Report No. 58

April, 1952

THE COLLEGE OF AERONAUTICS

CRANFIELD

Dynamic Aero-elasticity of Aircraft with swept wings

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SUMMARY

Using oblique coordinates, integro-differential equations of motion of aircraft with swept wings are deduced from Hamilton's Principle for the dynamic problems of aero-elasticity, i.e. for the problems of free vibrations, flutter, dynamic stability and gust loads. By use of a concise notation the final equations are presented in a form specially suited for fundamental as well as for practical investigations. They are discussed in some detail and their solution by numerical methods, conventional in aero-elastic work, is indicated. All important assumptions made are summarised and will be seen to agree with those commonly made.

BHF

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LIST OF CONTENTS

NOTATION

A, B, C	Axial Moments of inertia of aircraft referred to axes O_1x_1, O_1y_1, O_1z_1 (see 1A2, section 1.2)
$C_{ij}(y_1)$	Influence functions for wings (see 4.6)
C_f	Torsional stiffness of fuselage
E, F, G	Products of inertia of aircraft (see 1A2)
E_f	Young's modulus of material of fuselage
$\overline{I}_{y_1f}$, $\overline{I}_{y_1z_1f}$, $\overline{I}_{z_1f}$	Second moments and product of area of fuselage sections
I_{x_1}, $I_{x_1z_1}$, I_{z_1}	Moments and product of inertia of wing sections
J_{y_1}	Polar moment of inertia of wing sections
J_{x_1f}	Polar moment of inertia of fuselage sections
$K_\eta = \dfrac{cC_L}{\overline{c}\overline{C}_L}$	Loading coefficient for lift distribution under steady conditions
K_a, M_a etc.	Aerodynamic coefficients defined by (6.4.8) - (6.4.27)
L_1, M_1	Oblique components of couple applied to wings about O_1X_{1s}, O_1Y_{1s} (see Fig.3)
dL, dM	Aerodynamic lift and moment
M	Total mass of aircraft
$N = K_\eta \dfrac{\overline{c}\overline{a}_1}{c}$	Lift slope distribution of finite wing (see 5.3.10)
$O_ox_oy_oz_o$ $O_1x_1y_1z_1$	Orthogonal rectilinear right-handed coordinate systems fixed in (space / aircraft)
$O_1x_1y_{1s}z_1$ $O_1x_1y_{1P}z_1$	Oblique rectilinear coordinate systems fixed in aircraft (right handed for starboard wings / left handed for portside wings)
P, Q, R	Angular body motion of aircraft referred to $O_1x_1y_1z_1$
Q_r	Non-conservative forces
T	Kinetic energy

/ $\dot{U}_o$ ......

$\hat{U}_o$	Constant mean forward velocity
U, V, W	Coordinates of O_1 relative to $O_o x_o Y_o z_o$
U	Potential energy
W_g	Gust velocity
$\bar{\bar{x}}_1$, $\bar{\bar{Y}}_1$, $\bar{\bar{z}}_1$	Position of centre of mass of aircraft relative to O_1 (see 1A2)
$\bar{Y}_{1f}$, $\bar{z}_{1f}$	Position of centre of mass of fuselage sections
Z	Downward load at wing section
a_1, $\bar{a}_1$	Lift slope of wing section and of whole aircraft
c	Wing chord
c_r, c_t	Root and tip chord of wing
$\bar{c}$	Mean chord of wing
i, i_1, j, j_1, k	Unit vectors (see Fig.3)
k_a, k_b	Lift flutter derivatives
$m(y_1)$	Mass distribution of wing
$m_f(x_1)$	Mass distribution of fuselage
m_a, m_b	Moment flutter derivates
k_1, k_2	Wagner and Küssner functions
p_f	Torsional displacement of fuselage
p, q p_s, q_s p_p, q_p	Components of rotation of wing sections
s	Non-dimensional time for gust loads
t	Time
u, v, w u_s, v_s, w_s u_p, v_p, w_p	Displacements of points of wing sections relative to $O_1 x_1 Y_1 z_1$
v_f, w_f	Lateral displacements of fuselage relative to $O_1 x_1 Y_1 z_1$

/ $\bar{x}_1$ ......

$\bar{x}_1$, $\bar{z}_1$	Position of centre of mass of wing sections
z, θ	General displacements of wing section (see section 5.3)
Γ_{ij}	Influence functions (see 4.7)
Φ	Bending deflection of wing section
α	Complement of angle of sweep back of reference axis O_1y_1
β	Free stream Mach number
δ U, etc.	Arbitrary displacements, forces etc.
ν	Frequency of oscillation
ρ	Air density
ϕ	Shear deflection of wing section
ω	Reduced frequency

Additional notation of a more special character is defined in sections 6.2 and 6.3

Time } derivates are indicated by { dots
Space } { strokes

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1. INTRODUCTION

1.1 General Remarks

All the subjects which will here be classed as dynamic problems of aero-elasticity have been developed independently over many years, mostly as the result of the discovery of actual phenomena requiring theoretical explanations. For this reason, aero-elasticity still lacks a unified notation, and the transference of results from one problem to another often presents great difficulties. The lack of a common notation is probably also to some degree responsible for the fact that up to date no attempt has been made to develop a general theory of dynamic aero-elasticity.

The advent of swept wings once again has introduced into aero-elasticity a new aspect which this time is due to affect all the problems coming under that heading. This event thus appears to offer a good opportunity of filling the gap and it is one of the objects of this report to present a unified theoretical treatment of dynamic aero-elasticity.

For this purpose dynamic aero-elasticity will be conceived as the theory of the free and forced vibrations of aircraft, so that ab initio there exists no need for further sub-dividing the subject. Nevertheless a natural subdivision is suggested by the different types of forcing functions which give the problems of aero-elasticity their special character. In this way one arrives at the following three types of problems:-

i) Free vibrations in vacuo
i.e. no external forces exist

ii) Flutter and dynamic stability
i.e. external forces occur due to harmonic motion of the wings

iii) Gust loads
i.e. external forces occur due to arbitrary motion of the wings.

Obviously the basic equations of motion corresponding to these groups will only differ by terms representing the external forces. Only reasons of tradition and convenience make it advisable to distinguish between the problems (ii) and (iii).

On the whole, the notation of the present report will differ from that used in almost all other aero-elastic work. Apart from the general reason indicated above, this state of affairs has been caused by two facts: the use of integral equations and of oblique coordinates. Both of these may be considered unnecessary innovations, and thus it becomes the final object of this report to offer convincing evidence that their introduction leads to a clear and simple form of the relevant equations of motion. In the remaining part of this sub-section an attempt will be made to justify these steps on more general grounds.

/ Aircraft ....

Aircraft like any other bodies occurring in nature, are continuous mass systems which exhibit from the theoretical as well as from the practical point of view features typical for such systems. In mathematical terms this means that the equations of motion will be integro-differential equations, the kernels of which involve the continuous distributions of mass, aerodynamic loading, etc. The normal type of equations will contain time and space derivatives of the independent variables representing displacements. Combined with initial and boundary conditions such equations will define uniquely the state of the aircraft at any one instant.

However there exists yet a simpler set of equations of motion, the solutions of which automatically satisfy the boundary conditions. These integro-differential equations can be obtained in various ways; for example by integrating the traditional equations and using the boundary conditions. Here a different approach has been adopted in that these equations have been deduced directly from Hamilton's Principle. For this purpose it is only necessary to make a judicious choice of the independent variables, some of which will be seen to be space derivatives of the displacements used normally. Obviously this last step will only be possible if the continuous character of the structure is preserved. In physical terms these new variables will be seen to represent the curvatures and space rates of twist of the deformed structure. With these new variables the final integro-differential equations of motion will be found to involve only time derivatives. Thus in the case of the problems (i) and (ii) above, when the aircraft is subject to harmonic motion the characteristic equations determining frequencies, modes and flutter speeds are Fredholm integral equations of the second kind. Exact solution of these last equations in any practical application may present unsurmountable difficulties, so that it will become necessary to introduce some approximation such as replacement of the integrals by finite sums. It is easily realized that by this method the equations of motion become matrix equations of a type well known in any applied dynamic or aero-elastic work. Alternatively, experimentally determined modes may be used as in the standard practice of flutter investigations. But it should be understood that either of these methods are computational expedients which are little related to the fundamental problems.

As far as the introduction of oblique coordinates is concerned, there can be no doubt that they are in the first place suggested by the special nature of swept wings, and that any objections to their use will arise from the fact that they have not been used before. W.S. Hemp (4,5) has developed an elastic theory of structures consisting of spars, ribs and stressed skins which uses oblique coordinates, and which is very well suited for application to problems of the type considered here. It will be shown that the use of oblique coordinates for the wings does not introduce great complications, and that it eliminates the necessity of distinguishing between swept and straight wings, the latter simply being a special case of the former.

/ In Ref.6 ......

In Ref.6 a comparison has been made between the theoretical and experimental influence functions, indicating that the agreement will be quite satisfactory. Hence the present aero-elastic theory can already be applied at the design stage.

The remaining part of this section outlines the most important assumptions underlying the theory of this Report, and it will be seen that they fully agree with those customarily made. Section 2 gives a detailed discussion of the choice of coordinate axes and the basic notation referring to these axes. The following three sections are concerned with the deduction of expressions for the potential and kinetic energies and for the external forces. Section 6 gives the equations of motion for the three groups of problems stated above. Section 7 indicates possible ways of solving the equations of motion, and it will be seen that most methods of solution used hitherto in work of this kind are applicable.

1.2 Assumptions

A complete statement of all assumptions in the case of work as complex as the present offers great difficulties. Dynamic aero-elasticity, as its title suggests, draws on results and methods of the following three well defined branches of mathematical physics:-

1) General Dynamics
2) Aerodynamics
3) Elasticity

so that such a summary of assumptions would be widely spread, and demand very thorough knowledge of three large subjects, although it may be said that only certain special parts of some of these subjects are relevant to aero-elastic work. For this reason when giving a list of assumptions for the composite subject of aero-elasticity, those customarily made in the component subjects will not be stated in detail.

These assumptions will be grouped under the three headings given above:

1) Dynamics

1A1 The theory of small vibrations is applicable.

1A2 The quantities depending on the mass distribution of the aircraft as a whole do not vary as a result of deformation.

1A3 Rotations of the aircraft as a whole are small

1A4 Structural damping is neglected. (Introduction of a dissipation function could be easily effected. See section 5).

1A5 The effect of gravity is neglected.

1A6 The potential energy of the deformed aircraft is satisfactorily given by the strain energy (see elasticity).

/ 1A7 ....

1A7 The mass distribution is adequately described by piecewise continuous functions of the coordinates along the span and the fuselage.

1A8 The fuselage and the wings are rigidly connected and no internal vibrations in the direction of flight occur.

1A9 The mass of the tail unit is included with that of the fuselage.

2) <u>Aerodynamics</u>

2A1 The results of linear two-dimensional unsteady aerofoil theory are applicable in connection with a weight function which is based on the steady state span wise distribution of lift-slope for finite wings.

2A2 The forward speed of the aircraft is constant and no yawing takes place.

2A3 The gusts are assumed to be uniform across the span of the wings.

2A4 The fuselage does not contribute to the lift and the wings form a lifting surface extending from tip to tip.

2A5 Aerodynamic terms due to the tail plane have been omitted. (They can however easily be added provided due consideration can be given to 2A1).

2A6 The effect of ailerons, flaps, etc. has been neglected. (Obviously consideration of these effects introduces further complications which do not involve new principles.)

2A7 The forward speed of the aircraft is such that the wing leading edges are "subsonic".

2A8 The reference points for the moments lie on the theoretical axis Oy_1 of the wing. (In the work here the axis has been assumed along the mid chord line.)

2A9 Effects of twist of wing on lift + p.m. ignored (see p.20)

3) <u>Elasticity</u>

3A1 The results of the theory of Refs. 4 and 5 are applicable to the wing structure.

3A2 Simple beam theory is applicable to the fuselage.

/ 2 ....

2. COORDINATE SYSTEMS AND NOTATION

Because of the proposed use of oblique coordinates, two coordinate systems will be required as far as the wings are concerned. In this way it becomes unnecessary to introduce from the start assumptions restricting the permissible type of deformation of the wings. An alternative approach would have been to deal only with one half of the aircraft and to consider separately symmetric and anti-symmetric deformations as has been done for example in Refs. 1 and 2.

In order to obtain the most general equations of motion, body motion has to be taken into account. Hence two further orthogonal coordinate systems will be required, one of which will be fixed in space and the other at a suitable reference point in the aircraft. The latter will be placed at the intersection of the wing axes, and will also be used to describe the deformations of the fuselage.

Thus use will be made of the following coordinate systems:-

i) $O_o x_o y_o z_o$, an orthogonal rectilinear right-handed system fixed in space to which the translatory motion of the reference point O_1 of the aircraft will be referred by the displacement vector (U, V, W), the components of which are functions of the time t.

ii) $O_1 x_1 y_1 z_1$, an orthogonal rectilinear right-handed system with its axes fixed in the aircraft to which the angular motion of the aircraft will be referred by the angular displacement vector (P, Q, R), the components of which are functions of the time t.

iii) $O_1 x_1 y_{1s} z_1$, an oblique rectilinear right-handed system to which the "internal" motion of the starboard wing about a mean position will be referred by the displacement vector (u_s, v_s, w_s), the components of which are functions of x_1, y_{1s}, z_1, t.

iv) $O_1 x_1 y_{1p} z_1$, an oblique rectilinear left-handed system to which the "internal" motion of the portside wing about a mean position will be referred by the displacement vector (u_s, v_s, w_s), the components of which are functions of x_1, y_{1p}, z_1, t.

These systems are shown in Fig.1 in which are also indicated the positive directions of rotation about the "internal" axes of the aircraft which are given by the vectors $(p_s, q_s, 0)$ and p_p, q_p, 0) respectively. These directions of rotation are in agreement with the convention for rectilinear systems by which cyclic clockwise or anticlockwise rotation gives the positive directions for right or left handed systems respectively.

