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Theoretical Stability Derivatives of a Highly Swept Delta Wing and Slender Body Combination

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SUMMARY

The stability derivatives of a delta wing of small aspect ratio, mounted on a cylindrical body with a slender pointed head, are derived by considering the flow in planes perpendicular to the body axis to be uninfluenced by the change in the streamwise component of the air velocity (the so-called 'slender body' theory).

The results are tabulated, and the variations of the derivatives with the ratio of body diameter to wing span are shown in the form of graphs.

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NOTATION (see also Figure 1)

A	Wing gross aspect ratio	$(= b^2/S)$.
C_x	Longitudinal force coefficient forward along body axis.	$(= X/\frac{1}{2}\rho U^2 S)$
C_y	Side force coefficient in starboard direction.	$(= Y/\frac{1}{2}\rho U^2 S)$
C_z	Normal force coefficient downwards.	$(= Z/\frac{1}{2}\rho U^2 S)$
C_l	Rolling moment coefficient	$(= \text{moment}/\frac{1}{2}\rho U^2 Sb)$.
C_m	Pitching moment coefficient	$(= \text{moment}/\frac{1}{2}\rho U^2 S\bar{c})$.
C_n	Yawing moment coefficient	$(= \text{moment}/\frac{1}{2}\rho U^2 Sb)$.
H	Complex potential, with $\varphi + \emptyset$ as real part.	
S	Gross wing area.	
U, V, W	Velocity components of body relative to air along x, y and z axes.	
X	Longitudinal force on surface ahead of plane x = constant.	
Y	Side force on surface ahead of plane x = constant.	
Z	Normal force on surface ahead of plane x = constant.	
a	Body radius.	
a_o	Maximum body radius.	
b	Maximum gross wing span.	
c	Gross wing chord on centre line.	
$\bar{c}$	Geometric mean chord of wing.	
c_p	(pressure - free stream pressure)$/\frac{1}{2}\rho U^2$.	
h	Length of curved head of body ahead of plane $x = -\ell$.	
ℓ	Distance between shoulder of body head and wing apex on centre line.	
Δp	Difference in air pressure between upper and lower surfaces of wing or body along the line x = constant, y = constant.	
p	Rate of roll	of body-wing combination.
q	Rate of pitch	of body-wing combination.
r	Rate of yaw.	of body-wing combination.

Notation (contd.)

s Local wing gross semi-span.

s_o Maximum gross semi-span ($= \frac{1}{2}b$).

u, v, w Velocity components of air along x, y and z axes relative to body.

x, y, z System of orthogonal coordinates, origin at wing apex on centre line, x-axis backwards along body axis, y-axis to starboard in plane of wing, and z-axis normal (upwards) to plane of wing.

(R,θ,x) Polar cylindrical coordinates ($y = R\cos\theta$, $z = R\sin\theta$).

Γ Distance of geometric centroid of body head in front of shoulder ($x = -\ell$) $\div$ h.

Ω Volume of body head in front of shoulder ($x = -\ell$) $\div h\pi a_o^2$.

α Incidence of body wing combination to main stream.

α_e Effective incidence of a plane x = constant (= W/U).

α_p $= pb/2U$.

α_q $= q\bar{c}/U$.

α_r $= rb/2U$.

β Angle of yaw.

β_e Effective angle of yaw of a plane x = constant (= -V/U).

ζ $= y + iz$.

ξ Distance of arbitrary reference point downstream of origin.

ρ Air density.

φ Potential due to incidence and sideslip in plane x = constant.

ϕ Potential due to roll in plane x = constant.

ψ Stream function.

Primed symbols denote reference to wind axes (not body axes) - see Figure 2.

Suffix B denotes values on body surface.

Suffix W denotes values on wing surface.

The term 'gross wing' applies to that plan form produced by extending the lines of the leading and trailing edges to meet, on the centre-line.

1. Introduction

A large number of reports have recently been published, dealing with the flow past slender wings and wing-body combinations, using the assumption that the flow may adequately be described by neglecting its variation in the stream direction. This approach was apparently originated by Munk in studying the aerodynamics of slender airships[1]. R.T. Jones extended it to the calculation of the lift of triangular wings[2] of small aspect ratio, and more recently Ribner applied the method to the study of the stability derivatives of such wings[3]. Munk's work on bodies has been extended by various investigators[4,5] and, in particular, Spreiter[6] has considered slender wing-body combinations.

The object of this note is to apply the method to the calculation of the stability derivatives of a slender wing-body combination, having a triangular wing mounted on the cylindrical portion of a body with a pointed nose (of arbitrary shape). No attempt is made to examine the flow behind the wing trailing edge. Figure 1 is a diagram of the layout explaining the notation which we shall employ.

After the work was completed it became known that part had already been considered by American authors - in particular, the damping-in-roll estimation[7,8]; however, the present treatment has been left as a complete survey so that its reference value is not impaired. Sufficient discussion is given in each paragraph to enable the general lines of the calculation to be followed.

To simplify the presentation of the work, we deal separately with

(i) the forces and moments on the triangular wing mounted on a cylindrical body (i.e. downstream of a lateral plane corresponding to the wing leading edge at the body junction);

(ii) the forces and moments on the body upstream of the wing.

The first calculations are said to relate to a 'Wing on Cylindrical Body' and the second to a 'Cylindrical Body with Pointed Nose'. The forces and moments from each are, of course, additive.

The body axis is assumed to lie in the plane of the wing (taken to be of zero thickness) and at an incidence (α radians) to the direction of undisturbed motion. The stability derivatives are referred to body axes, as shown in Figure 1, except where otherwise stated.

2. The Potential of the Flow about a Wing-Body Combination

If u, v and w are the velocity components of the air relative to the body in the direction of the x, y and z axes respectively (see Figure 1); and if in a chosen lateral plane the apparent components of the free-stream velocity at infinity (relative to these axes fixed in the body) are U, -V and W: then we define the effective sideslip angle as

$$\alpha_e = W/U \qquad \text{and} \qquad \beta_e = -V/U.$$

These quantities are assumed to be small, as also are (u-U), v, w. According to Bernoulli's Theorem the pressure coefficient is then given by

$$c_p = -\frac{2}{U}\left[(u-U) - \beta_e(v-V) + \alpha_e(w-W)\right], \qquad \ldots\ldots(2.1)$$

where, in addition to the usual term (u-U), the terms of lowest order in α_e and β_e have also been retained. Although these are, strictly speaking, of second order, they nevertheless yield solutions of the lowest order in certain circumstances - as will later be apparent - where the derivatives required are due to sideslip or roll, for example, but vary also in proportion to α.

If the flow is irrotational, we may define a perturbation potential function φ such that

$$\varphi_x = u - U, \quad \varphi_y = v - V, \quad \text{and} \quad \varphi_z = w - W,$$

where the suffices denote partial differentiations.

It is the assumption of the 'slender body' theory, upon which our results will be based, that this potential function satisfies the equation

$$\varphi_{yy} + \varphi_{zz} = 0. \qquad \ldots\ldots(2.2)$$

In the present instance, we are considering the flow due to a body-wing combination, whose section in any transverse plane (x = constant) is a combination of a circle and a straight line. Spreiter[6] has given the potential which satisfies equation (2.2) together with the boundary condition (i) that the flow is tangential to the surface and (ii) that the velocity at infinity is given by v = 0 and w = Uα; and Heaslet and Lomax[8] have calculated that due to a rotation of the body-wing combination about the body axis. Unfortunately, the latter treatment contains one or two important typographical errors, and so - to avoid possible confusion - and for the sake of completeness, both this and the other potential functions are derived independently in Appendices I and II.

The potential function to which we shall most often refer is that due to a combination of upwash $U\alpha_e$ and sidewash $U\beta_e$: it is given by

$$\varphi = \mathrm{sgn}(z)\,\frac{U\alpha_e}{\sqrt{2}}\sqrt{\left[-(1+\frac{a^4}{R^4})R^2\cos 2\theta + s^2(1+\frac{a^4}{s^4})\right.}$$

$$+\sqrt{\left[R^4(1+\frac{a^8}{R^8}) + 2a^4\cos 4\theta + s^4(1+\frac{a^4}{s^4})^2\right.}$$

$$\left.\left.- 2s^2(1+\frac{a^4}{s^4})(1+\frac{a^4}{R^4})R^2\cos 2\theta\right]\right\}$$

$$- U\beta_e\frac{a^2y}{R^2} - U\alpha_e z, \qquad \ldots\ldots(2.3)$$

where

$$y + iz = Re^{i\theta}$$

$$\mathrm{sgn}\ z = z/|z|$$

$$a = \text{body radius}$$

$$s = \text{semi wing-span.}$$

From (2.1), the pressure coefficient may be written

$$c_p = -\frac{2}{U}\left(\frac{\partial\varphi}{\partial x} - \beta_e\frac{\partial\varphi}{\partial y} + \alpha_e\frac{\partial\varphi}{\partial z}\right). \qquad \ldots\ldots(2.4)$$

The variation of φ with x is implicit in the variation of the geometry of the surface, i.e.

$$\frac{\partial}{\partial x} = \frac{da}{dx}\frac{\partial}{\partial a} + \frac{ds}{dx}\frac{\partial}{\partial s} + \frac{d\alpha_e}{dx}\frac{\partial}{\partial\alpha_e} + \frac{d\beta_e}{dx}\frac{\partial}{\partial\beta_e}.$$

The value of this derivative on the wing body combination may be found most simply from the separate expressions for the surface potential on the wing and the body

$$(\varphi)_W = \mathrm{sgn}(z)U\alpha_e\sqrt{\left[s^2(1+\frac{a^4}{s^4}) - y^2(1+\frac{a^4}{y^4})\right]} + U\beta_e\frac{a^2}{y}$$

$$(\varphi)_B = \mathrm{sgn}(z)U\alpha_e\sqrt{\left[s^2(1+\frac{a^2}{s^2})^2 - 4y^2\right]} - U\alpha_e\sqrt{a^2-y^2} + U\beta_e y \qquad \ldots\ldots(2.5)$$

where the suffices 'W' and 'B' refer to conditions at the wing and body surfaces respectively.

3. The Derivatives due to Incidence

3.1. Wing on cylindrical body

For a wing-body combination at incidence α and at zero sideslip, the pressure difference between the upper and lower surfaces may be found from equations (2.3) and (2.4), putting $\alpha_e = \alpha$ and $\beta_e = 0$.

Since φ is an odd function of z, and $\frac{\partial\varphi}{\partial z}$ is an even function, it follows that the pressure difference Δp is given by

$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = -\frac{4}{U}\left(\frac{\partial\varphi}{\partial x}\right)_{\text{upper surface}},$$

where $\Delta p = (p)_{\text{upper surface}} - (p)_{\text{lower surface}}$.

But, from (2.5), performing the necessary differentiations we find that, since $a = a_o$ (a constant),

$$\left.\begin{aligned}\left(\frac{\partial\varphi}{\partial x}\right)_W &= \frac{ds}{dx}\left(\frac{\partial\varphi}{\partial s}\right)_W = \frac{ds}{dx}\,\mathrm{sgn}(z)U\alpha\left(1-\frac{a_o^4}{s^4}\right)\Bigg/\sqrt{\left(1+\frac{a_o^4}{s^4}\right)-\frac{y^2}{s^2}\left(1+\frac{a_o^4}{y^4}\right)}\\ \left(\frac{\partial\varphi}{\partial x}\right)_B &= \frac{ds}{dx}\left(\frac{\partial\varphi}{\partial s}\right)_B = \frac{ds}{dx}\,\mathrm{sgn}(z)U\alpha\left(1-\frac{a_o^4}{s^4}\right)\Bigg/\sqrt{\left(1+\frac{a_o^2}{s^2}\right)^2-\frac{4y^2}{s^2}}\end{aligned}\right\} \quad\ldots\ldots(3.1)$$

and these expressions enable us to describe the variation of $\Delta p/\frac{1}{2}\rho U^2$.

The longitudinal distribution of normal force is given by the integral

$$\frac{dZ}{dx} = \rho U^2\left[\int_0^{a_o}\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)_B dy + \int_{a_o}^{s}\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)_W dy\right]. \quad\ldots\ldots(3.2)$$

Using (3.1) we may then perform the integrations, and recalling the identity

$$\arcsin\frac{1-\sigma^2}{1+\sigma^2} + \arcsin\frac{2\sigma}{1+\sigma^2} = \frac{\pi}{2},$$

the final expression may be simplified* to

$$\frac{dZ}{dx} = -\,2\pi\alpha\rho U^2 s\,\frac{ds}{dx}\left(1-\frac{a_o^4}{s^4}\right). \quad\ldots\ldots(3.3)$$

Hence, integrating with respect to x from $x = a_o c/s_o$ to $x = c$ (the body-wing junction to the wing trailing

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*The definition of all coefficients used here will be found in the list of symbols on p. 1.

edge), we find that

$$\frac{\partial C_z}{\partial \alpha} = -\frac{\pi A}{2}(1-\sigma^2)^2 \qquad \ldots\ldots(3.4)$$

where $\sigma = a_o/s_o$, the ratio of body diameter to wing span.

The pitching moment coefficient about the point $(\xi,0,0)$ is given by

$$C_m = 2\int_{\sigma c}^{c}\left(\frac{x}{c}-\frac{\xi}{c}\right)\frac{dC_z}{dx}\,dx \qquad \ldots\ldots(3.5)$$

so that using (3.3) and integrating

$$\frac{\partial C_m}{\partial \alpha} = -\frac{2\pi}{3}A(1-4\sigma^3+3\sigma^4) - \frac{2\xi}{c}\frac{\partial C_z}{\partial \alpha}. \qquad \ldots\ldots(3.6)$$

The side force and the rolling and yawing moments are all zero since, from (2.3) and (2.4), φ (and so c_{ℓ}) is an even function of y,

$$\text{i.e.}\quad \frac{\partial C_y}{\partial \alpha} = \frac{\partial C_n}{\partial \alpha} = \frac{\partial C_\ell}{\partial \alpha} = 0. \qquad \ldots\ldots(3.7)$$

The longitudinal force - in the direction of the x-axis, which is also the body axis - is due only to the distribution of suction along the wing leading edge, there being no resolved component of the normal pressures in this direction (since it is evidently in the plane of the wing). This suction force may be estimated if we note that its component in the direction of the y-axis may be found from the flow in the transverse plane x = constant. This is a side force (outwards from the wing) per unit length of amount

$$\left.\begin{aligned} &Y = \pm\pi\rho|\gamma|^2 \quad \text{at} \quad \zeta=\zeta_o \\ &\text{where} \\ &\frac{dH}{d\zeta} \rightarrow \frac{\gamma}{(\zeta-\zeta_o)^{\frac{1}{2}}} \quad \text{as} \quad \zeta\rightarrow\zeta_o \\ &\zeta = y+iz \quad \text{and} \quad H = \varphi + i\psi \end{aligned}\right\} \qquad \ldots\ldots(3.8)$$

and where ζ_o is the value of ζ corresponding to the wing leading edge. In fact, H is the complex potential and, in Appendix I, its value due to a uniform upwash $U\alpha$ is shown to be

$$H = -iU\alpha\sqrt{\left(\zeta+\frac{a_o^2}{\zeta}\right)^2 - \left(s+\frac{a_o^2}{s}\right)^2}\,.$$

The real part of this expression appears in (2.3).

