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A potential flow model for the flow about a nacelle with jet

-by-

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SUMMARY

The inviscid incompressible flow round a thin nacelle from which a jet is issuing is considered. It is shown that the inhomogeneous motion can be transformed into an equivalent homogeneous motion which may be represented by two semi-infinite distributions of vortices in the two-dimensional case and by a semi-infinite distribution of circular vortex rings in the axi-symmetric case. By assuming constant vorticity in the wake and constant or linearly increasing vorticity to represent the duct, the duct shape and the pressure distribution over its outer surface are calculated for given ratios of jet speed to free stream speed. Assuming a slender duct the vorticity representing it is expressed, to a first approximation, as a function of the duct width or diameter and the volume flow through the duct. Methods of extending the treatment to a thick-walled duct with a wake of finite thickness and for including the viscous effects are suggested.

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LIST OF SYMBOLS

c	chord length of duct
d	half width of assumed duct shape
E	complete elliptic integral of the second kind
F	complete elliptic integral of the first kind
k	the modulus of the elliptic integrals
p	pressure
p_o	pressure in undisturbed stream
Q_o	volume flow through duct
r	radial distance
R	radius of circular vortex ring
U	undisturbed free stream velocity
u_o, u_i	longitudinal components of the perturbation velocity on the outside and inside surfaces of the duct respectively
v	normal component of the perturbation velocity on duct surface
x, x_1	longitudinal distances
y, y_1	normal distances
z, z_1	distances in axial direction
α	angle between duct wall and x-axis
α_o	angle of incidence of duct
γ	strength per unit length of the vortex distribution in two dimensions
γ_w	strength per unit length of wake vorticity distribution
δ	0 for x upstream of duct leading edge; 1 for x downstream of duct leading edge

Θ vector potential for a system of circular vortex rings

K strength per unit length of vortex ring distribution

ρ density

1. Introduction

In the past most work on jet flow has been confined to the free jet and to the problem of the flow in the mixing region downstream of the jet exit. Attention has been focussed upon the velocity, and in some cases the density and temperature distributions in the mixing region.

Little has been written, however, on the effect of the jet upon the flow around the body from which it issues, and in particular the effect upon the flow at the rear end of the body. This paper contains an attempt to set up potential flow models for both two-dimensional and axi-symmetric ducts from which jets are issuing. Some consideration is also given to the staggered two-dimensional duct and the two-dimensional duct at incidence. Further, the approximations of slender body theory are used to obtain expressions for the vorticity distributions representing the ducts.

The detailed mathematical analysis is given in separate appendices to this paper, only methods and results being discussed in the main body. Reference to equations in appendices is made by quoting the letter denoting the appendix and the equation number e.g. (A.32). A number in the position of an index in the text denotes a reference included in the list on page 19. An asterisk * signifies a footnote.

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2. A potential flow model representing the flow about a nacelle from which a jet is issuing. The general problem.

In this problem we can recognise two distinct fields of flow, the gases in each being incompressible and inviscid. There is the flow through the duct in which the total head of the flow is changed by some external action (e.g. a jet engine) and the flow exterior to the duct. Downstream of the duct these two flows are separated by a wake across which the pressure must be continuous. Although the effect of the thickness of the duct is discussed later, the main arguments are developed for a thin duct for which the wake is thin and can be regarded as a dividing or wake streamline, or in the axi-symmetric case a stream surface. Since, in general, the fluid velocity is different on either side of this wake, there will be, lying on this streamline or surface, a distribution of vorticity of strength per unit length equal to the difference between the velocities, (fig. 1).

Küchemann and Weber [1] have published a series of monographs dealing with duct flow, and mention this problem amongst others concerning ducts and cowlings for propulsive units. They give in detail the solution of the problem of an axi-symmetric duct without wake, i.e. a duct from which there is no jet velocity. Since the total heads of jet and stream are different we are considering an inhomogeneous flow problem which cannot be treated by normal potential flow methods without modification.

2.1. The equivalent homogeneous flow

The complete flow can be rendered homogeneous by considering the total heads put equal without changing the velocity in either the jet or the surrounding stream. In consequence there will be a constant pressure difference between the jet and the stream, and we must therefore assume that the wake streamline or stream surface is replaced by an infinitesimally thin surface separating the jet and stream, and which can support the supposed pressure difference (fig. 2).

The validity of this argument can be appreciated by considering the equation of steady motion for an incompressible inviscid fluid, which can be written

$$q.\ \text{grad}\ q = -\frac{1}{\rho}\ \text{grad}\ p \qquad \ldots\ldots\ldots\ldots(1)$$

for zero body force. This equation contains only the pressure gradient and hence a constant change of pressure as advocated above will not affect the velocity field.

This equivalent flow can now be treated by the usual methods. Subsequently the conditions in the original flow can be found by cancelling the pressure change across the wake.

2.2. The distribution of vorticity

Since, in the equivalent flow, the total heads are the same on both sides of the dividing surface, the pressure difference across this thin surface is (A.1)

$$\Delta p = \tfrac{1}{2}\rho \left(V_2^2 - V_1^2\right)$$

where V_1 is the speed on the outside, and V_2 the speed on the inside, of the dividing surface.
Now this dividing surface is a streamline so that the direction of flow on both sides of it are parallel to it. This enables us to represent the surface as a distribution of vorticity equal in strength per unit length to the difference in velocity. Thus the vorticity representing the dividing surface has strength per unit length

$$\gamma_W = V_2 - V_1 = \frac{2\Delta p}{V_1 + V_2} \qquad \text{............(2)}$$

The duct itself is idealised to a thin surface so that in two dimensions it is represented, as for two thin aerofoils, by two distributions of vortices, or in the axi-symmetric case, by a distribution of circular vortex rings.

The problem of the establishment of a potential model for the jet flow now resolves itself into that of determining the strength and position of the vortex distribution representing the wake, and also of determining the vorticity distribution which represents the duct.

3. The two-dimensional duct

The two-dimensional duct is considered both at zero incidence and at a small incidence. The mathematical derivation of results quoted in this section are given in Appendix A and Appendix B for the zero incidence case and the case of the duct at small incidence respectively.

3.1. The symmetrical duct at zero incidence

The two-dimensional duct is taken as being formed by two similar thin aerofoils of chord length c, given by the equation

$$y = \pm y_1(x) \quad ; \quad -\frac{c}{2} \leqslant x \leqslant \frac{c}{2}$$

placed symmetrically about the x-axis in a uniform stream of speed U parallel to the x-axis. Following thin aerofoil theory, we replace the two thin aerofoils by distributions of vortices, which must be continuous both in magnitude and position with the vortex distribution representing the wake so that the Joukowski condition of no flow round the duct trailing edges is satisfied. From the symmetry of the configuration it is clear that, if the vorticity representing the upper thin aerofoil at $x = x_1$ is $\gamma(x_1)$, then that representing the lower is $-\gamma(x_1)$.

3.1.1. The perturbation velocity

It is shown that these distributions of vorticity lead to a perturbation velocity at any point (x,y) which has longitudinal and normal components given by (A.4,5) as

$$u(x,y) = -\frac{1}{2\pi}\int_{-c/2}^{\infty} \gamma(x_1)\left\{\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \frac{y+y_1}{(x-x_1)^2+(y+y_1)^2}\right\} dx_1 \qquad \text{...........(3)}$$

and

$$v(x,y) = \frac{1}{2\pi}\int_{-c/2}^{\infty} \gamma(x_1)(x-x_1)\left\{\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_1)^2}\right\} dx_1 \qquad \text{...........(4)}$$

The normal component is continuous through $y = y_1$ whereas the longitudinal component is not.

If the equation for the position of the wake streamline were known, i.e. if we knew the relation between y_1 and x_1 over the wake, then by using the boundary condition that the duct walls are streamline

$$\frac{dy}{dx} = \frac{v}{U + u} \qquad \text{...........(5)}$$

equations 3 and 4 give a singular integral equation for the vorticity distribution $\gamma(x_1)$. The integral equation will be singular since at the point considered on the duct or wake surface $y = y_1$ and $x = x_1$. In section 8 of Appendix A this problem is considered in slightly more detail, and it is shown that even when the problem is simplified by assuming that u is very small compared with U, the vorticity in the wake is assumed constant, and that the vorticity distributions representing the duct and wake lie along the lines $y = \pm d$, the integral equation becomes (A.30)

$$V(x,d) = \int_{-c/2}^{c/2} \gamma(x_1)\left[\frac{4d^2}{(x-x_1)^2+4d^2}\right]\frac{dx_1}{x-x_1} \qquad \ldots\ldots\ldots\ldots(6)$$

The equation is an extension of the equation

$$g(\mu) = \frac{1}{2\pi}\int_{-1}^{1} \frac{f(\eta)}{\mu-\eta}\, d\eta$$

which Schröder[2] and Söhngen[3] have shown to have the solution

$$f(\eta) = -\frac{2}{\pi}\sqrt{\frac{1-\eta}{1+\eta}}\int_{-1}^{1} g(\mu)\sqrt{\frac{1+\mu}{1-\mu}}\,\frac{d\mu}{\eta-\mu}$$

As yet (6) has not been solved, but it is believed that a solution may be possible following the method of Schröder and Söhngen.

Assuming a slender duct, i.e. the duct chord large compared with its width, with leading edges at $x = 0$, the longitudinal component of the perturbation velocity can be written

i) just outside the duct and wake (A.35)

$$u_o(x) = -\tfrac{1}{2}\gamma(x)\left[1 - \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right] + \int_0^{\infty} \frac{\gamma(x_1)y_1(x_1)}{(x-x_1)^2}\, dx_1 \qquad (7)$$

and

ii) just inside the wake (A.36)

$$u_i(x) = u_o + \gamma(x)\left[1 - \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right] \qquad \ldots\ldots\ldots\ldots(8)$$

where $\int$ denotes the Cauchy finite principal part of the singular integral.

The normal component can then be calculated from

$$v = (U + u)\frac{dy}{dx}.$$

Furthermore, the vorticity distribution representing a slender duct can be found (section 3.1.4).

3.1.2. The perturbation velocity in special cases

To proceed further it is necessary to assume forms for the vorticity distributions representing the duct and wake streamlines, and to assume that the duct and wake have a chosen shape. Ordinary thin aerofoil theory applies only to aerofoils with small camber, and hence it is reasonable to suppose that the argument developed here will apply only to a duct whose walls have small curvature and a wake which is approximately straight. Thus the chosen vortex distributions are assumed to lie on the parallel lines $y_1(x) = \pm d$, where d is a constant.

In Appendix A, section 4, the vortex distributions considered are

i) a constant vorticity, of strength $\gamma = K$ per unit length, along the duct and its wake.

ii) a linearly increasing strength of vorticity along the duct, and constant strength in the wake, i.e.

$$\gamma(x) = \frac{K}{c}\left(x + \frac{c}{2}\right) \quad ; \quad -\frac{c}{2} \leqslant x \leqslant \frac{c}{2}$$

$$\gamma_w(x) = K \quad ; \quad x \geqslant \frac{c}{2}$$

and

iii) a sinusoidal variation of vorticity over the duct and constant vorticity in the wake, i.e.

$$\gamma(x) = K \sin\left[\frac{\pi}{2c}\left(x + \frac{c}{2}\right)\right] \quad ; \quad -\frac{c}{2} \leqslant x \leqslant \frac{c}{2}$$

$$\gamma_w(x) = K \quad ; \quad x \geqslant \frac{c}{2}$$

The assumption of constant vorticity in the wake does place a slight restriction on the solution since it implies, from (2) that $V_1 - V_2$ is constant. Now $\Delta p = \frac{1}{2}\rho(V_2^2 - V_1^2)$ is (to a first order) constant, and hence $V_2 + V_1$ is constant. Therefore, in the wake, we are assuming that V_1 and V_2 are constants which is strictly only true if the wake streamlines are straight and parallel to the x-axis. While this is not exactly the case figs. 3 and 5 show that it is very nearly so. If however the accuracy required demand it the variation in V_1 and V_2 due to the curving of the wake streamline can be calculated by considering the balance between the centrifugal force and the pressure on the boundary streamline. A new vorticity distribution can then be found to replace the constant distribution assumed at first and the modified shape of the wake calculated. This iterative process can be repeated to any degree of accuracy.

The perturbation velocity components given by a constant overall vorticity distribution are, from (A.8,9), on the outside of the upper surface of the duct and wake

$$u_o(x,d) = \frac{K}{2\pi}\left\{\frac{\pi}{2} + \tan^{-1}\left(\frac{x+c/2}{2d}\right)\right\} - \frac{\delta K}{2} \; ; \; \delta = 0 \text{ for } x < -\frac{c}{2}, \quad 1 \text{ for } x \geq -\frac{c}{2} \qquad \ldots\ldots\ldots\ldots(9)$$

on the inside of the upper surface

$$u_i(x,d) = \frac{K}{2\pi}\left\{\frac{\pi}{2} + \tan^{-1}\left(\frac{x+c/2}{2d}\right)\right\} + \frac{\delta K}{2} \qquad \ldots\ldots(10)$$

and

$$v(x,d) = -\frac{K}{2\pi}\log_e\left[1 + \left(\frac{2d}{x+\frac{c}{2}}\right)^2\right] \qquad \ldots\ldots\ldots\ldots(11)$$

The normal component $v(x)$ is continuous across the duct and wake, and infinite at the leading edge - a parallel with thin aerofoil theory. It also tends to zero far upstream and far downstream. It is seen that at $x = +\infty$, where

$$\tan^{-1}\left(\frac{x + c/2}{2d}\right) = \frac{\pi}{2},$$

$$u_o = 0 \; ; \; u_i = K$$

Now, at infinity, $K = V_2 - V_1$ and hence the velocity on the jet side of the wake streamline is $U + K = V_1 + (V_2 - V_1) = V_2$ and V_1 on the freestream side. Also, far upstream at $x = -\infty$, $\delta = 0$ and hence,

$$u_o = u_i = 0$$

as required. Thus the conditions of the problem are satisfied by the solution found.

Similar expressions for the other two assumed vorticity distributions are considerably more complicated and are given in Appendix A, section 4.2 and 4.3.

3.1.3. The shape of the duct and wake

When the components of the perturbation velocity have been found to a first approximation it is then possible to find approximately the shape corresponding to the assumed vortex distribution by using the condition that the surface of the duct and wake must be a streamline, i.e.

$$\frac{dy}{dx} = \frac{v}{U+u}$$

to give the slope of the profile at any value of x. The shape is then found by integration.

These calculations have been performed in the case of the constant vorticity distribution and for the linearly increasing distribution. The corresponding duct shapes are shown in figs. 3 and 5 respectively.

To obtain the second approximation shown in these figures the assumed vorticity distribution was taken to lie on the shape calculated before. The components of the perturbation velocity are then found numerically and the duct shape is re-calculated from them. This process can be repeated until the difference between two successive shapes is negligible, giving the correct duct shape for the assumed vorticity distribution.