/ Compatibility .....

Compatibility of the coordinate systems is obviously insured, and it is easily seen that all internal displacements and their first derivatives will be zero at the reference point O_1. Further, it is seen from Fig.1 that if p_s, q_s and p_p, q_p are equal in magnitude and have equal or opposite signs, the corresponding displacements will be symmetric or anti-symmetric respectively. Throughout most of the work of this Report there will be no need to distinguish between the starboard and portside wings, so that the subscripts s, p can be omitted.

As mentioned earlier in this section, the system (ii) may also serve as reference system for deformations of the fuselage and tail unit, provided the latter does not involve swept tail planes. However, in the present Report only the fuselage will be taken into account in order to illustrate the inclusion of such additional components in the analysis, and all quantities referring to the fuselage will have the subscript f.

3. KINETIC ENERGY

Using the notation of section 2, the velocity of a point of the aircraft is given by

$$\left\{(\mathring{U}, \mathring{V}, \mathring{W}) + (\mathring{P}, \mathring{Q}, \mathring{R}) \times (x_1, Y_1, z_1)\right\} + (\mathring{u}, \mathring{v}, \mathring{w}) \quad \text{----} \ (3.1)$$

where it should be remembered that each of the vectors is referred to a different coordinate system; in particular the vector $(\mathring{u}, \mathring{v}, \mathring{w})$ is referred to one of the oblique coordinate systems whenever the wings are being considered and, further, notice must be taken of the fact that the forward component of the translatory body motion is large. The kinetic energy of the aircraft will involve the scalar product of this vector with itself, and it is most easily determined in four steps.

The first of these requires the square of the vector in curly brackets. As in the later steps, multiplying this square by $\frac{\mu(x_1, Y_1, z_1)}{2}$, the local mass density, and integrating over all points of the aircraft, one finds, using 1A2, the kinetic energy of the aircraft considered as a rigid body:

$$\begin{aligned} T_1 = {} & \frac{M}{2}\left\{\mathring{U}^2 + \mathring{V}^2 + \mathring{W}^2\right\} + \frac{1}{2}\left\{A\mathring{P}^2 + B\mathring{Q}^2 + C\mathring{R}^2 - 2E\mathring{P}\mathring{Q} - 2F\mathring{Q}\mathring{R} - 2G\mathring{R}\mathring{P}\right\} \\ & + M\Big[\mathring{P}(\bar{\bar{Y}}_1\mathring{W} - \bar{\bar{z}}_1\mathring{V}) + \mathring{Q}(\bar{\bar{z}}_1\mathring{U} - \bar{\bar{x}}_1\mathring{W}) + \mathring{R}(\bar{\bar{x}}_1\mathring{V} - \bar{\bar{Y}}_1\mathring{U}) \\ & + \dot{U}_o\left\{R(\bar{\bar{z}}_1\mathring{P} - \bar{\bar{x}}_1\mathring{R}) + Q(\bar{\bar{Y}}_1\mathring{P} - \bar{\bar{x}}_1\mathring{Q})\right\}\Big] \end{aligned} \quad \text{----} \ (3.2)$$

where higher order terms have been neglected.

/ Although .....

Although several of the constants above will be zero in most practical cases, they will be retained here in order to ensure that the final equations of motion are as general as possible.

Next consider the kinetic energy of the wings vibrating when the aircraft is in steady motion (or at rest). In Ref.3 the kinetic energy of a swept box has been deduced using oblique coordinates. For application to the present problem, the work of Ref. 3 requires some minor modifications which are necessitated by the use of different coordinate systems and notation. The system used in Refs. 3-5 is shown in Fig.2. The system (iii) of section 2 is obtained from it by an interchange of the parts played by the axes Ox and Oy, and by increasing the angle α of Fig.2 beyond $\pi/2$. It will be seen from the first part of Ref.4 that the general work done there on kinematics in oblique coordinates is still applicable. With the sign convention of section 2 it is obviously unnecessary to distinguish between starboard and portside wings.

Comparing Figs. 1 and 2, it is seen that the α of Refs. 3 and 4 has now to be replaced by $\pi - \alpha$. Making the appropriate changes in notation in equation (9) of Ref. 4, the displacements of a point $P(x_1, y_1, z_1)$ are given by

$$\begin{aligned} u &= z_1 \left\{ - p \cot\alpha + q \operatorname{cosec}\alpha \right\} \\ v &= - z_1 \left\{ p \operatorname{cosec}\alpha - q \cot\alpha \right\} \\ w &= \Phi + \emptyset - q x_1 \sin\alpha \end{aligned} \qquad \text{---- (3.3)}$$

where p, q, Φ and $\emptyset$ will now be assumed to be functions of y_1 and t only. The corresponding velocity components are

$$\begin{aligned} \dot{u} &= z_1 \left\{ - \dot{p} \cot\alpha + \dot{q} \operatorname{cosec}\alpha \right\} \\ \dot{v} &= - z_1 \left\{ \dot{p} \operatorname{cosec}\alpha - \dot{q} \cot\alpha \right\} \\ \dot{w} &= \dot{\Phi} + \dot{\emptyset} - \dot{q} x_1 \sin\alpha \end{aligned} \qquad \text{---- (3.4)}$$

Since this velocity vector is referred to an oblique coordinate system (e.g. (iii) of section 2), the square of this vector is given by

$$\begin{aligned} \dot{u}^2 + \dot{v}^2 + \dot{w}^2 - 2\dot{u}\dot{v}\cos\alpha &= z_1^2 \left\{ \dot{p}^2 - 2\dot{p}\dot{q}\cos\alpha + \dot{q}^2 \right\} + x_1 \dot{q}^2 \sin^2\alpha \\ &\quad - 2x_1 \dot{q} \left\{ \dot{\Phi} + \dot{\emptyset} \right\} \sin\alpha + \left\{ \dot{\Phi} + \dot{\emptyset} \right\}^2 \end{aligned} \qquad \text{---- (3.5)}$$

/ Multiplying .....

Multiplying (3.5) by $\frac{\mu(x_1, y_1, z_1)}{2}$ and integrating with regard to x_1 and z_1 over a wing section y_1 = const. one finds the kinetic energy dT_2 of a wing element:

$$dT_2 = \tfrac{1}{2}\Big\{ I_{x_1}(y_1)(\mathring{p}^2 - 2\mathring{p}\mathring{q}\cos\alpha + \mathring{q}^2) + I_{z_1}(y_1)\mathring{q}^2 \sin^2\alpha$$

$$- 2m(y_1)\bar{x}_1(y_1)\mathring{q}(\mathring{\phi} + \mathring{\emptyset})\sin\alpha + m(y_1)(\mathring{\phi} + \mathring{\emptyset})^2 \Big\} dy_1 \quad \text{-- (3.6)}$$

Using (3.6) the required kinetic energy of the wings becomes

$$T_2 = \int_0^{\ell} (dT_{2s} + dT_{2p}) \quad \text{-- (3.7)}$$

where the subscripts indicate that in (3.6) y_1 is to be replaced by y_{1s} or y_{1p} respectively. In the sequel it will often be convenient to write integrals of the type (3.7) in the following manner

$$T_2 = \int_0^{\ell} dT_2 \quad \text{-- (3.7')}$$

where it will be assumed that the integral extends over both wings.

Next consider the fuselage vibrating under the same conditions as the wings above. Using 1A8 the relevant kinetic energy is:-

$$T_3 = \tfrac{1}{2}\int_{-\ell_1}^{\ell_2} \Big\{ m_f(x_1)(\mathring{w}_f^2 + \mathring{v}_f^2 + 2\bar{Y}_{1f}(x_1)\mathring{w}_f\mathring{p}_f - 2\bar{z}_{1f}(x_1)\mathring{v}_f\mathring{p}_f)$$

$$+ J_{x_1 f}(x_1)\,\mathring{p}_f^2 \Big\} dx_1 \quad \text{-- (3.8)}$$

Finally the contributions to the total kinetic energy arising from arbitrary motion of the reference point of the aircraft will be deduced. In the case when normal coordinates are being used, such contributions will not occur but it is easily seen that the variables p, q are not such coordinates and that the introduction of normal coordinates in terms of these variables would be difficult.

First consider the contribution coming from the wings. Using 1A3 it may be assumed that the axes $O_o x_o Y_o z_o$ and $O_1 x_1 Y_1 z_1$ for the purpose of the present calculation are parallel. The relevant cross term in the square of the velocity vector (3.1) is then:

/ 2{........

$$2\left\{\mathring{U}(\mathring{u}-\mathring{v}\cos\alpha)+\mathring{U}_o(Q\mathring{w}-R\mathring{v}\sin\alpha)+\mathring{V}\mathring{v}\sin\alpha+\mathring{W}\mathring{w}+\begin{vmatrix}\mathring{P} & \mathring{Q} & \mathring{R}\\ x_1-y_1\cos\alpha & y_1\sin\alpha & z_1\\ \mathring{u}-\mathring{v}\cos\alpha & \mathring{v}\sin\alpha & \mathring{w}\end{vmatrix}\right\} \quad \text{---}\ (3.9)$$

where use has been made of the formula

$$j = -i\cos\alpha + j_1\sin\alpha \quad \text{---}\ (3.10)$$

expressing the oblique unit vector j in terms of the orthogonal unit vectors i and j_1 (see Fig.3).

Substituting for $\mathring{u}$, $\mathring{v}$, $\mathring{w}$ from (3.4), multiplying by $\frac{\mu(x_1, y_1, z_1)}{2}$ and integrating as before when deducing (3.6) one finds:

$$dT_4=\Big[m\Big\{\overline{z}_1\mathring{U}\mathring{q}\sin\alpha+\overline{z}_1(\mathring{q}\cos\alpha-\mathring{p})(\mathring{V}-\mathring{U}_oR)+(\mathring{\phi}+\mathring{\emptyset}-\overline{x}_1\mathring{q}\sin\alpha)(\mathring{W}+\mathring{U}_oQ)\Big\}$$

$$+\mathring{P}\Big\{y_1 m\sin\alpha(\mathring{\phi}+\mathring{\emptyset}-\overline{x}_1\mathring{q}\sin\alpha)+I_{x_1}(\mathring{p}-\mathring{q}\cos\alpha)\Big\}$$

$$+\mathring{Q}\Big\{m\overline{x}_1 y_1\mathring{q}\sin\alpha\cos\alpha+J_{y_1}\mathring{q}\sin\alpha-m(\overline{x}_1-y_1\cos\alpha)(\mathring{\phi}+\mathring{\emptyset})\Big\}$$

$$+\mathring{R}\Big\{my_1\overline{z}_1\mathring{p}\cos\alpha+(I_{x_1z_1}\cos\alpha-my_1\overline{z}_1)\mathring{q}-I_{x_1z_1}\mathring{p}\Big\}\Big]dy_1$$

$$\text{---}\ (3.11)$$

And hence the required kinetic energy

$$T_{4w} = \int_0^{\ell} dT_4 \quad \text{---}\ (3.12)$$

Corresponding to (3.9) the cross term for the fuselage is

$$2\left\{(\mathring{V}-R\mathring{U}_o)(\mathring{v}_f-z_1\mathring{p}_f)+(\mathring{W}+\mathring{U}Q)(\mathring{w}_f+Y_1\mathring{p}_f)+\begin{vmatrix}\mathring{P} & \mathring{Q} & \mathring{R}\\ x_1 & Y_1 & z_1\\ O & \mathring{v}_f-\mathring{p}_f z_1 & \mathring{w}_f+\mathring{p}_f Y_1\end{vmatrix}\right\} \quad \text{---}\ (3.13)$$

and its contribution to the kinetic energy:

/T_{4f} ......

$$T_{4f} = \int_{-\ell_1}^{\ell_2} \left[m_f \left\{ (\mathring{V} - R\mathring{U}_o)(\mathring{v}_f - \bar{z}_{1f}\mathring{p}_f) + (\mathring{W} + \mathring{U}_o Q) \quad (\mathring{w}_f + \bar{Y}_{1f}\mathring{p}_f) \right\} \right.$$

$$+ \mathring{P}\left\{ m_f \mathring{w}_f \bar{Y}_{1f} - m_f \mathring{v}_f \bar{z}_{1f} + J_{x_1 f}\, \mathring{p}_f \right\}$$

$$- \mathring{Q}\left\{ m_f x_1 \mathring{w}_f + m_f x_1 \bar{Y}_{1f}\, \mathring{p}_f \right\}$$

$$\left. + \mathring{R}\left\{ m_f \mathring{v}_f x_1 - m_f x_1 \bar{z}_{1f}\, \mathring{p}_f \right\} \right] dx_1 \qquad \text{--- (3.14)}$$

The total kinetic energy of the aircraft is thus given by:

$$T = T_1 + T_2 + T_3 + T_{4w} + T_{4f} \qquad \text{--- (3.15)}$$

using (3.2), (3.7'), (3.8), (3.12) and (3.14)

4. POTENTIAL ENERGY

The potential energy of the deformed aircraft, using 1A5 and 1A6 consists of two parts:

i) U_f the strain energy of the fuselage

ii) U_w the strain energy of the wings

By 3A2 the first of these is given by

$$U_f = \tfrac{1}{2}\int_{-\ell_1}^{\ell_2} \left[E_f \left\{ \bar{I}_{Y_1 f}\, w_f''^2 + 2 I_{Y_1 z_1 f} w_f'' v_f'' + \bar{I}_{z_1} v_f''^2 \right\} + C_f p_f'^2 \right] dx_1$$

--- (4.1)

The strain energy of the wings may be obtained in a manner similar to that adopted in Ref.3; however the shear energy will not be included and use will be made of formulae, analogous to those given in Ref.4, which have lately been developed for the case of a uniform two-cell swept box (Ref.5). By assuming the influence coefficients given there to vary along the axis Oy_1, the theory of Ref. 5 may be extended to non-uniform structures by the same reasoning used in the similar extension of simple beam theory. (For experimental evidence see Ref. 6)

/ In order .....

