Analysis of Two-Cell Swept Box with Ribs parallel to the Line of Flight under Loading by Constant Couples

- by -

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SUMMARY

The method of oblique co-ordinates\(^{(1)}\) is used to analyse the problem associated with the strength and deformation of a uniform, rectangular, two-cell swept box beam having ribs parallel to the line of flight. The case of loading by constant couples is considered, but no account of root effects is taken.

The ribs are assumed to be continuously distributed, the rib boom area, together with the stringer area, being distributed over the skins. A degree of flexibility is allowed to the rib webs.

Results are presented in the form of cross sectional rotations and stress resultants.

BHF

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NOTATION

Oxyz   Main system of oblique Cartesian Coordinates
OXYz   Auxiliary system of oblique coordinates

\( A_{i,j}, A'_{i,j} \) \((i=1,2,3, j=1,2,3)\) Matrix inversions for rear and front skins respectively.

\( A_1 \) \((i=1,2,3)\) Areas of rear, main and frontspar booms respectively.

\( \sum A = A_1 + A_2 + A_3 \)

\( A_R, A_R' \) Rib boom areas in rear and front cells respectively

\( A_s, A_s' \) Stringer areas in " " " "

\( a_R, a_R' \) Rib pitch in " " " 

\( a_s, a_s' \) Stringer pitch " " " "

\( b \) Half depth of box in direction of \( z \) axis.

\( C_{i,j} \) Coefficients used in expression of rates of section rotation.

\( c \) Half width of box in direction of \( y \) axis

\( c_1, c_2 \) Width of rear and front cells respectively

\( E \) Young's Modulus of Elasticity

\[
\left( \frac{e_{xx}}{\frac{\partial}{\partial x}} \right); \left( \frac{e_{yy}}{\frac{\partial}{\partial y}} \right); \left( \frac{e_{xy}}{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}} \right); \left( \frac{e'_{xx}}{\frac{\partial}{\partial x}} \right); \left( \frac{e'_{yy}}{\frac{\partial}{\partial y}} \right);
\]

\[
\left( \frac{e'_{xy}}{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}} \right); \text{ Strain components referred to axes } Oxy; \text{ in rear and front cells respectively.}
\]

\( G = \frac{E}{2(1 + \nu)} \) Shear Modulus

\( K_{i,j} \) Coefficients used in expression of stress resultants.

\( (L_i, M_i) \) Oblique components of couple, axes OXY.

\( (p, q, r) \) Oblique components of rotation about axes Oxyz.

\( (p_i, q_i) \) Rates of oblique components of rotation

\( S, S' \) Shear stress resultants in rear and front cells respectively.

\( S_i \) \((i=1,2,3)\) Shear flows in rear, main, and front-spar webs respectively

\( T, T' \) Direct stress resultants in \( x \) direction in the rear and front cells respectively.

\( T_2, T_2' \) Direct stress resultants in \( y \) direction in rear and front cells respectively.
Skin thicknesses of rear and front cells respectively

t, t'  (i = 1, 2, 3) Thickness of rear, main and front spar webs respectively

U, U'  Displacements of rear and front cell skins in x direction

u_1  (i = 1, 2, 3) Displacements of rear, main and front spar webs in x direction

V, V'  Displacements of rear and front cell skins in y direction

W_1  (i = 1, 2, 3) Displacements of rear, main and front spar webs in z direction

Displacements of rib web on y axis in z direction

(X, Y, Z) Components of force, axes Oxyz

α  Angle between Ox and Oy axes

Δ, Δ'  Distortion of section in rear and front cells respectively

Δ_1, Δ'_1, Δ_2, Δ'_2  Constants in equations for distortion of section

Terms used in expression of coefficients

κ  Poisson's Ratio

ω, ω'  Warping of section in rear and front cells respectively

ω_1, ω'_1, ω_2, ω'_2  Constants in equations for warping of section
1. INTRODUCTION

The method used in this analysis is essentially that developed by Hemp in Part 3 of his work (1) on the application of oblique coordinates to swept wing structures. Fig. 1 shows the construction of the box and the notation used.