3.1.4. The vortex distribution for a given shape

In fig. 3 the duct shapes given by a constant vorticity distribution for both $V_2/V_1 = 1.2$ and $V_2/V_1 = 1.3$ are shown. It is seen immediately that the duct shape is different in the two cases, and thus we can infer that a given duct shape is represented by different vorticity distributions for different speeds, but it is only by a process of trial and error that a suitable vortex distribution can be found without recourse to numerical solutions of equations of the form of (6). Birnbaum[4] has shown that, for the finite duct with no wake (i.e. straight through flow) three basic vorticity distributions can be used which give the shapes of a flat plate, a parabolic arc, and an S-shaped profile. In each case the vorticity at the trailing edge is zero since there is no wake.

It is thought that the three vortex distributions given in section 3.1.2 above give a good basis for calculating the shapes of ducts from which a jet is issuing. It must be remembered, however, that whatever combination of basic distributions is used, the vorticity at infinity downstream must be that given by (2) where V_1 takes the free stream value and V_2 is the jet speed at infinity.

If the duct is assumed slender then an expression for the vorticity, representing the duct and wake, can be found explicitly. It is shown (A.41) that, if Q_o is the volume flow of fluid through unit span of the duct, and U is the free stream velocity, then the vorticity at $y(x)$ on the duct surface is

$$\gamma(x) = \frac{Q_o}{2y(x)} - U + O\left(\frac{y}{c}\right) \qquad \ldots\ldots\ldots\ldots(12)$$

where c is the length of the duct in the streamwise direction and y/c is small.

As an example an arbitrary duct shape with wake was chosen (fig. 15) and the jet and free stream speeds taken such that $V_2/V_1 = 1.25$. Using the vorticity distribution given by (12) the duct shape corresponding to this distribution was calculated from equations 3, 4 and 5 and the result compared with the assumed shape (fig. 15). The agreement between assumed and calculated shape is close except near the leading edge when, due to the nature of equation 4, a singularity is to be expected.

3.1.5. The pressure distribution

The pressure distribution over the outside surface of the duct can be immediately calculated from the perturbation velocity. Having found u and v as, for example, in (9) and (11), the difference in pressure between a point on the duct surface and in the undisturbed stream can be expressed as

$$\delta p = p - p_o = \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho\left[(U+u)^2 + v^2\right]$$

or

$$\frac{\delta p}{\tfrac{1}{2}\rho U^2} = -\frac{1}{U^2}\left[u^2 + 2uU + v^2\right] \qquad \ldots\ldots\ldots\ldots(13)$$

The pressure distributions for the ducts represented by the constant vorticity distribution and the linearly increasing vorticity distribution are given in Figs. 4 and 6. In both cases the pressure distributions corresponding to the first and second approximations to the duct shape are shown.

3.2. The staggered duct

If the x-coordinates of the leading edges of the upper and lower thin aerofoils are different, but those of the trailing edges are the same, we have a staggered duct of a form often met in practice. This case is considered in Appendix A, section 7, where it is pointed out that there is no longer symmetry about the x-axis, and hence it is unlikely that the same vorticity distribution can be used in this case, even when the different limits of integration are taken. The perturbation velocity can be calculated as before and from it the shape and pressure distribution can be found, as for the symmetrical duct.

3.3. The two-dimensional duct at incidence (Appendix B)

Although the duct is supposed symmetrical about the x-axis (fig. 7) the complete flow is not symmetrical about a line drawn parallel to the undisturbed stream. Thus it is unlikely that the duct shape, obtained by methods outlined in section 3.1, will be exactly symmetrical since the cross flow would be neglected. However, provided the incidence is small, such a solution will give some information about the flow round a duct with jet at incidence.

When a physically likely shape (B.7) is assumed for the shape of the wake, the integrals for the perturbation velocity are formidable. If, however, for small incidences the wake streamlines are assumed to leave the duct parallel to the free stream (fig. 8) the perturbation velocity can be calculated as in Appendix B, section 3.3, and the shape and pressure distribution can be found as before.

The equations for the components of the perturbation velocity both show logarithmic singularities at the trailing edge of the duct. These are due to the sudden change in flow direction, which is assumed at the trailing edge.

4. The thin axi-symmetric duct

The thin axi-symmetric duct at zero incidence is represented by a distribution of circular vortex rings the centres of which lie on the z-axis, (the axis of symmetry)ˣ

It is shown in Appendix C that the magnitude of the vector potential at a point P (z,r,ϕ') due to the

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ˣ fig. 9.

distribution of vortex rings representing the duct is

$$\Theta = \frac{1}{2\pi} \int_0^\infty K(z') \left(\frac{R(z')}{r}\right)^{\frac{1}{2}} \left\{\frac{2-k^2}{k} F - \frac{2}{k} E\right\} dz'$$

where $R(z')$ is the radius of the vortex ring at $z = z'$ and F and E are elliptic integrals of the first and second kind respectively of modulus

$$k^2 = \frac{4r\,R(z')}{(z-z')^2+\left[r+R(z')\right]^2}$$

It is further shown that the perturbation velocity at P has components

$$u(z,r) = \frac{1}{2\pi} \int_0^\infty K(z') \left[F + \frac{R^2(z')-r^2-(z-z')^2}{(z-z')^2+\left[R(z')-r\right]^2} E\right] \frac{dz'}{\left[(z-z')^2+ (R(z')+r)^2\right]^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots(14)$$

and

$$v(z,r) = -\frac{1}{2\pi} \int_0^\infty \frac{K(z')(z-z')}{r} \left[k^2F - \frac{2r\,R(z')E}{(z-z')^2+\left[R(z')-r\right]^2}\right] \frac{dz'}{\left\{(z-z')^2+(R(z)+r\right.} \qquad \ldots\ldots\ldots\ldots(15)$$

From these the duct shape given in fig. 11 has been calculated.

If a slender duct is assumed then it is shown that the strength per unit length of the vorticity distribution is

$$K(z) = \gamma(z) = \frac{Q}{S(z)} - U + O\left(\frac{r}{c}\right) \qquad \ldots\ldots\ldots\ldots(16)$$

As for the two-dimensional duct, which is not assumed to be thin, the further solution of the problem of the non-slender axi-symmetric duct proceeds by assuming that a chosen vorticity distribution lies on a circular cylinder of constant radius. The perturbation velocity is determined along the duct surface, and the first approximation to the duct shape then calculated from the boundary condition of no flow across the surface, i.e.

$$\frac{dr}{dz} = \frac{v}{U + u}$$

More accurate approximations can then be found as in 3.1.3.

Fig. 11 shows a duct shape calculated for the case $V_2/V_1 = 0.3$ which corresponds to a ducted radiator. Constant vorticity has been assumed on the duct and in the wake. This solution is compared with an intake duct calculated by Kuchemann and Weber[5] and shown to be in close agreement. The corresponding pressure distribution is shown in fig. 12. Returning to jet flow where $V_2/V_1 > 1$ the duct shape and pressure distribution over the duct induced by the jet are given in figs. 13 and 14 for the cases of constant vorticity and $V_2/V_1 = 1.5$ and 1.2.

5. The extension to a duct with walls of finite thickness

Previously the ducts considered have been formed either by two thin aerofoils or by a thin cylinder, and no consideration has been given to the thickness of the walls. Thick profiles can be represented by a distribution of sources and sinks superimposed upon the vortex distribution found previously. The calculation of the appropriate system of sources and sinks is complicated, not only by the fact that their strength and location are not known on the duct, but also by the need to have a source distribution to represent the thickness of the wake.

Kuchemann[6] has suggested that the effect of thickness is of minor importance compared with the camber line of the profile (i.e. the shape of the corresponding thin profile), and that the thickness is only important when determining the rate of flow through the duct. A correction to the velocity inside the duct is given as

$$V_\alpha - V_\beta = \left(\frac{R_\beta^{\,2}}{R_\alpha^{\,2}} - 1\right)\left(1 - U + V_\beta\right)$$

where V_α and V_β are the speed on the inner surface of the thick duct (radius R_α) and thin duct (radius R_β) respectively. By inference we can use this to determine the speed inside a wake of finite thickness.

A general theory for a two-dimensional finite duct, taking account of thickness, has been given by Wittich[7]. He considers a disturbance velocity potential $Z(x,y)$ which must satisfy Laplace's equation

$$\nabla^2 Z(x,y) = 0$$

and the boundary condition

$$\frac{\partial Z}{\partial n} = - U \frac{\partial x}{\partial n}$$

over both aerofoils. He then deduces the appropriate source distribution $\sigma(s)$ in the form of a singular integral equation, the integral being taken round two closed curves. The restriction that the curves must be closed and finite prevents the method being used without considerable modification for a duct with wake.

Another method of including the thickness of the duct walls in two dimensions is to represent them, as for aerofoils, by the region of fluid at rest enclosed by a system of vortices arranged on the boundary of the duct and its wake. The determination of the actual distribution of vorticity representing the duct and wake is extremely difficult as again the integral equations are not solvable except by numerical methods. In a similar way, the thickness of an axi-symmetric duct can be considered by taking a distribution of vortex rings on the inside and outside surfaces of the duct.

In an attempt to correct the over-large values of lift on aerofoils found by the classical Joukowski circulation theory, Witosynski and Thompson[8] have evolved a discontinuous potential. They recognise a wake behind the aerofoil bounded by the streamline $\psi = 0$ which is split at the trailing edge into two branches. In transforming the aerofoil to a circle the wake region is excluded from the transformation. The corresponding complex potential is single-valued, and defines the flow completely, except in the wake region. A development of this theory may perhaps assist in the solution of the problem of a duct with wake from which a jet is issuing.

6. The effect of the flow in the mixing region

The discussions in the previous parts of this section have dealt with only inviscid fluids. In practice viscous mixing processes in the wake will affect the flow over the duct. Much has been written on the mixing in the wake, but the effect of the mixing on the viscous flow about the duct has not yet been investigated.

It is clear, however, that the viscous jet will entrain some of the fluid from the surrounding stream, and thereby increase the normal component of the perturbation velocity near the duct trailing edge. This will cause the now finite width wake, or mixing region, to curve further inwards. This increase in velocity also causes a reduction in pressure at the rear of the duct, and therefore the duct will experience a drag due to the presence of the mixing region. Further, the reduction in pressure will tend to prevent separation of flow over the rear of the nacelle and to restrain the formation of a turbulent boundary layer over the nacelle.

Viscous forces in the mixing region will also cause the jet to be slowed up and thus the assumption of constant vorticity in the wake, which was made earlier, is not realisable in practice.

To cover the effects of viscosity it is necessary to extend the treatment given earlier in this section to consider a distribution of sinks in the wake, and also to include the effect of the boundary layer attached to the duct walls. The actual flow outside the jet can be regarded as very nearly equivalent to that produced by a system of sinks along the jet axis, the strength of the sinks being sufficient to induce the correct inflow at the jet boundary. Now the inflow velocity is equal to

i) $\frac{\partial \psi}{\partial x}$ in two dimensions

ii) $\frac{1}{r} \frac{\partial \Psi}{\partial x}$ in axi-symmetric flow,

where ψ, Ψ are the appropriate stream functions for the additional radial flow. The system of sinks has strength per unit length proportional to $\partial \psi / \partial x$. Squire and Trouncer[9] have shown that the required sink strength can be made up of a combination of constant and linear sink distributions taken over successive small lengths of the jet axis. The exact analytic solution for the source strength involves an integral equation, which is not readily solvable.

If the inflow has been found from the associated problem of the flow in the mixing region, the value of

$\frac{\partial \psi}{\partial x}$, $\frac{\partial \Psi}{\partial x}$ can be found immediately. Comparison between

these values and those calculated for a source distribution of strength m extending over a small length give the required value of the source strength at the appropriate point of the jet axis.

7. Concluding remarks

The potential flow model for the jet flow from a nacelle is limited in its application by the method used in deriving it. The representation of a duct by a simple vortex distribution is only permissible when the duct is thin, the difference in the speeds of the jet and stream is small, and the speeds are low. As explained in section 5, the thickness of the duct and wake lead to additional problems which have yet to be investigated when the added complication of the jet flow is included. Further, the effects of temperature and density difference between the jet and the free stream must be included.

Viscous effects, resulting in the development of the mixing region, have been discussed fully in the literature (e.g. ref. 12). A complete solution to the effect of the jet on the viscous flow around the body from which it is issuing is, in fact, a union of the solutions of this paper and those of the problem in the mixing region. The viscous effects themselves can be represented by a system of sinks in the wake, the strength of the sink distribution being found from the velocity of inflow into the jet. An added complication is the presence, before mixing begins, of a developed boundary layer. This also must be considered and included in the determination of the sink distribution.

Knowing the temperature distribution in the mixing region it should be possible to represent the flow of a hot jet by a system of heat sources distributed, in the first instance, along the jet axis. A first step in this study has been made by Squire[13] who has calculated the temperature distribution in the laminar mixing region of a hot jet by considering the temperature field due to a single heat source at the centre of the orifice. It still remains to determine the effect of the temperature field on the flow round the body from which the jet is issuing. Since the pressure across the wake streamline must be continuous it should then be possible to allow for the effects of different density.

The calculation of the forces and moments on the duct is complicated by the problem of representing the change in the total head of the flow through the duct by a system of singularities. For a symmetrical duct at zero incidence there will be only a force in the free stream direction but at incidence there will be lift and pitching moments in addition. The incidence case is further complicated by the cross flow around the duct and part of the wake. This has not been considered in Appendix B and further investigation is required to find the change in the vorticity distribution necessary to represent the symmetrical duct at incidence.

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APPENDIX A

A potential-flow model for two dimensional duct flow

1. The two-dimensional duct is considered formed by two similar thin aerofoils, chord length c, placed symmetrically about the x-axis in a uniform stream parallel to the x-axis. We shall assume that the flow through the duct is accelerated or slowed down by external action, so that there is a difference in total head between the free stream and the flow from the duct, giving an inhomogeneous flow. This also implies that there is a wake in which resides a distribution of vorticity of strength per unit length equal to the difference in velocity.

2. The equivalent homogeneous flow

To render the problem tractable, the flow is made homogeneous by assuming the wake streamline replaced by an infinitesimally thin solid boundary, a continuation of the duct, upon which the vorticity lies. This wall supports the pressure difference between jet and stream, which must then be assumed if the total heads are put equal without changing the velocities (see fig. 2).
Thus, if V_1 and V_2 are respectively the speeds just outside and inside the wake streamline at any point of its length, the pressure difference supported by the solid boundary is

$$\Delta p = p_1 - p_2 = \tfrac{1}{2}\rho(V_2^2 - V_1^2) \qquad \text{.............(1)}$$

This pressure difference is assumed constant along the wake, and the strength of the bound vorticity distribution in the wake replacing the solid boundary is

$$\gamma_W = V_2 - V_1 = \frac{2\Delta p}{\rho(V_1+V_2)} \qquad \text{.............(2)}$$

3. The induced velocity

As in thin aerofoil theory, the two thin aerofoils, and here the wakes, are replaced by vortex distributions. The distribution representing the upper aerofoil and wake is taken to be of strength $\gamma(x_1)$ per unit length along the x-axis at (x_1, y_1), and that representing the lower aerofoil and wake then has strength $-\gamma(x_1)$ at $(x_1, -y_1)$. The vorticity is bound on the aerofoil and wake boundary and hence $y_1 = f(x_1)$.