In order to illustrate the modifications required by the introduction of the oblique systems of section 2, Fig.3 shows the system referring to the starboard wing together with the auxiliary axes used in refs. 4 and 5. With the unit vectors of Fig.3, the moment and force acting at a wing section may be written:

$$L_1 i_1 + M_1 j_1 \quad , \quad Zk \qquad \text{--- (4.2)}$$

To this load system correspond the displacement vectors

$$dpi + dqj \quad , \quad d\phi k \qquad \text{--- (4.3)}$$

so that the strain energy is given by

$$dU = \frac{1}{2}\left[\left\{L_1 i_1 + M_1 j_1\right\} \cdot \left\{dpi + dqj\right\} + Zd\phi k^2\right]$$

$$= \frac{1}{2}\left[\sin\alpha \left\{L_1 \frac{dp}{dy_1} + M_1 \frac{dq}{dy_1}\right\} + Z\frac{d\phi}{dy_1}\right] dy_1 \qquad \text{--- (4.4)}$$

since by Fig.3

$$i_1 \cdot i = j_1 \cdot j = \sin \ , \ k \cdot k = 1 \ , \ i_1 \cdot j = j_1 \cdot i = 0 \qquad (4.5)$$

But by the theory of Ref.5, after appropriate changes in the notation,

$$\begin{vmatrix} \frac{dq}{dy_1} \\ \frac{dp}{dy_1} \\ \frac{d\phi}{dy_1} \end{vmatrix} = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & 0 \\ C_{33} & 0 & C_{31} \end{vmatrix} \begin{vmatrix} M_1 \\ L_1 \\ Z \end{vmatrix} , \quad \frac{dp}{dy_1} = -\text{cosec}\alpha \frac{d^2\Phi}{dy_1^2} \qquad \text{--- (4.6)}$$

where physical considerations will easily verify that the C_{ij} play the same parts as in Ref.5. Solving (4.6) for L_1, M_1 and Z one finds

$$\begin{vmatrix} L_1 \\ M_1 \\ Z \end{vmatrix} = (\Gamma_{ij}) \cdot \begin{vmatrix} \frac{dp}{dy_1} \\ \frac{dq}{dy_1} \\ \frac{d\phi}{dy_1} \end{vmatrix} \qquad \text{--- (4.7)}$$

/ where ....

where

$$\Gamma \cdot \Gamma_{11} = C_{11}C_{33} - C_{13}C_{31}\,, \quad \Gamma \cdot \Gamma_{12} = -C_{21}C_{33}\,, \quad \Gamma \cdot \Gamma_{13} = C_{21}C_{13}$$

$$\Gamma \cdot \Gamma_{21} = -C_{12}C_{33}\,, \quad \Gamma \cdot \Gamma_{22} = C_{22}C_{33}\,, \quad \Gamma \cdot \Gamma_{23} = -C_{22}C_{13}$$

$$\Gamma \cdot \Gamma_{31} = C_{12}C_{31}\,, \quad \Gamma \cdot \Gamma_{32} = -C_{22}C_{31}\,, \quad \Gamma \cdot \Gamma_{33} = C_{11}C_{22} - C_{12}C_{21} \qquad \text{--- (4.8)}$$

and

$$\Gamma = \left\| \Gamma_{ij} \right\| = C_{33}(C_{11}C_{22} - C_{12}^2) - C_{22}C_{13}C_{31} \qquad \text{--- (4.9)}$$

Substituting from (4.7) in (4.4) gives the strain energy stored at a station y_1

$$dU = \frac{1}{2}\Big[\sin\alpha \left\{ \Gamma_{11}\, p'^2 + (\Gamma_{12} + \Gamma_{21}) p'q' + \Gamma_{22}\, q'^2 \right\} + \Gamma_{33}\, \phi'^2 + (\Gamma_{13} \sin\alpha + \Gamma_{31}) p'\phi' + (\Gamma_{23} \sin\alpha + \Gamma_{32})\, q'\phi' \Big] dy_1 \qquad \text{--- (4.10)}$$

and hence the total potential energy of the wings

$$U_W = \int_0^{\ell} dU \qquad \text{--- (4.11)}$$

By (4.1) and (4.11) the total potential energy of the aircraft is

$$U = U_f + U_W \qquad \text{--- (4.12)}$$

/ 5. ....

5. NON-CONSERVATIVE FORCES

5.1 General Remarks

In the Introduction the dynamic problems of aero-elasticity have been divided into three groups which differ from each other by the character of the external forces acting on the aircraft. It is also indicated there that these problems will be studied under the two headings of free and forced vibrations, and it becomes the object of this section to write down expressions for the impressed forces in the different cases.

However, before turning the attention to the external forces, there is one other feature which is common to all the dynamic problems of aero-elasticity, i.e. elastic dissipation or structural damping. Although by 1A4 this phenomenon will be neglected, it will be worthwhile to give it some consideration here. In most applied problems the existence of a dissipation function is assumed, nevertheless this discussion has been intentionally included in this section which deals with non-conservative forces, because the real mechanism of structural damping has not yet been satisfactorily explained. Thus the use of a dissipation function is mainly justified by the convenience it offers. In actual problems the coefficients of such a function are subject to estimates based on experience and, if possible, experiments, and in general their values are unreliable, since they even tend to vary among aircraft of the same type. The use of oblique coordinates therefore does not in any way affect the general position, and a dissipation function may be defined in exactly the same way as it is done in other work.

5.2 Vibrations in Vacuo and Still Air

It is customary to refer to vibrations taking place in the absence of external forces as still air vibrations. However, this terminology is not quite correct, and it has been realised for some time that the effect of still air damping may be quite important. For example, when performing fatigue tests (Ref.7) of wings by exciting them at one of their natural frequencies, knowledge of the amount of energy absorbed by this type of damping would offer the possibility of estimating the energy absorbed by the wing structure during the tests. Knowledge of the latter quantity in its turn would not only help in the study of fatigue but also provide a more rational approach to the problem of structural damping. However, to the author's knowledge, no theoretical or experimental evidence on still air damping is at present available, and for this reason its effect will not be studied here.

/ 5.3 ....

5.3 Flutter and Dynamic Stability

In Ref.8 a strong case has been made for the use of unsteady derivatives in problems of dynamic stability, since the use of quasi-static derivatives which are independent of the frequency of oscillation is no longer justifiable. Once this point of view has been accepted it is obvious that the equations of motion for the two hitherto separate problems become identical. However since the frequency of oscillation in stability problems is usually very low, it will be possible to use values of the derivatives obtained from the simple approximate formulae of Ref.9 which deals with the problem of a two-dimensional aerofoil oscillating slowly in a subsonic air stream. For the problem of flutter mostly the exact derivatives of Ref.12 will have to be used since the frequency will be too high.

Apart from the link due to the use of common derivatives, there is also a physical reason for which the problems of stability and flutter should no longer be treated separately. Because of the presence of swept wings, new types of flutter have arisen which by former standards would have been considered to belong to the domain of stability work, i.e. phenomena involving body motion. Thus the only distinction between the two problems remains the fact that usually the resilience of the aircraft is neglected in stability work while it is essential to flutter, but even this difference no longer holds entirely.

The use of different notations in stability and flutter work so far has been the strongest impediment to the union of these two subjects. This point was raised strongly during the Anglo-American Aeronautical Conference in 1951 only to draw the comment that apart from notational conflict between subjects there was also one between countries such as England and the U.S.A. In particular, there exists a great diversity in notation and presentation of flutter derivatives. Because Refs.9 and 12 contain all the necessary numerical data, which would be required in applications of the present work, and because the notation used in these reports is the simplest possible, it will be adopted in this report. This notation has been in use for many years in Holland as well as Germany, and it can easily be shown that in a disguised manner it has also been applied elsewhere.

After this introduction consider the thin aerofoil shown in Fig.4. Let the translatory displacement of the half-chord point

$$z = A \frac{c}{2} e^{i\nu t} \qquad \text{-- (5.3.1)}$$

and its rotational displacement (nose up) about that point

$$\theta = B e^{i\nu t} \qquad \text{-- (5.3.2)}$$

/ then ......

then the corresponding aerodynamic forces on a unit strip of the infinite aerofoil are given by

$$\text{lift} = 2\pi\rho \frac{c}{4} v^2 e^{i\nu t} \left[Ak_a + Bk_b\right] \quad \text{--- (5.3.3)}$$

$$\text{moment} = 2\pi\rho \frac{c^2}{8} v^2 e^{i\nu t} \left[Am_a + Bm_b\right] \quad \text{--- (5.3.4)}$$

The derivatives k_a, k_b, m_a, m_b are complex functions of the reduced frequency

$$\omega = \frac{\nu c}{2v} \quad \text{--- (5.3.5)}$$

and the free stream Mach number

$$\beta = \frac{v}{a} \quad \text{--- (5.3.6)}$$

exact and approximate values of which for various values of ω and β are tabulated in Refs.12 and 9 respectively. It will be seen in these references that the general expressions for the derivatives reduce for $\beta = 0$ to Küssner's formulae for incompressible flow. In Ref.11 the exact theory has been extended to the case of wings with flaps with open and closed gaps. But in the present report by 2Λ6 no consideration will be given to such effects.

Before giving attention to the manner in which it will be proposed to use the above results for three-dimensional wings, consider one term of the expressions above, e.g.

$$2\pi\rho \frac{c}{4} v^2 \ A e^{i\nu t} \ k_a$$

It has already been noted that k_a is complex, i.e. let

$$k_a = k'_a + i\, k''_a \quad \text{--- (5.3.7)}$$

as it is done in Refs.9 and 12 for the purpose of tabulation. By (5.3.1) the translatory velocity and acceleration are

$$\dot{z} = A \frac{c}{2} i \nu e^{i\nu t} \quad , \quad \ddot{z} = - A \frac{c}{2} \nu^2 e^{i\nu t}$$

respectively, so that the corresponding lift can be written

$$2\pi\rho\frac{v^2}{2}\left[k'_a\, z + k''_a \frac{\dot{z}}{\nu}\right] \text{ or } 2\pi\rho\frac{v^2}{2}\left[- k'_a \frac{\ddot{z}}{\nu^2} + k''_a \frac{\dot{z}}{\nu}\right] \quad \text{--- (5.3.8)}$$

both expressions being equivalent and real.

/ Admittedly .....

Admittedly this presentation of the flutter derivatives no longer retains the division into aerodynamic stiffness, damping and inertia, but in any case such a division increases the notation without offering any advantages. In addition, it is easily seen from the exact theory of Ref.11 that for subsonic derivatives such a division is even more artificial than for the incompressible derivatives, since their general expressions involve series of Mathieu functions, so that the separation of a term representing e.g. aerodynamic inertia would not necessarily be unique.

Next consider the problem of finite wings. The great difficulty experienced in obtaining easily interpolated results applicable to all types of wings for any Mach number and frequency of oscillation causes the use of the two-dimensional derivatives in most practical applications. The procedure to be adopted here is indicated in 2A1, and the weight function to be used will be based on the lift distribution along the span of a wing under steady conditions. By the help of Ref.13 such a function can be estimated in a matter of minutes with an accuracy, comparable with that obtained from lifting surface theory after computations which may extend over weeks or even months. From the loading function

$$K_\eta = \frac{c\, C_L}{\bar{c}\, \bar{C}_L} \qquad \text{-- (5.3.9)}$$

of Ref.13 follows immediately that

$$a_1 = K_\eta \frac{\bar{c}\bar{a}_1}{c} \qquad \text{-- (5.3.9')}$$

But it is known that the lift slope of the two-dimensional thin aerofoil is 2π and that the factor 2π in the expressions (5.3.3) and (5.3.4) may be interpreted as referring to this quantity. Hence when using the two-dimensional results for finite wings, the factor 2π will be replaced by what will now be called the weight function

$$N = K_\eta \frac{\bar{c}\bar{a}_1}{c} \qquad \text{-- (5.3.10)}$$

The use of the weight function N in actual fact implies that under unsteady conditions the character of the lift distribution does not vary. Its application brings about compatibility with the work of the next section which deals with the problem of gustloads, because it can be shown that the indicial lift function $k_1(s)$, describing the growth of lift on a wing subsequent to a sudden lateral or rotational movement of the wing, can be obtained from the unsteady derivatives corresponding to harmonic motion by means of a Fourier integral (Ref.14). Thus the expression to be used here for the lift on the oscillating wing, after application of such a Fourier transform, will lead to an indicial lift function which eventually produces the steady lift slope distribution on the wing.

/ There ....

There is still one point which requires mentioning although little can be done here towards its solution. This concerns the fact that the wing will be deformed and, in particular, twisted. A partial solution of this problem could be obtained by complementing the weight function N by a corresponding expression deduced from the basic lift distribution of Ref.15, which assumed the wing to have uniform twist (Ref.6). But normally the twist experienced by a wing in flutter will neither be uniform nor known beforehand, so that such a procedure would require a step by step process, and hence would be lengthy.

Under these conditions the lift and moment at a wing station y_1 are given by

$$dL = \rho \frac{v^2}{2} \left[z\, k_a + \frac{c}{2}\, \theta\, k_b \right] N \sin\alpha \, dy_1 \qquad \text{-- (5.3.11)}$$

$$dM = \rho \frac{v^2}{2} \left[z\, m_a + \frac{c}{2}\, \theta\, m_b \right] N \frac{c}{2} \sin\alpha \, dy_1 \qquad \text{-- (5.3.12)}$$

The corresponding arbitrary displacements to the first order are by (3.1) and (3.3)

$$\begin{gathered} \delta W + y_1 \sin\alpha\, \delta P + y_1 \cos\alpha\, \delta Q + \delta \bar{\Phi} + \delta \phi \\ \delta Q + \sin\alpha\, \delta q \end{gathered} \qquad \text{-- (5.3.13)}$$

Note that in this work it has been assumed that the axis Oy_1 lies along the half-chord of the wing. If this condition is not satisfied, the aerodynamic moment will have to be transferred to the position of that axis in the wing (see 2A8).