The sweep back angle is \((\pi/2 - \alpha)\), and the box is defined by a set of oblique axes Oxyz. An auxiliary set of oblique axes OXYZ are also used. The upper and lower surfaces are given by \(z = \pm b\) respectively, and they are assumed to be reinforced by closely spaced stringers parallel to the x axis, and closely spaced ribs parallel to the y axis. The skin thickness in the rear cell is \(t\), the rib boom area \(A_R\), rib pitch \(a_R\), stringer area \(A_S\), and stringer pitch \(a_S\). The comparable dimensions in the front cell are \(t', A'_R, a'_R, A'_S\) and \(a'_S\) respectively. The rear spar web is defined by \(y = c\) and has thickness \(t_1\), the main spar web by \(y = (c - c_1)\) and has thickness \(t_2\), whilst the front spar is given by \(y = -c\) and has thickness \(t_3\). The areas of the rear, main, and front spar booms are \(A_1, A_2, \) and \(A_3\) respectively.

Where the spar and rib webs are capable of carrying end loads, their effective area is considered to be included in the appropriate boom area, the webs themselves being assumed to carry only shear loads. All the materials have a Young's Modulus of \(E\), and Poisson Ratio \(\nu\). The rib webs are considered to be rigidly connected to the spar webs, but are allowed a limited flexibility in themselves.

The effect of root constraint is not investigated, and the box is considered to be loaded by constant couples.

2. THEORY

Assume a linear variation of the rotation of the box with \(x\).

Rotation component about x axis \(p = p_1x\) \(\cdots (1)\)
Rotation component about y axis \(q = q_1x\)

/ The warping \(\cdots\)
The warping and distortion of a cross section of the box are assumed to be linear in y:

Warping:
- Rear Cell: \( w = w_1 y + w_2 \)  
- Front Cell: \( w' = w_1 y + w'_2 \)

Similarly distortion:
- \( \Delta = \Delta_1 y + \Delta_2 \)  
- \( \Delta' = \Delta_1 y + \Delta'_2 \)

Using Eqs (1) to (3), the displacements become:

For the Skins:
- \( U = q_1 x \sin \alpha + \omega(y) \)  
- \( U' = q_1 x \sin \alpha + \omega'(y) \)  
- \( V = -p_1 x \sin \alpha + \Delta(y) \)  
- \( V' = -p_1 x \sin \alpha + \Delta'(y) \)

For the Spar Webs:
- \( u_1 = q_1 x z \sin \alpha + (\omega)_{y=c} \frac{x^2}{2b} \)  
- \( u_2 = q_1 x z \sin \alpha + (\omega)_{y=(c-c_1)} \frac{x^2}{2b} \)  
- \( u_3 = q_1 x z \sin \alpha - e_{xx} \frac{x^2}{2b} \)

The displacement in the z direction, of the rib webs, on the centreline of the box is given by Ref. 1 Eqs. (94) and (98) as:

\[
(w_R)_{y=0} = -e_{xx} \frac{x^2}{2b}
\]

Hence:
- \( w_1 = p_1 x \sin \alpha - e_{xx} \frac{x^2}{2b} \)  
- \( w_2 = p_1 x (c-c_1) \sin \alpha - e_{xx} \frac{x^2}{2b} \)  
- \( w_3 = -p_1 x c \sin \alpha - e_{xx} \frac{x^2}{2b} \)

/ Eqs. ........
Eqs. (4) to (7) are used to obtain the strains in the skins.

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial U}{\partial x} = a_1 b \sin \alpha \\
\varepsilon_{yy} &= \frac{\partial V}{\partial y} = \Delta_1 \\
\varepsilon_{xy} &= \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = -p_1 b \sin \alpha + \omega_1
\end{align*}
\]

\[
\varepsilon'_{xx} = a_1 b \sin \alpha \\
\varepsilon'_{yy} = \Delta_1 \\
\varepsilon'_{xy} = -p_1 b \sin \alpha + \omega_1
\]

\[... (8)\]

The box is loaded by constant couples

\[X = Y = Z = 0 \]

The Stress Resultants are restricted.

\[T_1 \text{ and } T'_1 \text{ are functions of } y \text{ only} \]
\[T_2 \text{ and } T'_2 \text{ are zero} \]
\[S \text{ and } S' \text{ are constant} \]

\[
\begin{align*}
\varepsilon_{xx} &= A_{11} T_1 + A_{13} S \\
\varepsilon_{yy} &= A_{21} T_1 + A_{23} S \\
\varepsilon_{xy} &= A_{31} T_1 + A_{33} S
\end{align*}
\]

\[
\begin{align*}
\varepsilon'_{xx} &= A'_{11} T'_1 + A'_{13} S' \\
\varepsilon'_{yy} &= A'_{21} T'_1 + A'_{23} S' \\
\varepsilon'_{xy} &= A'_{31} T'_1 + A'_{33} S'
\end{align*}
\]

\[... (9)\]