At any point z the complex perturbation potential $\omega(z)$ due to these vortex distributions is given by

$$\omega(z) = \frac{i}{2\pi}\int_{-c/2}^{\infty} \gamma(x_1)\left\{\log_e(z-z_1) - \log_e(z-\bar{z}_1)\right\} dx_1 \qquad \ldots\ldots\ldots\ldots(3)$$

where $z_1 = x_1 + iy_1 = x_1 + iy_1(x_1)$.

Now $\frac{d\omega}{dz} = -u + iv$ where u and v are the longitudinal and normal components of the induced velocity. Thus, at any point (x,y) from (3) taking real and imaginary parts

$$u = -\frac{1}{2\pi}\int_{-c/2}^{\infty}\gamma(x_1)\left\{\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \frac{y+y_1}{(x-x_1)^2+(y+y_1)^2}\right\} dx_1 \qquad \ldots\ldots\ldots\ldots(4)$$

$$v = \frac{1}{2\pi}\int_{-c/2}^{\infty}\gamma(x_1)(x-x_1)\left\{\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_1)^2}\right\} dx_1 \qquad \ldots\ldots\ldots\ldots(5)$$

As the aerofoils are thin, y will be approximately constant for all values of x, and hence to a first approximation we put $y_1 = d$ giving

$$u = -\frac{1}{2\pi}\int_{-c/2}^{\infty}\gamma(x_1)\left\{\frac{y-d}{(x-x_1)^2+(y-d)^2} - \frac{y+d}{(x-x_1)^2+(y+d)^2}\right\} dx_1 \qquad \ldots\ldots\ldots\ldots(6)$$

$$v = \frac{1}{2\pi}\int_{-c/2}^{\infty}\gamma(x_1)(x-x_1)\left\{\frac{1}{(x-x_1)^2+(y-d)^2} - \frac{1}{(x-x_1)^2+(y+d)^2}\right\} dx_1 \qquad \ldots\ldots\ldots\ldots(7)$$

To proceed without further approximation it is necessary to assume vorticity distributions in terms of x and then the corresponding perturbation velocity components can be calculated.

4. The Perturbation Velocity in special cases

4.1. Constant vorticity along aerofoil and wake

If the vorticity along the aerofoils and their wake is constant so that $\gamma(x) = K$ along $y = \pm d$, then from (7) the normal velocity component at any point x on the duct is

$$v = \frac{K}{2\pi}\left\{\int_{-c/2}^{\infty} \frac{1}{x-x_1}\,dx_1 - \int_{-c/2}^{\infty} \frac{x-x_1}{(x-x_1)^2 + 4d^2}\,dx_1\right\}$$

and hence $v = -\dfrac{K}{4\pi}\log_e\left[1 + \left(\dfrac{2d}{x+\frac{c}{2}}\right)^2\right]$ ....(8)

Thus v is infinite at the leading edge and tends to zero as x tends to infinity.

The longitudinal component at any point on the duct is obtained by evaluating u on $y = d$.

The longitudinal component of the perturbation velocity on the outside of the duct and wake is, from (6)

$$-\frac{K}{2\pi}\lim_{y\to d+}\left\{(y-d)\int_{-c/2}^{\infty}\frac{1}{(x-x_1)^2+(y-d)^2}\,dx_1\right\} + \frac{K}{2\pi}\int_{-c/2}^{\infty}\frac{}{(x-x_1)^2+4d^2}\,dx_1$$

i.e.

$$u = \frac{K}{2\pi}\tan^{-1}\left[\frac{x_1-x}{2d}\right]_{-\frac{c}{2}}^{\infty} - \frac{K}{2\pi}\lim_{y\to d+}\left\{(y-d)\int_{-c/2}^{\infty}\frac{1}{(x-x_1)^2+(y-d)^2}\,dx_1\right\}$$

or $$u = \frac{K}{2\pi}\left\{\frac{\pi}{2} + \tan^{-1}\frac{\left(x+\frac{c}{2}\right)}{2d}\right\} - \frac{K}{2}\delta \quad \text{.............(9)}$$

$$\delta = 0,\ \text{for } x < -\frac{c}{2}$$

$$\delta = 1,\ \text{for } x \geqslant -\frac{c}{2}$$

and, just inside the wake, taking the limit as y tends to $d-$

$$u = \frac{K}{2\pi}\left\{\frac{\pi}{2} + \tan^{-1}\left(\frac{x+\frac{c}{2}}{2d}\right)\right\} + \frac{K}{2}\delta\ ;\ \text{for } x \geqslant -\frac{c}{2}, \delta = 1$$

$$\text{for } x < -\frac{c}{2}, \delta = 0$$

............(10)

4.2. Linear vorticity distribution on the aerofoil

As a second case, we consider a vorticity distribution such that

$$\gamma(x_1) = \frac{K}{c}\left(x + \frac{c}{2}\right) \quad ; \quad -\frac{c}{2} \leqslant x \leqslant \frac{c}{2}$$

$$\gamma(x_1) = K \quad ; \quad x \geqslant \frac{c}{2}$$

i.e. the vorticity is linearly increasing in the duct and constant in the wake.
From (7), the normal perturbation velocity component at a point x, on the duct is

$$v = \frac{1}{2\pi}\left[\frac{K}{c}\int_{-\frac{c}{2}}^{c/2}(x-x_1)\left(x_1+\frac{c}{2}\right)\left[\frac{1}{(x-x_1)^2} - \frac{1}{(x-x_1)^2+4d^2}\right]dx_1 + K\int_{c/2}^{\infty}(x-x_1)\left[\frac{1}{(x-x_1)^2} - \frac{1}{(x-x_1)^2+4d^2}\right]dx_1\right]$$

and after performing the integrations

$$v = \frac{K}{2\pi c}\left\{\frac{1}{2}\left(x-\frac{c}{2}\right)\log_e\left[1+\frac{4d^2}{\left(x-\frac{c}{2}\right)^2}\right] - \frac{1}{2}\left(x+\frac{c}{2}\right)\log_e\left[1+\frac{4d^2}{\left(x+\frac{c}{2}\right)^2}\right] - 2d\tan^{-1}\left[\frac{2\,cd}{x^2+4d^2-\frac{c^2}{4}}\right]\right\} \qquad \ldots\ldots\ldots\ldots(11)$$

The longitudinal perturbation velocity component on the outside surface is given, from (6), by

$$\frac{1}{2\pi}\left\{\int_{-\frac{c}{2}}^{\frac{c}{2}}\frac{\frac{2dK}{c}\left(x_1+\frac{c}{2}\right)dx_1}{(x-x_1)^2+4d^2} + \int_{\frac{c}{2}}^{\infty}\frac{2dK\,dx_1}{(x-x_1)^2+4d^2}\right\}$$

$$-\frac{K}{2\pi}\lim_{y\to d+}(y-d)\left[\int_{-c/2}^{c/2}\frac{\frac{1}{c}\left(x_1+\frac{c}{2}\right)dx_1}{(x-x_1)^2+(y-d)^2} + \int_{\frac{c}{2}}^{\infty}\frac{dx_1}{(x-x_1)^2+(y-d)^2}\right]$$

or

$$u(x) = \frac{K}{2\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x - \frac{c}{2}}{2d}\right)\right] + \frac{Kd}{2\pi c}\log_e \frac{\left(x + \frac{c}{2}\right)^2 + 4d^2}{\left(x - \frac{c}{2}\right)^2 + 4d^2}$$

$$- \frac{K}{2\pi c}\left(x + \frac{c}{2}\right)\tan^{-1}\frac{2dc}{x^2 + 4d^2 - \frac{c^2}{4}} - \frac{\gamma(x)}{2}\,\delta$$

where $\delta = 0$ for $x \geqslant -\frac{c}{2}$

$\delta = 1$ for $x < -\frac{c}{2}$ ..(12)

and just inside the wake

$$u(x) = \frac{K}{2\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x - \frac{c}{2}}{2d}\right)\right] + \frac{Kd}{2\pi c}\log_e \frac{\left(x + \frac{c}{2}\right)^2 + 4d^2}{\left(x - \frac{c}{2}\right)^2 + 4d^2}$$

$$- \frac{K}{2\pi c}\left(x + \frac{c}{2}\right)\tan^{-1}\frac{2dc}{x^2 + 4d^2 - \frac{c^2}{4}} + \frac{\gamma(x)}{2}\,\delta$$

..............(13)

4.3. <u>The vorticity distribution in the form of a sine function</u>

If the strength of vorticity distribution on the aerofoil is assumed to be proportional to a sine function of x we take

$$\gamma(x_1) = K\sin\left[\frac{\pi}{2c}\left(x_1 + \frac{c}{2}\right)\right] \quad ; \quad -\frac{c}{2} \leqslant x_1 \leqslant \frac{c}{2}$$

and

$$\gamma(x_1) = K; \quad x \geqslant \frac{c}{2}$$ ....(14)

giving a vorticity distribution which increases from zero at the leading edge sinusoidally along the chord of the thin aerofoil and is constant in the wake.

From (7) the normal component of the perturbation velocity is

$$v = \frac{K}{2\pi}\int_{-c/2}^{c/2} \sin\left[\frac{\pi}{2c}\left(x_1 + \frac{c}{2}\right)\right] (x-x_1)\left[\frac{1}{(x-x_1)^2} - \frac{1}{(x-x_1)^2+4d^2}\right] dx_1$$

$$+ \frac{K}{2\pi}\int_{c/2}^{\infty} (x-x_1)\left[\frac{1}{(x-x_1)^2} - \frac{1}{(x-x_1)^2+4d^2}\right] dx_1$$

$$= \frac{K}{2\pi}\log_e\left[1 + \frac{(2d)^2}{\left(x - \frac{c}{2}\right)^2}\right]$$

$$+ \frac{K}{2\sqrt{2}\pi}\int_{-c/2}^{c/2}\left[\sin\frac{\pi x_1}{2c} + \cos\frac{\pi x_1}{2\ c}\right](x-x_1)\left[\frac{1}{(x-x_1)^2} - \frac{1}{(x-x_1)^2+4d^2}\right] dx_1$$

Putting

$$D(\chi)^{*} = \frac{1}{2}\left[\text{Si}\ \frac{\pi}{2c}(\chi+2id) + \text{Si}\ \frac{\pi}{2c}(\chi-2id)\right]\cosh\frac{\pi d}{c}$$

$$- \frac{i}{2}\left[\text{Ci}\ \frac{\pi}{2c}(\chi+2id) - \text{Ci}\ \frac{\pi}{2c}(\chi-2id)\right]\sinh\frac{\pi d}{c} \qquad \text{...........(15)}$$

and

$$E(\chi) = \frac{1}{2}\left[\text{Ci}\ \frac{\pi}{2c}(\chi+2id) + \text{Ci}\ \frac{\pi}{2c}(\chi-2id)\right]\cosh\frac{\pi d}{c}$$

$$- \frac{i}{2}\left[\text{Si}\ \frac{\pi}{2c}(\chi+2id) - \text{Si}\ \frac{\pi}{2c}(\chi-2id)\right]\sinh\frac{\pi d}{c} \qquad \text{...........(16)}$$

where $\chi = x_1 - x$,

$$\text{Si}(u) = \int_0^u \frac{\sin t}{t}\,dt = u - \frac{u^3}{3!.3} + \frac{u^5}{5!.5} - \frac{u^7}{7!.7} + \cdots$$

and

$$\text{Ci}(u) = -\int_u^{\infty}\frac{\cos t}{t}\,dt = \log_e \gamma - \frac{u^2}{2!.2} - \frac{u^4}{4!.4} + \frac{u^6}{6!.6} - \cdots$$

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* Reference 10

where $$\log_e \gamma = \int_0^1 \frac{1 - e^{-t}}{t}\, dt - \int_1^{\infty} \frac{e^{-t}}{t}\, dt$$

$$= .57721(6) \;\ldots\ldots$$

(D and E are seen to be real functions of x, x_1, and d)

v becomes

$$v = \frac{K}{2\pi} \log \left[1 + \left(\frac{2d}{x - \frac{c}{2}} \right)^2 \right] + \frac{K}{2\sqrt{2\pi}} \left(\cos \frac{\pi x}{2c} - \sin \frac{\pi x}{2c} \right)$$

$$\left[D\left(\frac{c}{2} - x\right) - D\left(\frac{c}{2} + x\right) - Si\, \frac{\pi}{4}\left(1 - \frac{2x}{c}\right) + Si\, \frac{\pi}{4}\left(-1 - \frac{2x}{c}\right) \right]$$

$$+ \frac{K}{2\sqrt{2\pi}} \left(\cos \frac{\pi x}{2c} + \sin \frac{\pi x}{2c} \right) \left[E\left(\frac{c}{2} - x\right) - E\left(\frac{c}{2} + x\right) - Ci\, \frac{\pi}{4}\left(1 - \frac{2x}{c}\right) \right.$$

$$\left. + Ci\, \frac{\pi}{4}\left(-1 - \frac{2x}{c}\right) \right] \qquad \ldots\ldots\ldots\ldots(17)$$

From (6), using the vorticity distribution given in (14), the normal component of the perturbation velocity just outside the duct is

$$u = \frac{1}{2\pi} \left\{ \int_{-c/2}^{c/2} \frac{2d\, K \sin\left[\frac{\pi}{2c}\left(x_1 + \frac{c}{2}\right)\right] dx_1}{(x-x_1)^2 + 4d^2} + \int_{c/2}^{\infty} \frac{2Kd \quad dx_1}{(x-x_1)^2 + 4d^2} \right\}$$

$$- \frac{1}{2\pi} \lim_{y \to d+} (y-d) \left\{ \int_{-c/2}^{c/2} \frac{K \sin\left[\frac{\pi}{2c}\left(x_1 + \frac{c}{2}\right)\right] dx_1}{(x-x_1)^2 + (y-d)^2} \right.$$

$$\left. + \int_{c/2}^{\infty} \frac{K\, dx_1}{(x-x_1)^2 + (y-d)^2} \right\}$$

which reduces to

$$u = \frac{K}{2\sqrt{2\pi}} \left\{ \left(\cos\frac{\pi x}{2c} + \sin\frac{\pi x}{2c}\right)\left[F(\tfrac{c}{2}-x) - F(\tfrac{c}{2}+x)\right] + \left(\cos\frac{\pi x}{2c} - \sin\frac{\pi x}{2c}\right)\left[G(\tfrac{c}{2}-x) - G(\tfrac{c}{2}+x)\right]\right\}$$

$$+ \frac{K}{2\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{x + \frac{c}{2}}{2d}\right) - \frac{\gamma(x)}{2}\,\delta\ ;\ \delta = 0 \text{ for } x < -\frac{c}{2}, \quad \delta = 1 \text{ for } x \geq -\frac{c}{2} \qquad \text{.............(18)}$$

where

$$F(\chi) = i\left[\mathrm{Ci}\,\frac{\pi}{2c}(\chi+2id) - \mathrm{Ci}\,\frac{\pi}{2c}(\chi-2id)\right]\cosh\frac{\pi d}{c} + \left[\mathrm{Si}\,\frac{\pi}{2c}(\chi+2id) + \mathrm{Si}\,\frac{\pi}{2c}(\chi-2id)\right]\sinh\frac{\pi d}{c}$$

and

$$G(\chi) = i\left[\mathrm{Si}\,\frac{\pi}{2c}(\chi+2id) - \mathrm{Si}\,\frac{\pi}{2c}(\chi-2id)\right]\cosh\frac{\pi d}{2c} - \left[\mathrm{Ci}\,\frac{\pi}{2c}(\chi+2id) + \mathrm{Ci}\,\frac{\pi}{2c}(\chi-2id)\right]\sinh\frac{\pi d}{2c}$$

again F and G are real functions of x, x_1, and d.