When applying (5.3.11) and (5.3.12) in the problems considered here one has to substitute for z and θ expressions corresponding to (5.3.13), viz:

$$\begin{aligned} z &= W + y_1 \sin\alpha\, P + y_1 \cos\alpha\, Q + \bar{\Phi} + \phi \qquad \text{-- (5.3.14)} \\ \theta &= Q + \sin\alpha\, q \end{aligned}$$

while by 2A2

$$v = \dot{U} = \text{const} \qquad \text{-- (5.3.15)}$$

along the span of the wing

The assumption 2A2 is required as one of the basic assumptions of unsteady aerofoil theory which takes only account of small lateral motions. Thus the present theory does not allow for yawing motion of the aircraft since it would introduce spanwise variation of the forward speed. This limitation is due to the lack of a suitable unsteady theory by which such effects could be superimposed on those due to lateral motion.

/ 5.4 ......

5.4 Gust Loads

In the work of the present section assumptions 2A1 and 2A2 will apply as well as the remarks made above. Thus the same weight functions will be used in conjunction with the two-dimensional results referring to arbitrary unsteady motion of an aerofoil in a flow disturbed by gusts. To the author's knowledge the relevant incompressible results have not yet been fully extended to the subsonic range, although they are known for the supersonic range (Ref.17). Only the indicial lift function $k_1(s)$ has been investigated throughout the entire range (Refs.14-16). For this reason the present section will in principle be confined to cases in which the effect of compressibility may be neglected. However this restriction only applies to the actual gust case involving disturbances in the free air stream.

An important feature of the arbitrary unsteady motion is the introduction of the non-dimensional time variable

$$s = \frac{2\,v\,t}{c} \qquad \text{-- (5.4.1)}$$

which in the case of a non-rectangular wing will vary along the span. In particular, when dealing with a tapered wing, one has

$$s(y_1, t) = \frac{2vt}{c_r - \left(\frac{c_r - c_t}{\ell}\right)y_1} = \frac{2vt}{c_r\left(1 - (1 - \lambda)\eta\right)} \qquad \text{-- (5.4.2)}$$

where $\lambda = c_t/c_r$, $\eta = \dfrac{y_1}{\ell}$

so that

$$s(y_1, t) = \frac{s_r}{1 - (1 - \lambda)\eta} \qquad \text{-- (5.4.3)}$$

The presence of sweepback introduces a further complication in that it will be necessary to allow for the fact that the gust reaches the tip of the wing some time after its root. By Fig.5 the intersection of the leading edges of the wings, i.e. the point $(g_r, 0)$ is $(\ell\cos\alpha - g_t + g_r)$ ahead of the leading edge at the tips, and hence any point $(y_1, c/2)$ of the leading edge will be reached

$$\Delta s = \frac{2}{c_r}\,\frac{(\ell\cos\alpha - g_t + g_r)\eta}{1 - (1 - \lambda)\eta} \qquad \text{-- (5.4.4)}$$

later by the disturbance than the point $(g_r, 0)$.

/ Hence ......

Hence by Ref.18, the lift and moment at a station y_1 are given by

$$dL = -\frac{\rho v^2}{2}\left[\frac{d^2z}{ds^2} + \frac{c}{2}\frac{d\theta}{ds} + \int_0^s k_1(s-\sigma)\left\{\frac{d^2z}{d\sigma^2} + \frac{c}{2}\left(\frac{d\theta}{d\sigma} + \tfrac{1}{2}\frac{d^2\theta}{d\sigma^2}\right)\right\}d\sigma\right.$$

$$\left. - \frac{c}{2v}\int_0^{s-\Delta s} k_2(s-\sigma)\frac{dW_g}{d\sigma}\,d\sigma\right] N \sin\alpha \; dy_1 \qquad \text{-- (5.4.5)}$$

$$dM = -\frac{\rho v^2}{2}\left[\frac{c}{4}\frac{d\theta}{ds} + \frac{c}{16}\frac{d^2\theta}{ds^2} - \int_0^s k_1(s-\sigma)\left\{\frac{d^2z}{d\sigma^2} + \frac{c}{2}\left(\frac{d\theta}{d\sigma} + \tfrac{1}{2}\frac{d^2\theta}{d\sigma^2}\right)\right\}d\sigma\right.$$

$$\left. + \frac{c}{4v}\int_0^{s-\Delta s} k_2(s-\sigma)\frac{dW_g}{d\sigma}\,d\sigma\right] N \frac{c}{2} \sin\alpha \; dy_1 \qquad \text{-- (5.4.6)}$$

In (5.4.5) and (5.4.6) the displacements z and θ are again given by (5.3.14) and their corresponding arbitrary displacements by (5.3.13).

6. THE INTEGRAL EQUATIONS OF MOTION

6.1 General Remarks

In the earlier sections the foundation has been laid for the deduction of the equations of motion by one of the analytical methods of general dynamics. In view of the complexity of the expressions for the energies, it is not proposed to present here the analysis in full detail. In Appendix 1 the most important steps will be described in order to allow a better understanding of this part of the work.

It has already been pointed out in the introduction that the equations of motion will be deduced in the form of integro-differential equations in terms of independent variables, some of which will represent the curvatures and rates of twist of the deformed aircraft. Such equations are most easily obtained from Hamilton's Principle which in the presence of non-conservative forces takes the form

$$\int_{t_1}^{t_2}\left\{\delta(T - U) + \sum_r Q_r\,\delta q_r\right\} dt = 0 \qquad \text{-- (6.1.1)}$$

/ The independent ....

The independent variables q_r to be used here will be those describing the rigid body motion of the aircraft

$$U, V, W \quad , \quad P, Q, R \qquad \text{-- (6.1.2)}$$

the first spanwise derivatives of the functions pertaining to the wing deformation

$$\frac{\partial p}{\partial y_1} = p' \ , \quad \frac{\partial q}{\partial y_1} = q' \ , \quad \frac{\partial \phi}{\partial y_1} = \phi' \qquad \text{-- (6.1.3)}$$

the curvatures of the lateral fuselage displacements

$$\frac{\partial^2 v_f}{\partial x_1^2} = v_f'' \ , \quad \frac{\partial^2 w_f}{\partial x_1^2} = w_f'' \qquad \text{-- (6.1.4)}$$

and the rate of twist along the fuselage

$$\frac{\partial p_f}{\partial x_1} = p_f' \qquad \text{-- (6.1.5)}$$

The equations obtained from (6.1.1) using these variables will only involve these functions and their time derivatives and the solutions of these equations will automatically satisfy the boundary conditions normally required in connection with differential equations. The latter equations can be obtained from the integral equations by differentiations and by integrating certain terms by parts.

In the next subsection the above mentioned integro-differential equations will be given for the case of natural vibrations of the aircraft in vacuo. Various special cases of these equations will be discussed which arise as the result of simplifying assumptions, in particular, one of these will illustrate application of the present work to aircraft with straight wings.

The remaining subsections give the terms which must be added to the equations of section 6.2 in order to obtain the equations of motion for the aero-elastic problems of stability, flutter and gust load.

6.2 Free Vibrations in Vacuo

The most general equations of motion of the type discussed above and obtained by the analytical process, some details of which are explained in appendix 1, are:

$$M\left\{\ddot U + \ddot Q\,\bar z_1 - \ddot R\,\bar Y_1\right\} + \int_0^{\ell} \sin\alpha\,\ddot q'\,\mu_z\,dy_1 = 0 \qquad \text{--- (6.2.1)}$$

$$M\left\{\ddot V - \ddot P\bar z_1 + \ddot R\bar x_1\right\} + \int_0^{\ell}\left\{\ddot q'\cos\alpha - \ddot p'\right\}\mu_z\,dy_1 + \int_{-\ell_1}^{\ell_2}\left\{\ddot v_f''\mu_f^1 - \ddot p_f'\mu_{zf}\right\}dx_1 = 0 \qquad \text{--- (6.2.2)}$$

$$M\left\{\ddot W + \ddot P\bar Y_1 - \ddot Q\bar x_1\right\} + \int_0^{\ell}\left\{\ddot\phi'\mu - \ddot q'\sin\alpha\mu_x - \ddot p'\sin\alpha\mu^1\right\}dy_1 + \int_{-\ell_1}^{\ell_2}\left\{\ddot w_f''\mu_f + \ddot p_f'\mu_{Yf}\right\}dx_1 = 0 \qquad \text{--- (6.2.3)}$$

$$A\ddot P - E\ddot Q - G\ddot R + M\left\{\bar Y_1(\ddot W + \dot U_o\dot Q) - \bar z_1(\ddot V - \dot U_o\dot R)\right\} + \int_0^{\ell}\left\{\ddot\phi'\sin\alpha\mu_1 - \ddot q'\sin^2\alpha\mu_{1x} - \ddot p'\sin^2\alpha\mu_1^1 + (\ddot p' - \ddot q'\cos\alpha)i_x\right\}dy_1$$
$$+ \int_{-\ell_1}^{\ell_2}\left\{\ddot w_f''\mu^1_{Yf} - \ddot v_f''\mu^1_{zf} + \ddot p_f' j_f\right\}dx_1 = 0 \qquad \text{--- (6.2.4)}$$

$$B\ddot Q - E\ddot P - F\ddot R + M(\bar z_1\ddot U - \bar x_1\ddot W - \bar Y_1\dot U_o\dot P) + \int_0^{\ell}\Big\{\ddot q'\sin\alpha(j + \cos\alpha\mu_{1x}) - \ddot\phi'(\mu_x - \cos\alpha\mu_1) + \ddot p'\sin\alpha(\mu_x^1 - \cos\alpha\mu_1^1)$$
$$+ \dot U_o(\dot p'\mu^1\sin\alpha - \dot\phi'\mu + \dot q'\mu_x\sin\alpha)\Big\}dy_1 - \int_{-\ell_1}^{\ell_2}\left\{\ddot w_f''\mu^1_{1f} + \ddot p_f'\mu_{1Yf} + \dot U_o(\dot w_f''\mu^1_f + \dot p_f'\mu_{Yf})\right\}dx_1 = 0 \qquad \text{--- (6.2.5)}$$

$$C\ddot R - F\ddot Q - G\ddot P + M(\bar x_1\ddot V - \bar Y_1\ddot U - \bar z_1\dot U_o\dot P) + \int_0^{\ell}\left\{(\ddot q'\cos\alpha - \ddot p')i_{xz} + (\ddot p'\cos\alpha - \ddot q')\mu_{1z} + \dot U_o\mu_z(\dot q'\cos\alpha - \dot p')\right\}dy_1$$
$$+ \int_{-\ell_1}^{\ell_2}\left\{\ddot v_f''\mu^1_{1f} - \ddot p_f'\mu_{zf} + \dot U_o(\dot v_f''\mu_f^1 - \dot p_f'\mu_{zf})\right\}dx_1 = 0 \qquad (6.2.6)$$

$$\int_0^{\ell}\left\{\ddot p'(\bar i_x + \sin^2\alpha\bar\mu^2) - \ddot q'(\cos\alpha\,\bar i_x - \sin^2\alpha\bar\mu_x^1) - \ddot\phi'\sin\alpha\bar\mu^1\right\}d\eta - \ddot V\mu_z + \ddot P(i_x - \sin^2\alpha\mu_1^1)$$
$$- \ddot W\sin\alpha\,\mu^1 + \ddot Q\sin\alpha(\mu_x - \cos\alpha\,\mu_1) + \ddot R(\cos\alpha\,\mu_{1z} - i_{xz})$$
$$- \dot U_o(\dot Q\sin\alpha\,\mu^1 - \dot R\mu_z) + \left(\Gamma_{11}p' + \frac{\Gamma_{12} + \Gamma_{21}}{2}q'\right)\sin\alpha + \frac{\Gamma_{13}\sin\alpha + \Gamma_{31}}{2}\phi' = 0 \qquad (6.2.7)$$

$$\int_0^{\ell}\left\{\ddot q'(\bar i_x + \sin^2\alpha\,\bar i_z) - \ddot p'(\cos\alpha\,\bar i_x - \sin^2\alpha\bar\mu_x^1) - \ddot\phi'\sin\alpha\mu_x\right\}d\eta + \ddot U\sin\alpha\mu_z + \ddot V\cos\alpha\mu_z - \ddot W\sin\alpha\mu_x$$
$$- \ddot P(\sin^2\alpha\,\mu_{1x} + \cos\alpha\,i_x) + \ddot Q(\sin\alpha\,j + \cos\alpha\,\sin\alpha\,\mu_{1x}) + \ddot R(\cos\alpha\,i_{xz} - \mu_{1z})$$
$$- \dot U_o(\dot Q\sin\alpha\mu_x + \dot R\cos\alpha\mu_z) + \left(\frac{\Gamma_{12} + \Gamma_{21}}{2}p' + \Gamma_{22}q'\right)\sin\alpha + \frac{\Gamma_{23}\sin\alpha + \Gamma_{32}}{2}\phi' = 0$$

