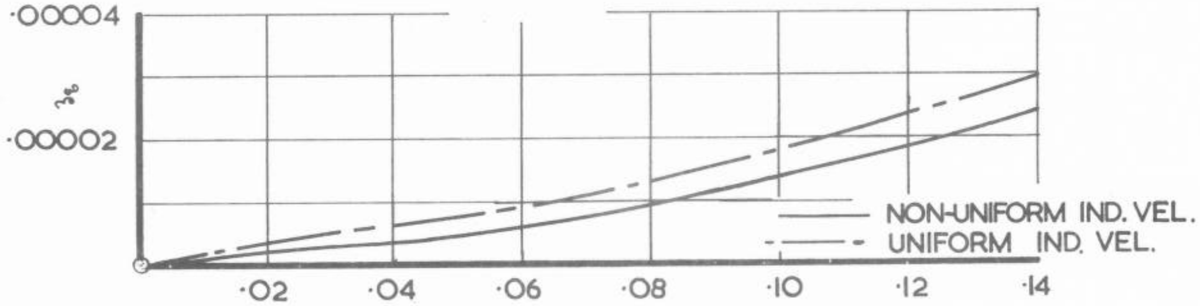

FIG 14 z_q vs μ

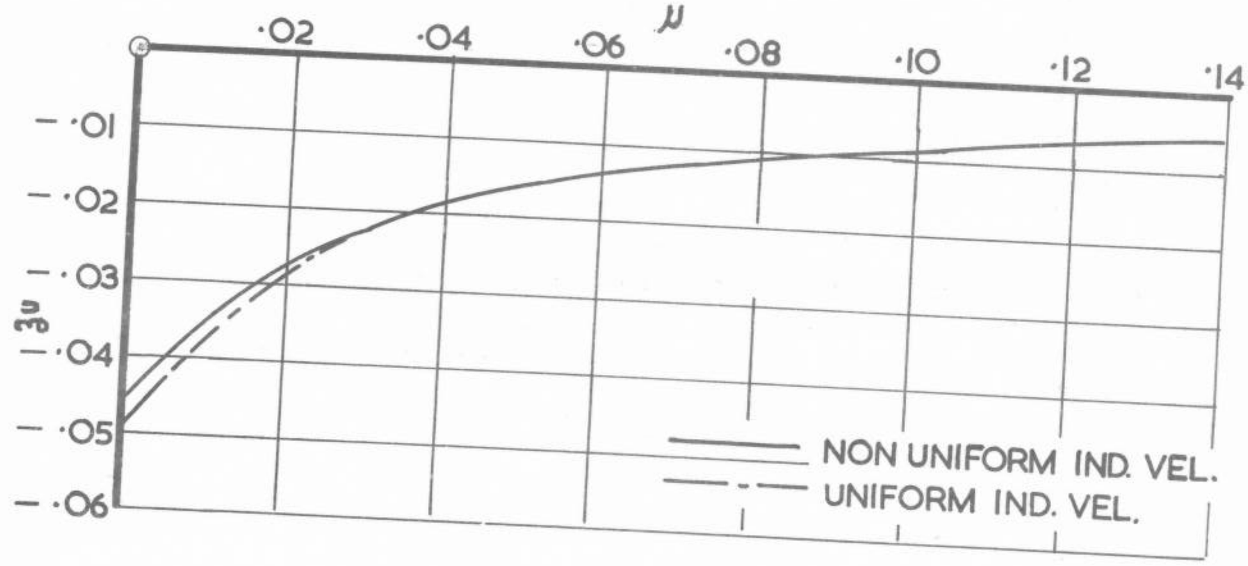

FIG 15 z_μ vs μ

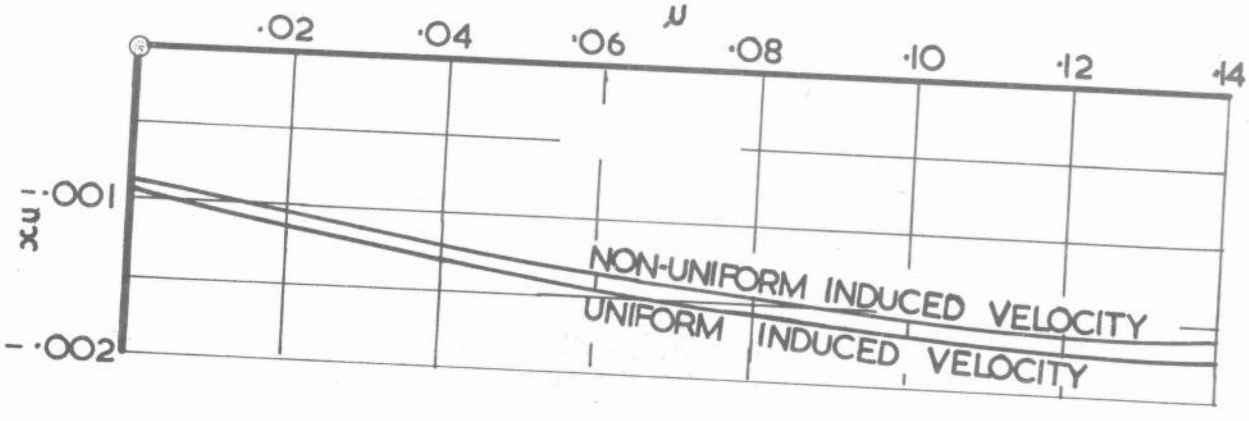

FIG 16 x_μ vs μ