Thus

$$\frac{dH}{d\zeta} = - iU\alpha\zeta\left(1-\frac{a_o^4}{\zeta^4}\right)\Big/\sqrt{\left[\left(\zeta+\frac{a_o^2}{\zeta}\right)^2-\left(s+\frac{a_o^2}{s}\right)^2\right]}$$

$$\rightarrow \mp iU\alpha\sqrt{\frac{s}{2}\left(1-\frac{a_o^4}{s_o^4}\right)}\Big/(\zeta\mp s)^{\frac{1}{2}} \quad \text{as } \zeta\rightarrow \pm s.$$

In (3.8), with $\zeta_o = s$, we find that

$$|Y| = U\alpha\sqrt{\frac{s}{2}\left(1-\frac{a_o^4}{s^4}\right)} \; ,$$

whence

$$Y = \pm\frac{\pi}{2}\rho U^2\alpha^2 s\left(1-\frac{a_o^4}{s^4}\right). \qquad \text{......(3.9)}$$

This is the resolved part of a force normal to the leading edge, and it follows that its component in the x-direction is X where

$$\frac{dX}{dx} = \frac{ds}{dx}|Y| .$$

Hence X is obtained by integrating $|Y|$ with respect to s and the contribution from both half-wings gives

$$C_x = \frac{\pi A}{4}\alpha^2(1-\sigma^2)^2 = -\tfrac{1}{2}C_z\alpha,$$

i.e.

$$\frac{\partial C_x}{\partial\alpha} = -\alpha\frac{\partial C_z}{\partial\alpha} . \qquad \text{......(3.10)}$$

3.2. Cylindrical body with pointed nose

The relevant potential may be formed from (2.3) if we put s = a where now a must be treated as a variable with x: thus, if $\alpha_e = \alpha$ and $\beta_e = 0$ (i.e. there is an upwash due to the body incidence but no sideslip), then from (2.3)

$$\varphi = U\alpha a^2 \sin\theta/R . \qquad \text{......(3.11)}$$

This is an odd function of z whereas $\partial\varphi/\partial z$ is an even function, so that from (2.4), the pressure difference between the upper and lower surfaces is $\triangle p$ where

$$\frac{\triangle p}{\frac{1}{2}\rho U^2} = -\frac{4}{U}\left(\frac{\partial\varphi}{\partial x}\right)_{\text{upper surface}} .$$

Differentiating φ from (3.11) and putting R = a, it then follows that

$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = 8\alpha\frac{da}{dx}\sqrt{1-\frac{y^2}{a^2}}$$

where da/dx is an arbitrary function of x, describing the nose shape.

Since

$$\frac{dZ}{dx} = \rho U^2 \int_0^a \left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right) dy \qquad \ldots\ldots (3.12)$$

we find on performing the integration that

$$\frac{dZ}{dx} = -2\pi\alpha\rho U^2 a\frac{da}{dx} \qquad \ldots\ldots (3.13)$$

so that integrating from $x = -(\ell + h)$ to $x = \sigma c$ (from the nose to the body-wing junction), and noting that

$$\int_{-(\ell+h)}^{\sigma c} a\frac{da}{dx}\,dx = \tfrac{1}{2}a_o^2$$

we find

$$\frac{\partial C_z}{\partial \alpha} = -\frac{\pi A}{2}\sigma^2 . \qquad \ldots\ldots (3.14)$$

The pitching moment coefficient about the point $(\xi,0,0)$ is given by

$$C_m = 2\int_{-(\ell+h)}^{\sigma c} \left(\frac{x}{c}-\frac{\xi}{c}\right)\frac{dC_z}{dx}\,dx , \qquad \ldots\ldots (3.15)$$

i.e. from (3.13), if we integrate by parts we find

$$\frac{\partial C_m}{\partial \alpha} = -\frac{8\pi}{Sc}\left(\tfrac{1}{2}a_o^2\sigma c - \frac{1}{2\pi}\int_{-(\ell+h)}^{\sigma c}\pi a^2\,dx\right) - \frac{2\xi}{c}\frac{\partial C_z}{\partial \alpha} .$$

But since πa^2 is the cross-sectional area of the body, if $\Omega\pi a_o^2 h$ is the volume of the head of the body (i.e. ahead of the plane $x = -\ell$), then evidently

$$\frac{\partial C_m}{\partial \alpha} = \pi A\sigma^2\left(\frac{\ell + \Omega h}{c}\right) - \frac{2\xi}{c}\frac{\partial C_z}{\partial \alpha} . \qquad \ldots\ldots (3.16)$$

For a conical head $\Omega = 1/3$, and for a slender ogive $\Omega = 8/15$, approximately.

From symmetry, the side force and the rolling and yawing moments are all zero,

$$\text{i.e.}\quad \frac{\partial C_y}{\partial \alpha} = \frac{\partial C_n}{\partial \alpha} = \frac{\partial C_l}{\partial \alpha} = 0. \qquad \ldots\ldots(3.17)$$

Finally, the longitudinal force is obtained from the suction over the nose, caused by the acceleration of the flow normal to the axis of the body, i.e. in planes x = constant. The excess velocity at the surface relative to the body is obtained from (3.8) as equal to $2U\alpha\cos\theta$, whence it follows that there is a component of pressure in the plane x = constant, equal to

$$\tfrac{1}{2}\rho U^2\alpha^2(1 - 4\cos^2\theta)$$

on the body surface. This must, in fact, be the resolved part of a pressure acting normally to the body surface which is (if the body is slender) to the first order of approximation exactly as given above. It follows that the resolved component of the normal pressures in the direction of the x-axis contribute to a longitudinal force given by the integral

$$X = -\tfrac{1}{2}\rho U^2\alpha^2 \int_0^{a_0}\int_0^{2\pi} a(1 - 4\cos^2\theta)\,d\theta\,da,$$

whence, as for the body-wing combination,

$$C_x = \frac{\pi}{4} A\sigma^2\alpha^2 = -\tfrac{1}{2} C_z \alpha,$$

i.e.

$$\frac{\partial C_x}{\partial \alpha} = -\alpha \frac{\partial C_z}{\partial \alpha}. \qquad \ldots\ldots(3.18)$$

This result is, of course, fundamental and has been demonstrated, quite generally, by Ward[9] for all body shapes.

4. The Derivatives due to Pitching

4.1. Wing on cylindrical body

If there is an angular velocity of pitch q about the line $x = \xi$, $z = 0$, then the effective incidence of the flow relative to the body in any plane x = constant is given by

$$\alpha_e = \alpha + \left(\frac{x - \xi}{U}\right) q \qquad \ldots\ldots(4.1)$$

where α is the angle of attack. Since, for a wing on a cylindrical body, $a = a_0$, a constant, we have in (2.4) for $\beta_e = 0$

$$c_p = -\frac{2}{U}\left(\frac{ds}{dx}\frac{\partial\varphi}{\partial s} + \frac{d\alpha_e}{dx}\frac{\partial\varphi}{\partial\alpha_e} + \alpha_e\frac{\partial\varphi}{\partial z}\right).$$

The value of $\partial\varphi/\partial\alpha_e$ follows from (2.5), very simply, and since the term involving $\partial\varphi/\partial z$ is an even function of z, the pressure difference Δp between the upper and lower surfaces is given from (4.1) by

$$\frac{\Delta p}{(\frac{1}{2}\rho U^2)} = -\frac{4}{U}\left(\frac{ds}{dx}\frac{\partial\varphi}{\partial s} + \frac{q}{U\alpha_e}\varphi\right)_{\text{upper surface}}$$

Thus, the pressure difference caused by pitching is

$$\frac{\partial}{\partial q}\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right) = -\frac{4}{U^2\alpha_e}\left[(x-\xi)\frac{ds}{dx}\frac{\partial\varphi}{\partial s} + \varphi\right]_{\text{upper surface}}$$

$$= -\frac{4}{U^2\alpha_e}\left[s\frac{\partial\varphi}{\partial s} + \varphi\right]_{\text{upper surface}} - \frac{\xi}{U}\frac{\partial}{\partial\alpha}\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)$$

......(4.2)

where the operator $\partial/\partial\alpha$ is here meant to imply conditions due to incidence alone as described in paragraph 3; and using equations (2.5) and (3.1) we may describe the variation of $\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)$ due to pitch.

The longitudinal distribution of normal force is given by (3.2) so that from equations (2.5), (3.1) and (4.2) we find on integration that

$$\frac{d}{dx}\left(\frac{\partial Z}{\partial q}\right) = -\left[\frac{\pi}{U}s^2\left(3 - \frac{a_o^2}{s^2} - \frac{a_o^4}{s^4}\right) + \frac{\xi s}{2U}\frac{d}{dx}\left(\frac{\partial C_z}{\partial\alpha}\right)\right]\rho U^2.$$

.......(4.3)

Thus, integrating from $x = \sigma c$ to $x = c$, we find that

$$\frac{\partial C_z}{\partial\alpha_q} = -\pi A(1 - \sigma^2 - \sigma^3 + \sigma^4) - 2\frac{\xi}{c}\frac{\partial C_z}{\partial\alpha}. \quad ......(4.4)$$

Again, the pitching moment coefficient is given by the integral expressed in (3.5), and from (4.3) on evaluation this gives

$$\frac{\partial C_m}{\partial\alpha_q} = -\frac{3\pi A}{2}\left(1 - \frac{2}{3}\sigma^2 - \frac{1}{3}\sigma^4 + \frac{4}{3}\sigma^4\ln\sigma\right) - \frac{2\xi}{c}\left(\frac{\partial C_z}{\partial\alpha_q} + \frac{\partial C_m}{\partial\alpha}\right)\Bigg|_{\xi=0}$$

$$+ \frac{4\xi^2}{c^2}\frac{\partial C_z}{\partial\alpha}. \quad ......(4.5)$$

The side force, and the yawing and rolling moments are all zero, since φ (and so c_p) is an even function of y, i.e.

$$\frac{\partial C_y}{\partial \alpha_q} = \frac{\partial C_n}{\partial \alpha_q} = \frac{\partial C_l}{\partial \alpha_q} = 0. \qquad \ldots\ldots(4.6)$$

The longitudinal force is derived exactly as in paragraph 3.1.on p.6, except that in the expression for the sideways component of the suction force on the wing leading edge given in equation (3.9), α_e^2 has to be substituted for α^2, and is a function of x according to equation (4.1). It therefore follows that if X is the longitudinal component of the suction force

$$\frac{d}{dx}\frac{\partial X}{\partial q} = \pi\rho U^2 \alpha_e \left(\frac{x-\xi}{U}\right) s \left(1 - \frac{a_o^4}{s^4}\right)\frac{ds}{dx},$$

whence by comparison with equations (3.3) and (3.5), it follows that in the limit as $q \to 0$, the contribution from both half wings gives

$$\frac{\partial C_x}{\partial \alpha_q} = -\alpha \frac{\partial C_m}{\partial \alpha}. \qquad \ldots\ldots(4.7)$$

4.2. <u>Cylindrical body with pointed nose</u>

The effective incidence α_e of any plane section of the body is given as in equation (4.1), and from (2.3) we have, therefore

$$\varphi = U\alpha_e a^2 \sin\theta / R \qquad \ldots\ldots(4.8)$$

where this expression is equivalent to (3.11) if $\alpha_e = \alpha$ simply. Since we may now write s = a, and treat <u>a</u> as a variable with x, it follows that the pressure coefficient is

$$c_p = -\frac{2}{U}\left(\frac{da}{dx}\frac{\partial\varphi}{\partial a} + \frac{d\alpha_e}{dx}\frac{\partial\varphi}{\partial\alpha_e} + \alpha_e\frac{\partial\varphi}{\partial z}\right)$$

Also, the function φ is odd in z whereas $\partial\varphi/\partial z$ is an even function of z so that the pressure difference

$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = -\frac{4}{U}\left(\frac{da}{dx}\frac{\partial\varphi}{\partial a} + \frac{d\alpha_e}{dx}\frac{\partial\varphi}{\partial\alpha_e}\right)_{\text{upper surface}}$$

or from (4.8),

$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = -\frac{4\sin\theta}{a}\frac{d(\alpha_e a^2)}{dx}.$$

Thus substituting in (3.12) and integrating we have for the normal force:

$$\frac{dZ}{dx} = -\pi\rho U^2 \frac{d(\alpha_e a^2)}{dx}, \qquad \ldots\ldots(4.9)$$

- a general expression of which (3.13) is a particular case.

Whence, from (4.1), integrating from $x = -(\ell+h)$ to $x = \sigma c$, we may calculate

$$\frac{\partial C_z}{\partial \alpha_q} = -\pi A \sigma^3 - \frac{2\xi}{c}\frac{\partial C_z}{\partial \alpha}\,. \qquad \ldots\ldots (4.10)$$

Again, with the pitching moment coefficient given as in (3.15) we may use (4.9) to show that

$$\frac{\partial C_m}{\partial \alpha_q} = -\frac{8\pi}{sc}\left(a_o{}^2\sigma^2 c - \frac{1}{\pi}\int_{-(\ell+h)}^{\sigma c}\frac{x}{c}\pi a^2 dx\right) - \frac{2\xi}{c}\left(\frac{\partial C_z}{\partial \alpha_q} + \frac{\partial C_m}{\partial \alpha}\right)\Bigg|_{\xi=0} + \frac{4\xi^2}{c^2}\frac{\partial C_z}{\partial \alpha}.$$

If Γh is the distance of the centroid of the (solid) nose ahead of the shoulder at $x = -\ell$, then

$$\Gamma h = \frac{-\int_{-(\ell+h)}^{-\ell}(\ell+x)\pi a^2 dx}{\int_{-(\ell+h)}^{-\ell}\pi a^2 dx} = \frac{-\int_{-(\ell+h)}^{-\ell}\pi a^2 x dx}{-\Omega h \pi a_o{}^2} - \ell$$

so that we may write

$$\frac{\partial C_m}{\partial \alpha_q} = -A\pi\sigma^2\left(\sigma^2 + \frac{2\Gamma\Omega h^2 + 2\Omega h\ell + \ell^2}{c^2}\right) - \frac{2\xi}{c}\left(\frac{\partial C_z}{\partial \alpha_q} + \frac{\partial C_m}{\partial \alpha}\right)\Bigg|_{\xi=0} + \frac{4\xi^2}{c^2}\frac{\partial C_z}{\partial \alpha}. \qquad \ldots\ldots (4.11)$$

The value of Γ for a cone is 1/4, whereas for a slender ogive we find that Γ = 5/16 approximately.