(6.2.8)

$$\int_0^{\ell}\left\{\ddot{\phi}'\bar{\mu} - \ddot{p}'\sin\alpha\,\bar{\mu}^1 - \ddot{q}'\sin\alpha\,\bar{\mu}_x\right\}d\eta + \ddot{W}\mu + \ddot{P}\sin\alpha\,\mu_1 + \ddot{Q}(\cos\alpha\,\mu_1 - \mu_x) + \dot{U}_o\dot{Q}\,\mu$$

$$+ \Gamma'_{33}\,\phi' + \frac{\Gamma'_{13}\sin\alpha + \Gamma'_{31}}{2}\,p' + \frac{\Gamma'_{23}\sin\alpha + \Gamma'_{32}}{2}\,q' = 0 \qquad \text{-- (6.2.9)}$$

$$\int_{-\ell_1}^{\ell_2}\left\{\ddot{v}''_f\bar{\mu}^2_f - \ddot{p}'_f\bar{\mu}^1_{zf}\right\}d\eta + \ddot{V}\mu^1_f - \ddot{P}\mu^1_{zf} + \ddot{R}\mu^1_{1f} - \dot{U}_o\dot{R}\mu^1_f + E_f(\bar{I}_{Y_1z_1f}w''_f + \bar{I}_{z_1f}\,v''_f) = 0 \qquad \text{-- (6.2.10)}$$

$$\int_{-\ell_1}^{\ell_2}\left\{\ddot{w}''_f\bar{\mu}^2_f + \ddot{p}'_f\bar{\mu}^1_{Yf}\right\}d\eta + \ddot{W}\mu^1_f + \ddot{P}\mu^1_{Yf} - \ddot{Q}\mu^1_{1f} + \dot{U}_o\dot{Q}\mu^1_f + E_f(\bar{I}_{Y_1f}\,w''_f + \bar{I}_{Y_1z_1f}\,v''_f) = 0 \qquad \text{-- (6.2.11)}$$

$$\int_{-\ell_1}^{\ell_2}\left\{\ddot{p}'_f\bar{j}_f + \ddot{w}''_f\bar{\mu}^1_{Yf} - \ddot{v}''_f\bar{\mu}^1_{zf}\right\}d\eta - \ddot{V}\mu_{zf} + \ddot{W}\mu_{Yf} + \ddot{P}j_f - \ddot{Q}\mu_{1Yf} - \ddot{R}\mu_{1zf} + \dot{U}_o(\dot{R}\mu_{zf} + \dot{Q}\mu_{Yf}) + C_f p'_f = 0 \qquad \text{-- (6.2.12)}$$

where the integrals $\int_0^{\ell}$ are to be extended over both wings (see 3.7'). The coefficients μ, j and i are functions of the relevant coordinates and physical data, as defined by the following formulae:

$$\mu(y_1) = \int_{y_1}^{\ell} m\,d\zeta \qquad (6.2.13)$$

$$\mu_1(y_1) = \int_{y_1}^{\ell} m\,\zeta\,d\zeta \qquad (6.2.14)$$

$$\mu^1(y_1) = \int_{y_1}^{\ell} m(\zeta - y_1)\,d\zeta \qquad (6.2.15)$$

$$\mu_x(y_1) = \int_{y_1}^{\ell} m\,\bar{x}_1\,d\zeta \qquad (6.2.16)$$

$$\mu_z(y_1) = \int_{y_1}^{\ell} m\,\bar{z}_1\,d\zeta \qquad (6.2.17)$$

$$\mu_{1x}(y_1) = \int_{y_1}^{\ell} m\,\zeta\,\bar{x}_1\,d\zeta \qquad (6.2.18)$$

$$\mu_{1z}(y_1) = \int_{y_1}^{\ell} m\,\zeta\,\bar{z}_1\,d\zeta \qquad (6.2.19)$$

$$\mu^1_x(y_1) = \int_{y_1}^{\ell} m\,\bar{x}_1(\zeta - y_1)\,d\zeta \qquad (6.2.20)$$

$$\mu^1_1(y_1) = \int_{y_1}^{\ell} m\,\zeta\,(\zeta - y_1)\,d\zeta \qquad (6.2.21)$$

$$i_x(y_1) = \int_{y_1}^{\ell} I_{x_1}\,d\zeta \qquad (6.2.22)$$

$$i_{xz}(y_1) = \int_{y_1}^{\ell} I_{x_1z_1}\,d\zeta \qquad (6.2.23)$$

$$j(y_1) = \int_{y_1}^{\ell} J_{y_1}\,d\zeta \qquad (6.2.24)$$

$$\bar{\mu}(y_1,\ \eta) = \int_{\overline{y_1,\eta}}^{\ell} m \ d\ \zeta \qquad (6.2.25)$$

$$\bar{\mu}_x(y_1,\ \eta) = \int_{\overline{y_1,\eta}}^{\ell} m\ \bar{x}_1\ d\ \zeta \qquad (6.2.26)$$

$$\bar{\mu}^1(y_1,\ \eta) = \int_{\overline{y_1,\eta}}^{\ell} m(\zeta - y_1)\ d\ \zeta \qquad (6.2.27)$$

$$\bar{\mu}_x^1(y_1,\ \eta) = \int_{\overline{y_1,\eta}}^{\ell} m\ \bar{x}_1\,(\zeta - y_1) d\ \zeta \qquad (6.2.28)$$

$$\bar{\mu}^2(y_1,\ \eta) = \int_{\overline{y_1,\eta}}^{\ell} m(\zeta - \eta)(\zeta - y_1) d\zeta \qquad (6.2.29)$$

$$\bar{I}_x(y_1,\ \eta) = \int_{\overline{y_1,\eta}}^{\ell} I_{x_1}\ d\ \zeta \qquad (6.2.30)$$

$$\bar{I}_z(y_1,\eta) = \int_{\overline{y_1,\eta}}^{\ell} I_{z_1}\ d\ \zeta \qquad (6.2.31)$$

$$\mu_{Yf}(x_1) = \int_a^b m_f\ \bar{Y}_{1f}\ d\ \zeta \qquad (6.2.32)$$

$$\mu_{zf}(x_1) = \int_a^b m_f\ \bar{z}_{1f}\ d\ \zeta \qquad (6.2.33)$$

$$\mu_{1Yf}(x_1) = \int_a^b m_f\ \zeta\ \bar{Y}_{1f}\ d\ \zeta \qquad (6.2.34)$$

$$\mu_{1zf}(x_1) = \int_a^b m_f\ \zeta\ \bar{z}_{1f}\ d\ \zeta \qquad (6.2.35)$$

$$\mu_f^1(x_1) = \int_a^b m_f(\zeta - x_1) d\ \zeta \qquad (6.2.36)$$

$$\mu_{1f}^1(x_1) = \int_a^b m_f\zeta(\zeta - x_1)\ d\ \zeta \qquad (6.2.37)$$

$$\mu_{Yf}^1(x_1) = \int_a^b m_f\bar{Y}_{1f}(\zeta - x_1)\ d\ \zeta \qquad (6.2.38)$$

$$\mu_{zf}^1(x_1) = \int_a^b m_f\bar{z}_{1f}(\zeta - x_1) d\ \zeta \qquad (6.2.39)$$

$$j_f(x_1) = \int_a^b J_{x_1 f}\ d\ \zeta \qquad (6.2.40)$$

where $\begin{cases} a = x_1\ ,\ b = \ell_2 \\ a = -\ell_1,\ b = x_1 \end{cases}$ for $\begin{cases} x_1 > 0 \\ x_1 < 0 \end{cases}$

$$\bar{\mu}_{Yf}^1(x_1,\eta) = \int_c^d m_f\bar{Y}_{1f}(\zeta - x_1) d\ \zeta \qquad (6.2.41)$$

$$\bar{\mu}_{zf}^1(x_1,\eta) = \int_c^d m_f\bar{z}_{1f}(\zeta - x_1) d\ \zeta \qquad (6.2.42)$$

$$\bar{\mu}_f^2(x_1,\eta) = \int_c^d m_f(\zeta - \eta)(\zeta - x_1) d\ \zeta \qquad (6.2.43)$$

$$\bar{j}_f(x_1,\eta) = \int_c^d J_{x_1 f}\ d\ \zeta \qquad (6.2.44)$$

where $\begin{cases} c = \overline{x_1,\eta}\ ,\quad d = \ell_2 \\ c = -\ell_1\ ,\quad d = \underline{x_1,\eta} \end{cases}$ for $\begin{cases} x_1 > 0 \\ x_1 < 0 \end{cases}$

It may be noted that the coefficient functions just defined are Green's functions for the mass system. The notation introduced here is quite self-explanatory in that the sub- and super scripts contain all the essential information required for the re-construction of the integrals defining these functions. Thus

the base letters μ, i, j refer to physical data such as mass and section moments of inertia

subscript numerals indicate the presence of the integration variable as a factor.

subscript letters indicate the positions of the centres of gravity or the axes to which the moments of inertia refer at each section.

superscript numerals indicate the number of linear terms involving differences of the integration and actual variables.

bars indicate the special type of integration limits, the cause of which is explained in Appendix 1.

The same principle will be applied to the notation in the later parts of section 6.

For the purpose of the subsequent discussion, the equations of motion have been written in Table 1 in the form of a scheme using matrix representation.

It may be reasoned that these equations are too general and therefore too complex to permit a clear understanding of the meaning of the various terms. The main reason for retaining full generality lies with the fact that over and over again it has been found necessary in recent years to extend aero-elastic investigations to take account of special features, formerly considered unimportant. The availability of completely general equations may therefore be of considerable assistance at such occasions. For example, it is customary to assume that the aircraft is symmetrical with respect to the plane Ox_1z_1 and for this reason to put the product of inertia E equal to zero. On the other hand, an occasion may arise when interest will be concentrated on the behaviour of an aircraft which carries all its fuel in one wing; under those conditions the assumption $E = 0$ is obviously no longer satisfied. While this example is of a rather simple nature so that the extension of the equations to cover this case could be easily effected, other more complicated problems could be thought of for which this is no longer the case.

In all aero-elastic investigations which are known to the author an assumption has been introduced by which shear deflections have been neglected. The only reason for the inclusion here of the relevant terms, as far as the wings are concerned, is that they might be of interest in dealing with wing vibrations. It is easily seen that introduction of this assumption will cause the disappearance of equation (6.2.9) and of other relevant terms.

/ Next .....

Next to this assumption the most reasonable one refers to the terms involving the position $\bar{Y}_{1f}$ of the centres of mass of the fuselage sections. Asymmetry of the fuselage of this kind will rarely arise. If in addition it is assumed that the axis Ox_1 is one of mass symmetry of the fuselage all along its length, the terms involving $\bar{Z}_{1f}$ and $I_{Y_1 Z_1 f}$ will likewise disappear, i.e. the corresponding columns of Table 1 will be free from inertia coupling between the various modes of deformation of the fuselage. In addition several terms involving the body motion will vanish in equations (6.2.10) to (6.2.12).

Since in many investigations the fuselage is of secondary importance, the above assumptions will very often be made. Although the introduction of similar assumptions for the wings, i.e. $\bar{x}_1 = \bar{z}_1 = I_{x_1 z_1} = 0$, will rarely be justifiable, it will be of interest to compare the equations, corresponding to these conditions, with those of Ref.3. For this purpose it will also be assumed that the origin O_1 of the system $O_1 x_1 y_1 z_1$ is at rest, and that there is no fuselage present. The dynamic equations of a two-cell box with a fixed root then become:-

$$\Gamma_{11} \sin\alpha \, p' + \frac{\Gamma_{12} + \Gamma_{21}}{2} \sin\alpha \, q' = \int_0^{\ell} \left\{ - \ddot{p}'(\bar{i}_x + \sin^2\alpha \, \bar{\mu}^2) + \ddot{q}' \, \bar{i}_x \cos\alpha \right\} dy_1$$

$$\frac{\Gamma_{12} + \Gamma_{21}}{2} \sin\alpha \, p' + \Gamma_{22} \sin\alpha \, q' = \int_0^{\ell} \left\{ \ddot{p}' \, \bar{i}_x \cos\alpha - \ddot{q}'(\bar{i}_x + \bar{i}_z \sin^2\alpha) \right\} dy_1 \qquad (6.2.45)$$

It is easily seen that these equations are in agreement with the equations (4.9) of Ref.3, if due notice is given to the changed notation, coordinate system and the fact that in the present work the angle between the coordinate axes is $\pi - \alpha$ instead of α. Note, however, that the function $f_4(\xi_1 \, x)$ defined by (4.10) of Ref.3 should read

$$f_4(\xi_1 \, x) = \int_{\overline{\xi_1 x}}^{\ell} \left\{ k_y^2 \operatorname{cosec}\alpha + \sin\alpha (\eta - x)(\xi - x) \right\} d\eta$$

Finally consider the case of straight wings when $\alpha = \frac{\pi}{2}$ In Ref.19 the integro-differential equations have been deduced for the case of gust loads on aircraft with straight wings. The assumptions made there with reference to the aircraft structure are similar to those made above when obtaining (6.2.45). In addition, only symmetrical motion is considered, but body motion is allowed for, although the rotation of the body fixed system relative to the space fixed system has been neglected. Under those conditions one finds from (6.2.3), (6.2.5), (6.2.7) and (6.2.8):-

/ $M\ddot{W}$ ......

$$M\ddot{W} - \int_0^{\ell} \mu^1 \ddot{p}' \, dY_1 = 0$$

$$B\ddot{Q} + \int_0^{\ell} j \, \ddot{q}' \, dY_1 = 0$$

$$-\mu^1 \ddot{W} + \int_0^{\ell} \bar{\mu}^2 \ddot{p}' \, dY_1 + \Gamma_{11} p' + \frac{\Gamma_{12} + \Gamma_{21}}{2} q' = 0$$

$$j\ddot{Q} + \int_0^{\ell} \bar{j} \, \ddot{q}' \, dY_1 + \frac{\Gamma_{12} + \Gamma_{21}}{2} p' + \Gamma_{22} q' = 0 \qquad (6.2.46)$$

Since by (4.6) in the present case

$$p' = -\Phi''$$

it is easily seen that the above equations agree with the dynamical terms of the equations (5.215) - (5.218) of Ref.19, where it has been assumed that there is also no elastic coupling between the translatory and rotational deformations of the wing. Comparison with the equations of Ref.19 also gives a straight forward interpretation of the elastic constants Γ_{ij} which could also have been obtained from Ref.4. Thus Γ_{11} is the bending and Γ_{22} is the torsional stiffness of the wing when there is no sweep back.