Compatibility of warping at the mainspar requires:

\[\omega_1 (c - c_1) + \omega_2 = \omega'_1 (c - c_1) + \omega'_2 \]

or rewriting:

\[\omega_1 = \omega'_1 + \frac{(\omega'_2 - \omega_2)}{(c - c_1)} \]

\[... (10)\]

Using Eqs. (8) and (9):

\[
\begin{align*}
q_1 b \sin \alpha &= A_{11} T_1 + A_{13} S = A'_{11} T'_1 + A'_{13} S' \\
A_{21} T_1 + A_{23} S - \Delta_1 &= A'_{21} T'_1 + A'_{23} S' - \Delta'_1 = 0 \\
-p_1 b \sin \alpha &= A_{31} T_1 + A_{33} S - \omega_1 = A'_{31} T'_1 + A'_{33} S' - \omega'_1
\end{align*}
\]

\[... (11)\]

/ Using ........
Using the stress-strain relation for the spar webs:

\[
S_1 = Gt_1 \left( \frac{3u}{3z} + \frac{3w}{2x} \right)
\]

\[
S_1 = Gt_1 \left\{ p_1 c \sin \alpha + \frac{(\omega_1 c + \omega_2)}{b} \right\}
\]

\[
S_2 = Gt_2 \left\{ p_1 (c-c') \sin \alpha + \frac{\omega_1 (c-c') + \omega_2}{b} \right\}
\]

\[
S_3 = Gt_3 \left\{ -p_1 c \sin \alpha + \frac{(-\omega_1 c + \omega_2)}{b} \right\}
\]

Equilibrium of the spar flange joints requires that:

\[
\begin{align*}
S_1 - S & = 0 \\
S_3 - S' & = 0 \\
S - S_2 - S' & = 0
\end{align*}
\]

Eq. (13) implies that:

\[
S_1 + S_2 + S_3 = \frac{2}{2b} = 0
\]

For overall equilibrium:

\[
L = 2bc S_1 - 2bc S_3 + 2b(c-c')S_2 - 2b^2 Sc + S'c_2
\]

and using Eq. (13):

\[
\frac{L}{4b} = -c_1 S - c_2 S'
\]

or:

\[
S' = -S \frac{c_1}{c_2} - \frac{L}{4bc_2}
\]

\[
S = -S \frac{c_2}{c_1} - \frac{L}{4bc_1}
\]

\[
M_4 = 2bE(\Sigma A)(q_4 b \sin \alpha) + 2b(T_4 c_1 + T_4 c_2)
\]

where \( \Sigma A = A_1 + A_2 + A_3 \)

/ Substituting *****
Substituting from Eq. (13) into Eq. (12):

\[
\frac{S'}{\partial t^2} = -p_1 \sin a + \frac{(-a_1 \phi + \omega_2)}{bc}
\]

\[
\frac{(S - S')}{\partial t} = p_1 \sin a + \frac{[\frac{a_1 (c - c_1) + \omega_2}{b (c - c_1)}]}{bc}
\]

\[
\frac{-S}{\partial t^2} = p_1 \sin a + \frac{(\omega_1 c + \omega_2)}{bc}
\]

Eqs. (16) and (10) give \( \omega_2 \) and \( \omega'_2 \):

\[
\omega_2 = \frac{bc(c - c_1)}{c_4} \left[ \frac{(S - S')}{{t_2}'(c - c_1)} + \frac{s}{t_1 c} \right]
\]

\[
\omega'_2 = \frac{bc(c - c_1)}{c_4} \left[ \frac{(S - S')}{{t_2}'(c - c_1)} + \frac{s'}{t_3 c} \right]
\]

Using Eqs. (11) and (16):

\[
\frac{S_h}{\partial t^2} = A_{31} T_1 + A_{33} S - 2\omega_4 - \frac{\omega_2}{c}
\]

\[
\text{and} \quad \frac{S'_h}{\partial t^2} = A_{31} T'_1 + A_{33} S' - 2\omega'_4 + \frac{\omega'_2}{c}
\]

Elimination of \( \omega_4 \) and \( \omega'_4 \) from Eq. (18) by using Eq. (10) yields:

\[
\frac{b}{c} \left[ \frac{S'}{t^2} - \frac{S}{t^2} \right] = A_{31} T'_1 + A_{33} S' - A_{31} T_1 - A_{33} S + \omega_2 \left[ \frac{(c - c_1) + 2c}{c_4 (c - c_1)} \right] + \omega'_2 \left[ \frac{(c - c_1) - 2c}{c_4 (c - c_1)} \right]
\]