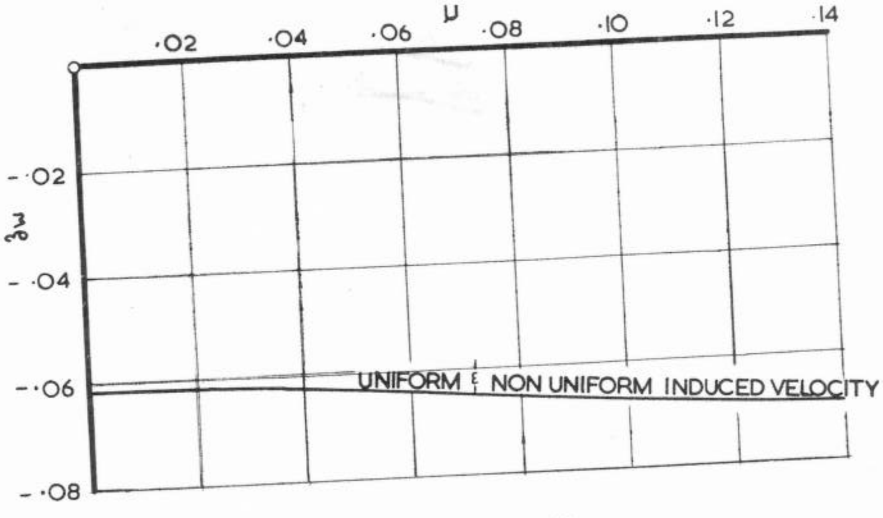

FIG 17 $\mathfrak{z}_w$ vs μ

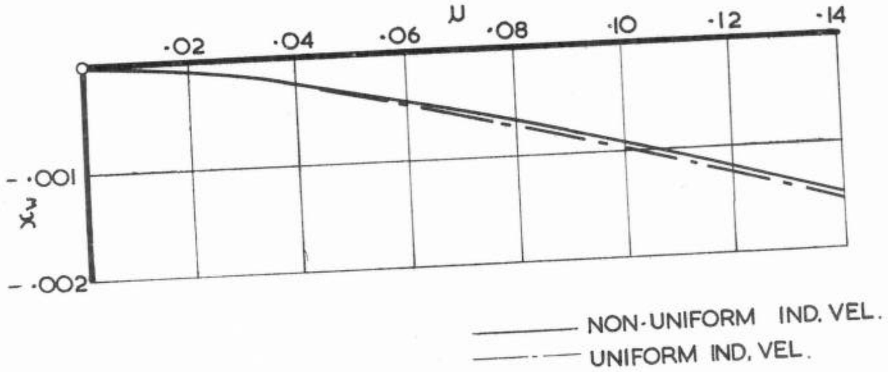

FIG. 18. x_w vs μ

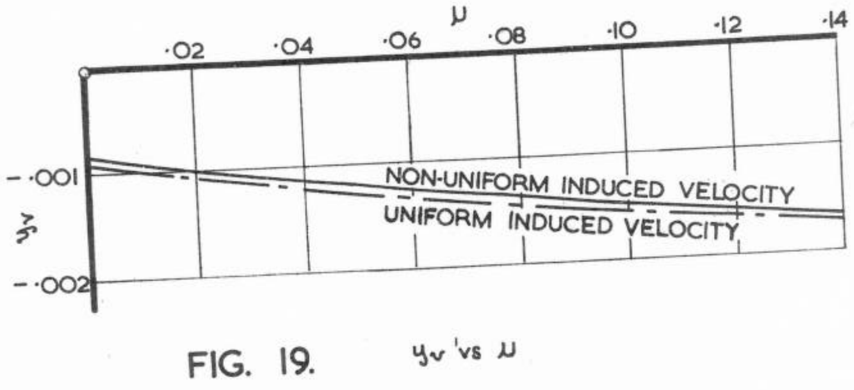

FIG. 19. y_v' vs μ