From symmetry, it follows that

$$\frac{\partial C_y}{\partial \alpha_q} = \frac{\partial C_n}{\partial \alpha_q} = \frac{\partial C_\ell}{\partial \alpha_q} = 0 \qquad \ldots\ldots (4.12)$$

and the longitudinal force is derived exactly as in paragraph 3.2 on p.8., except that now the normal pressure distribution on the nose due to the upwash is given by

$$\tfrac{1}{2}\rho U^2 \alpha_e{}^2 (1 - 4\cos^2\theta),$$

where, of course, α_e is a function of x. Hence, the resolved parts of the normal pressure in the direction of the x-axis contribute to a longitudinal force given by the integral

$$X = -\tfrac{1}{2}\rho U^2 \int_{-(\ell+h)}^{\sigma c} \alpha_e^{\,2} a \frac{da}{dx}\,dx \int_0^{2\pi} (1-4\cos^2\theta)\,d\theta$$

so that as $q \to 0$,

$$\frac{d}{dx}\frac{\partial X}{\partial q} = \pi\rho U^2 \alpha \left(\frac{x-\xi}{U}\right)\frac{d(a^2)}{dx}$$

whence by comparison with equations (3.13) and (3.15) we find

$$\frac{\partial C_x}{\partial \alpha_q} = -\alpha \frac{\partial C_m}{\partial \alpha} \; . \qquad \ldots\ldots(4.13)$$

5. The Derivatives due to Sideslip

5.1. Wing on cylindrical body

We consider now a wing-body combination, at incidence, sideslipping with uniform velocity - V so that the angle of sideslip is β. Evidently, then $\alpha_e = \alpha$ and $\beta_e = \beta$ for all planes x = constant, and so from (2.1) if $a = a_o$, a constant

$$c_p = -\frac{2}{U}\left[\frac{ds}{dx}\frac{\partial\varphi}{\partial s} + \alpha\frac{\partial\varphi}{\partial z} - \beta\frac{\partial\varphi}{\partial y}\right].$$

The contribution to the pressure distribution due to sideslip alone is found from

$$\frac{\partial c_p}{\partial\beta} = -\frac{2}{U}\left[\frac{ds}{dx}\frac{\partial^2\varphi}{\partial\beta\partial s} + \alpha\left(\frac{\partial^2\varphi}{\partial\beta\partial z} - \frac{\partial^2\varphi}{\partial\alpha\partial y}\right)\right] \qquad \ldots\ldots(5.1)$$

where we have given the limiting value of the expression $\beta\frac{\partial\varphi}{\partial y}$ for $\beta \to 0$.

Now, from (2.3),

$$\frac{\partial\varphi}{\partial\beta} = -Ua_o^{\,2}y\Big/(y^2+z^2), \quad \text{so that } \frac{\partial^2\varphi}{\partial\beta\partial s} = 0 \ldots.(5.2)$$

whereas, after differentiation, we find for $a = a_o$,

$$\left.\begin{aligned}\left(\frac{\partial^2\varphi}{\partial\alpha\partial y}\right)_W &= -\,\mathrm{sgn}(z)U\,\frac{y}{s}\left(1-\frac{a_o^{\,4}}{y^4}\right)\Bigg/\sqrt{\left(1+\frac{a_o^{\,4}}{s^4}\right)-\frac{y^2}{s^2}\left(1+\frac{a_o^{\,4}}{y^4}\right)}\\ \left(\frac{\partial^2\varphi}{\partial x\partial y}\right)_B &= -\,\mathrm{sgn}(z)U\,4\,\frac{y}{s}\left(1-\frac{y^2}{a_o^{\,2}}\right)\Bigg/\sqrt{\left(1+\frac{a_o^{\,2}}{s^2}\right)^2-4y^2}\,.\end{aligned}\right\}$$

$$\ldots\ldots(5.3)$$

From (5.2) and (5.3) it follows that although in (5.1) the terms

$$\frac{\partial c_p}{\partial \beta} = -\frac{2\alpha}{U}\left(\frac{\partial^2 \varphi}{\partial \beta \partial z} - \frac{\partial^2 \varphi}{\partial \alpha \partial y}\right) \qquad \ldots\ldots(5.4)$$

are odd functions of z, they are also odd functions of y. Also, the suction force at the wing leading edge (arising from the singularity there in the potential function) is influenced only by the upwash $U\alpha_e$, and so is unaffected by sidewash. Hence the only effect of sidewash is to cause a rolling moment but no other force or moment, i.e.

$$\frac{\partial C_x}{\partial \beta} = \frac{\partial C_y}{\partial \beta} = \frac{\partial C_z}{\partial \beta} = \frac{\partial C_m}{\partial \beta} = \frac{\partial C_n}{\partial \beta} = 0. \qquad \ldots\ldots(5.5)$$

No rolling moment can, however, be contributed by the pressure distribution upon the body, so that the rolling moment is from (5.1), (5.2) and (5.3),

$$\frac{\partial \mathcal{L}}{\partial \beta} = 2\rho U^2 \int_{\sigma c}^{c} dx \int_{a_o}^{s} \left(\frac{\partial c_p}{\partial \beta}\right)_{\text{upper surface}} y\, dy.$$

But $\left(\frac{\partial^2 \varphi}{\partial \beta \partial z}\right)_W = 0$, from (5.2), so that from (5.4)

$$\frac{\partial C_{\mathcal{L}}}{\partial \beta} = \frac{4\alpha}{Ss} \int_{\sigma c}^{c} dx \int_{a_o}^{s} \frac{1}{U}\left(\frac{\partial^2 \varphi}{\partial \alpha \partial y}\right)_{z=0+} y\, dy. \qquad \ldots\ldots(5.6)$$

Using (5.3) and performing the first integration with respect to y, we find

$$+\ \frac{\partial C_{\mathcal{L}}}{\partial \beta} = -\frac{2\alpha}{Ss}\frac{dx}{ds}\int_{a_o}^{s_o}\left[sa_o\left(1 - \frac{a_o^2}{s^2}\right) + \frac{\pi}{4}s^2\left(1 - \frac{a_o^2}{s^2}\right) + \frac{s^2}{2}\left(1 + \frac{a_o^2}{s^2}\right)\arcsin\frac{s^2 - a^2}{s^2 + a^2}\right] ds$$

and upon further integration we may calculate finally

$$\frac{\partial C_{\mathcal{L}}}{\partial \beta} = -\frac{\pi\alpha}{3}\left[(1 + 4\sigma^3 - 3\sigma^4) - \frac{1}{\pi}(1 + 6\sigma^2 - 3\sigma^4)\arcsin\left(\frac{2\sigma}{1+\sigma^2}\right) + \frac{2}{\pi}\sigma(1 - \sigma^2) - \frac{8}{\pi}\sigma^3 \ln\left(\frac{1+\sigma^2}{2\sigma^2}\right)\right]. \qquad \ldots\ldots(5.7)$$

5.2. Cylindrical body with pointed nose

The derivatives due to sideslip for a body alone are, of course, analogous to those due to incidence and accordingly, we may immediately write down - using equations (3.14), (3.16) and (3.17)

$$\left.\begin{aligned}
&\frac{\partial C_z}{\partial \beta} = \frac{\partial C_m}{\partial \beta} = \frac{\partial C_\ell}{\partial \beta} = 0 \\
&\frac{\partial C_y}{\partial \beta} = \frac{\partial C_z}{\partial \alpha} = -\frac{\pi A}{2}\sigma^2 \\
&\frac{\partial C_n}{\partial \beta} = -\frac{1}{A}\frac{\partial C_m}{\partial \alpha} = -\pi\sigma^2\frac{\ell + \Omega h}{c} + \frac{2\xi}{Ac}\frac{\partial C_y}{\partial \beta}.
\end{aligned}\right\} \quad \ldots\ldots (5.8)$$

Similarly, from (3.18),

$$\frac{\partial C_x}{\partial \beta} = \beta\,\frac{\partial C_z}{\partial \alpha},$$

but this gives in the limiting condition of $\beta \to 0$,

$$\frac{\partial C_x}{\partial \beta} = 0. \quad \ldots\ldots (5.9)$$

6. The Derivatives due to Yaw

If yawing occurs about the point $(\xi,0,0)$ with angular velocity r, the longitudinal velocity at a point (x,y,z) is (U - ry) and the sideslip velocity is $-r(x - \xi)$. We shall assume that the pressure coefficient at a point (x,y,z) for the yawing wing is the same as the pressure coefficient for a non-yawing wing in a free stream of velocity U, sideslipping with velocity $-r(x - \xi)$. We shall neglect the variation in longitudinal velocity at spanwise stations off the centre line of the body, since this is evidently negligible if the body and wing plan form is indeed slender, as assumed. This is the same assumption as implicit in the treatment of pitching, (paragraph 4), where the longitudinal velocity was taken as U everywhere. Thus, putting

$$\beta_e = -\frac{4}{A}\alpha_r\left(\frac{x-\xi}{c}\right) \text{ and } \alpha_e = \alpha,$$

we find from equation (2.4)

$$c_p = -\frac{2}{U}\left[\frac{\partial\varphi}{\partial x} + \frac{4\alpha_r}{A}\left(\frac{x-\xi}{c}\right)\frac{\partial\varphi}{\partial y} + \alpha\,\frac{\partial\varphi}{\partial z}\right]$$

whence, for $\alpha_r \to 0$, we have correct to the first order in α,

$$\frac{\partial c_p}{\partial \alpha_r} = \frac{8}{UA}\left\{\left(\frac{x-\xi}{c}\right)\left[\frac{\partial^2\varphi}{\partial\beta_e\partial x} - \alpha\left(\frac{\partial^2\varphi}{\partial\alpha\partial y} - \frac{\partial^2\varphi}{\partial\beta_e\partial z}\right)\right] + \frac{1}{c}\frac{\partial\varphi}{\partial\beta_e}\right\}. \quad \ldots\ldots (6.1)$$

In this report we are considering only the effects that arise in isentropic and inviscid flow. In practice, of course, the differential change in profile drag of the two half wings, resulting from yaw, will contribute a yawing moment; in fact, this might well be the most important effect, but despite this it will not be considered here.

6.1. Wing on cylindrical body

Of those terms that are contained in the expression for $\partial c_p/\partial\alpha_r$, all have at one time or the other already been obtained, thus

(i) the term $\partial^2\varphi/\partial\beta\partial x$ was shown to be zero by eqn.(5.2); and whereas $\partial^2\varphi/\partial\beta\partial z$ is zero on the wing, it is non-zero on the body - where, however, as an odd function of both y and z it contributes nothing to the resultant forces and moments. Thus, both these terms may be neglected.

(ii) The term involving $\partial^2\varphi/\partial\alpha\partial y$ is expanded in equation (5.3) where - as an odd function of y and z it contributes to a rolling moment on the wing surface, but not, of course, on the body surface.

(iii) The term $\partial\varphi/\partial\beta_e$ is given in (5.2) as an odd function of y and as an even function of z. Accordingly, this contributes to a side force on the body and a yawing moment.

To sum up, (ii) gives rise to a rolling moment and (iii) to a yawing moment and side force. There are no other additional forces or moments due to yaw; this includes also the longitudinal force contributed by leading edge suction, which, inasmuch as it depends only on the upwash $U\alpha$, is uninfluenced by yaw. Thus

$$\frac{\partial C_z}{\partial\alpha_r} = \frac{\partial C_m}{\partial\alpha_r} = \frac{\partial C_x}{\partial\alpha_r} = 0. \qquad \ldots\ldots(6.2)$$

The side force, as we have just seen, is derived from the pressure distribution

$$\frac{\partial c_p}{\partial\alpha_r} = \frac{8}{UAc}\,\frac{\partial\varphi}{\partial\beta_e} + \ldots$$

i.e.

$$\frac{\partial c_p}{\partial\alpha_r} = -\frac{8a_o^2\,y}{A\,c\,R^2} + \ldots$$

from (5.2), as applied to the body. Hence

$$\frac{\partial Y}{\partial\alpha_r} = \frac{16\,\rho U^2}{Ac}\int_{\sigma c}^{c} dx \int_0^{a_o}\sqrt{a_o^2 - z^2}\;dz\,.$$

Performing the integrations it follows that

$$\frac{\partial C_y}{\partial\alpha_r} = 2\pi\sigma^2(1-\sigma). \qquad \ldots\ldots(6.3)$$

The yawing moment is similarly given by

$$-\frac{16\,\rho U^2}{A\,c}\int_{\sigma c}^{c}\frac{(x-\xi)}{c}\,dx\int_0^{a_o}\sqrt{a_o^2 - z^2}\;dz,$$

whence again quite simply we may calculate

$$* \quad \frac{\partial C_n}{\partial \alpha_r} = -\frac{2\pi}{A}\sigma^2(1-\sigma^2) + \frac{2}{A}\frac{\xi}{c}\left(\frac{\partial C_y}{\partial \alpha_r}\right). \quad \ldots\ldots (6.4)$$

In (ii) above, it was shown that the rolling moment arises from the terms in the pressure distribution -

$$\frac{\partial c_p}{\partial \alpha_r} = -\frac{8\alpha}{UA}\left(\frac{x-\xi}{c}\right)\frac{\partial^2 \varphi}{\partial \alpha \partial y} + \ldots$$

as applied to the wing surface. Thus, by comparison with (5.6), it will be seen that the rolling moment is given by

$$\frac{\partial \mathcal{l}}{\partial \alpha_r} = 2\rho U^2 \int_{\sigma c}^{c} dx \int_{a_o}^{s} \left(\frac{\partial c_p}{\partial \alpha_r}\right)_{\text{upper surface}} y\, dy,$$

i.e.

$$* \quad \frac{\partial C_{\mathcal{l}}}{\partial \alpha_r} = -\frac{4\alpha}{s^3}\int_{\sigma c}^{c}\frac{x}{c}\,dx \int_{a_o}^{s}\frac{1}{U}\left(\frac{\partial^2 \varphi}{\partial \alpha \partial y}\right)_{z=0+} y\,dy + \frac{4}{A}\frac{\xi}{c}\frac{\partial C_{\mathcal{l}}}{\partial \beta}.$$