Substitution from the first pair of Eqs. (11) into Eq. (15) and elimination between the resulting two equations gives:

\[
T_1 = c \left( A_{13} S' - A_{13} S \right) + A_{14} \left( \frac{M}{2b} - EA_{13} S A \right) / A_{11} \left( EA_{11} S \right) + \phi \left( C_{2} A_{11} \right)
\]

\[
T'_1 = c \left( A_{13} S' - A_{13} S \right) + A_{14} \left( \frac{M}{2b} - EA_{13} S A \right) / A_{11} \left( EA_{11} S \right) + \phi \left( C_{2} A_{11} \right)
\]
Using Eqs. (17) and (19) and rearranging:

\[ A_{31} T_1' - A_{31} T_1 = S \left[ A_{33} + \frac{2bc}{G_{t_2 c_1 c_2}} + \frac{b}{G_{t_1 c_1}} \right] - S' \left[ A_{33}' + \frac{2bc}{G_{t_2' c_1' c_2'} + \frac{b}{G_{t_3' c_2'}} \right] \]

Eqs. (20) and (21) give:

\[
\begin{bmatrix}
A_{34} A_{34} c_1 + A_{34}^2 (c_2 + A_{11} E^2 A) \\
A_{41} \left[ E A_{11} E A + c_1 \right] + A_{44} c_2
\end{bmatrix} = \left[ A_{33} + \frac{2bc}{G_{t_2 c_1 c_2}} + \frac{b}{G_{t_1 c_1}} \right] S
\]

\[
\begin{bmatrix}
A_{34} A_{34} c_2 + A_{34}^2 (c_1 + A_{11} E^2 A) \\
A_{41} \left[ E A_{11} E A + c_1 \right] + A_{44} c_2
\end{bmatrix} = \left[ A_{33}' + \frac{2bc}{G_{t_2' c_1' c_2'} + \frac{b}{G_{t_3' c_2'}} \right] S'
\]

\[ \frac{A_{31} A_{41} - A_{34} A_{41}}{A_{41} \left[ E A_{11} E A + c_1 \right] + A_{44} c_2} \]

Eqs. (22) and (14) enable \( S, S' \) to be found and hence \( T_1, T_1' \) from Eq. (20).

\[ \omega' \text{ and } \omega \text{ follow from Eqs. (17) and (18):} \]

\[ \omega_1' = \frac{A_{31} T_1'}{2} + \frac{b S}{2 G_{t_2' c_2}} + \frac{3}{2} \left[ A_{33} - \frac{b}{G_{c_2}} \left[ \frac{1}{t_3} + \frac{1}{t_2} \right] \right] \]

\[ \omega_1 = \frac{A_{31} T_1}{2} + \frac{b S_1}{2 G_{t_2 c_2}} + \frac{3}{2} \left[ A_{33} - \frac{b}{G_{c_1}} \left[ \frac{1}{t_1} + \frac{1}{t_2} \right] \right] \]

From Eqs. (23) and (11):

\[ p_1 = \frac{d}{dx} = -\frac{\cossec \frac{c}{2b} \left[ A_{31} T_1 + S \left[ A_{33} + \frac{b}{G_{c_1}} \left[ \frac{1}{t_2} + \frac{1}{t_1} \right] \right] - \frac{b S_1}{G_{t_2' c_2}} \right]}{A_{11} T_1 + A_{13} S} \]

The displacements can be found using Eqs. (24), (23) and (17). The strains follow from Eq. (9)
3. RESULTS

\[ T_1 = K_{11} L_1 + K_{12} M_1 \]
\[ S = K_{21} L_1 + K_{22} M_1 \]
\[ S_1 = K_{31} L_1 + K_{32} M_1 \]
\[ S_2 = K_{41} L_1 + K_{42} M_1 \]
\[ S_3 = K_{51} L_1 + K_{52} M_1 \]
\[ q_1 = \frac{dx}{da} = c_{12} L_1 + c_{12} M_1 \]