5. The shape of the duct

Having calculated the components u, v of the perturbation velocity on the outside of the duct and wake for a given vorticity distribution, the slope of the duct surface can be found since

$$\frac{dy}{dx} = \frac{v}{u+U} \qquad \text{.............(19)}$$

where U is the free stream velocity.

This slope, integrated, gives the shape of the duct.

As can be seen from section 4 of this Appendix, the components of the perturbation velocity are complicated even for simple vorticity distributions, and an exact integral is unlikely to be found. Thus a numerical integration is suggested in all cases.

To obtain a second approximation to the shape of the duct, the values of u and v are found with the assumed vorticity shape, and the shape recalculated as shown above. Better approximations can be obtained by further iterations.

There is, obviously, just one shape for a given vorticity distribution but, provided the total vorticity at the trailing edge is equal to the constant value of the vorticity in the wake, different shapes can be obtained by combining any number of vorticity distributions over the chord of the thin aerofoils replacing the duct.

6. The Pressure Distribution over the duct

From the perturbation velocity components it is also possible to calculate the pressure distribution over the duct surface. If p_o is the free stream pressure and $p(x)$ the pressure at any point x of the duct surface

$$p(x) - p_o = -\tfrac{1}{2}\rho(u^2 + 2uU + v^2)$$

or

$$\frac{p(x) - p_o}{\frac{1}{2}\rho U^2} = -\frac{1}{U^2}(u^2 + 2uU + v^2) \qquad \ldots\ldots\ldots\ldots(20)$$

The effect of compressibility can be included by using the result due to Glauert;

$$\frac{c_{pc}}{c_{pi}} = \frac{1}{\sqrt{1-M^2}} \qquad \ldots\ldots\ldots\ldots(21)$$

where c_{pc} is the pressure coefficient in the compressible flow

c_{pi} is the pressure coefficient in the corresponding incompressible flow

and M is the free stream Mach number.

7. The 'staggered' duct

If the x-coordinates of the leading edges of the upper and lower aerofoils are different, representing a staggered duct, the perturbation velocity can still be calculated although here there is not the simplification which previously was the case due to symmetry.

Consider the duct formed by the upper aerofoil $y = y_1(x)$, $- c_1/2 \leq x \leq c/2$ and the lower aerofoil $y = - y_2(x)$; $- c_2/2 \leq x \leq c/2$ where $y_1\left(\frac{c}{2}\right) = y_2\left(\frac{c}{2}\right) = d$.
For such a configuration the perturbation velocity at any point (x,y) can be written in terms of its longitudinal and normal components as

$$u = - \frac{1}{2\pi}\left\{\int_{-c_1/2}^{\infty} \gamma_1(x_1) \frac{y - y_1}{(x-x_1)^2+(y-y_1)^2} dx_1 + \int_{-c_2/2}^{\infty} \gamma_2(x_1) \frac{y + y_2}{(x-x_1)^2+(y+y_2)^2} dx_1\right. \qquad \text{.............}(23)$$

and

$$v = \frac{1}{2\pi}\left\{\int_{-c_1/2}^{\infty} \frac{\gamma_1(x_1)(x-x_1)dx_1}{(x-x_1)^2+(y-y_1)^2} + \int_{-c_2/2}^{\infty} \frac{\gamma_2(x_1)(x-x_1)dx_1}{(x-x_1)^2+(y+y_2)^2}\right. \qquad \text{.............}(24)$$

where $\gamma_1(x_1)$ and $+\gamma_2(x_2)$ are the vorticity distributions representing the upper and lower thin aerofoils respectively, and y_1 and y_2 are functions of x.

Since constant vorticity K is assumed in the wake, (23) and (24) can be rewritten as

$$u = - \frac{1}{2\pi}\left\{\int_{-c_1/2}^{c/2} \frac{\gamma_1(x_1)(y-y_1)dx}{(x-x_1)^2+(y-y_1)^2} + \int_{-c_2/2}^{c/2} \frac{\gamma_2(x_1)(y+y_2)dx}{(x-x_1)^2+ (y+y_2)^2}\right.$$

$$- \frac{K}{2\pi} \int_{c/2}^{\infty}\left\{\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \frac{y+y_2}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1 \qquad \text{.............}(25)$$

$$v = \frac{1}{2\pi}\left\{\int_{-c_1/2}^{c/2} \frac{\gamma_1(x_1)(x-x_1)dx_1}{(x-x_1)^2+(y-y_1)^2} + \int_{-c_2/2}^{c/2} \frac{\gamma_2(x_1)(x-x_1)dx_1}{(x-x_1)^2+(y+y_2)^2}\right\}$$

$$+ \frac{K}{2\pi}\int_{c/2}^{\infty}(x-x_1)\left\{\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1 \qquad \ldots\ldots\ldots\ldots(26)$$

y_1 and y_2 are not, in general, equal even in the wake, but as a first approximation, particularly in the wake, y_1 and y_2 would be taken equal to a constant d.

The shape of each side of the duct and wake can be calculated as in section 5 of this appendix, and the pressure distribution as in section 6.

8. The inverse problem

8.1. In the previous sections of this appendix a vorticity distribution has been assumed, and the perturbation velocity components calculated, from which the shape of a duct and its pressure distribution have been found. A more direct approach is to take a given shape and from it to find the vorticity distribution required to represent the duct.

8.2. The exact integral equation for the vorticity distribution

The slope of the thin profiles making up the duct is given in terms of the perturbation velocity components as

$$\frac{dy}{dx} = \frac{v}{U+u} \qquad \ldots\ldots\ldots\ldots(27)$$

Now, for a symmetrical duct at zero incidence, the normal component of the perturbation velocity at (x,y) is given in terms of the vorticity distribution as, (equation 5),

$$v(x,y) = \frac{1}{2\pi}\int_{-c/2}^{\infty}\gamma(x_1)(x-x_1)\left(\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_1)^2}\right) dx_1$$

where $y = \pm y_1(x_1)$ is the locus of the vorticity distribution.

Assume now that the vorticity distribution is constant in the wake, and thus the effect at (x,y) of the wake vorticity is

$$v_W(x,y) = \frac{K}{2\pi} \int_{c/2}^{\infty} (x-x_1) \left(\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_1)^2} \right) dx_1 \qquad \ldots\ldots\ldots\ldots(28)$$

Substituting from (28) into (27) we have

$$v(x,y) - v_W(x,y) = \frac{1}{2\pi} \int_{-c/2}^{c/2} \gamma(x_1)(x-x_1) \left(\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_1)^2} \right) dx_1 \qquad \ldots\ldots\ldots\ldots(29)$$

and thus we obtain an integral equal for the vorticity distribution representing the aerofoils in terms of the normal perturbation velocity given by the shape of the profile, and the known constant vorticity in the wake.

As it stands (29) is not solvable exactly. If, however, we assume the vorticity distribution to lie on the lines $y_1 = \pm d$, (29) can be simplified to

$$V(x,d) \equiv v(x,d) - v_W(x,d) = \int_{-c/2}^{c/2} \gamma(x_1) \left[\frac{4d^2}{(x-x_1)^2+4d^2} \right] \frac{dx_1}{x-x_1} \qquad \ldots\ldots\ldots\ldots(30)$$

which, again, is not readily solvable.

9. An approximate solution using slender body theory

9.1. The perturbation velocity

Consider a duct made up of two thin aerofoils $y = \pm y(x)$, $0 \leqslant x \leqslant c$ and their wake. At any point (x,y) the longitudinal perturbation velocity is, from (4)

$$u(x,y) = -\frac{1}{2\pi} \int_{0}^{\infty} \gamma(x_1) \left\{ \frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \frac{y+y_1}{(x-x_1)^2+(y+y_1)^2} \right\} dx_1$$

To obtain the longitudinal component of the perturbation velocity on the duct surface we must evaluate this integral as $y \longrightarrow y_1(x_1)$

i.e.

$$u(x,y_1) = \frac{1}{2\pi}\int_0^\infty \gamma(x_1)\,\frac{2y_1}{(x-x_1)^2+4y_1^2}\,dx_1 - \frac{1}{2\pi}\lim_{y\rightarrow y_1}\int_0^\infty \gamma(x_1)\,\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2}\,dx \qquad \ldots\ldots\ldots\ldots(31)$$

It is immediately obvious that the second integral is a singular integral of the form encountered in thin aerofoil theory and hence (31) may be written

$$u(x,y_1) = \frac{1}{2\pi}\int_0^\infty \gamma(x_1)\,\frac{2y_1}{(x-x_1)^2+4y_1^2}\,dx_1 \mp \frac{1}{2}\,\frac{\gamma(x)}{\sqrt{1+\left(\frac{dy_1}{dx}\right)^2}} \qquad \ldots\ldots(32)$$

the negative or positive sign being taken according as we consider the outer or inner surface of the duct.

Now, taking the length of the duct as c, we can express all the lengths as non-dimensional multiples of c in the form

$$x = Xc, \quad x_1 = X_1 c \quad \text{and} \quad y_1 = \varepsilon Y(X_1)c$$

Substituting these into (32) we have

$$u(X,Y) = \frac{1}{2\pi}\int_0^\infty \gamma(X_1)\,\frac{2Y\varepsilon}{(X-X_1)^2+4\varepsilon^2Y^2}\,dX_1 \mp \frac{1}{2}\,\gamma(X)\Big/\left(1+\varepsilon^2\left[Y'(X)\right]^2\right)^{\frac{1}{2}}$$

which, expanded in ascending powers of ε, becomes

$$u(X,Y) = \frac{1}{2\pi}\int_0^\infty \frac{\gamma(X_1)2\varepsilon Y}{(X-X_1)^2}\left\{1 - \frac{4\,\varepsilon^2Y^2}{(X-X_1)^2} + \left[\frac{4\,\varepsilon^2Y^2}{(X-X_1)^2}\right]^2 - \ldots\right\}\,dX_1$$

$$\mp \frac{1}{2}\,\gamma(X)\left\{1 - \frac{1}{2}\,\varepsilon^2\left[Y'(X)\right]^2 + \frac{3}{8}\,\varepsilon^4\left[Y'(X)\right]^4 \ldots\right\} \qquad \ldots\ldots(33)$$

If we now assume a slender duct, i.e. ε small compared with unity, the terms of third and higher order of ε in (33) can be neglected giving the longitudinal component of the perturbation velocity on the duct surface as

$$u(X,Y) = \frac{\varepsilon}{\pi}\int_0^{\infty}\frac{\gamma(X_1)Y\,dX_1}{(X-X_1)^2} \mp \frac{1}{2}\gamma(X)\left\{1-\frac{1}{2}\varepsilon^2\left[Y'(X)\right]^2\right\} \quad ..(34)$$

Thus the actual longitudinal component of the perturbation velocity on the outside surface of the duct and wake is

$$u_o(X) = -\frac{1}{2}\gamma(X)\left[1-\frac{1}{2}\varepsilon^2\left\{Y'(X)\right\}^2\right] + \frac{\varepsilon}{\pi}\int_0^{\infty}\frac{\gamma(X_1)Y(X_1)dX_1}{(X-X_1)^2} \quad (35)$$

and on the inside

$$u_i = u_o + \gamma(X)\left\{1-\frac{1}{2}\varepsilon^2\left[Y'(X)\right]^2\right\} \quad ....(36)$$

neglecting terms of third and higher order in ε.

The normal component of the perturbation velocity can be found immediately since the flow must be tangential to the duct surface, i.e.

$$v = (U+u)\frac{dy}{dx} .$$

9.2. The stream functions and vorticity

The complex perturbation potential for the fluid motion about the duct is

$$\omega = \frac{i}{2\pi}\int_0^{\infty}\gamma(x_1)\left\{\log(z-z_1) - \log(z-\bar{z}_1)\right\}dx_1$$

where $z_1 = x_1 + iy_1$ and $y_1 = y_1(x)$

Thus the stream function for the perturbation is

$$\psi = \frac{1}{2\pi}\int_0^{\infty}\gamma(x_1)\left\{\log\sqrt{\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}}\,dx_1\right.$$

or

$$\psi = \frac{1}{4\pi}\int_0^{\infty}\gamma(x_1)\left\{\log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}\right\}dx_1 \quad ...(37)$$

It is seen that on the x-axis,

$$\psi = 0 \quad ....(38)$$

Thus the stream function for the flow about such a duct in a uniform stream in the x-direction of speed U is

$$- Uy + \psi \qquad \text{.............}(39)$$

Now on the duct surface, as on any streamline, the stream function is constant and equal, in magnitude, to half the volume flow (Q_o) through the duct. Thus, from (37) and (39), on the duct surface

$$\frac{1}{2} Q_o = - Uu + \frac{1}{4\pi} \int_0^\infty \gamma(x_1) \log \frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2} dx_1.$$

or in dimensionless form, using (32),

$$\frac{Q_o}{2U\varepsilon c} = - Y(X) + \frac{1}{4\pi U\varepsilon} \int_0^\infty \gamma(X_1) \log \frac{(X-X_1)^2+\varepsilon^2(Y-Y_1)^2}{(X-X_1)^2+\varepsilon^2(Y+Y_1)^2} dX_1 \qquad (40)$$

The integral equation (40) can be satisfied by

$$\gamma(x) = \frac{Q_o}{2\varepsilon c Y(x)} - U + O(\varepsilon)$$

giving the vortex distribution as

$$\gamma(x) = \frac{Q_o}{2y} - U + O(\varepsilon) \qquad \text{.............}(41)$$

APPENDIX B

The two-dimensional duct at incidence

1. The methods of thin aerofoil theory used in the previous appendix are here applied to a two-dimensional duct at incidence a_0 to a uniform stream of speed U at infinity.