The first integration is exactly the same as that required for (5.6) and the second, with respect to x, yields the expression

$$\frac{\partial C_{\mathcal{l}}}{\partial \alpha_r} = \frac{\pi\alpha}{A}\left[1 + \frac{2\sigma}{\pi}(1-7\sigma^2+6\sigma^3) - \frac{1+4\sigma^2}{\pi}\arcsin\left(\frac{2\sigma}{1+\sigma^2}\right)\right.$$

$$\left. + \frac{6\sigma^4}{\pi}\operatorname{arccot}\sigma + \frac{8\sigma^4}{\pi}I(\sigma)\right] + \frac{4}{A}\frac{\xi}{c}\frac{\partial C_{\mathcal{l}}}{\partial \beta} \quad \ldots\ldots (6.5)$$

where* $I(\sigma) = \int_{\sigma}^{1}\frac{1}{t}\operatorname{arccot} t\, dt.$

6.2. <u>Cylindrical body with pointed nose</u>

The derivatives due to yaw for a body alone are, of course, analogous to those due to pitch, and we may accordingly write down - using equations (4.10),

---

*The value of $I(\sigma)$ may most easily be computed by a series expansion: thus, if $\tau = \sigma^2/1+\sigma^2$ we find

$$I(\sigma) = \frac{\pi}{2}\ln\frac{1}{\sigma} - 0.9160 + \sigma - \frac{1}{3^2}\sigma^3 + \frac{\sigma^3\tau}{5^2}\left\{1 + \frac{8.3}{7^2}\tau + \frac{8.143}{7^2.9^2}\tau^2\right.$$

$$\left. + \frac{8.11328}{7^2.9^2.11^2}\tau^3 + \ldots\right\}$$

(4.11) and (4.12) -

$$
\left.
\begin{aligned}
&\frac{\partial C_z}{\partial \alpha_r} = \frac{\partial C_m}{\partial \alpha_r} = \frac{\partial C_\ell}{\partial \alpha_r} = 0 \\
&\frac{\partial C_y}{\partial \alpha_r} = -\frac{1}{A}\frac{\partial C_m}{\partial \alpha_q} = \pi\sigma^3 + \frac{2}{A}\frac{\xi}{c}\frac{\partial C_y}{\partial \beta} \\
&\frac{\partial C_n}{\partial \alpha_r} = \frac{1}{A^2}\frac{\partial C_m}{\partial \alpha_q} = -\frac{\pi\sigma^2}{A}\left(\sigma^2 + \frac{2\Gamma\Omega h^2 + 2\Omega h\ell + \ell^2}{c^2}\right) \\
&\qquad\qquad + \frac{2}{A}\frac{\xi}{c}\left(\frac{\partial C_y}{\partial \alpha_r} + \frac{\partial C_n}{\partial \beta}\right)\Bigg|_{\xi=0} + \frac{4\xi^2}{A^2c^2}\frac{\partial C_y}{\partial \beta} \, .
\end{aligned}
\right\} \quad \ldots\ldots (6.6)
$$

Similarly, from (4.13)

$$\frac{\partial C_x}{\partial \alpha_r} = \beta \frac{\partial C_n}{\partial \beta} ,$$

but since $\beta = 0$

$$\frac{\partial C_x}{\partial \alpha_r} = 0 \, . \quad \ldots\ldots (6.7)$$

7. The Derivatives due to a Longitudinal Change in Speed

If U is changed from U to U + dU, keeping the upwash constant, α will be altered to $\alpha - \alpha\, dU/U$. Also, the pressure everywhere will be increased in the proportion U + dU: U. Evidently then

$$
\left.
\begin{aligned}
U \frac{\partial C_z}{\partial U} &= \alpha \frac{\partial C_z}{\partial \alpha} \\
U \frac{\partial C_m}{\partial U} &= \alpha \frac{\partial C_m}{\partial \alpha}
\end{aligned}
\right\} \quad \ldots\ldots (7.1)
$$

whereas all the other derivatives vanish, i.e.

$$\frac{\partial C_x}{\partial U} = \frac{\partial C_y}{\partial U} = \frac{\partial C_\ell}{\partial U} = \frac{\partial C_n}{\partial U} = 0. \quad \ldots\ldots (7.2)$$

These results are valid either for the wing on a cylindrical body, or for the cylindrical body with a pointed nose.

8. The Derivatives due to Roll

Obviously, a rolling motion has no effect on a body alone, and we shall therefore consider only the combination of wing mounted on cylindrical body.

The potential ϕ, in any plane x = constant, induced by a rolling motion with angular velocity p, of a wing-body combination is derived in Appendix I. This potential satisfies the condition that on the surface of a wing, the upwash is

$$\phi_z = -py. \qquad \ldots\ldots (8.1)$$

The value of the surface potential is (as given in equation (D) of the Appendix I)

$$(\phi)_W = \mathrm{sgn}(z)\,\frac{p}{4}\left\{\left[1+\frac{2}{\pi}\arccos\left(\frac{2a_o s}{a_o^2+s^2}\right)\right]\left(y+\frac{a_o^2}{y}\right)\sqrt{s^2\left(1+\frac{a_o^4}{s^4}\right)-y^2\left(1+\frac{a_o^4}{y^4}\right)+\frac{2}{\pi}\,y^2\left(1-\frac{a_o^2}{y^2}\right)^2}\times\right.$$

$$\left.\times\ \mathrm{argtanh}\left[\frac{2a_o ys\sqrt{s^2\left(1+\frac{a_o^4}{s^4}\right)-y^2\left(1+\frac{a_o^4}{y^4}\right)}}{(y^2+a_o^2)\,(s^2-a_o^2)}\right]\right\},$$

$$(\phi)_B = \mathrm{sgn}(z)\,\frac{p}{2}\left\{\left[1+\frac{2}{\pi}\arccos\left(\frac{2a_o s}{a_o^2+s^2}\right)\right]y\sqrt{s^2\left(1+\frac{a_o^2}{s^2}\right)^2-4y^2}\right.$$

$$\left.-\,2y\sqrt{a_o^2-y^2}-\frac{4}{\pi}(a_o^2-y^2)\,\mathrm{argcoth}\left[\frac{a_o s\sqrt{s^2\left(1+\frac{a_o^2}{s^2}\right)^2-4y^2}}{y(s^2-a_o^2)}\right]\right\}.$$

$$\ldots\ldots (8.2)$$

Equation (2.4) gives, as before, the pressure coefficient whose component due to roll is (in the absence of sideslip)

$$\frac{\partial c_p}{\partial p} = -\frac{2}{Up}\left(\frac{\partial\phi}{\partial x}+\alpha\,\frac{\partial\phi}{\partial z}\right). \qquad \ldots\ldots (8.3)$$

From (8.2) it will be seen that $\partial\phi/\partial x$ is an odd function of both y and z, whereas from (8.1) $\partial\phi/\partial z$ is an odd function of y only. The normal force and pitching moment are therefore zero, i.e.

$$\frac{\partial C_z}{\partial \alpha_p} = \frac{\partial C_m}{\partial \alpha_p} = 0. \qquad \ldots\ldots (8.4)$$

The rolling moment is derived only from the term involving $\partial\phi/\partial x$ in (8.3) since the other term does not contribute to a difference of pressure between the upper and lower surfaces of the wing, i.e. since $(\phi)_W = 0$ at the wing leading edge,

$$\frac{\partial l}{\partial p} = -\frac{4\rho U}{p}\int_{a_o}^{s_o} y\,dy \int_{\frac{yc}{s_o}}^{c}\left(\frac{\partial\phi}{\partial x}\right)_{z=0+} dx = -\frac{4\rho U}{p}\int_{a_o}^{s_o} y\,(\phi)_{\substack{x=c\\ z=0+}}\,dy.$$

The integral may be evaluated using equation (8.2) and yields the answer

$$\frac{\partial l}{\partial p} = -\frac{\rho U s_o^4}{\pi}\left\{\tfrac{1}{2}\sigma^2(1-\sigma^2)^2 + \tfrac{1}{2}\sigma(1-\sigma^2)(1-6\sigma^2+\sigma^4)\left(\frac{\pi}{2}+\arccos\frac{2\sigma}{1+\sigma^2}\right)\right.$$

$$-\sigma^4\pi\left(\frac{\pi}{2}+\arccos\frac{2\sigma}{1+\sigma^2}\right)+\frac{1}{8}(1+\sigma^2)^4\left(\frac{\pi}{2}+\arccos\frac{2\sigma}{1+\sigma^2}\right)^2$$

$$\left.+\,\sigma^4\int_1^{1/\sigma^2}\frac{1}{t}\,\mathrm{argtanh}\left[\frac{2\sigma}{1-\sigma^2}\,\frac{\sqrt{\left(\frac{1+\sigma^4}{\sigma^2}\right)t-t^2-1}}{(t+1)}\right]dt\right\}.$$

But

$$\frac{\partial}{\partial\sigma}\left\{\int_1^{1/\sigma^2}\mathrm{argtanh}\left[\frac{2\sigma}{1-\sigma^2}\,\frac{\sqrt{\left(\frac{1+\sigma^4}{\sigma^2}\right)t-t^2-1}}{(t+1)}\right]\frac{dt}{t}\right\}$$

$$= -\frac{2}{1+\sigma^2}\int_1^{1/\sigma^2}\frac{1+t}{t}\,\frac{dt}{\sqrt{\left(\frac{1+\sigma^4}{\sigma^2}\right)t-t^2-1}},$$

$$= -\frac{2\pi}{1+\sigma^2} = \pi\frac{\partial}{\partial\sigma}\left\{\arccos\frac{2\sigma}{1+\sigma^2}\right\}.$$

Hence, substituting this value for the definite integral in the above expression for $\partial l/\partial p$, we have after some rearrangement - on using the identity -

$$2\,\mathrm{arccot}\,\sigma = \frac{\pi}{2} + \arccos\frac{2\sigma}{1+\sigma^2},$$

the final expression*

$$\frac{\partial C_{\ell}}{\partial \alpha_p} = -\frac{A}{8\pi}\Big\{(1+\sigma^2)^4(\text{arccot }\sigma)^2 - \pi^2\sigma^4 + 2\sigma(1-\sigma^2)(\sigma^4-6\sigma^2+1)\times$$

$$\times \text{ arccot }\sigma + \sigma^2(1-\sigma^2)^2\Big\} \quad \ldots\ldots (8.5)$$

In evaluating the side and longitudinal force we must now consider the suction along the wing leading edge, which now is not distributed symmetrically about the body - being larger on one side than the other. The outwards force is derived as given in (3.8), but now H is the complex potential corresponding to the real potential $\phi + \varphi$, ϕ being that part due to roll and φ being that part due to incidence. Evidently in (3.8), since $\varphi_z = 0$,

$$\left.\frac{dH}{d\zeta}\right|_{z=0} = (\phi_y)_W + (\varphi_y)_W - i(\phi_z)_W$$

where, from (8.2),

$$\lim_{y\to\pm s}(\phi_y)_W = -\,\text{sgn}(z)\,\frac{p}{4}\sqrt{\frac{s^4-a_o^4}{s^2-y^2}}\left\{\left[1+\frac{2}{\pi}\arccos\frac{2a_o s}{a_o^2+s^2}\right]\left(1+\frac{a_o^2}{s^2}\right)\right.$$

$$\left. + \frac{4}{\pi}\left(\frac{s^2-a_o^2}{s^2+a_o^2}\right)\frac{a_o}{s}\right\}$$

and from (2.5), if $\beta_e = 0$ and $\alpha_e = \alpha$,

$$\lim_{y\to\pm s}(\varphi_y)_W = \mp\,\text{sgn}(z)\,\frac{U\alpha}{s}\sqrt{\frac{s^4-a_o^4}{s^2-y^2}}\,,$$

whilst, from (8.1), it follows that ϕ_z is finite on the wing surface. Hence the singularity in $dH/d\zeta$ at the leading edge is such that in (3.8),

$$\gamma = -\,\text{sgn}(z).U\alpha\,\frac{s^{\frac{1}{2}}}{\sqrt{2}}\sqrt{1-\frac{a_o^4}{s^4}}\left\{(\text{sgn } y)\right.$$

$$\left. + \frac{ps}{2U\alpha}\left[\left(1+\frac{a_o^2}{s^2}\right)\text{arccot}\,\frac{a_o}{s} + \frac{2a_o}{\pi s}\,\frac{s^2-a_o^2}{s^2+a_o^2}\right]\right\},$$

and so we find that, if Y is the side force at the leading edge, the increment due to roll is

$$\frac{\partial Y}{\partial p} = \frac{\pi}{2}\rho U\alpha s^2\left(1-\frac{a_o^4}{s^4}\right)\left[\frac{2}{\pi}\left(1+\frac{a_o^2}{s^2}\right)\text{arccot}\,\frac{a_o}{s} + \frac{2a_o}{\pi s}\,\frac{s^2-a_o^2}{s^2+a_o^2}\right].$$

$$\ldots\ldots (8.6)$$

---

*This is equivalent to equation (20) of Ref.8, with which it agrees, although the numerical calculation of this reference appears in error, as well as some of the other equations used in the discussion.