6.3 Flutter and Dynamic Stability

Using the results of section (5.3) one obtains after the appropriate transformations, analogous to those explained in Appendix 1, the following aerodynamic terms which have to be added to the corresponding equations of section (6.2), which have been indicated in square brackets

$$W K_a(0) + P \sin\alpha \, K_{a1}(0) + Q\left\{\cos\alpha \, K_{a1}(0) + K_b(0)\right\} + \int_0^{\ell}\left\{- p'\sin\alpha \, K_a^1 + q' \sin\alpha \, K_b + \emptyset' K_a\right\} dy_1 \qquad (6.3.1)\ [6.2.3]$$

$$\left[W K_{a1}(0) + P \sin\alpha \, K_{a2}(0) + Q\left\{\cos\alpha \, K_{a2}(0) + K_{b1}(0)\right\} + \int_0^{\ell}\left\{- p'\sin\alpha \, K_{a1}^1 + q' \sin\alpha \, K_{b1} + \emptyset' K_{a1}\right\} dy_1\right] \sin\alpha \qquad (6.3.2)\ [6.2.4]$$

/ W .....

$$W\left\{\cos\alpha\, K_{a1}(0) + M_a(0)\right\} + P\sin\alpha\left\{\cos\alpha\, K_{a2}(0) + M_{a1}(0)\right\}$$
$$+ Q\left\{\cos^2\alpha\, K_{a2}(0) + \cos\alpha\, K_{b1}(0) + \cos\alpha\, M_{a1}(0) + M_b(0)\right\}$$
$$+ \int_0^{\ell}\left\{- p'(\sin\alpha\cos\alpha\, K^1_{a1} + \sin\alpha\, M^1_a) + q'(\sin\alpha\cos\alpha\, K_{b1} + \sin\alpha\, M_b)\right.$$
$$\left. + \emptyset'(\cos\alpha\, K_{a1} + M_a)\right\} dy_1 \qquad \text{-- (6.3.3)}$$

[6.2.5]

$$-\left[W K^1_a + P\sin\alpha\, K^1_{a1} + Q\left\{\cos\alpha\, K^1_{a1} + K^1_b\right\} + \int_0^{\ell}\left\{- p'\sin\alpha\, \bar{K}^2_a + q'\sin\alpha\, \bar{K}^1_b\right.\right.$$
$$\left.\left. + \emptyset'\ \bar{K}^1_a\right\} d\eta\right]\sin\alpha \qquad \text{-- (6.3.4)}$$

[6.2.7]

$$\left[W M_a + P\sin\alpha\, M_{a1} + Q\left\{\cos\alpha\, M_{a1} + M_b\right\} + \int_0^{\ell}\left\{- p'\sin\alpha\, \bar{M}^1_a + q'\, \bar{M}_b\right.\right.$$
$$\left.\left. + \emptyset'\ \bar{M}_a\right\} d\eta\right]\sin\alpha \qquad \text{-- (6.3.5)}$$

[6.2.8]

$$W K_a + P\sin\alpha\, K_{a1} + Q\left\{\cos\alpha\, K_{a1} + K_b\right\} + \int_0^{\ell}\left\{- p'\sin\alpha\, \bar{K}^1_a + q'\sin\alpha\, \bar{K}_b + \emptyset\, \bar{K}^1_a\right\} d\eta \qquad \text{-- (6.3.6)}$$

[6.2.9]

Putting $\frac{\rho\, \dot{U}^2}{2} N \sin\alpha = \chi$ -- (6.3.7)

the coefficients appearing in the above terms are defined by the following formulae

$$K_a(y_1) = \int_{y_1}^{\ell} \chi k_a \, d\zeta \qquad (6.3.8) \qquad K^1_{a1}(y_1) = \int_{y_1}^{\ell} \zeta \chi k_a (\zeta - y_1) d\zeta \qquad (6.3.12)$$

$$K_{a1}(y_1) = \int_{y_1}^{\ell} \zeta \chi k_a \, d\zeta \qquad (6.3.9) \qquad M_a(y_1) = \int_{y_1}^{\ell} \frac{c}{2} \chi m_a \, d\zeta \qquad (6.3.13)$$

$$K_{a2}(y_1) = \int_{y_1}^{\ell} \zeta^2 \chi k_a \, d\zeta \qquad (6.3.10) \qquad M_{a1}(y_1) = \int_{y_1}^{\ell} \frac{c}{2} \zeta \chi m_a \, d\zeta \qquad (6.3.14)$$

$$K^1_a(y_1) = \int_{y_1}^{\ell} \chi\, k_a (\zeta - y_1) d\zeta \qquad (6.3.11) \qquad M^1_a(y_1) = \int_{y_1}^{\ell} \frac{c}{2} \chi m_a (\zeta - y_1) d\zeta \qquad (6.3.15)$$

/ $\bar{K}$ ....

$$\bar{K}_a(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \chi k_a \, d\zeta \qquad (6.3.16)$$

$$\bar{K}_a^1(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \chi k_a (\zeta - y_1) \, d\zeta \qquad (6.3.17)$$

$$\bar{K}_a^2(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \chi k_a (\zeta - y_1)(\zeta - \eta) \, d\zeta \qquad (6.3.18)$$

$$\bar{M}_a(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \frac{c}{2} \chi m_a \, d\zeta \qquad (6.3.19)$$

$$\bar{M}_a^1(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \frac{c}{2} \chi m_a (\zeta - y_1) \, d\zeta \qquad (6.3.20)$$

$$K_b(y_1) = \int_{y_1}^{\ell} \frac{c}{2} \chi k_b \, d\zeta \qquad (6.3.21)$$

$$K_{b1}(y_1) = \int_{y_1}^{\ell} \frac{c}{2} \zeta \chi k_b \, d\zeta \qquad (6.3.22)$$

$$K_b^1(y_1) = \int_{y_1}^{\ell} \frac{c}{2} \chi k_b (\zeta - y_1) \, d\zeta \qquad (6.3.23)$$

$$M_b(y_1) = \int_{y_1}^{\ell} \left(\frac{c}{2}\right)^2 \chi m_b \, d\zeta \qquad (6.3.24)$$

$$\bar{K}_b(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \frac{c}{2} \chi k_b \, d\zeta \qquad (6.3.25)$$

$$\bar{K}_b^1(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \frac{c}{2} \chi k_b (\zeta - y_1) \, d\zeta \qquad (6.3.26)$$

$$\bar{M}_b(y_1, \eta) = \int_{\overline{y_1, \eta}}^{\ell} \left(\frac{c}{2}\right)^2 \chi m_b \, d\zeta \qquad (6.3.27)$$

When evaluating these coefficients for any particular case notice must be taken of the fact that the derivatives

$$k_a, \; m_a, \; k_b, \; m_b$$

are functions of the reduced frequency (cf 5.3.5)

$$\omega = \frac{\nu c}{2 U} \qquad (6.3.28)$$

which normally will vary along the wing. Further, these derivatives will be complex functions of ω and the Mach number β and therefore when writing down the final equations in real form, for example, the term $\bar{M}_a^1 p'$ has to be presented in analogy with (5.3.8), i.e.

$$\bar{M}_a^1 p' = \bar{M}_a^{1\prime} p' + \bar{M}_a^{1\prime\prime} \frac{\dot{p}'}{\nu} \qquad (6.3.29)$$

/ 6.4 .....

6.4 Gust Loads

The formal work which leads to the terms which have to be added to equations (6.2.1) - (6.2.12) is very similar to that necessary for the deduction of the expressions given in the preceding sections. The fact that the non-dimensional time coordinate varies along the span does not introduce any principal difficulties, although it tends to complicate the analysis. Due to the complex nature of the theoretical expressions for the aerodynamic forces (5.3.5) and (5.3.6) the most general case corresponding to the freedom of motion considered in the earlier sections leads to a large number of terms. For this reason and because it is customary in the study of gust loads to introduce additional assumptions restricting the mode of motion of the aircraft, it is not proposed to give here the general expressions. On the basis of Appendix 1 and of the earlier work given in this report, it is felt that these expressions could be deduced without great difficulty by anyone requiring them.

However a few remarks will be made with regard to the choice of a suitable time variable since this may be of great help to anyone wanting to obtain these expressions. In Ref.20 the problem of gust loads has been treated for the case of straight wings and a dimension less time variable s_m has been introduced by using for c of (5.4.2) the mean chord of the wing. As a result the time derivatives of the equations of section 6.2 had to be multiplied by appropriate conversion factors. It is shown in that reference that such a procedure tends to over-estimate the loads inboard of the wing station corresponding to the mean chord and to under-estimate them further outboard. This is easily seen to be true because $k_1(s)$ is a monotomic increasing function and s is inversely proportional to the wing chord (assuming, of course, that the wing is of conventional plan form with outward taper).

When it is desired to avoid the above simplification referring to the time variable, it will in general be preferable to use the dimensional time t throughout instead of s. It will be easily seen that one can transform the integrals involving the function k_1 in the following manner:

$$\int_0^s k_1(s-\sigma)\frac{\partial^2 z}{\partial\sigma^2}\,d\sigma = \int_0^t k_1\left((t-\tau)\frac{2\mathring{U}}{c(y_1)}\right)\frac{\partial^2 z(y_1,\tau)}{\partial\tau^2}\,\frac{c(y_1)}{2U}\,d\tau$$

Thus when integrating expressions containing integrals of the above type over the span, the order of integration can be inverted without difficulty as the integration variables will refer to time and space coordinates which are independent of each other. Such a procedure then leads to special types of Wagner and Küssner functions allowing for three-dimensional effects. These modified functions will result from the integration of all terms, depending on the spanwise coordinate, over parts of the span, since, as before, further inversions of the order of integration will be caused by the introduction of the new independent variables discussed in section 1.

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7. METHODS OF SOLUTION

In all but the very simplest cases an exact solution of the equations of motion, deduced here, cannot be expected. However, this is a fact which holds true in whatever form the equations of motion may have been obtained. Thus in most practical cases approximate methods of solution have to be used and it has already been pointed out earlier that all such methods used in Aero-elasticity when dealing with the conventional types of equations will likewise be applicable to integral equations.

In Ref.6 some space has been devoted to the conversion of integro-differential equations into matrix equations, and it has been shown there that the latter can be solved by the common iteration processes. The main difference between the matrix equations obtained from the integral equations and those obtained by sectioning the aircraft from the start lies with the variables occurring in both types of equations.

On the other hand, if it is proposed to use natural modes which either have been chosen suitably in the form of polynomials or which have been obtained from vibration tests, there is no need to perform the transition to matrix equations. The modes which are to be used may be introduced directly into the integral equations, although it may become necessary to evaluate the integrals using approximate methods such as Simpson's rule.

Consider, for example, the procedure for a binary flutter investigation of the wings. If it is assumed that all the relevant coefficient functions of sections 6.2 and 6.3 are known and that the principal response will be shown by the wings, it will only be necessary to investigate equations (6.2.7) and (6.2.8), augmented by the aerodynamic terms (6.3.4) and (6.3.5). Substituting in these equations for the independent variables linear combinations of the appropriate modes, obtained from experiment, i.e. replacing, for example, p' by

$$r_1 p_1' + r_2 p_2' \qquad \text{-- (7.1)}$$

where p_1', p_2' are now known functions of the spanwise coordinate and r_1 and r_2 are the corresponding normal coordinates, one deduces finally two simultaneous differential equations in r_1 and r_2 and their time derivatives. Assuming harmonic motion, these equations will lead to a characteristic equation involving flutter speeds and frequencies. Since the aerodynamic terms depend on these latter quantities, some of the coefficients of the characteristic equation may have to be calculated several times before it is satisfied.

/ In a ....

In a similar manner, a procedure for ternary calculations can be developed by introducing, for example, also equation (6.2.3) + (6.3.1), referring to the vertical body motion, into the consideration. A similar approach has often also been applied to the problem of gust loads, but in that case the normal coordinates will be arbitrary functions of the time and systems of differential equations have to be solved. However, it should be realised that such a procedure introduces a constraint, since normally an infinite number of modes will be involved.

Finally a short remark will be made with regard to the specification of the gust loading process. It is customary to make certain assumptions with regard to the "gust profile", although all investigators are well aware of the fact that it is very difficult or almost impossible to reproduce such theoretical gusts in order to check on the results of the theory. On the other hand, it is possible to use the vertical acceleration for this purpose and to consider the gust velocity one of the unknown quantities to be determined in the process of solution. The main advantages of such a step are that it is easy to measure accelerations suffered by the aircraft and that in this way gust structure can be investigated.

8. CONCLUSIONS

The equations of motion of aircraft deduced in this report are the most general within the assumptions stated in section 1.2. They are integro-differential equations involving only time derivatives of the independent variables, some of which are themselves space derivatives of the displacements commonly used. The use of such variables has only been possible because the continuous character of the aircraft structure has been retained throughout. In addition, this fact has led to a concise notation in terms of the physical data specifying the aircraft as an elastic mass system and aerodynamically, which will be found to be very lucid and suitable for fundamental aero-elastic work. The final equations of motion are obtained by combining the dynamic terms given in section 6.2 with the relevant aerodynamic ones of section 6.3 as far as problems of flutter and dynamic stability are concerned. In section 6.4 a few remarks have been made explaining how similar equations can be deduced for the problem of gust loads.

In section 7, procedures have been outlined for solving these equations in any practical case, and it is seen there that all conventional methods used in aero-elastic work are applicable. In particular, the equations lend themselves to iterative processes and to the use of experimentally determined modes.

/ References .....

LIST OF REFERENCES

No.	Author	Title, etc.
1.	R.W. Trail-Nash	The symmetric vibrations of aircraft. Aeronautical Quarterly Vol.III 1951 pp 1-21.
2.	R.W. Traill-Nash	The anti-symmetric vibrations of aircraft Aeronautical Quarterly Vol.III 1951 pp 145-160.
3.	J.R.M. Radok	Vibration of a swept box. College of Aeronautics, Report No.47 1951
4.	W.S. Hemp	On the application of oblique coordinates to problems of plane elasticity and swept back wings. College of Aeronautics, Report No.31 1950
5.	W.S. Hemp	Theory of the uniform two-cell swept box. Lectures given at the College of Aeronautics, Cranfield, December 1951.
6.	J.R.M. Radok	Aileron reversal and divergence of swept wings with special consideration of the relevant aerodynamic and elastic characteristics. College of Aeronautics, Report No.55 1952
7.	A.O. Payne	A theory for the dynamic testing of large structures. Aeronautical Research Laboratories, Melbourne. (To be published).
8.	William Milliken Jr.	Dynamic stability and control research. Proc. Anglo-American Aeronautical Conference, Brighton 1951, pp 447-524.
9.	J.R.M. Radok	An approximate theory of the oscillating wing in a compressible subsonic flow for low frequencies. Nationaal Luchtvaartlaboratorium F.97 1951
10.	H.G. Küssner	Zusammen fassender Bericht über den instationären Autrieb von Flügeln Luftfahrtforschung Vol.13 (1936) pp 410-424.
11.	R. Timman and A.I.van de Vooren	Theory of the oscillating wing with aerodynamically balanced control surface in a two-dimensional subsonic compressible flow. Nationaal Luchtvaartlaboratorium F.54. 1949.