The duct considered is formed by two thin aerofoils $y = \pm y(x)$, $-c \leqslant x \leqslant 0$; $y(x)$ being always positive. The wakes upon which the vorticity distributions lie are not now mirror images of each other in the x-axis and hence we take the wake streamline of the upper thin aerofoil $y = y(x)$ to be given by

$$y = y_1(x) \qquad 0 \leqslant x \leqslant \infty$$

where $\quad y(0) = y_1(0)$

and that of the lower aerofoil to be given by

$$y = y_2(x) \qquad 0 \leqslant x \leqslant \infty$$

where $\quad -y(0) = y_2(0)$

such that $y_1(x)$ and $y_2(x)$ are both parallel to $y = x \tan a_0$ for large x.

The configuration is shown in figure 7.

2. The vorticity distribution

2.1. The vorticity in the wake

For reasons given in the previous appendix the vorticity distributions on the two wake streamlines are taken to be constant and of equal magnitude but of different sign i.e.

$$\gamma_1(x) = -\gamma_2(x) = K \; ; \; 0 \leqslant x \leqslant \infty \quad \ldots\ldots\ldots\ldots(1)$$

2.2. The vorticity representing the duct

Previously a vortex distribution has been assumed and this has dictated the shape of the duct. If, however, distributions of vorticity $\gamma(x)$ and $-\gamma(x)$ are assumed to lie on $y = + y(x)$ and $y = - y(x)$ respectively, then due to the bending of the wake, the duct represented by these vorticity distributions will not in general be symmetrical. It is therefore assumed that the symmetrical duct is represented by distributions of vorticity $\gamma_1(x)$ replacing the upper thin aerofoil and $-\gamma_2(x)$ replacing the lower.

3. The perturbation velocity

3.1. General expressions

As in section 3 of Appendix A, the longitudinal and normal components of the perturbation velocity at any point (x,y) are

$$u = -\frac{1}{2\pi}\int_{-c}^{\infty}\left\{\frac{(y-y_1)\gamma_1(x_1)}{(x-x_1)^2+(y-y_1)^2} - \frac{(y+y_2)\gamma_2(x_1)}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1 \quad \ldots(2)$$

and

$$v = \frac{1}{2\pi}\int_{-c}^{\infty}(x-x_1)\left\{\frac{\gamma_1(x_1)}{(x-x_1)^2+(y-y_1)^2} - \frac{\gamma_2(x_1)}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1 \quad (3)$$

or, on expanding the vorticity distributions,

$$u(x,y) = -\frac{1}{2\pi}\int_{-c}^{\infty}\left\{\frac{(y-y_1)\gamma_1(x_1)}{(x-x_1)^2+(y-y_1)^2} - \frac{(y+y_2)\gamma_2(x_1)}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1$$

$$-\frac{K}{2\pi}\int_{0}^{\infty}\left\{\frac{(y-y_1)}{(x-x_1)^2+(y-y_1)^2} - \frac{(y+y_2)}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1 \quad (4)$$

Following the assumptions of thin aerofoil theory the vorticity representing the duct walls is taken as lying upon the lines $y = \pm d, \; -c \leqslant x \leqslant 0$.

Hence, (4) becomes

$$u(x,y) = -\frac{1}{2\pi}\int_{-c}^{0}\left\{\frac{(y-d)\gamma_1(x_1)}{(x-x_1)^2+(y-d)^2} - \frac{(y+d)\gamma_2(x_1)}{(x-x_1)^2+(y+d)^2}\right\} dx_1$$

$$-\frac{K}{2\pi}\int_{0}^{\infty}\left\{\frac{(y-y_1)}{(x-x_1)^2+(y-y_1)^2} - \frac{y+y_2}{(x-x_1)^2+(y+y_2)^2}\right\} dx_1 \quad (5)$$

and likewise (3) becomes

$$v(x,y) = \frac{1}{2\pi}\int_{-c}^{0}(x-x_1)\left\{\frac{\gamma_1(x_1)}{(x-x_1)^2+(y-d)^2} - \frac{\gamma_2(x_1)}{(x-x_1)^2+(y+d)^2}\right\}dx_1$$

$$+\frac{K}{2\pi}\int_{0}^{\infty}(x-x_1)\left\{\frac{1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{(x-x_1)^2+(y+y_2)^2}\right\}dx_1 \qquad \ldots\ldots\ldots\ldots(6)$$

Once forms for $\gamma_1(x)$ and $\gamma_2(x)$ are known the first integral in each of (5) and (6) are calculable with no more difficulty than that incurred in section 4 of the previous appendix.

3.2. The shape of the wake

To proceed with the integrals over the wake some form for y_1 and y_2 must be assumed. Equation (18) shows that the wake streamlines will have very nearly exponential form. Also both wake streamlines must be parallel to $y = x \tan a_o$ for large x and be continuous with their particular thin aerofoil,

i.e. at $x = 0$ $\quad y_1 = d$; $y_2 = -d$; $\frac{dy_1}{dx} = \frac{dy_2}{dx} = 0$

and for large x $\quad \frac{dy_1}{dx} = \frac{dy_2}{dx} = \tan a_o$

The simplest form for y_1 and y_2 fulfilling these conditions is

$$y_1 = \tan a_o\left[e^{-x} + x - 1\right] + d$$
$$y_2 = \tan a_o\left[e^{-x} + x - 1\right] - d \qquad \ldots\ldots\ldots\ldots(7)$$

and, substituting from (7) into the second integrals of (5) and (6), we have integrals of the form

$$\int_{0}^{\infty}\frac{y - d - \tan a_o\left[e^{-x_1}+x_1-1\right]}{(x-x_1)^2+\left[d+\tan a_o(e^{-x_1}+x_1-1)-y\right]^2}$$

The integrals over the wake can be simplified by assuming that the wake streamlines are straight and parallel to the free stream direction. Thus we take

$$y_1 = d + x \tan \alpha_o$$

$$y_2 = -d + x \tan \alpha_o \qquad \text{..........(8)}$$

and the wake integral in (5) becomes

$$\int_o^\infty \left\{ \frac{y - d - x_1\theta_o}{(x-x_1)^2+(y-d-x_1\theta_o)^2} - \frac{y + d - x_1\theta_o}{(x-x_1)^2+(y+d-x_1\theta_o)^2} \right\} dx_1 \qquad \text{..........(9)}$$

where $\theta_o = \tan \alpha_o$.

The idealised configuration is shown in figure 8.

3.3. <u>The perturbation velocity for the idealised duct</u>

If we put

$$A = 1 + \theta_o^2$$

$$B = -2(x+\theta_o [y-d]) \;;\; B' = -2(x+\theta_o [y+d])$$

$$C = x^2+(y-d)^2 \;;\; C' = x^2+(y+d)^2 \qquad \text{.......(10)}$$

(9) becomes

$$\int_o^\infty \frac{\theta_o x_1-(y-d)}{Ax_1^2+Bx_1+C} dx_1 - \int_o^\infty \frac{\theta_o x_1 -(y+d)}{Ax_1^2+B'x_1+C'} dx_1$$

$$= -\frac{\theta_o}{2A} \log_e \frac{C}{C'} + T - T' \qquad \text{.......(11)}$$

where $$T = \left\{-(y-d) - \frac{B\theta_o}{2A}\right\} \int_o^\infty \frac{dx_1}{Ax_1^2+Bx_1+C}$$

and $$T' = \left\{-(y+d) - \frac{B'\theta_o}{2A}\right\} \int_o^\infty \frac{dx_1}{Ax_1^2+B'x_1+C}$$

Now $$B^2-4AC = 4\left[x+\theta_o(y-d)\right]^2 - 4(1+\theta_o^2)\left[x^2+(y-d)^2\right]$$

$$= -4\left[\theta_o x -(y-d)\right]^2$$

and is thus essentially negative or zero.

Thus (ref. 10) for B^2-4AC negative i.e. y not on the wake streamline

$$T = \frac{-(y-d) - \frac{B\theta_o}{2A}}{\theta_o x - [y-d]} \left[\frac{\pi}{2} - \tan^{-1} \frac{B}{\sqrt{4AC-B^2}}\right]$$

and

$$T' = \frac{-(y+d) - \frac{B'\theta_o}{2A}}{\theta_o x - (y+d)} \left[\frac{\pi}{2} - \tan^{-1} \frac{B'}{\sqrt{4AC'-B'^2}}\right] \quad \ldots\ldots(12)$$

Thus from (9), (10), (11) and (12) the wake integral becomes

$$\frac{-\theta_o}{1+\theta_o^2} \log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} + \frac{1}{1+\theta_o^2}\left[\tan^{-1} \frac{x+\theta_o[y-d]}{\theta_o x-(y-d)} - \tan^{-1} \frac{x+\theta_o[y+d]}{\theta_o x-[y+d]}\right]$$

which, when $\theta_o = 0$ and $y=d$ reduces to $\frac{\pi}{2} + \tan \frac{x}{2d}$ and agrees with (A9) and hence the longitudinal component of the perturbation velocity is, for the simplified wake

$$u(x,y) = -\frac{1}{2\pi} \int_{-c}^{0} \left\{\frac{(y-d)\gamma_1(x_1)}{(x-x_1)^2+(y-d)^2} - \frac{(y+d)\gamma_2(x_1)}{(x-x_1)^2+(y+d)^2}\right\} dx_1$$

$$+ \frac{K}{2\pi(1+\tan^2 \alpha_o)} \left\{\tan \alpha_o \log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} - \left[\tan^{-1} \frac{x+\theta_o(y-d)}{\theta_o x-(y-d)} - \tan^{-1} \frac{x+\theta_o(y+d)}{x\theta_o-(y+d)}\right]\right. \quad \ldots\ldots\ldots\ldots(13)$$

Similarly, the integral over the wake in (6) becomes, using (8) and (10)

$$\int_0^{\infty} (x-x_1) \left[\frac{1}{Ax_1^2 + Bx_1 + C} - \frac{1}{Ax_1^2 + B'x_1 + C'}\right] dx_1$$

which is equal to (ref. 10)

$$- \frac{1}{2A} \log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} - \frac{2\left(x + \frac{B}{2A}\right)}{\sqrt{4AC - B^2}} \left[\frac{\pi}{2} - \tan^{-1} \frac{B}{\sqrt{4AC-B^2}}\right]$$

$$+ \frac{2\left(x + \frac{B'}{2A}\right)}{\sqrt{4AC'-B'^2}} \left[\frac{\pi}{2} - \tan^{-1} \frac{B'}{\sqrt{4AC'-B'^2}}\right]$$

$$= - \frac{1}{2(1+\theta_o^2)} \log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} - \frac{\theta_o}{2(1+\theta_o^2)} \left[\tan^{-1} \frac{x+\theta_o(y-d)}{\theta_o x-(y-d)} - \tan^{-1} \frac{x+\theta_o(y+d)}{\theta_o x-(y+d)}\right]$$

and hence the normal component of the perturbation velocity is

$$v(x,y) = \frac{1}{2\pi} \int_{-c}^{0} (x-x_1) \left\{\frac{\gamma_1(x_1)}{(x-x_1)^2+(y-d)^2} - \frac{\gamma_2(x_1)}{(x-x_1)^2+(y+d)^2}\right\} dx_1$$

$$- \frac{K}{4\pi(1+\tan^2 a_o)} \left[\log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} - \tan a_o \left\{\tan^{-1} \frac{x+\tan a_o(y-d)}{x \tan a_o-(y-d)} - \tan^{-1} \frac{x+(y+d)\tan a_o}{x \tan a_o-(y+d)}\right\}\right] \quad \ldots\ldots\ldots\ldots(14)$$

If y lies on the wake streamline $y_1 = d + x \tan a_o$, $B^2 - 4AC$ is zero and hence

$$T = - \frac{2}{B} \left[(y-d) + \frac{B\,\theta_o}{2A}\right] \quad \ldots\ldots\ldots\ldots(15)$$

T' remaining unaltered.

Thus

$$u(x,y) = - \frac{1}{2\pi} \int_{-c}^{0} \left\{\frac{(y-d)\gamma_1(x_1)}{(x-x_1)^2+(y-d)^2} - \frac{(y+d)\gamma_2(x_1)}{(x-x_1)^2+(y+d)^2}\right\} dx_1$$

$$+ \frac{K}{2\pi(1+\tan^2 a_o)} \left\{\tan a_o \log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} + \tan^{-1} \left[\frac{x+\theta_o(y-d)}{x\theta_o-(y+d)} - \frac{\pi}{2}\right] + \frac{x\theta_o+(y-d)}{x+\theta_o(y-d)}\right\} \quad \ldots\ldots\ldots\ldots(16)$$

Also,

$$v(x,y) = \frac{1}{2\pi} \int_{-c}^{0} (x-x_1)\left\{\frac{\gamma_1(x_1)}{(x-x_1)^2+(y-d)^2} - \frac{\gamma_2(x_1)}{(x-x_1)^2+(y+d)^2}\right\} dx_1$$

$$- \frac{K}{4\pi(1+\tan^2 a_o)} \left[\log_e \frac{x^2+(y-d)^2}{x^2+(y+d)^2} - \tan a_o \left\{\frac{x\theta_o+(y-d)}{x+\theta_o(y-d)} + \tan^{-1} \frac{x+\theta_o(y+d)}{x\theta_o-(y+d)} - \frac{\pi}{2}\right]\right.$$

.............(17)

Similarly $u(x,y)$ and $v(x,y)$ can be calculated when y lies on the lower wake streamline in which case $B'^2 - 4AC'$ will be zero.

4. The duct shape

The duct shape is found from the modified boundary condition over a body at incidence

$$\frac{dy}{dx} = \frac{v + U \sin a}{u + U \cos a} .$$

APPENDIX C

The Axi-symmetric Duct in Potential Flow

1. The axi-symmetric duct is represented by a surface distribution of vorticity made up of an infinite number of coaxial circular vortex rings of radius R and strength γ per unit axial length. Both R and γ are functions of the axial distance z of the ring from the lip of the duct. Reasoning on parallel lines to that given in sections 1 and 2 of appendix A shows that the vorticity extends into the wake and that the equivalent homogeneous flow about the duct and its wake can be represented by a semi-infinite distribution of coaxial vortex rings. (See fig. 9).

2. The circular vortex ring

Consider a circular vortex ring of radius R and strength K, such that the normal through the centre is the z-axis of cylindrical polar coordinates x, r, ϕ. Consider also the point $P\ (x,r,\phi)$; see fig. 10.