The sign is the same for $y \gtrless 0$, because the increment has the same direction on both sides - adding to the suction on the starboard side $(y > 0)$ and decreasing it on the port side. There is thus no resultant longitudinal force, i.e.

$$\frac{\partial C_x}{\partial \alpha_p} = 0 \qquad \text{......(8.7)}$$

but there is a resultant side force due to this asymmetric distribution which is given by

$$2p\int_{\sigma c}^{c}\left(\frac{\partial Y}{\partial p}\right)dx = 2p\,\frac{dx}{ds}\int_{a_o}^{s_o}\left(\frac{\partial Y}{\partial p}\right)ds \qquad \text{......(8.8)}$$

where $\partial Y/\partial p$ is obtained from (8.6). However, in addition to this there is a contribution from the body to the side force which is due to the pressure distribution described in (8.3) as caused by the term $\partial \phi/\partial z$ - which is an odd function of y. In fact, on the body surface, since $(\partial \phi/\partial r)_B = 0$,

$$\left(\frac{\partial \phi}{\partial z}\right)_B = \frac{\cos\theta}{a_o}\left(\frac{\partial \phi}{\partial \theta}\right)$$

$$= \frac{py}{a^2}\Bigg\{\left[8y^2 - s^2\left(1+\frac{a_o^{\,2}}{s^2}\right)^2\right]\sqrt{\frac{a_o^{\,2}-y^2}{s^2\left(1+\frac{a_o^{\,2}}{s^2}\right)^2 - 4y^2}}\;\frac{2}{\pi}\,\text{arccot}\left(\frac{a_o}{s}\right)$$

$$+ a^2 - 2y^2 + \frac{2a_o}{\pi s}\,\frac{(s^2 - a_o^{\,2})\sqrt{a_o^{\,2}-y^2}}{\sqrt{s^2\left(1+\frac{a_o^{\,2}}{s^2}\right)^2 - 4y^2}}$$

$$- \frac{4}{\pi}\,y\sqrt{a_o^2 - y^2}\;\text{argcoth}\left[\frac{a_o s\sqrt{s^2\left(1+\frac{a_o^{\,2}}{s^2}\right)^2 - 4y^2}}{y(s^2 - a_o^{\,2})}\right]\Bigg\}$$

and, from (8.3), the side force on the body is

$$- 4\rho U\alpha\int_{\sigma c}^{c}dx\int_{0}^{a_o}\left(\frac{\partial \phi}{\partial z}\right)_B dz.$$

Substituting for $\partial\phi/\partial z$, and performing the integration with respect to z, this becomes

$$-\frac{2\rho U\alpha p}{a^2}\frac{dx}{ds}\int_{a_o}^{s_o}\left\{\frac{\pi s^4}{4}\left[\left(\frac{s^2+a_o^2}{2s^2}\right)^4\left(1-\frac{4}{\pi^2}\arccos^2\frac{2a_o s}{a_o^2+s^2}\right)-\left(\frac{a_o}{s}\right)^4\right]\right.$$

$$+\,a_o^3 s\left(1-\frac{a_o^2}{s^2}\right)-\frac{1}{4\pi}a_o^2 s^2\left(1-\frac{a_o^2}{s^2}\right)^2$$

$$\left.-\,\frac{1}{4\pi}a_o s^3\left(1+\frac{a_o^2}{s^2}\right)\left(1-\frac{a_o^2}{s^2}\right)\arccos\frac{2a_o s}{s^2+a_o^2}\right\}ds. \qquad \ldots\ldots(8.9)$$

This contribution from the body must now be added to that from the wing, given by equations (8.6) and (8.8). The integration with respect to s cannot then be evaluated formally, and it appears easiest to derive the value by numerical methods. Accordingly we write

$$\frac{\partial C_y}{\partial \alpha_p} = 4\alpha\sigma^3\int_1^{1/\sigma} f(t)\,dt \qquad \ldots\ldots(8.10)$$

where using equations (8.6), (8.8) and (8.9) we may show that, on changing the parameter of integration to $a_o/s \equiv t$, we have

$$f(t) = \frac{\pi}{4}\left\{(t^2-\frac{1}{t^2})\left[\left(1+\frac{2}{\pi}\arccos\frac{2t}{1+t^2}\right)(1+\frac{1}{t^2})+\frac{4}{\pi t}\frac{t^2-1}{t^2+1}\right]\right.$$

$$\left.-\left(\frac{t^2+1}{2t}\right)^4\left[1-\frac{4}{\pi^2}\left(\arccos\frac{2t}{1+t^2}\right)^2\right]+1\right\}$$

$$+\,\frac{1}{8}(t^2-1)\left[\frac{2}{\pi}(1-\frac{1}{t^2})+\frac{8}{t}+\frac{2t}{\pi}\left(1+\frac{1}{t^2}\right)^2\arccos\frac{2t}{1+t^2}\right]. \qquad \ldots\ldots(8.11)$$

We may notice that for $\sigma \to 0$, the integral tends to $\frac{\pi}{6}\frac{1}{\sigma^3}$, i.e. for $\sigma = 0$, we have $\frac{\partial C_y}{\partial \alpha_p} = \frac{2}{3}\pi\alpha$.

Similarly, also, the yawing moment may be evaluated, for

$$\frac{\partial C_n}{\partial \alpha_p} = -\frac{c}{b}\int_0^{\sigma c}\left(\frac{x-\xi}{c}\right)\frac{d}{dx}\left(\frac{\partial C_y}{\partial \alpha_p}\right)dx$$

or since, in (8.10), we may put $\sigma = a_o/s$ and regard it as a function of x describing the variation of $\frac{\partial C_y}{\partial \alpha_p}$ with x, we find that

$$\frac{\partial C_n}{\partial \alpha_p} = -\frac{8\alpha}{A}\sigma^4 \int_1^{1/\sigma} t\,f(t)\,dt + \frac{2}{A}\frac{\xi}{c}\left(\frac{\partial C_y}{\partial \alpha_p}\right) \quad \text{......(8.12)}$$

where f(t) is given, as before, in (8.11) as a function of t.

We may now calculate that as $\sigma \to 0$, $\frac{\partial C_n}{\partial \alpha_p} \to -\frac{\pi\alpha}{A}$

if we make $\xi = 0$. This completes our discussion of the derivatives.

9. Stability Derivatives relative to Wind Axes

To avoid confusion we shall write the forces, moments and velocities relative to wind axes, as shown in Figure 2, as primed symbols. The relation of these to the previously used system may be obtained quite simply from the geometry: and we list the transformation formulae below

$$X' = X + \alpha Z \qquad \ell' = \ell + \alpha n$$

$$Y' = Y \qquad m' = m$$

$$Z' = Z - \alpha X \qquad n' = n - \alpha\ell$$

$$\delta U' = \delta U + \alpha U \delta\alpha \qquad \delta p' = \delta p + \alpha\delta r$$

$$V' = -U\beta \qquad q' = q$$

$$\delta W' = U\delta\alpha - \alpha\delta U \qquad \delta r' = \delta r - \alpha\delta p.$$

Since $U' \equiv U$ correct to order α^2, we may write

$$\beta' = -\frac{V'}{U},\ \alpha' = \frac{W'}{U},\ \alpha'_p = \frac{p'b}{2U},\ \alpha'_q = \frac{q'\bar{c}}{U},\ \alpha'_r = \frac{r'b}{2U}.$$

Then the derivatives relative to wind axes may be expressed in terms of those relative to the body axes, as -

$$\frac{\partial}{\partial U'} = \frac{\partial}{\partial U} + \frac{\alpha}{U}\frac{\partial}{\partial \alpha},\quad \frac{\partial}{\partial \beta'} = \frac{\partial}{\partial \beta},\quad \frac{\partial}{\partial \alpha'} = \frac{\partial}{\partial \alpha} - \alpha U\frac{\partial}{\partial U}$$

$$\frac{\partial}{\partial \alpha'_p} = \frac{\partial}{\partial \alpha_p} + \alpha\frac{\partial}{\partial \alpha_r},\quad \frac{\partial}{\partial \alpha'_q} = \frac{\partial}{\partial \alpha_q},\quad \frac{\partial}{\partial \alpha'_r} = \frac{\partial}{\partial \alpha_r} - \alpha\frac{\partial}{\partial \alpha_p}$$

$$C_{x'} = C_x + \alpha C_z,\quad C_{y'} = C_y,\quad C_{z'} = C_z - \alpha C_x$$

$$C_{\ell'} = C_\ell + \alpha C_n,\quad C_{m'} = C_m,\quad C_n = C_n - \alpha C_\ell.$$

More particularly, taking account of zero derivatives, Table I lists the stability derivatives relative to wind axes of the body-wing combination here considered in terms of those relative to body axes. Here it should be noted that although strictly

$$\frac{\partial C_{l'}}{\partial \alpha'_r} = \frac{\partial C_l}{\partial \alpha_r} - \alpha \frac{\partial C_l}{\partial \alpha_p} + \alpha \frac{\partial C_n}{\partial \alpha_r}$$

and

$$\frac{\partial C_{n'}}{\partial \alpha'_p} = \frac{\partial C_n}{\partial \alpha_p} - \alpha \frac{\partial C_l}{\partial \alpha_p} + \alpha \frac{\partial C_n}{\partial \alpha_r} ,$$

the term $\frac{\partial C_l}{\partial \alpha_p}$ is small compared with the others, since it is of order A whereas the others are of order 1/A — A, the aspect ratio, being by inference, a small quantity if the theory is valid.

10. Presentation of Results

The results are summarised in Tables II-IV and in Figures 3-17. Table V is an index to the figures.

In the Tables II and III, the stability derivatives referred to body axes, with the wing apex on the centre-line as reference point, are given for:

(i) a triangular wing alone,

(ii) a body with a pointed conical nose.

In Table IV, those terms which have to be added to those for $\xi = 0$, to account for the choice of an arbitrary reference point at $(\xi,0,0)$, are given.

In the figures 5-13, the variation of these stability derivatives with (body diameter/wing span) ratio - again if referred to body axes, with wing apex as reference point - is shown for the non-zero derivatives. A complete list of where the variation of each particular derivative may be found is given in Table V.

The relevant results if wind axes are used may be obtained from Table I. In the figures the variation of these derivatives with respect to (body diameter/wingspan) ratio is included where appropriate, and only in four cases (Figs. 14-17) is there any variation of these different from those with respect to body axes.

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A P P E N D I X   I

Construction of the Potential due to Incidence and Sideslip

We require the potential due to upwash and sideslip in any transverse plane, x = constant, about a body-wing combination; the surface contours in any such $\zeta = y + iz$ plane will be a circle radius a and a straight line extending outwards from the surface of the circle to a distance s from the origin, along the horizontal y-axis. This contour is mapped conformally by the transformation

$$\zeta_t = y_t + iz_t = \zeta + \frac{a^2}{\zeta}$$

on the ζ_t-plane by a straight line joining the points

$y_t = \pm(s + \frac{a^2}{s})$, $z_t = 0$. The complex potential of the flow about such a line due to upwash $U\alpha_e$ and sideslip $U\beta_e$ is then simply,

$$H = - U\beta_e\zeta_t + iU\alpha_e\sqrt{\zeta_t^2 - \left(s + \frac{a^2}{s}\right)^2}$$

so that in terms of ζ,

$$H = - U\beta_e\left(\zeta + \frac{a^2}{\zeta}\right) + iU\alpha_e\sqrt{\left(\zeta + \frac{a^2}{\zeta}\right)^2 - \left(s + \frac{a^2}{s}\right)^2}.$$

The velocity potential due to upwash may be found by squaring the term

$$H + U\beta_e\left(\zeta + \frac{a^2}{\zeta}\right),$$

substituting $\zeta = R(\cos\theta + i\sin\theta)$, separating real and imaginary parts and solving for the real part of H. The result is given in equation (2.3) of the text - and in ref. 6 - where the expression for φ is adjusted to give zero velocity at $R = \infty$ according to definition.

APPENDIX II

Construction of the Potential due to Roll

We shall here outline briefly a method by which the potential in any (y,z) plane due to a rolling motion of the body-wing combination may be obtained.

Let $\zeta = y + iz$ and $\zeta_t = y_t + iz_t$, then if

$$\zeta_t = \zeta + \frac{a^2}{\zeta} \quad \text{......(A)}$$

the circular wing + body combination in the (y,z) plane becomes a straight line $z_t = 0$ for $0 \leq |y_t| \leq s + a^2/s$ in the (y_t, z_t) plane. In particular, the circular body transforms into that part of the line $z_t = 0$ where $0 \leq |y_t| \leq 2a$.

The complex potential we require in the (y,z) plane is such that the upwash w on the wing surface is $-py$, where p is the angular velocity of roll, since this cancels the effect of the upwash py induced by rolling.

Thus the stream function ψ is a constant ψ_o on the circle, and is given by $\psi = \psi_o + \frac{1}{2}p(y^2 - a^2)$ on the wing surface. The stream function is unchanged by the transformation, so that in the (y_t, z_t) plane, the upwash required on the line z_t is zero for $0 \leq |y_t| \leq 2a$, and is

$$-\frac{\partial \psi}{\partial y_t} = -\tfrac{1}{2}p\left[y_t + (\operatorname{sgn} y_t)\,\frac{y_t^2 - 2a^2}{\sqrt{y_t^2 - 4a^2}}\right] = w_t, \text{ say}$$

$$\text{......(B)}$$

for $2a \leq |y_t| \leq s + \frac{a^2}{s} = 2k$, say.

Let us assume that the complex potential of the flow which satisfies these boundary conditions along the line $z_t = 0$ is given by a line distribution of vortices such that the complex potential is

$$\frac{ki}{\pi}\int_{-1}^{+1} f(t)\,\ln\left(\frac{\zeta_t}{2k} - t\right) dt\,.$$

Then, on the line $z_t = 0$, if $(y_t/2k) = \eta$,

$$v_t = \tfrac{1}{2}f(\eta) \quad \text{and} \quad w_t = -\frac{1}{2\pi}\int_{-1}^{+1}\frac{f(t)}{\eta - t}\,dt,$$

where v_t and w_t are respectively the sidewash (on the upper surface) and upwash along the line $z_t = 0$.

The expression for w_t is already known so that we have to solve the integral equation for f(t). The solution is

$$f(t) = \frac{2}{\pi\sqrt{1-t^2}}\left[A - \int_{-1}^{+1} \frac{\sqrt{1-\eta^2}\, w_t(\eta)}{\eta - t}\, d\eta\right]$$

where A is a constant; that is, from (B)

$$f(t) = \frac{2}{\pi\sqrt{1-t^2}}\left[A + 2pk \int_{a/k}^{1} \frac{\sqrt{1-\eta^2}}{\eta^2 - t^2}\left(\eta + \frac{\eta^2 - a^2/2k^2}{\sqrt{\eta^2 - a^2/k^2}}\right)\eta\, d\eta\right]$$

$$= \frac{4}{\pi} pk\left[\frac{A'}{\sqrt{1-t^2}} - \left(\frac{\pi}{2} + \arccos\frac{a}{k}\right)\frac{t^2}{\sqrt{1-t^2}} + \frac{t}{2}\ln\left|\frac{t\sqrt{1-\frac{a^2}{k^2}} + \frac{a}{k}\sqrt{1-t^2}}{t\sqrt{1-\frac{a^2}{k^2}} - \frac{a}{k}\sqrt{1-t^2}}\right|\right.$$

$$\left. + \frac{\pi}{2}\mathcal{R}\left(\frac{t^2 - \frac{a^2}{2k^2}}{\sqrt{\frac{a^2}{k^2} - t^2}}\right)\right]. \quad \ldots\ldots (C)$$

The value of the constant A' is now chosen so that the potential $\emptyset$ is equal at $y_t = \pm 2k$, since this is a condition of symmetry, but

$$\emptyset_{y_t}\Big|_{z_t=0+} = v_t = \tfrac{1}{2} f(\eta),$$

and so

$$\emptyset\Big|_{z_t=0+} = k\int_{-1}^{\eta} f(\eta)\, d\eta.$$

Performing the integration and selecting the value of A' accordingly, from (C),

$$\emptyset\Big|_{z_t=0+} = pk^2\left\{\left(1 + \frac{2}{\pi}\arccos\frac{a}{k}\right)\eta\sqrt{1-\eta^2} - \eta\,\mathcal{R}\left(\sqrt{\frac{a^2}{k^2} - \eta^2}\right)\right.$$

$$\left. + \frac{\eta^2 - \frac{a^2}{k^2}}{\pi}\ln\left|\frac{\eta\sqrt{1-\frac{a^2}{k^2}} + \frac{a}{k}\sqrt{1-\eta^2}}{\eta\sqrt{1-\frac{a^2}{k^2}} - \frac{a}{k}\sqrt{1-\eta^2}}\right|\right\}.$$