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No.	Author	Title, etc.
12.	R. Timman, A.I. van de Vooren, and J.H. Greidanus	Aerodynamic coefficients of an oscillating aerofoil in two-dimensional subsonic flow. Nationaal Luchtvaartlaboratorium F.83 1951
13.	R. Stanton Jones	An empirical method for rapidly determining the loading distributions on swept back wings. College of Aeronautics, Report No.32
14.	J.R.M. Radok	The asymptotic behaviour of the indicial lift function in subsonic compressible flow. Nationaal Luchtvaarlaboratorium F.106
15.	R. Stanton Jones	A rapid method for estimating the basic loading due to linear twist of wings of any planform Saunders-Roe ASR.7
16.	H. Lomax, M.A. Heaslet and L. Sluder	The indicial lift and pitching moment for a sinking or pitching two-dimensional wing flying at subsonic or supersonic speeds NACA TN 2403 1951.
17.	J.R.M. Radok	Gust loads on two-dimensional aerofoils in supersonic flow. A.R.L. Australia. Rep. A.66, SM 142,1949
18.	J.R.M. Radok	Unsteady Aerofoil theory. Aeronautical Quarterly Vol. III Feb.1952 pp 297-320
19.	J.R.M. Radok and Lurline F. Stiles	The motion and deformation of aircraft in uniform and non-uniform atmospheric disturbances. ACA, Australia, 41. July 1948.
20.	J.R.M. Radok	An extension of the earlier theory of gust loads on wings. C.S.I.R. Div. of Aeronautics, Australia. SM Rep.138 September 1949.

APPENDIX 1

Remarks on use of Hamilton's Principle

In section 6 of the main part of this report the equations of motion have been given in the form of integro-differential equations obtained directly from Hamilton's Principle. It is the purpose of this Appendix to give some detail of the analytical work leading to these equations.

It has already been pointed out in the Introduction that a special feature of the way in which Hamilton's Principle has been applied here lay in the particular choice of independent variables. Instead of using actual displacements, as has been done in most work of this nature, it has been found preferable to introduce as new variables curvatures and rates of twist whenever the customary variables are functions of space coordinates as well as of time. The analytic procedure is best explained by the presentation of the complete process which leads to one of the equations of section 6.2, e.g. equation (6.2.7). This is the equation which involves the most complicated preparatory work, because of the fact that by (4.6)

$$p' = -\operatorname{cosec}\alpha\,\bar{\Phi}'' \qquad \text{-- (A1.1)}$$

so that in the application of Hamilton's Principle this equation arises from two arbitrary displacements $\delta p'$ and $\delta\bar{\Phi}''$. By considering the deduction of this equation it will be assured that all different steps, occurring in the deduction of the complete set of equations are demonstrated.

Applying Hamilton's Principle (6.1.1) and writing down only the terms relevant for equation (6.2.7) one finds from (3.15) and (4.12)

$$\int_{t_1}^{t_2} dt \Bigg[\int_0^{\ell} \Big\{ I_{x_1}(\dot{p} - \cos\alpha\,\dot{q}) - \bar{z}_1 m\dot{V} + I_{x_1}\dot{P} + \dot{R}(y_1\bar{z}_1 m\cos\alpha - I_{x_1 z_1}) + m\bar{z}_1 R\dot{U}_o \Big\} \delta\dot{p}\,dy_1$$

$$- \int_0^{\ell} \Big\{ \sin\alpha\Big(\Gamma_{11}p' + \frac{\Gamma_{12} + \Gamma_{21}}{2}q'\Big) + \Big(\frac{\Gamma_{13}\sin\alpha + \Gamma_{31}}{2}\Big)\phi' \Big\} \delta p'\,dy_1$$

$$+ \int_0^{\ell} \Big\{ m(\dot{\bar{\Phi}} + \dot{\phi}) - m\bar{x}_1\sin\alpha q' + m\dot{W} + y_1 m\sin\alpha\,\dot{P} - m(\bar{x}_1 - y_1\cos\alpha)\dot{Q} + mQ\dot{U}_o \Big\} \delta\dot{\bar{\Phi}}\,dy_1 \Bigg]$$

$$= 0 \qquad \text{-- (A1.2)}$$

/ But ......

But by a condition fundamental to Hamilton's Principle

$$\delta \Phi = 0 \quad \text{for } t = t_1 , \; t = t_2 \qquad \text{-- (A1.3)}$$

and hence, integrating in (A1.2) those terms, involving time derivatives of the arbitrary displacements, by parts with respect to time, one finds:

$$\int_{t_1}^{t_2} dt \Bigg[\int_0^{\ell} \Big\{ I_{x_1} (\ddot{p} - \cos\alpha \, \ddot{q}) - m\bar{z}_1 \ddot{V} + I_{x_1} \ddot{P} + (y_1 \bar{z}_1 m \cos\alpha - I_{x_1 z_1}) \ddot{R} + m\bar{z}_1 \dot{R}\dot{U}_o \Big\} \delta p \, dy_1$$

$$+ \int_0^{\ell} \Big\{ \sin\alpha \, (\Gamma_{11} \, p' + \frac{\Gamma_{12} + \Gamma_{21}}{2} q') + (\frac{\Gamma_{13} \sin\alpha + \Gamma_{31}}{2}) \phi' \Big\} \delta p' \, dy_1$$

$$+ \int_0^{\ell} m \Big\{ (\ddot{\Phi} + \ddot{\phi}) - \bar{x}_1 \sin\alpha \ddot{q} + \ddot{W} + y_1 \sin\alpha \ddot{P} - (\bar{x}_1 - y_1 \cos\alpha) \ddot{Q} + \dot{Q}\dot{U}_o \Big\} \delta \Phi \, dy_1 \Bigg] = 0$$

$$\text{-- (A1.4)}$$

since by (A1.3) and (A1.1) the integrated terms vanish.

Next the dependent displacements δp, $\delta \Phi$ will be transformed into $\delta p'$, so that the above equation only involves one independent arbitrary displacement $\delta p'$. By the choice of the coordinate systems and variables (see section 2) it is obvious that

$$\Phi(0) = \Phi'(0) = 0 \text{ and hence } p(0) = 0 \qquad \text{(A1.5)}$$

at all times. Therefore

$$\int_0^{\ell} F_1(y_1, t) \delta p \, dy_1 = \int_0^{\ell} F_1(y_1, t) dy_1 \int_0^{y_1} \delta p' \, d\xi = \int_0^{\ell} \delta p' dy_1 \int_{y_1}^{\ell} F_1(\xi, t) d\xi \qquad \text{(A1.6')}$$

$$\int_0^{\ell} F_2(y_1, t) \delta \Phi \, dy_1 = \int_0^{\ell} F_2(y_1, t) dy_1 \int_0^{y_1} d\xi \int_0^{\xi} \delta \Phi'' \, d\eta = \int_0^{\ell} F_2(y_1, t) dy_1 \int_0^{y_1} \delta \Phi'' \, (y_1 - \eta) d\eta$$

$$= \int_0^{\ell} \delta \Phi'' \, dy_1 \int_{y_1}^{\ell} F_2(\xi, t)(\xi - y_1) d\xi \qquad \text{-- (A1.6'')}$$

/ where .....

where several inversions of the order of integration have occurred. But by (A1.1) the last expression becomes

$$\int_0^{\ell} F_2(y_1,t)\delta\,\bar{\Phi}\,dy_1 = -\sin\alpha\int_0^{\ell}\delta p'\,dy_1\int_{y_1}^{\ell}F_2(\xi,t)(\xi-y_1)d\xi \qquad \text{-- (A1.7)}$$

Introducing the transformations (A1.6') and (A1.7) into (A1.4) one finds

$$\int_{t_1}^{t_2}dt\int_0^{\ell}dy_1\,\delta p'\Bigg[\int_{y_1}^{\ell}\Big\{I_{x_1}(\ddot{p}-\cos\alpha\,\ddot{q})-m\bar{z}_1\ddot{V}+I_{x_1}\ddot{P}+(y_1\bar{z}_1 m\cos\alpha-I_{x_1z_1})\ddot{R}+m\bar{z}_1R\dot{U}_o\Big\}d\xi$$

$$+\sin\alpha\Big\{\vec{\Gamma}_{11}p'+\frac{\vec{\Gamma}_{12}+\vec{\Gamma}_{21}}{2}q'\Big\}+\left(\frac{\vec{\Gamma}_{13}\sin\alpha+\vec{\Gamma}_{13}}{2}\right)\phi'$$

$$-\sin\alpha\int_{y_1}^{\ell}m\Big\{(\ddot{\Phi}+\ddot{\phi})-\bar{x}_1\sin\alpha\,\ddot{q}+\ddot{W}+y_1\sin\alpha\,\ddot{P}-(\bar{x}_1-y_1\cos\alpha)\ddot{Q}+\dot{Q}\dot{U}_o\Big\}(\xi-y_1)d\xi\Bigg]=0 \qquad \text{-- (A1.8)}$$

For (A1.8) to be true it is necessary and sufficient that the expression in square brackets is zero. However the equation of motion thus obtained still contains $\ddot{\Phi}$, $\ddot{p}$, $\ddot{q}$, $\ddot{\phi}$, and space derivatives of p, q and ϕ. For this reason transformations of the types (A1.6') and (A1.6'') will be applied under the integrals of the equation of motion just deduced. It is easily verified that this will lead to the following equation:

$$\int_0^{\ell}\Big\{\ddot{p}'\Big(\int_{y_1,\eta}^{\ell}I_{x_1}d\zeta+\sin^2\alpha\int_{y_1,\eta}^{\ell}m(\zeta-\eta)(\zeta-y_1)d\zeta\Big)+\ddot{q}'\Big(\sin^2\alpha\int_{y_1,\eta}^{\ell}m\bar{x}_1(\zeta-x_1)d\zeta-\cos\alpha\int_{y_1,\eta}^{\ell}I_{x_1}d\zeta\Big)$$

$$-\ddot{\phi}\sin\alpha\int_{y_1,\eta}^{\ell}m(\zeta-y_1)d\zeta\Big\}d\eta-\ddot{V}\int_{y_1}^{\ell}m\bar{z}_1\,d\zeta+\ddot{P}\Big(\int_{y_1}^{\ell}I_{x_1}d\zeta-\sin^2\alpha\int_{y_1}^{\ell}m\zeta(\zeta-y_1)d\zeta\Big)$$

$$-\ddot{W}\sin\alpha\int_{y_1}^{\ell}m(\zeta-y_1)d\zeta+\ddot{Q}\sin\alpha\Big(\int_{y_1}^{\ell}m\bar{x}_1(\zeta-y_1)d\zeta-\cos\alpha\int_{y_1}^{\ell}\zeta(\zeta-y_1)m\,d\zeta\Big)$$

$$+\ddot{R}\Big(\cos\alpha\int_{y_1}^{\ell}m\bar{z}_1\zeta\,d\zeta-\int_{y_1}^{\ell}I_{x_1z_1}d\zeta\Big)+\dot{U}_o\Big(\dot{R}\int_{y_1}^{\ell}m\bar{z}_1d\zeta-\sin\alpha\,\dot{Q}\int_{y_1}^{\ell}m(\zeta-y_1)d\zeta\Big)$$

$$+\sin\alpha\Big(\vec{\Gamma}_{11}p'+\frac{\vec{\Gamma}_{12}+\vec{\Gamma}_{21}}{2}q'\Big)+\frac{\vec{\Gamma}_{13}\sin\alpha+\vec{\Gamma}_{31}}{2}\phi'=0 \qquad \text{-- (A1.9)}$$

Equation (A1.9) agrees with (6.2.7) after introduction of the additional notation used there.

Finally it may still be of interest to explain in detail the

/origin ...

origin of the limit of integration $\overline{y_1,\eta}$ occurring above. For example, the first term is

$$\int_{y_1}^{\ell} I_{x_1} \ddot{p}\, d\xi = \int_{y_1}^{\ell} I_{x_1} d\xi \int_{0}^{\xi} \ddot{p}'\, d\zeta = \int_{0}^{\ell} \ddot{p}' d\zeta \int_{\overline{y_1,\eta}}^{\ell} I_{x_1} d\xi \qquad \text{-- (A1.10)}$$

as is easily confirmed by use of Fig. 6, showing the area over which the double integral extends.

/ Table 1 .....