In cases of axial symmetry the vector potential $\vec{\Theta}$ at P, as defined by Clebsch, has only one component which is along the normal to the axial plane (i.e. the plane through the axis and P). The unit vector along this normal is $\vec{n}$. The magnitude of the vector potential is

$$\Theta = \vec{n}.\vec{\Theta} = \frac{K}{2\pi} \int_c \frac{\vec{n}.d\vec{s}}{r}$$

$$= \frac{K}{4\pi} \int_0^{2\pi} \frac{R \cos (\phi' - \phi)}{PQ^2} \quad d\phi'$$

where Q is the point $(0, R, \phi')$.

Writing $\phi' - \phi = 2\chi$

$$\Theta = \frac{KR}{\pi} \int_0^{\pi/2} \frac{(2\cos^2\chi - 1)d}{\left(\frac{4Rr}{k^2}\right)^{\frac{1}{2}} (1 - k^2\cos^2\chi)^{\frac{1}{2}}} \quad \text{........(1)}$$

where $$k^2 = \frac{4rR}{z^2 + (r+R)^2} \quad \text{..............(2)}$$

which can be rewritten as

$$\Theta = \frac{K}{2\pi}\left(\frac{R}{r}\right)^{\frac{1}{2}}\left\{\frac{2-k^2}{k}F - \frac{2}{k}E\right\} \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

where F and E are complete elliptic integrals of the first and second kinds respectively.

The vector velocity $\vec{q} = u_r\vec{r} + u_z\vec{z}$ is given by

$$\vec{q} = \text{curl}\,(\theta\,\vec{n})$$

or

$$u_r = -\frac{\partial\Theta}{\partial z}\ ;\ u_z = \frac{1}{r}\frac{\partial}{\partial r}(r\,\Theta)\ ;\ u_{\phi} = 0 \qquad \ldots\ldots(4)$$

3. The vector potential for the duct and wake

Consider now the duct of length c and wake consisting of a semi-infinite distribution of such circular vortex rings stretching from zero to infinity. The magnitude of the vector potential at $P(z,r,\phi)$ due to the duct and wake is, from (3) by integration

$$\Theta = \frac{1}{2\pi}\int_0^{\infty} K(z')\left(\frac{R(z')}{r}\right)^{\frac{1}{2}}\left\{\frac{2-k^2}{k}F - \frac{2}{k}E\right\}dz' \quad ..(5)$$

where now $k^2 = \dfrac{4\,r\,R(z')}{(z-z')^2 + \left[r+R(z')\right]^2}$

and hence E and F are functions of z'.

4. The Perturbation Velocity

The axial component of the perturbation velocity at P is, from (4) and (5),

$$u_z(z,r) = \frac{1}{2\pi}\int_0^{\infty} K(z')\left[F + \frac{R^2(z')-r^2-(z-z')^2}{(z-z')^2+\left[R(z')-r\right]^2}E\right]\frac{dz'}{\left\{(z-z')^2+\left[R(z')+r\right]^2\right\}^{\frac{1}{2}}}$$

$$\ldots\ldots\ldots\ldots\ldots(6)$$

and the radial component is

$$u_r(z,r) = -\frac{1}{2\pi}\int_0^{\infty}\frac{K(z')(z-z')}{r}\left[k^2F - \frac{2+R(z')E}{(z-z')^2+(r-R(z'))^2}\right]$$

$$\frac{dz'}{\left\{(z-z')^2+\left[R(z')+r\right]^2\right\}^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots(7)$$

If P is taken on the duct surface then the mean axial velocity $(\bar{u}_z)$ is found by putting $r = R(z)$ in (6) and interpreting the improper integral so obtained by its Cauchy principal part. It has been shown (ref. 11) that

$$(\bar{u}_z)_{r=R(z)} = \frac{1}{4\pi}\int_0^{c}\left\{D_W + \frac{R(z')-R(z)}{2(1-k^2)R(z')}E_W\right\}\frac{k_W^3 K(z')dz'}{\left[R(z)R(z')\right]^{\frac{1}{2}}} \quad , \ldots(8)$$

where $$D_W = \int_0^{\pi/2}\frac{\sin^2\phi}{1-k_W^2\sin^2\phi}\,d\phi$$

and $$k_V^2 = \frac{4R(z)R(z')}{(z-z')^2+\left[R(z')+R(z)\right]^2} \quad , \quad \ldots\ldots\ldots\ldots(9)$$

for a finite thin duct of length c.

This result can be generalised to the case of a duct with wake. Since the vortex tube representing the wake is a stream tube we may write, for the duct and wake,

$$(\bar{u}_z)_{r=R(z)} = \frac{1}{4\pi}\int_0^{\infty}\left\{D_W + \frac{R(z')-R(z)}{2(1-k^2)R(z')}E_W\right\}\frac{k_W^3\ (z')dz'}{\left[R(z)\ R(z')\right]^{\frac{1}{2}}} \quad \ldots(10)$$

The actual longitudinal velocity is obtained by adding the contribution due to the local vorticity

i.e. $\pm\ \frac{1}{2}K\Big/\left\{1+\left[R'(z)\right]^2\right\}$

the negative sign referring to the flow outside the duct and wake, and the positive to the inside.
For the outside surface, assuming a slender duct, (i.e. $R(z)$ is small compared with c), we have, by expanding the elliptic integrals and taking the limit as $z \to z'$,

$$(u_z)_{r=R(z)} = 1 + \frac{1}{2} \int_0^{\infty} \frac{R^2(z')K(z')}{|z-z'|^3} \, dz'$$

$$- \left[\frac{3}{4} + \frac{1}{2} \log \frac{R(z)}{2c}\right] \frac{d^2}{dz^2} \left(R^2(z) K(z)\right) + 0 \left(\frac{R}{c}\right)^3 \quad (11)$$

The normal component of the perturbation velocity for a slender duct is

$$(u_r)_{r=R(z)} = \left\{(U + u_z) \frac{dr}{dz}\right\}_{r=R(z)} \quad \ldots\ldots\ldots\ldots (12)$$

5. The vorticity distribution

The vorticity distribution representing a slender axisymmetric duct and wake can be found by an extension of the method used in ref. 11 for a slender finite duct without wake.

The Stokes' stream function for the semi-infinite vorticity distribution in a uniform stream of speed U parallel to the axis of symmetry is

$$\psi(z,r) = - \frac{U r^2}{2}$$

$$+ \frac{1}{2\pi} \int_0^{\infty} \left\{\frac{(z-z')^2+R^2(z')+ r^2}{(z-z')^2+\left[R(z')+r\right]^2} F - E\right\}\left[(z-z')^2+\left\{R(z')+r\right\}^2\right]^{\frac{1}{2}} \gamma(z')dz'$$

Now the Stokes' stream function at any point (z,r) is related to the flux or volume flow Q_o per unit time through a circle of radius r normal to the axis of symmetry

i.e. $\qquad \psi = - \dfrac{Q_o}{2\pi}$

On the surface of the duct ψ must be constant since the duct surface is a stream-tube. Hence, the boundary condition is

$$-\frac{Q_o}{2\pi} = -U\frac{R^2(z)}{2} + \frac{1}{2\pi}\int_o^\infty \left[\frac{(z-z')^2+2R^2(z')}{(z-z')^2+4R^2(z')}F-E\right]\left[(z-z')^2+4R^2(z')\right]^{\frac{1}{2}}\gamma(z')dz'$$

which, for a slender duct, reduces to

$$-\frac{\varepsilon}{2\pi}\int_o^\infty k_W^2\, C_W\left[R(z)R(z')\right]^{\frac{1}{2}}\gamma(z')dz' = \frac{Q_o}{2\pi} - \tfrac{1}{2}\,UR^2(z) \quad ......(13)$$

where $$C_W = \int_o^{\pi/2}\frac{\sin^2\phi\,\cos^2\phi\, d\phi}{(1-k_W\sin^2\phi)^{3/2}}$$

and $\varepsilon = \frac{R(z)}{C}$ and is small.

Equation (13) is satisfied by

$$\gamma(z) = \frac{Q_o}{S(z)} - U + O(\varepsilon) \quad ....(14)$$

where $S(z) = \pi R^2(z)$ is the cross sectional area of the duct at $(z,R(z))$.

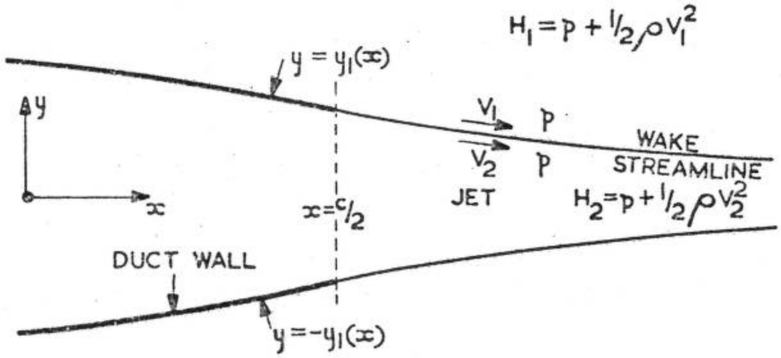


FIG. 1. THE POTENTIAL FLOW MODEL
INHOMOGENEOUS FLOW

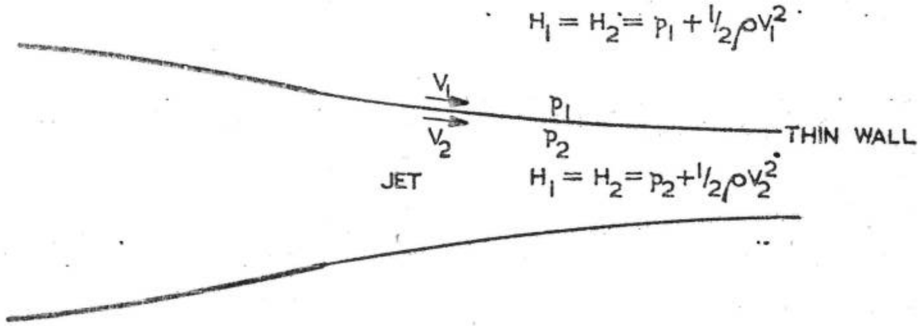
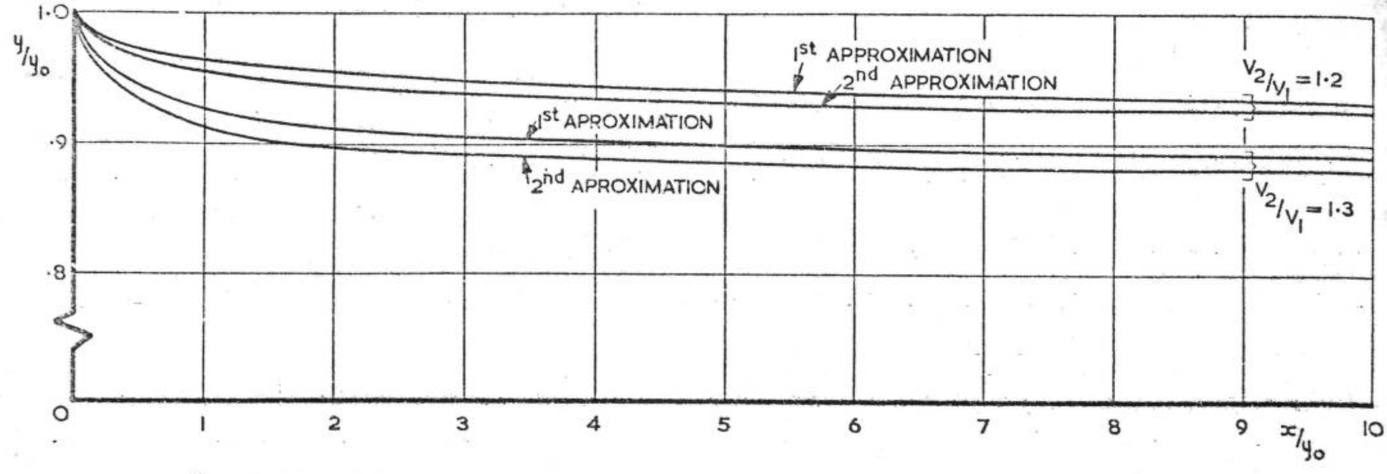


FIG. 2. THE POTENTIAL FLOW MODEL
EQUIVALENT HOMOGENEOUS FLOW



y = HALF DUCT WIDTH AT x

y_o = HALF DUCT WIDTH AT LEADING EDGE ($x = 0$)

FIG. 3. DUCT SHAPE FOR CONSTANT VORTICITY DISTRIBUTION.

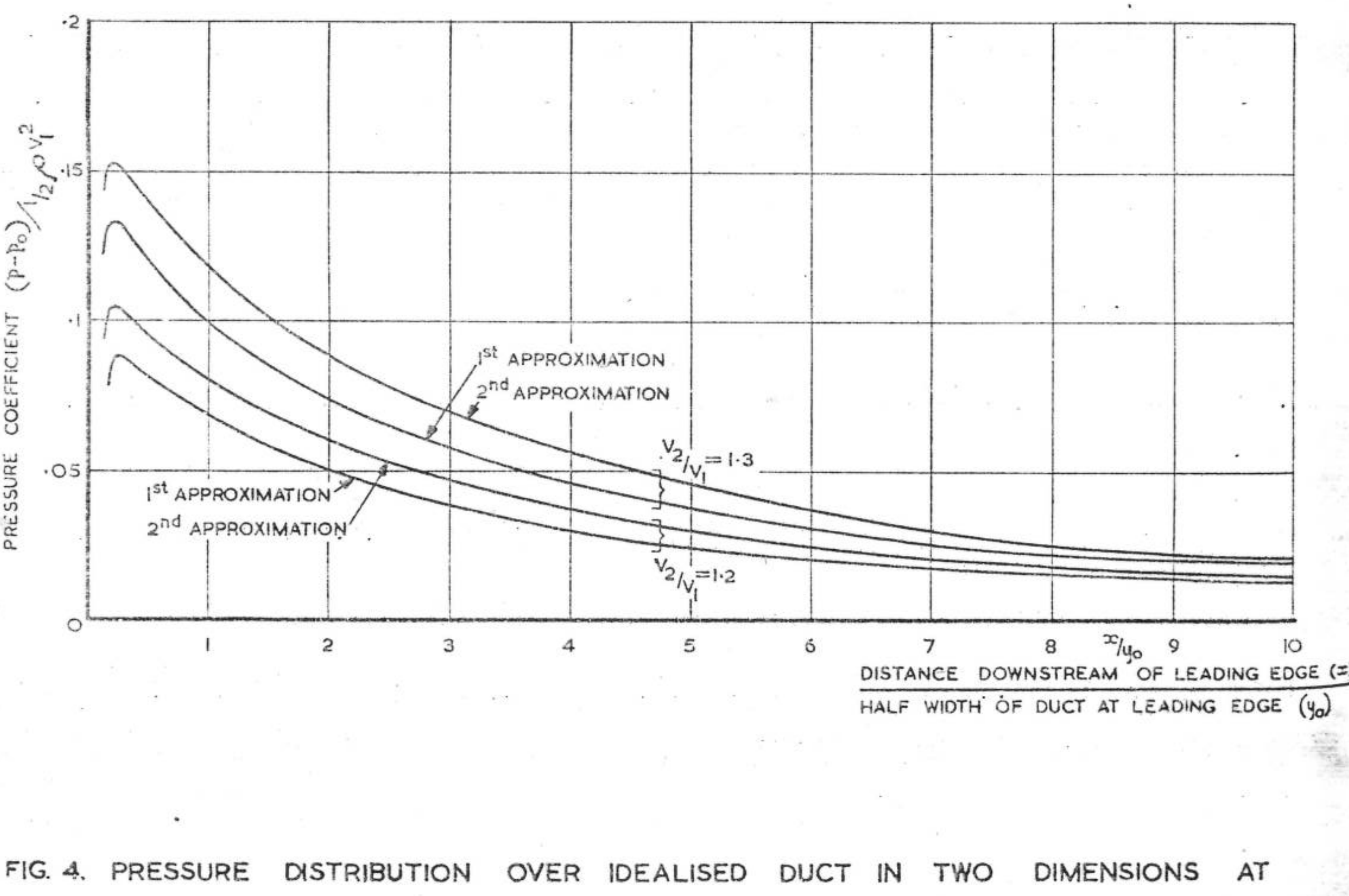


FIG. 4. PRESSURE DISTRIBUTION OVER IDEALISED DUCT IN TWO DIMENSIONS AT ZERO INCIDENCE. (CONSTANT VORTICITY DISTRIBUTION.)