But from (A),

$$y_t\Big|_{z_t=0+} = y + \frac{a^2}{y} \geqslant 2a$$

$$= 2a\cos\theta \leqslant 2a.$$

Hence, the potential on the surface in the (y,z) plane is given by

$$(\phi)_W = (\operatorname{sgn} z)\frac{p}{4}\left\{\left[1 + \frac{2}{\pi}\arccos\left(\frac{2as}{a^2+s^2}\right)\right]\left(y+\frac{a^2}{y}\right)\sqrt{s^2\left(1+\frac{a^4}{s^4}\right)-y^2\left(1+\frac{a^4}{y^4}\right)}\right.$$

$$\left. + \frac{2}{\pi}y^2\left(1-\frac{a^2}{y^2}\right)^2 \operatorname{argtanh}\left[\frac{2ays\sqrt{s^2\left(1+\frac{a^4}{s^4}\right)-y^2\left(1+\frac{a^4}{y^4}\right)}}{(y^2+a^2)(s^2-a^2)}\right]\right\}$$

$$(\phi)_B = (\operatorname{sgn} z)\frac{p}{2}\left\{\left[1 + \frac{2}{\pi}\arccos\left(\frac{2as}{a^2+s^2}\right)\right] y\sqrt{s^2\left(1+\frac{a^2}{s^2}\right)^2 - 4y^2} - 2y\sqrt{a^2-y^2}\right.$$

$$\left. - \frac{4}{\pi}(a^2-y^2)\operatorname{argcoth}\left[\frac{as\sqrt{s^2\left(1+\frac{a^2}{s^2}\right)^2-4y^2}}{y(s^2-a^2)}\right]\right\}.$$

......(D)

It will be noticed that if a = s, (i.e. if there is no wing) the potential due to roll is, of course, zero. On the other hand, if a = 0 (i.e. there is no body)

$$(\phi)_W = \tfrac{1}{2}(\operatorname{sgn} z)\, p\, y\sqrt{s^2 - y^2}$$

which is the result obtained from the statement of the potential due to the rotation of a flat plate - as a particular example of the potential due to the rotation of an elliptic cylinder - given, for example, by Lamb (Hydrodynamics, Arts. 71, 72).

TABLE I

Stability Derivatives with respect to fixed wind axes (Fig. 2) in terms of those with respect to fixed body axes (Fig. 1)

Derivative of Coefficient / with respect to	$C_{x'}$	$C_{y'}$	$C_{z'}$	$C_{l'}$	$C_{m'}$	$C_{n'}$
U'	$\frac{\alpha C_z}{U}$	0	$2\,\frac{\partial C_z}{\partial U}$	0	$2\,\frac{\partial C_m}{\partial U}$	0
β'	0	$\frac{\partial C_y}{\partial \beta}$	0	$\frac{\partial C_l}{\partial \beta} + \alpha\,\frac{\partial C_n}{\partial \beta}$	0	$\frac{\partial C_n}{\partial \beta}$
α'	C_z	0	$\frac{\partial C_z}{\partial \alpha}$	0	$\frac{\partial C_m}{\partial \alpha}$	0
α'_p	0	$\frac{\partial C_y}{\partial \alpha_p} + \alpha\,\frac{\partial C_y}{\partial \alpha_r}$	0	$\frac{\partial C_l}{\partial \alpha_p}$	0	$\frac{\partial C_n}{\partial \alpha_p} + \alpha\,\frac{\partial C_n}{\partial \alpha_r}$
α'_q	$\frac{\partial C_x}{\partial \alpha_q} + \alpha\,\frac{\partial C_z}{\partial \alpha_q}$	0	$\frac{\partial C_z}{\partial \alpha_q}$	0	$\frac{\partial C_m}{\partial \alpha_q}$	0
α'_r	0	$\frac{\partial C_y}{\partial \alpha_r}$	0	$\frac{\partial C_l}{\partial \alpha_r} + \alpha\,\frac{\partial C_n}{\partial \alpha_r}$	0	$\frac{\partial C_n}{\partial \alpha_r}$

T A B L E II

Stability Derivatives of Delta Wing (fixed body axes with reference point at wing apex).

Derivative of coefficient / with respect to	C_x	C_y	C_z	C_l	C_m	C_n
U	0	0	$-\frac{\pi A}{2}\frac{\alpha}{U}$	0	$-\frac{2\pi A}{3}\frac{\alpha}{U}$	0
β	0	0	0	$-\frac{\pi}{3}\alpha$	0	0
α	$\frac{\pi A}{2}\alpha$	0	$-\frac{\pi A}{2}$	0	$-\frac{2\pi A}{3}$	0
α_p	0	$\frac{2\pi}{3}\alpha$	0	$-\frac{\pi A}{32}$	0	$-\frac{\pi}{A}\alpha$
α_q	$\frac{2\pi A}{3}\alpha$	0	$-\pi A$	0	$-\frac{3\pi A}{2}$	0
α_r	0	0	0	$\frac{\pi}{A}\alpha$	0	0

T A B L E   III

Stability Derivatives of Cylindrical Body with Conical Nose
(fixed body axes with reference point at wing apex).

Derivative of coefficient / with respect to	C_x	C_y	C_z	C_ℓ	C_m	C_n
U	0	0	$\frac{\pi A\sigma^2}{2}\frac{\alpha}{U}$	0	$\pi A\sigma^2\left(\frac{\ell+\frac{1}{3}h}{c}\right)\frac{\alpha}{U}$	0
β	0	$\frac{\pi A\sigma^2}{2}$	0	0	0	$\pi A\sigma^2\left(\frac{\ell+\frac{1}{3}h}{c}\right)$
α	$-\frac{\pi A\sigma^2}{2}\alpha$	0	$\frac{\pi A\sigma^2}{2}$	0	$\pi A\sigma^2\left(\frac{\ell+\frac{1}{3}h}{c}\right)$	0
α_p	0	0	0	0	0	0
α_q	$\pi A\sigma^2\left(\frac{\ell+\frac{1}{3}h}{c}\right)\alpha$	0	$\pi A\sigma^3$	0	$-\frac{\pi}{2}\sigma^2 A\left[\frac{h^2+4h\ell+6\ell^2}{3c^2}+2\sigma^2\right]$	0
α_r	0	$2\pi\sigma^3$	0	0	0	$-\frac{\pi\sigma^2}{A}\left[\frac{h^2+4h\ell+6\ell^2}{3c^2}+2\sigma^2\right]$

T A B L E IV

Additional Term in Expression of Stability Derivatives expressed in Tables II and III, if body axes are taken with reference point distance ξ aft of the wing apex

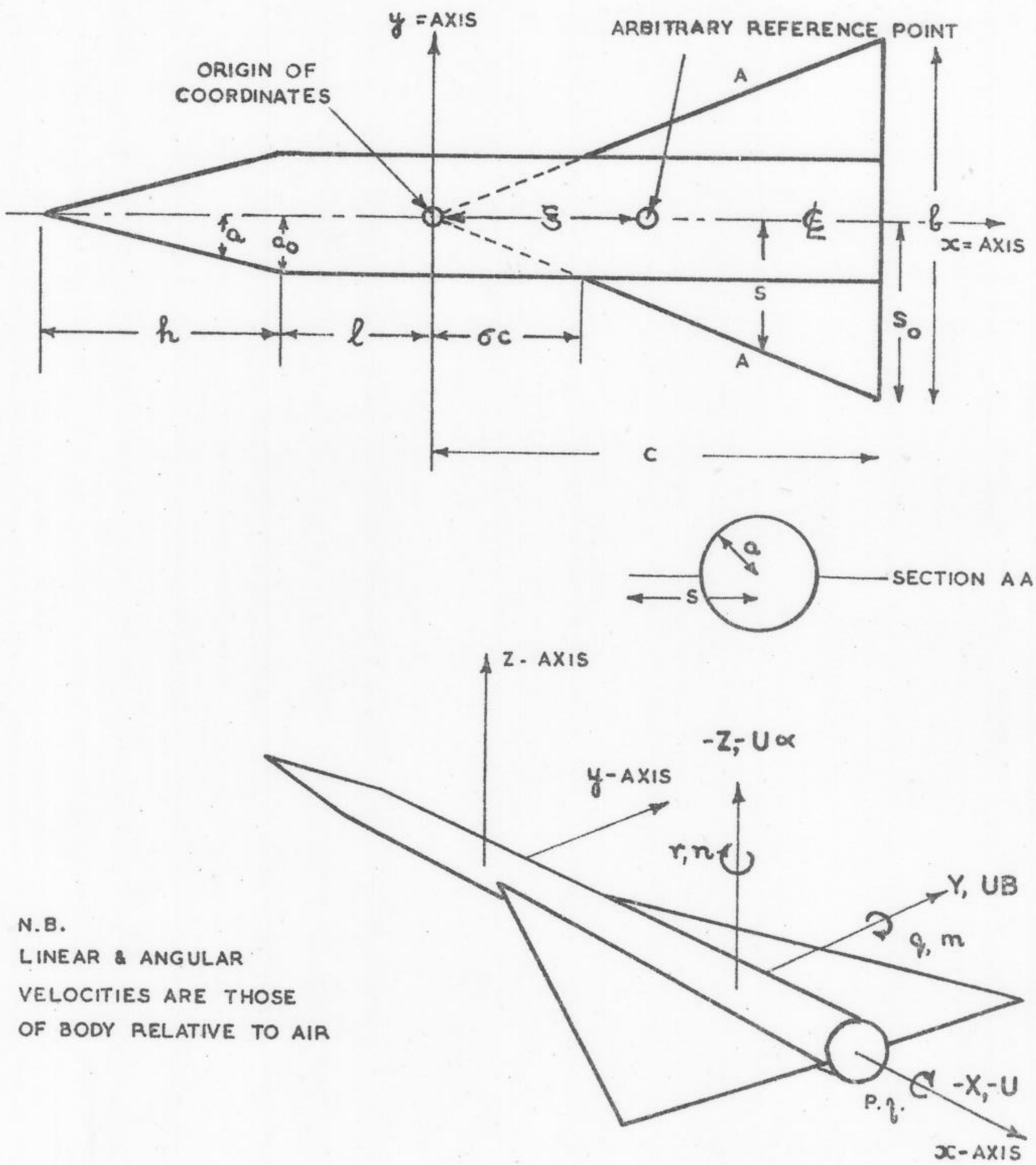
Addition to derivative of coefficient / with respect to	C_x	C_y	C_z	C_l	C_m	C_n
U	0	0	0	0	$-\frac{2\xi}{c}\frac{\partial C_z}{\partial U}$	0
β	0	0	0	0	0	$+\frac{2}{A}\frac{\xi}{c}\frac{\partial C_y}{\partial \beta}$
α	0	0	0	0	$-\frac{2\xi}{c}\frac{\partial C_z}{\partial \alpha}$	0
α_p	0	0	0	0	0	$+\frac{2}{A}\frac{\xi}{c}\frac{\partial C_y}{\partial \alpha_p}$
α_q	$+\frac{2\xi}{c}\frac{\partial C_x}{\partial \alpha}$	0	$-\frac{2\xi}{c}\frac{\partial C_z}{\partial \alpha}$	0	$-\frac{2\xi}{c}\left(\frac{\partial C_z}{\partial \alpha_q}-\frac{\partial C_m}{\partial \alpha}\right)\Big\vert_{\xi=0}+\frac{4\xi^2}{c^2}\frac{\partial C_z}{\partial \alpha}$	0
α_r		$\frac{2}{A}\frac{\xi}{c}\frac{\partial C_y}{\partial \beta}$		0	0	$\frac{2}{A}\frac{\xi}{c}\left(\frac{\partial C_y}{\partial \alpha_r}+\frac{\partial C_n}{\partial \beta}\right)\Big\vert_{\xi=0}+\frac{4}{A^2}\frac{\xi^2}{c^2}\frac{\partial C_y}{\partial \beta}$

T A B L E V

Index to Figures

Explanation: The figures in which will be found a graph showing the variation with (body diameter/wing span) ratio of each of the stability derivatives are enumerated below. The figures in brackets refer only to the stability derivatives with respect to wind axes, and the others are with respect to body axes.

Derivative of coefficient / with respect to	C_x	C_y	C_z	C_l	C_m	C_n
U	zero (3)		3 (3)		4 (4)	
β		zero		7 (7)		zero
α	3 (3)		3 (3)		4 (4)	
α_p		12 (14)		11 (11)		13 (15)
α_q	4 (16)		5 (5)		6 (6)	
α_r		9 (9)		8 (17)		10 (10)



N.B.
LINEAR & ANGULAR VELOCITIES ARE THOSE OF BODY RELATIVE TO AIR

FIG. 1. NOTATION USED IN ANALYSIS.