TABLE I

Matrix representation of equations (6.2.1) - (6.2.12)

$$\bar{A}.\begin{bmatrix}\ddot{U}\\ \ddot{V}\\ \ddot{W}\\ \ddot{P}\\ \ddot{Q}\\ \ddot{R}\end{bmatrix} + \dot{U}_o\,\bar{B}.\begin{bmatrix}\dot{P}\\ \dot{Q}\\ \dot{R}\end{bmatrix} + \int_o^{\ell}\left\{\bar{C}.\begin{bmatrix}\ddot{p}'\\ \ddot{q}'\\ \ddot{\phi}'\end{bmatrix} + \dot{U}_o\,\bar{D}.\begin{bmatrix}\dot{p}'\\ \dot{q}'\\ \dot{\phi}'\end{bmatrix}\right\}dy_1 + \int_{-\ell_1}^{\ell_2}\left\{\bar{E}.\begin{bmatrix}\ddot{v}_f''\\ \ddot{w}_f''\\ \ddot{p}_f'\end{bmatrix} + \dot{U}_o\,\bar{F}.\begin{bmatrix}\dot{v}_f''\\ \dot{w}_f''\\ \dot{p}_f'\end{bmatrix}\right\}dx_1$$

$$+\ \bar{G}.\begin{bmatrix}p'\\ q'\\ \phi'\end{bmatrix} + \bar{H}.\begin{bmatrix}v_f''\\ w_f''\\ p_f'\end{bmatrix} = 0$$

where the first integral $\int_o^{\ell}$ is to be taken over both wings and the coefficient matrices are:

$$\bar{A} = \begin{bmatrix}
M & 0 & 0 & 0 & M\bar{\bar{z}}_1 & -M\bar{\bar{Y}}_1\\
0 & M & 0 & -M\bar{\bar{z}}_1 & 0 & M\bar{\bar{x}}_1\\
0 & 0 & M & M\bar{\bar{Y}}_1 & -M\bar{\bar{x}}_1 & 0\\
0 & -M\bar{\bar{z}}_1 & M\bar{\bar{Y}}_1 & A & -E & -G\\
M\bar{\bar{z}}_1 & 0 & -M\bar{\bar{x}}_1 & -E & B & -F\\
-M\bar{\bar{Y}}_1 & M\bar{\bar{x}}_1 & 0 & -G & -F & C\\
0 & -\mu_z & -\mu^1\sin\alpha & i_x-\mu_1^1\sin^2\alpha & \sin\alpha(\mu_x^1-\mu_1^1\cos\alpha) & \mu_{1z}\cos\alpha-i_{xz}\\
\mu_z\sin\alpha & \mu_z\cos\alpha & \mu_x\sin\alpha & -(i_x\cos\alpha+\mu_{1x}\sin^2\alpha) & j\sin\alpha+\mu_{1x}\sin\alpha\cos\alpha & i_{xz}\cos\alpha-\mu_{1z}\\
0 & 0 & \mu & \mu_1\sin\alpha & \mu_1\cos\alpha-\mu_x & 0\\
0 & \mu_f^1 & 0 & -\mu_{zf}^1 & 0 & \mu_{1f}^1\\
0 & 0 & \mu_f^1 & \mu_{Yf}^1 & -\mu_{1f}^1 & 0\\
0 & \mu_{zf} & \mu_{Yf} & j_f & -\mu_{1Yf} & -\mu_{1zf}
\end{bmatrix}$$

$$\bar{B} = \begin{bmatrix}
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & M\bar{\bar{Y}}_1 & M\bar{\bar{z}}_1\\
-M\bar{\bar{Y}}_1 & 0 & 0\\
-M\bar{\bar{z}}_1 & 0 & 0\\
0 & -\mu^1\sin\alpha & \mu_z\\
0 & -\mu_x\sin\alpha & -\mu_z\cos\alpha\\
0 & \mu & 0\\
0 & 0 & -\mu_f^1\\
0 & \mu_f^1 & 0\\
0 & \mu_{Yf} & \mu_{zf}
\end{bmatrix}
\quad
\bar{C} = \begin{bmatrix}
0 & \mu_z\sin\alpha & 0\\
-\mu_z & \mu_z\cos\alpha & 0\\
-\mu_1\sin\alpha & -\mu_x\sin\alpha & \mu\\
i_x-\mu_1^1\sin^2\alpha & -(\mu_{1x}\sin^2\alpha+i_x\cos\alpha) & \mu_1\sin\alpha\\
(\mu_x^1-\mu_1^1\cos\alpha)\sin\alpha & (j+\mu_{1x}\cos\alpha)\sin\alpha & \mu_1\cos\alpha-\mu_x\\
\mu_{1z}\cos\alpha-i_{xz} & i_{xz}\cos\alpha-\mu_{1z} & 0\\
\bar{i}_x+\bar{\mu}^2\sin^2\alpha & \bar{\mu}_x^1\sin^2\alpha-\bar{i}_x\cos\alpha & -\mu_1^1\sin\alpha\\
\bar{\mu}_x^1\sin^2\alpha-\bar{i}_x\cos\alpha & \bar{i}_x+\bar{i}_z\sin^2\alpha & -\bar{\mu}_x\sin\alpha\\
-\bar{\mu}^1\sin\alpha & -\bar{\mu}_x\sin\alpha & \bar{\mu}\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0
\end{bmatrix}$$

/ $\bar{D}$ = ...

$$\bar{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_1^1 \sin\alpha & \mu_x \sin\alpha & -\mu \\ -\mu_z & \mu_z \cos\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{E} = \begin{bmatrix} 0 & 0 & 0 \\ \mu_f^1 & 0 & -\mu_{zf} \\ 0 & \mu_f^1 & \mu_{Yf} \\ -\mu_{zf}^1 & \mu_{Yf}^1 & j_f \\ 0 & -\mu_{1f}^1 & -\mu_{1Yf} \\ \mu_{1f} & 0 & -\mu_{1zf} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{\mu}_f^2 & 0 & -\bar{\mu}_{1f}^1 \\ 0 & \bar{\mu}_f^2 & \bar{\mu}_{Yf}^1 \\ -\bar{\mu}_{zf}^1 & \bar{\mu}_{Yf}^1 & \bar{j}_f \end{bmatrix} \quad \bar{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\mu_f^1 & -\mu_{Yf}^1 \\ \mu_f^1 & 0 & -\mu_{zf} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Gamma_{11}\sin\alpha & \frac{\Gamma_{12}+\Gamma_{21}}{2} & \frac{\Gamma_{13}\sin\alpha+\Gamma_{31}}{2} \\ \frac{\Gamma_{12}+\Gamma_{21}}{2} & \Gamma_{22}\sin\alpha & \frac{\Gamma_{23}\sin\alpha+\Gamma_{32}}{2} \\ \frac{\Gamma_{13}\sin\alpha+\Gamma_{31}}{2} & \frac{\Gamma_{23}\sin\alpha+\Gamma_{32}}{2} & \Gamma_{33} \end{bmatrix} \quad \bar{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_f\bar{I}_{z_1 f} & E_f\bar{I}_{Y_1 z_1 f} & 0 \\ E_f\bar{I}_{Y_1 z_1 f} & E_f\bar{\bar{I}}_{Y_1 f} & 0 \\ 0 & 0 & C_f \end{bmatrix}$$

where the coefficients are defined in section 6.2.

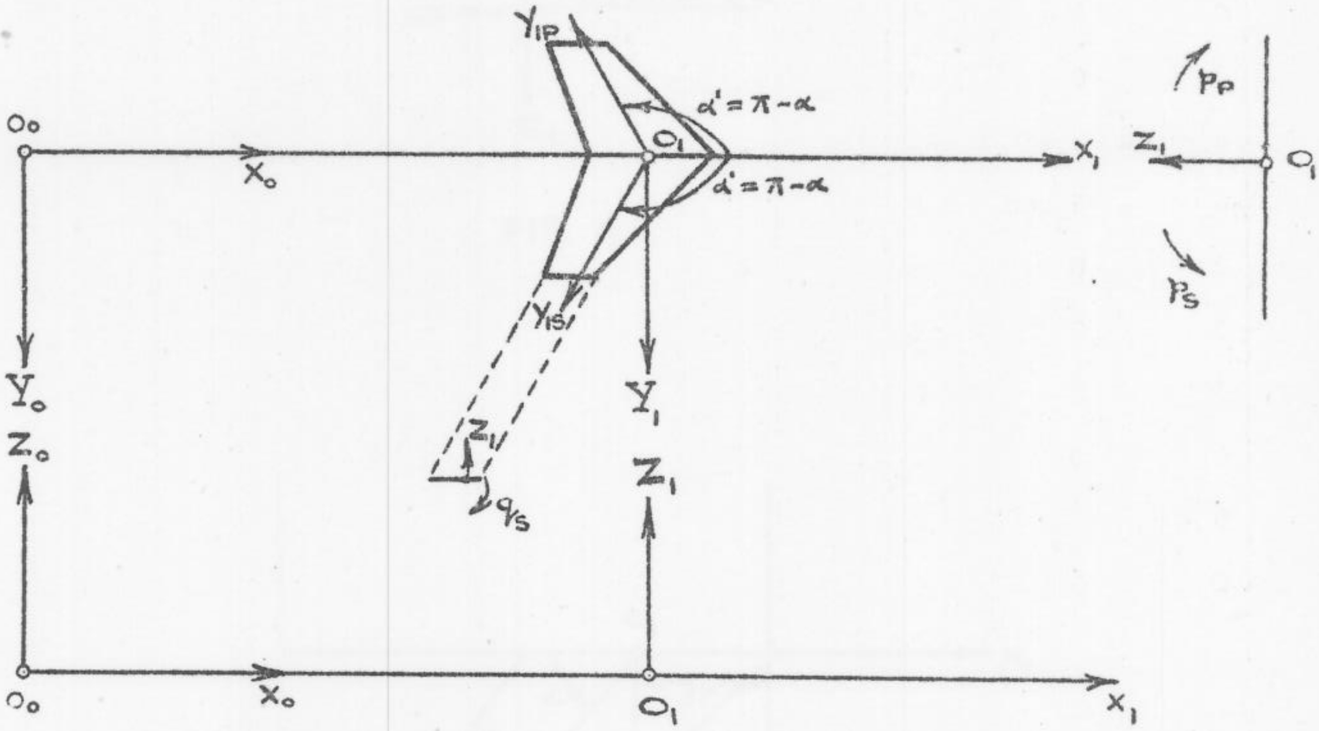

FIG. 1.

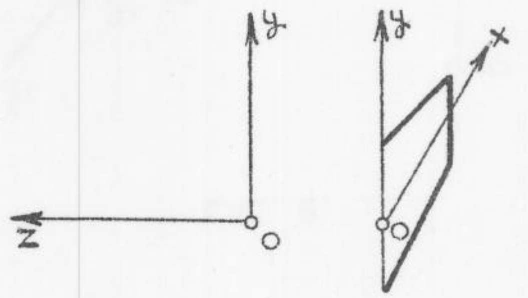

FIG. 2.

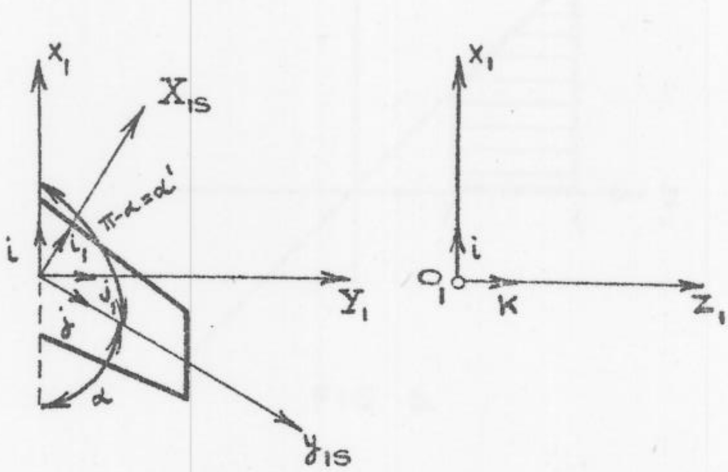

FIG. 3.

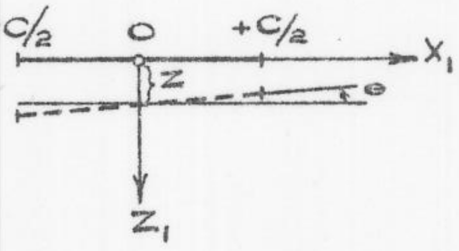

FIG. 4.

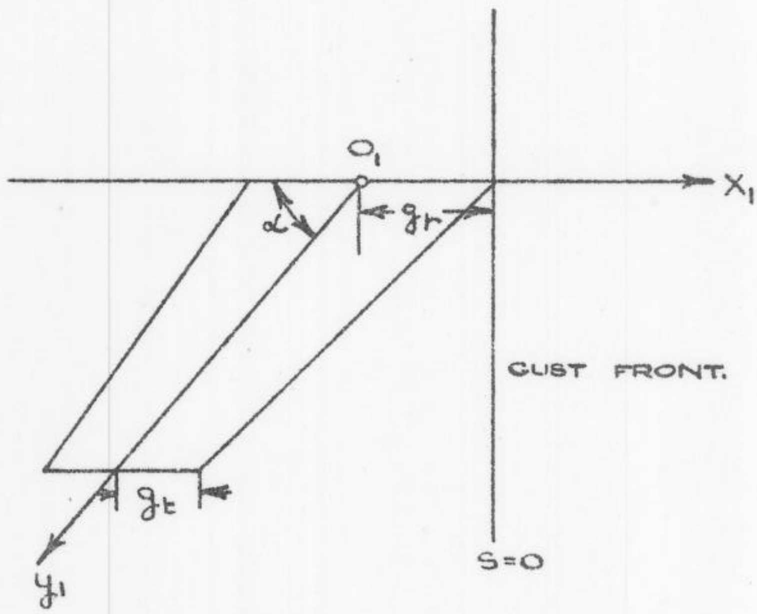

FIG. 5.

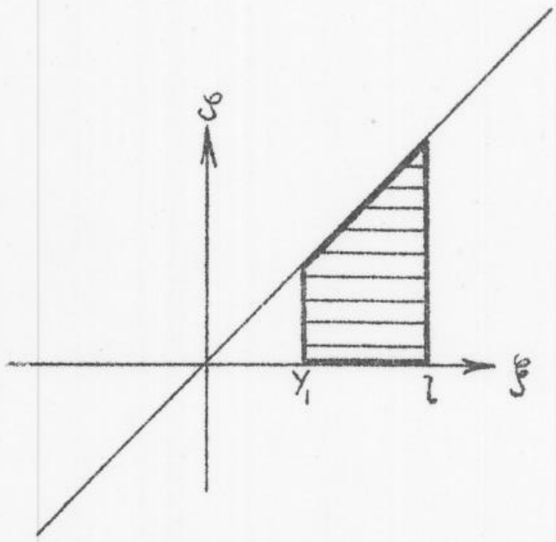

FIG. 6.