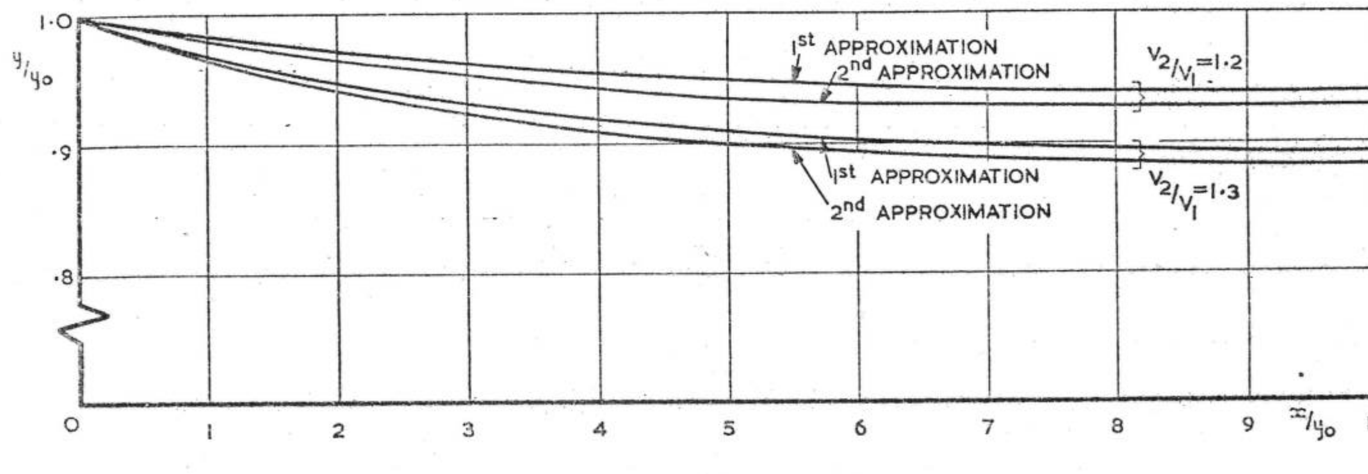


FIG. 5. DUCT SHAPE FOR LINEARLY INCREASING VORTICITY ON DUCT & CONSTANT IN WAKE (TRAILING EDGE CORRESPONDS TO $x/y_0 = 4$)

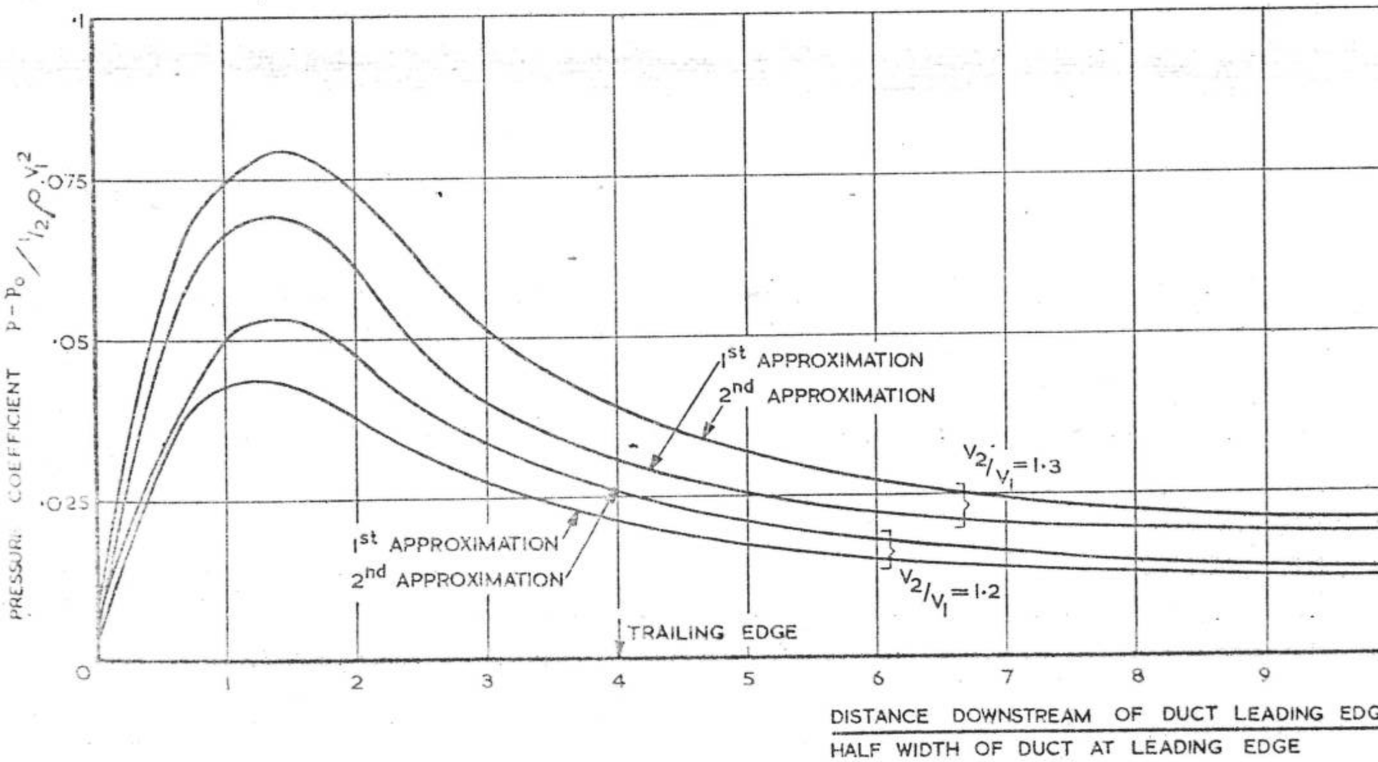


FIG. 6. PRESSURE DISTRIBUTION OVER IDEALISED DUCT IN TWO DIMENSIONS AT ZERO INCIDENCE. LINEAR VORTICITY ON DUCT CONSTANT VORTICITY IN WAKE

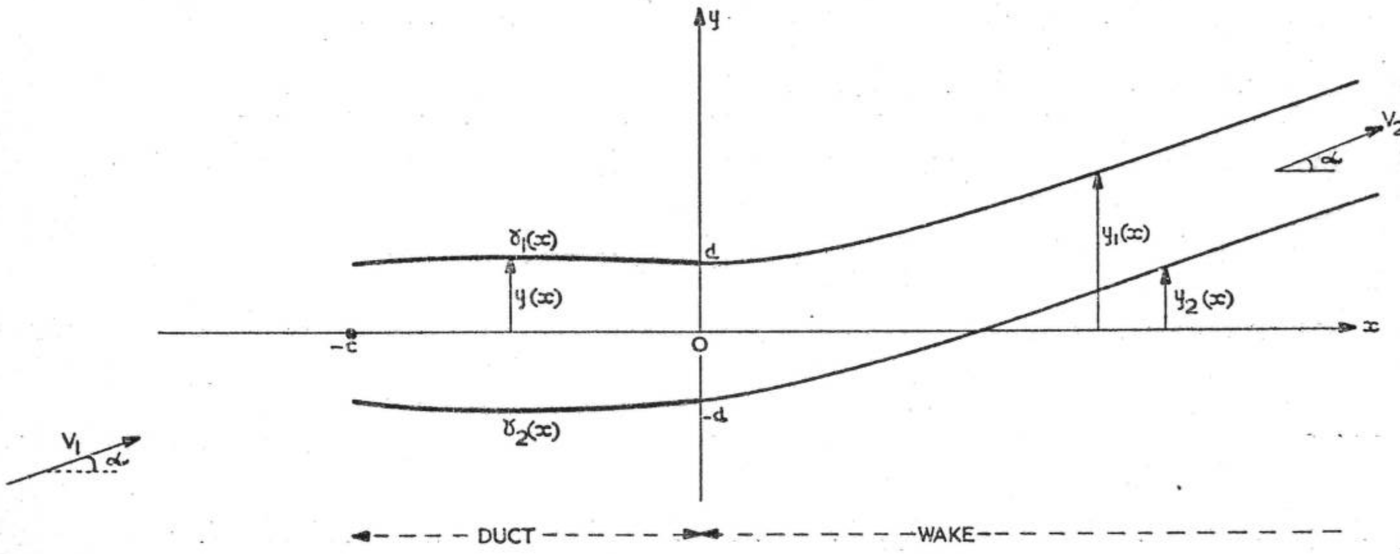


FIG 7. THE TWO DIMENSIONAL DUCT AT INCIDENCE.

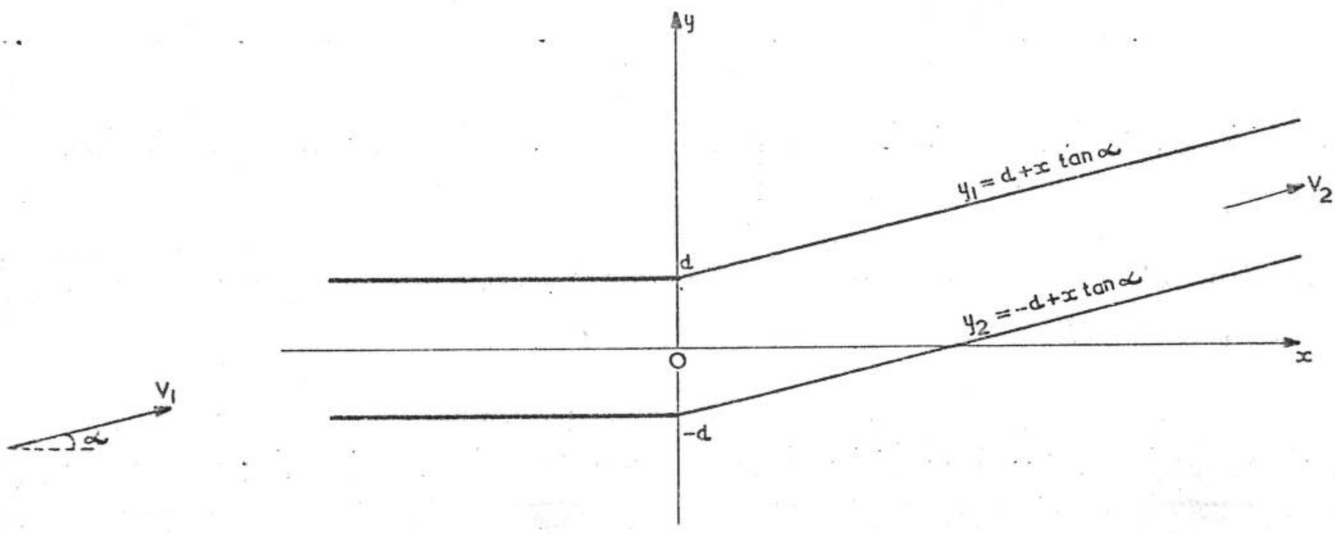


FIG. 8. THE IDEALISED TWO DIMENSIONAL DUCT AT INCIDENCE.

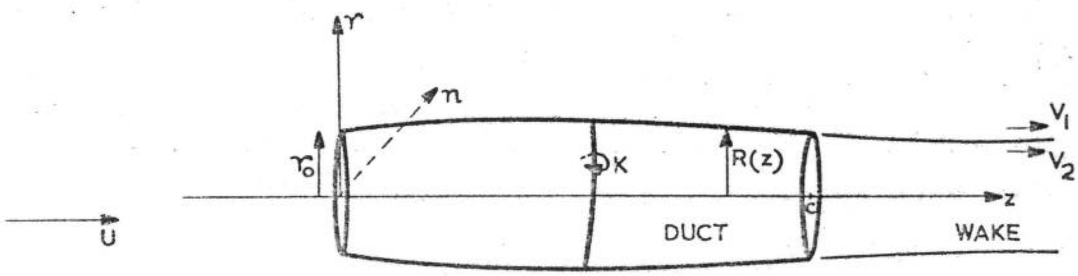


FIG. 9. THE AXISYMMETRIC DUCT & WAKE.

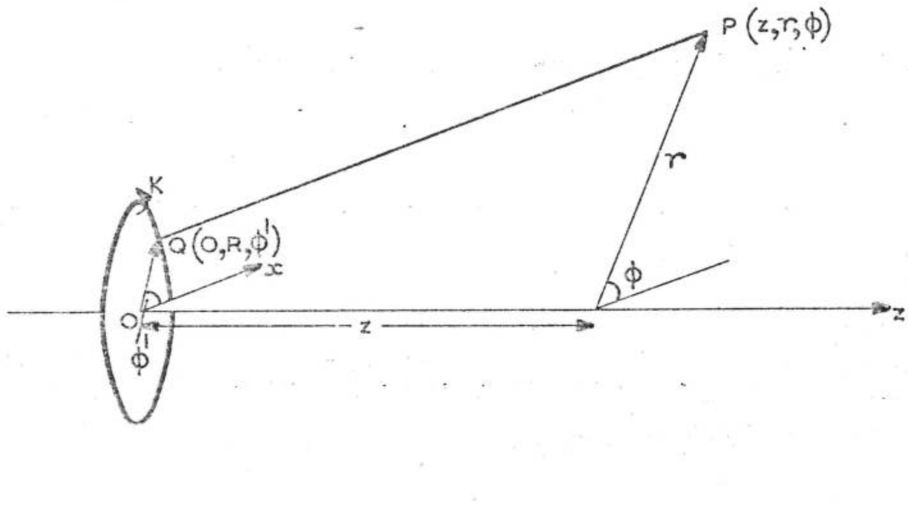


FIG. 10. THE CIRCULAR VORTEX RING.