(BODY AXES)

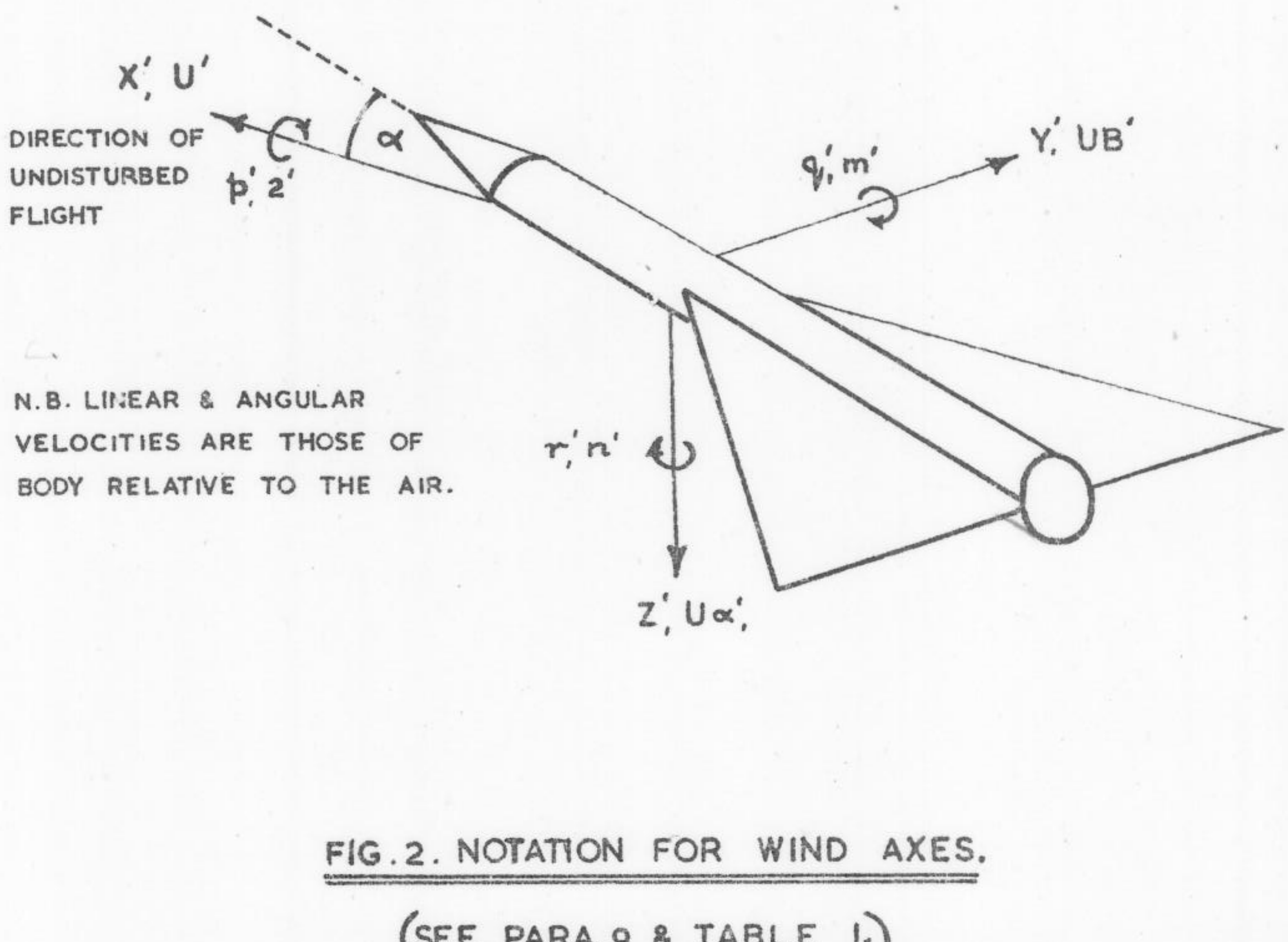


FIG. 2. NOTATION FOR WIND AXES.

(SEE PARA. 9 & TABLE I.)

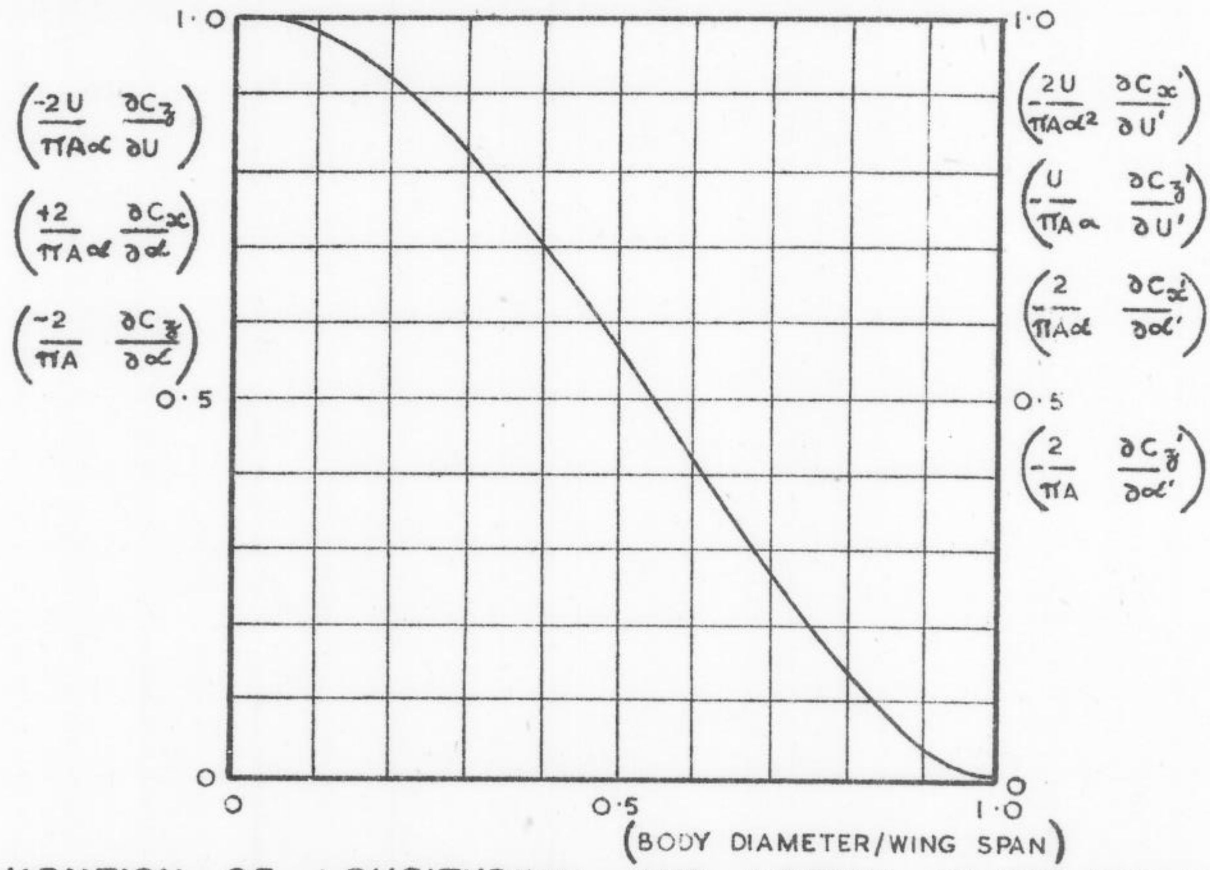


FIG. 3 VARIATION OF LONGITUDINAL AND NORMAL FORCE DUE TO INCIDENCE, AND OF NORMAL FORCE DUE TO A LONGITUDINAL DISTURBANCE, WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY

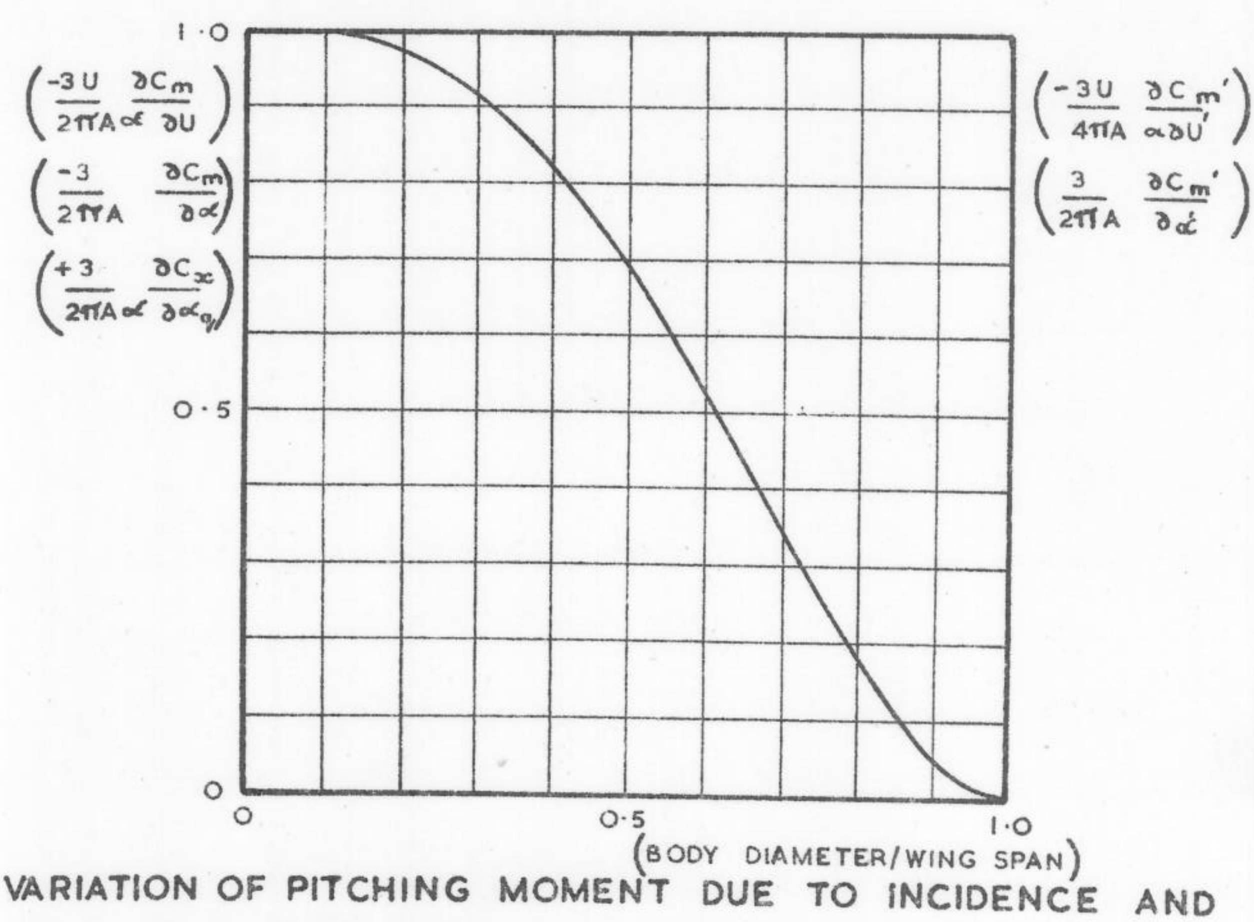


FIG. 4 VARIATION OF PITCHING MOMENT DUE TO INCIDENCE AND LONGITUDINAL DISTURBANCE, AND OF LONGITUDINAL FORCE DUE TO PITCH, WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY (ORIGIN AT WING APEX ON CENTRE LINE)

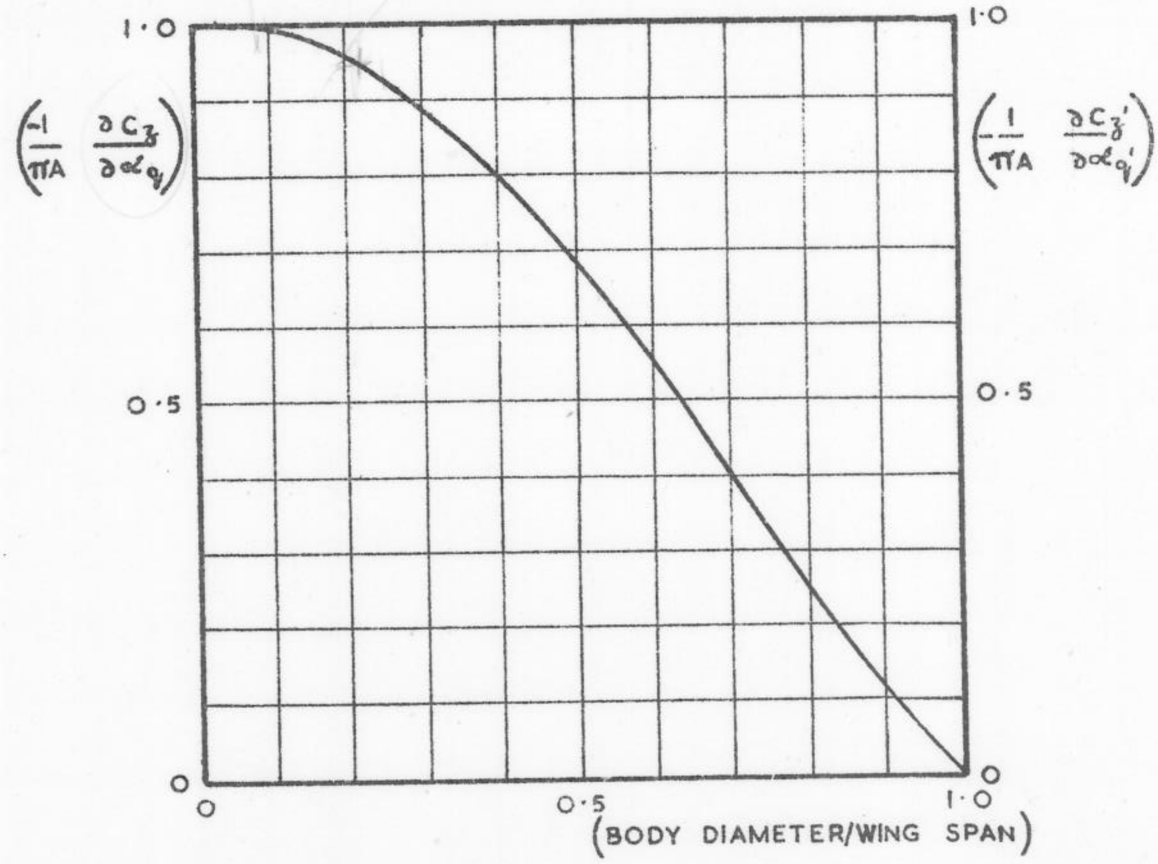


FIG. 5. VARIATION OF NORMAL FORCE DUE TO PITCH WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY (ORIGIN AT WING APEX ON CENTRE LINE)