Where:

\[ K_{11} = \left\{ \begin{array}{l} c_2 (A_{13} (A_{13} - A_{13}^{\mu})) + A_{13} (A_{13}^{\mu} E \Sigma A) / 4b \times \lambda \\ \frac{d}{dx} \end{array} \right\} \]
\[ K_{12} = \left\{ \begin{array}{l} \frac{c_2}{2b} \times \lambda \\ \frac{d}{dx} \end{array} \right\} \]
\[ K_{21} = \frac{C}{4b} \times \lambda 
\[ K_{22} = \frac{C}{4b} \times \lambda 
\[ K_{31} = \frac{C}{4b} \times \lambda 
\[ K_{32} = \frac{C}{4b} \times \lambda 
\[ K_{41} = \frac{(\lambda - \mu)}{4b} \times \lambda 
\[ K_{42} = \frac{(\lambda + \mu)}{4b} \times \lambda 
\[ K_{51} = \frac{(\lambda - \mu)}{4b} \times \lambda 
\[ K_{52} = \frac{(\lambda + \mu)}{4b} \times \lambda 

\[ \phi = \ldots \]
\[ K = A_{11} \left[ EA_{11} \sigma A + c_1 \right] + A_{11} c_2 \]

\[ \gamma = A_{31} A_{11} - A_{31} A_{11} \]

\[ \mu = A_{31} A_{31} c_2 + A_{31} (c_1 + A_{11} E \sigma A) - A_{31} \left( \frac{b(c_1 + c_2)}{2b(c_1 + c_2) - \frac{b}{2c_1 c_2}} \right) \]

\[ \lambda = \left( A_{31} A_{31} c_1 + A_{31} (c_2 + A_{11} E \sigma A) - A_{31} \left( \frac{b(c_1 + c_2)}{2b(c_1 + c_2) - \frac{b}{2c_1 c_2}} \right) \right) \]

\[ \varepsilon = \lambda c_2 + \mu c_1 \]

**SPECIAL CASE**

**Two Equal Cells**

\[ t_1 = t_2 = t_3 = t_w \quad c_1 = c_2 = c \quad t = t' \]

(Equal cells with constant web and skin thicknesses)

\[ T_i = T_i' = \left( A_{11} E \sigma A L_i \frac{L_i}{5bc} + M_i \right) \]

\[ \begin{align*}
S & = S' = - \frac{L_i}{5bc} \\
S_1 & = - \frac{L_i}{5bc} \\
S_2 & = 0 \\
K_{11} & = \frac{1}{5bc} \cdot \frac{A_{11} E \sigma A}{(2c + A_{11} E \sigma A)} \\
K_{12} & = \frac{1}{2b(2c + A_{11} E \sigma A)} \\
K_{21} & = - \frac{L_i}{5bc} \\
K_{22} & = 0 \\
K_{31} & = \frac{1}{5bc} \\
K_{32} & = K_{41} = K_{52} = 0 \\
K_{42} & = K_{51} = - \frac{1}{5bc} \\
\end{align*} \]

"/ 0_{11} ......."
\[
C_{11} = \csc A \left\{ \frac{(1+\sigma)}{Ew^c} + \frac{A_{13}^2}{2b} - \frac{A_{13}^2 E \Sigma A}{2b(2c+A_{14}E \Sigma A)} \right\} \\
C_{12} = C_{21} = -\csc A \frac{A_{13}}{4b^2(2c+A_{14}E \Sigma A)} \\
C_{22} = \csc A \frac{A_{11}}{2b^2(2c+A_{14}E \Sigma A)} \\
\gamma = 0 \\
\lambda = \mu \\
\varepsilon = 2\lambda c = 2\mu c \\
\lambda = 2A_{14}c + A_{14}^2 E \Sigma A
\]

5. DISCUSSION

It can be seen from Eqs (26), that there is a contribution to the oblique shear stress from the "bending couple" $M_1$. This contribution is dependent upon the value of $\gamma$, and it can be shown that $\gamma$ itself is dependent upon the relative values of $A_R$, $a_R$, and $t$ in the two cells.

For the case of constant rib pitch, with the ratio of the rib boom area to skin thickness the same in both cells, i.e. $\frac{A_R}{a_R^t} = \frac{A_R'}{a_R'^t}$, the value of $\gamma$ is zero. Under these conditions, there is no oblique shear stress due to $M_1$.

The results for the special case of two equal cells with constant web and skin thicknesses given in Eqs (29) to (32), are directly comparable to the single cell results, Eqs (78), (83) and (100) of Ref. 1.

Complete analysis of the box beam subjected to constant couples is achieved by using the above results in conjunction with the relevant parts of Ref. 1 § 3.2.

References ...
REFERENCES