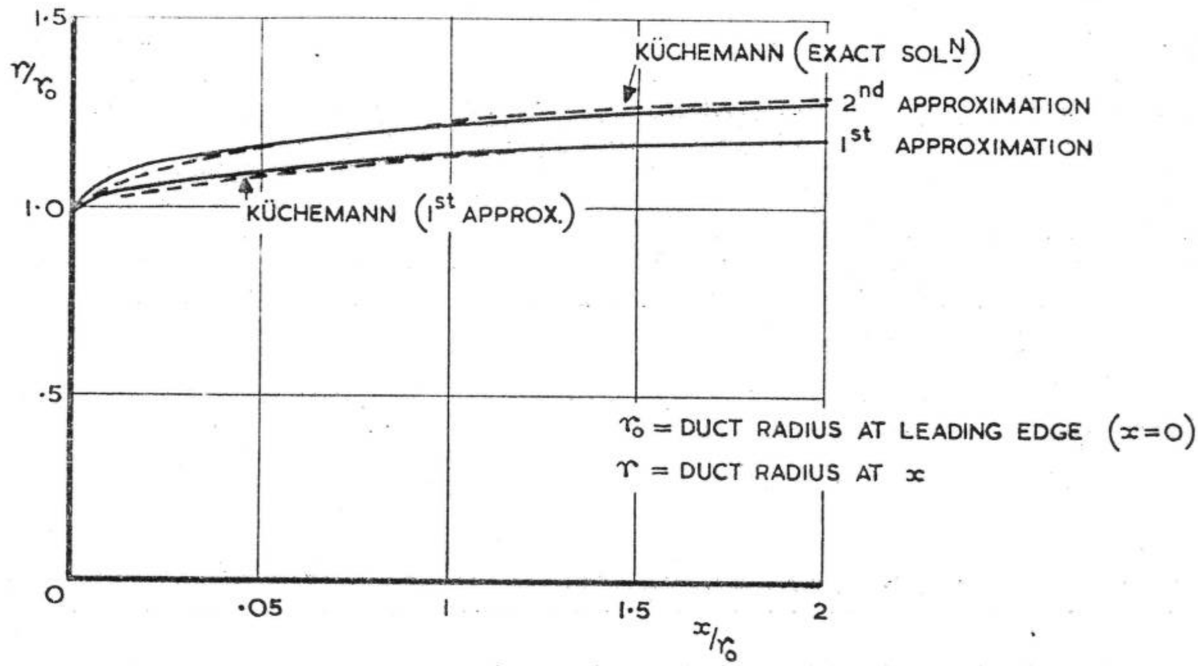


FIG 11. INTAKE DUCT SHAPE $V_2/V_1 = \cdot 3$

CONSTANT VORTICITY DISTRIBUTION.

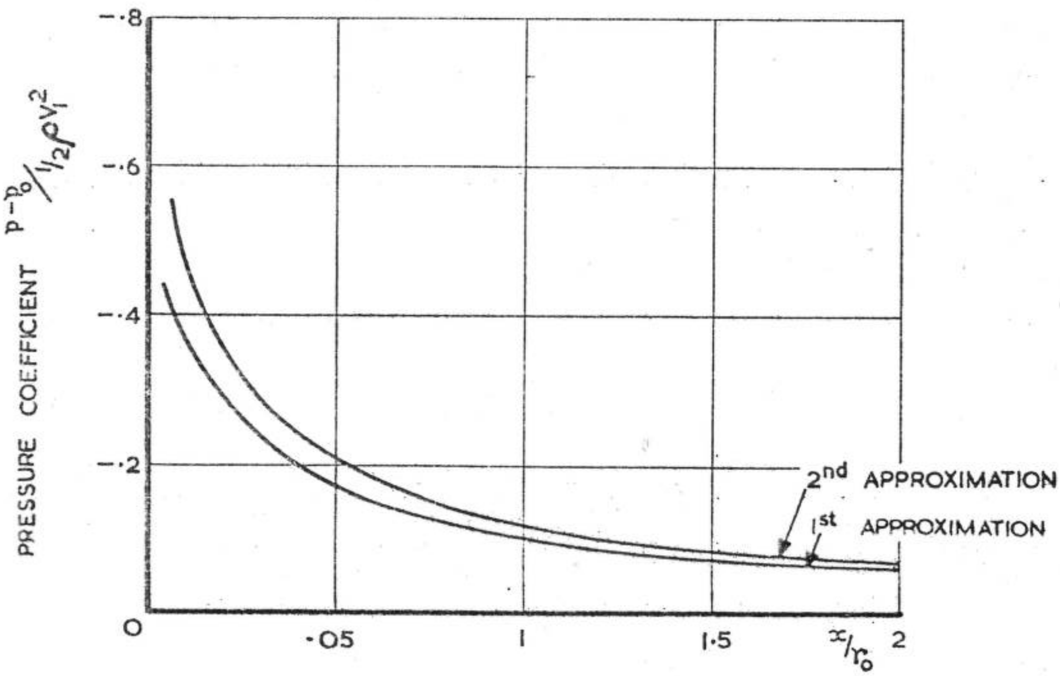


FIG. 12. PRESSURE DISTRIBUTION ALONG INTAKE DUCT $V_2/V_1 = \cdot 3$

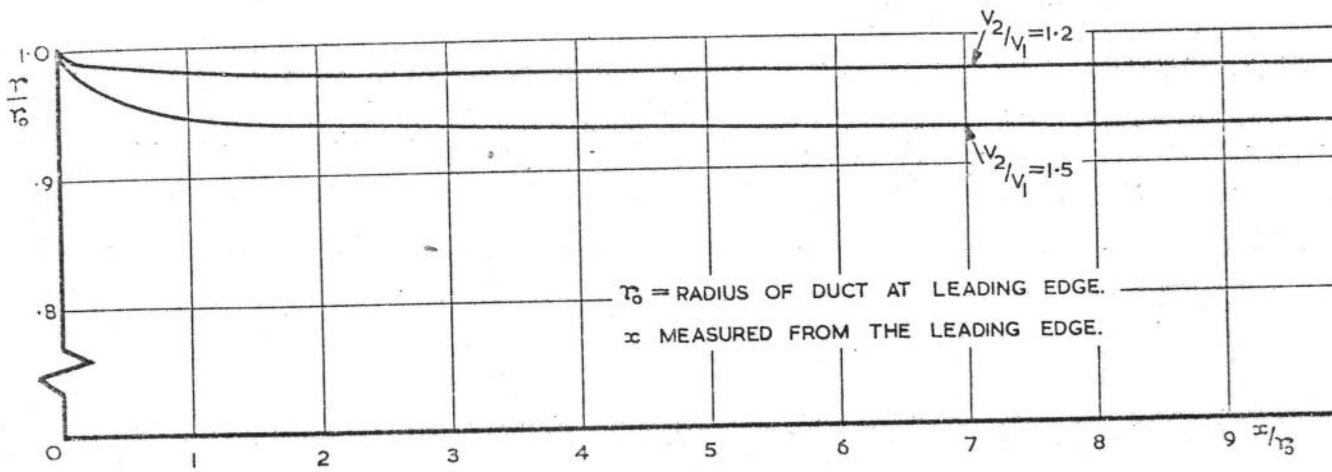


FIG. 13. DUCT & WAKE SHAPE FOR AXISYMMETRIC CASE, CONSTANT VORTICITY; FIRST APPROXIMATION.

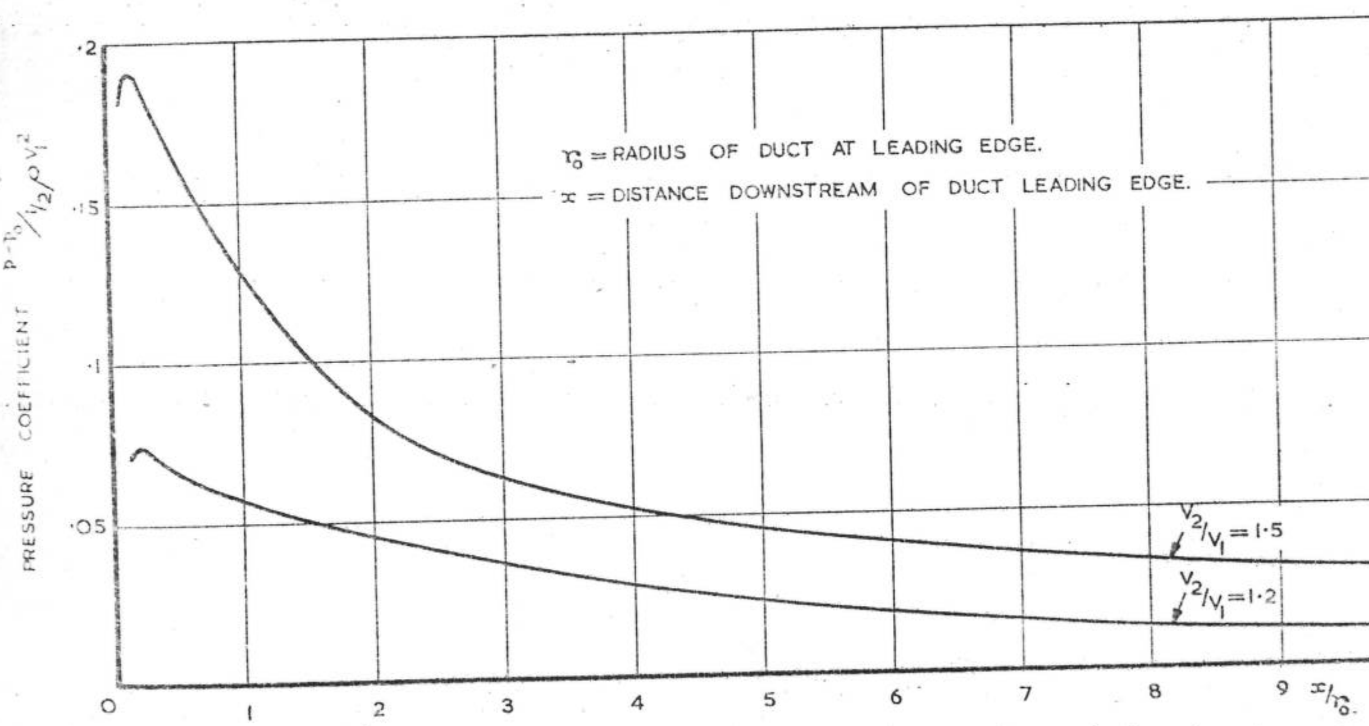


FIG. 14. PRESSURE DISTRIBUTION FOR AXISYMMETRIC DUCT & WAKE. CONSTANT VORTICITY DISTRIBUTION — FIRST APPROXIMATION.

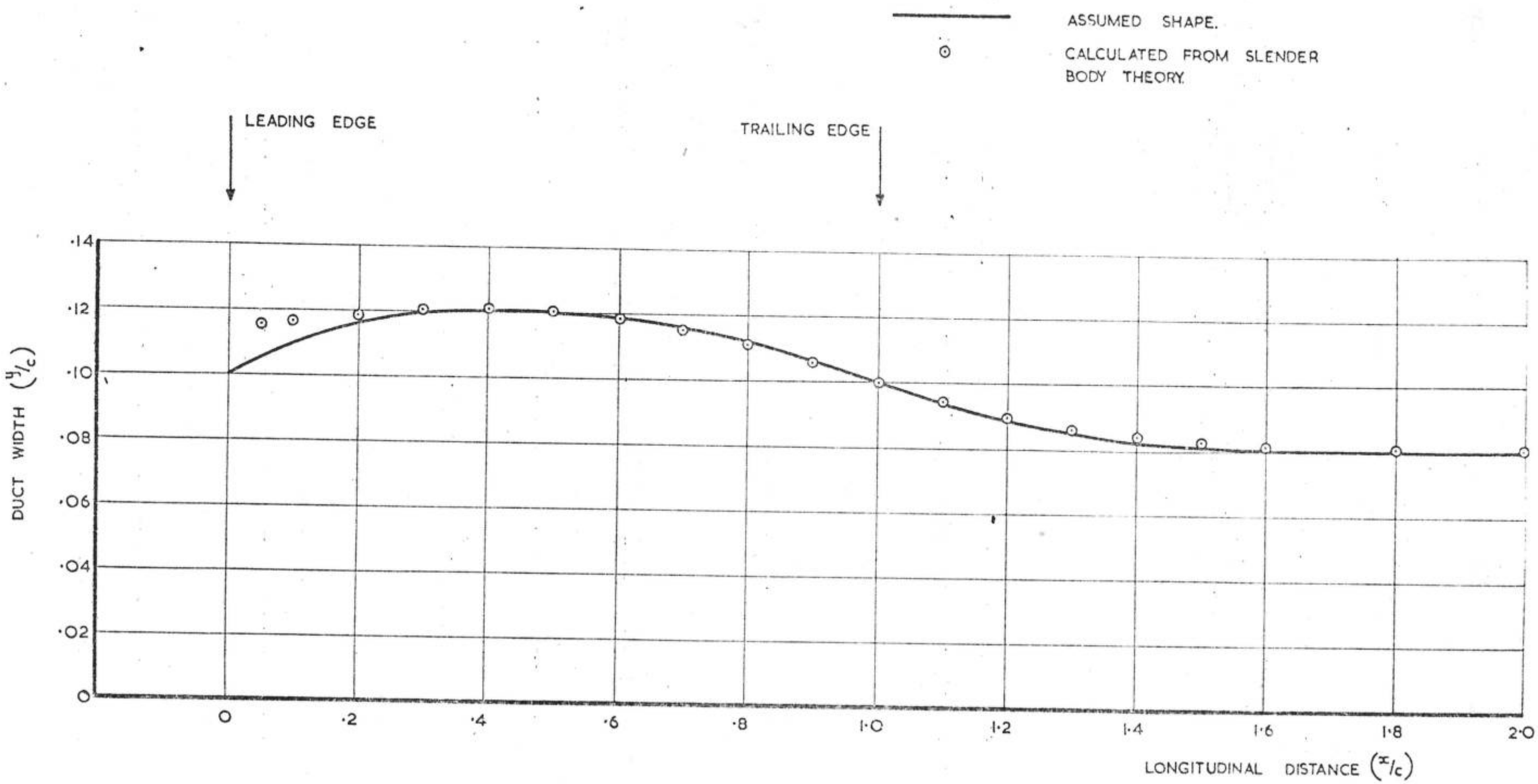


FIG.15. COMPARISON OF ASSUMED DUCT SHAPE & SHAPE CALCULATED FROM SLENDER BODY THEORY $\left(V_2/V_1 = 1{\cdot}25 \right)$