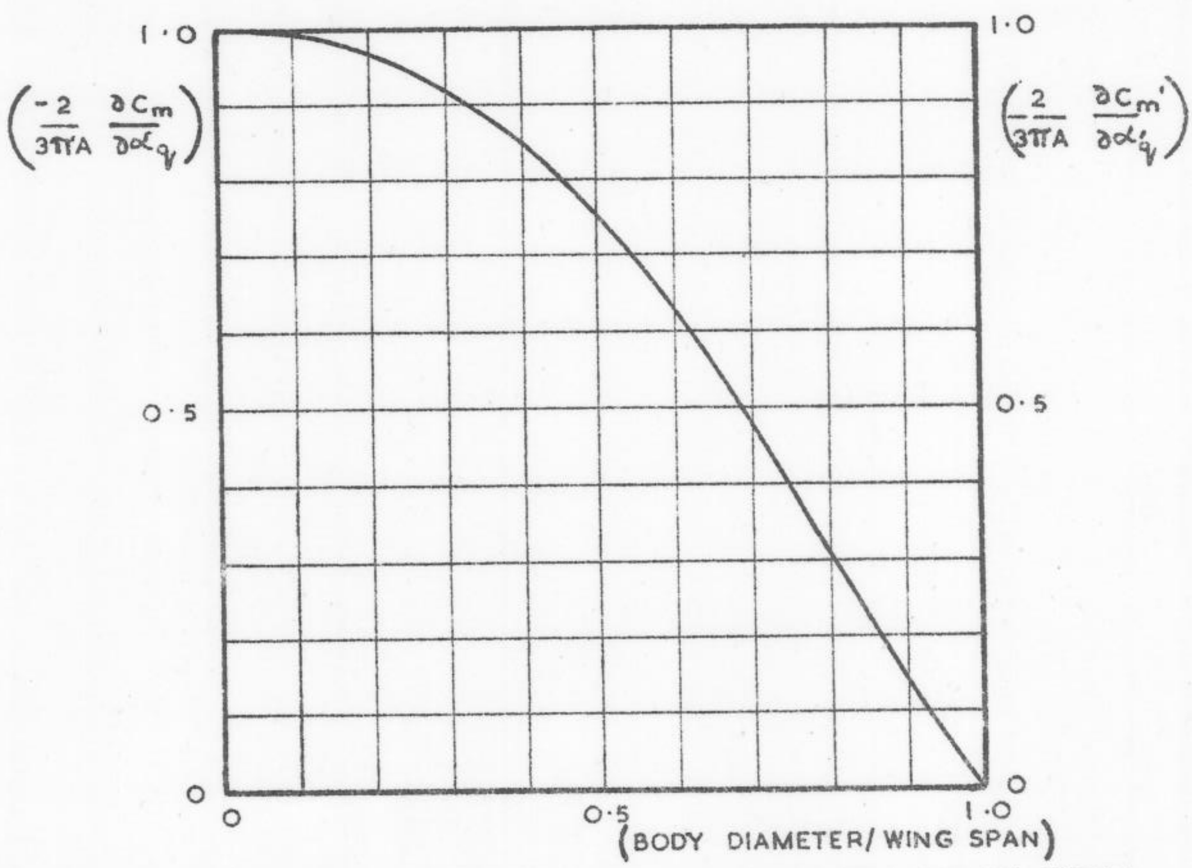


FIG. 6 VARIATION OF PITCHING MOMENT DUE TO PITCH WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY (ORIGIN AT WING APEX ON CENTRE LINE)

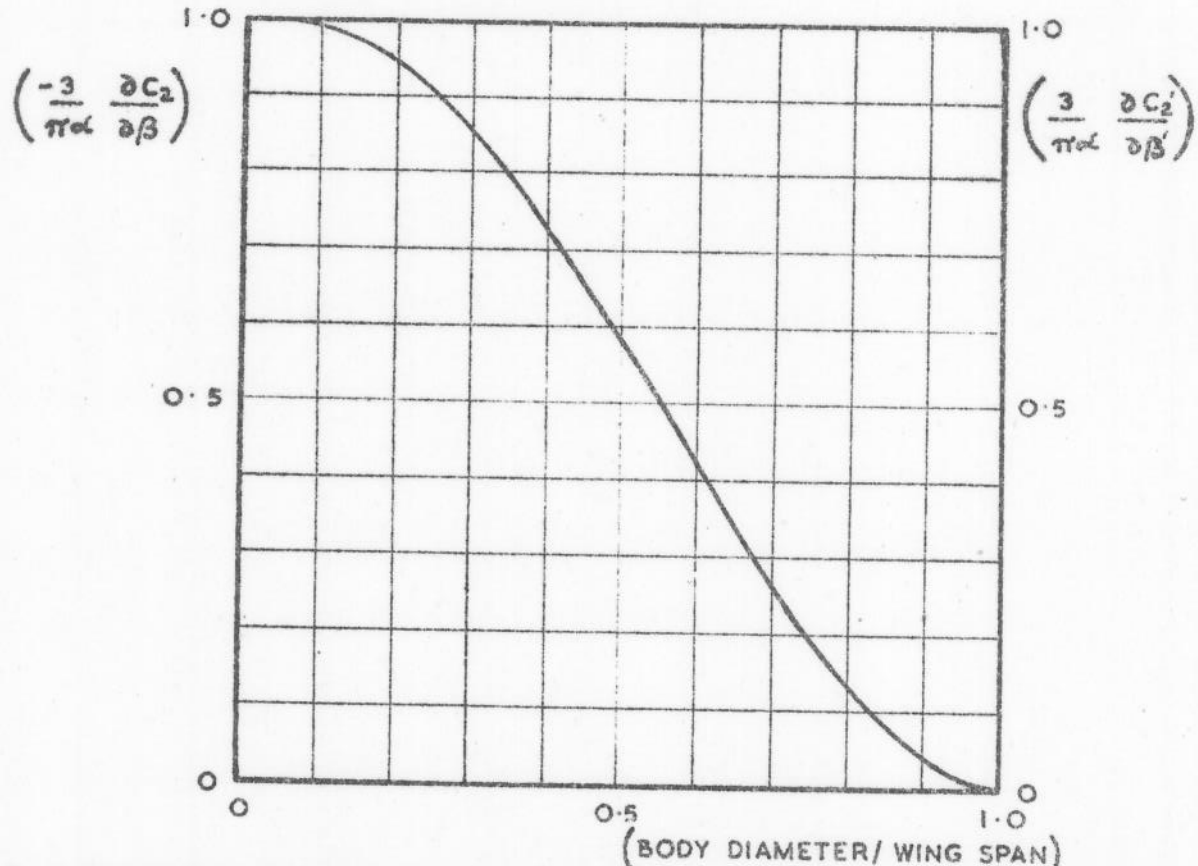


FIG 7 VARIATION OF ROLLING MOMENT DUE TO SIDESLIP WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY

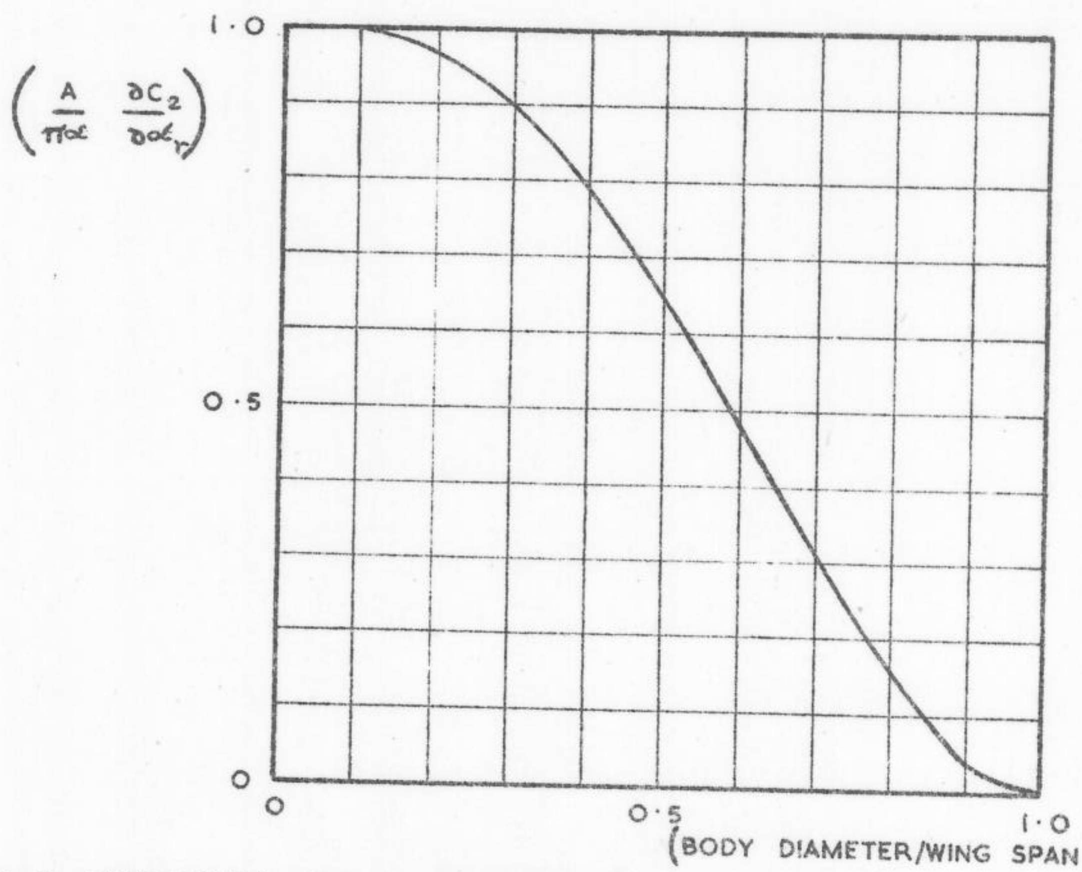


FIG. 8 VARIATION OF ROLLING MOMENT DUE TO YAW WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY (ORIGIN AT WING APEX ON CENTRE LINE)

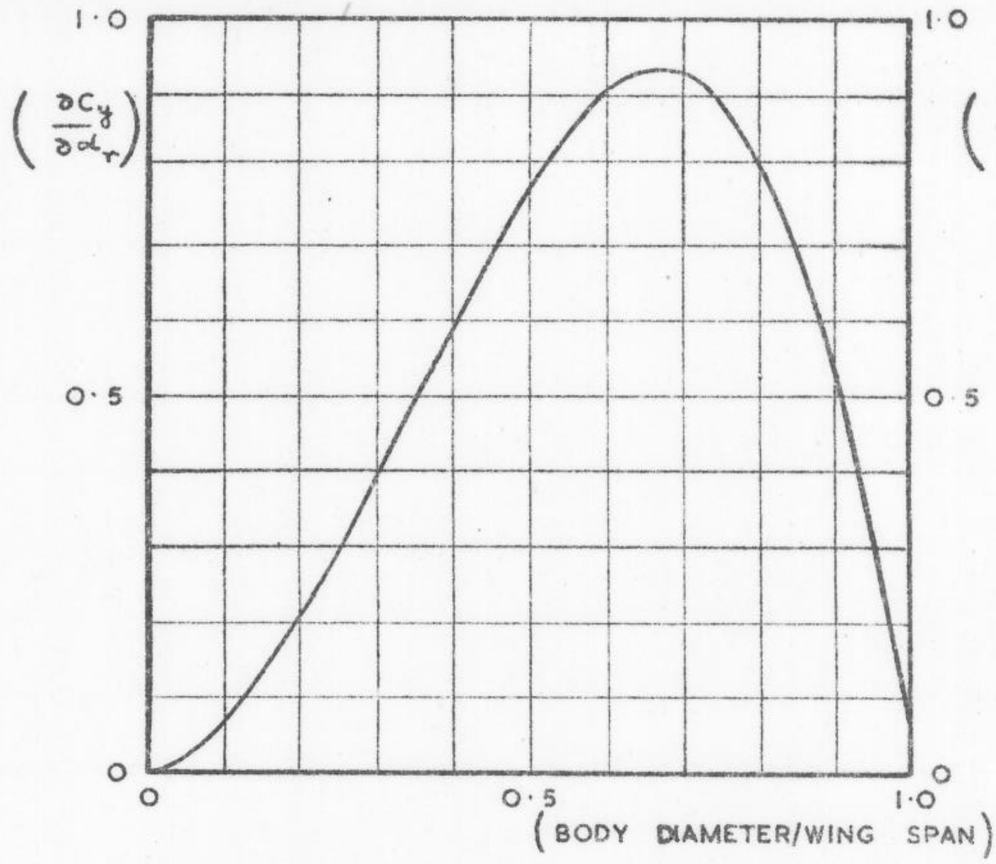


FIG. 9. SIDE FORCE DUE TO YAW FOR DELTA WING ON CYLINDRICAL BODY (ORIGIN OF AXES AT WING APEX ON CENTRE LINE)

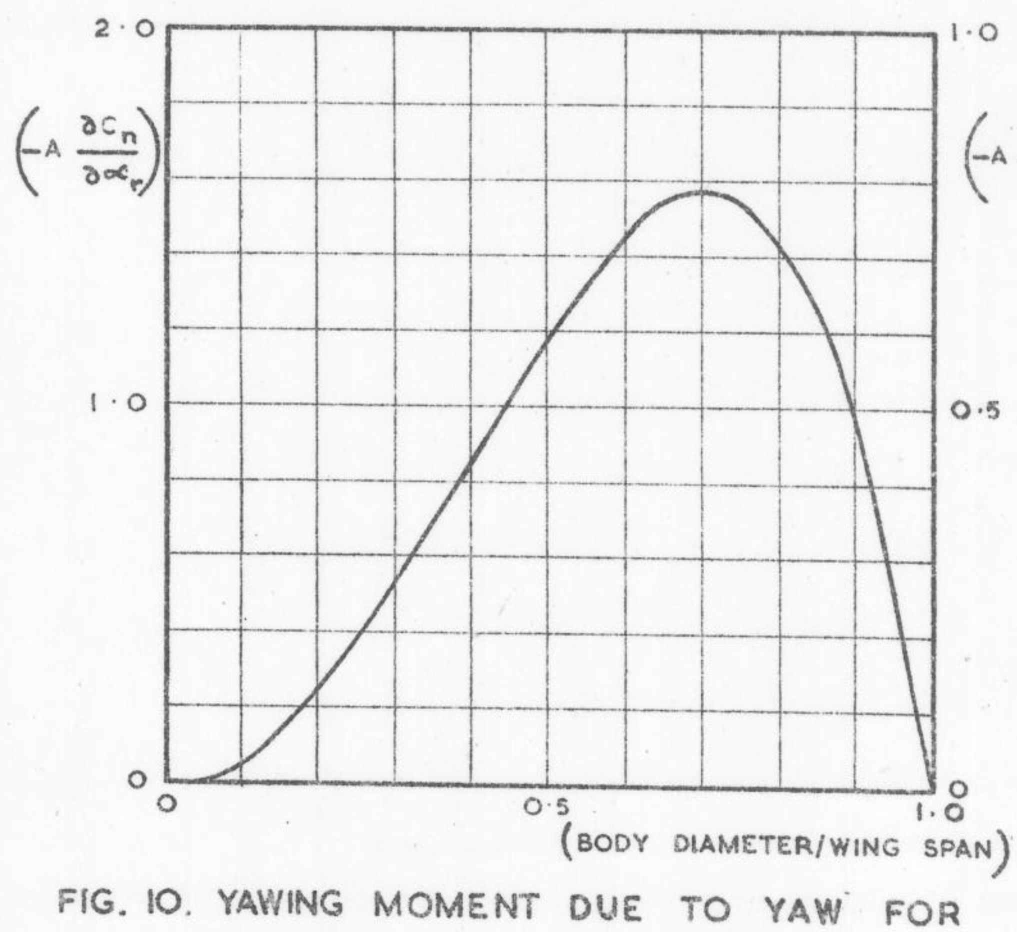


FIG. 10. YAWING MOMENT DUE TO YAW FOR DELTA WING ON A CYLINDRICAL BODY (ORIGIN OF AXES AT WING APEX ON CENTRE LINE)

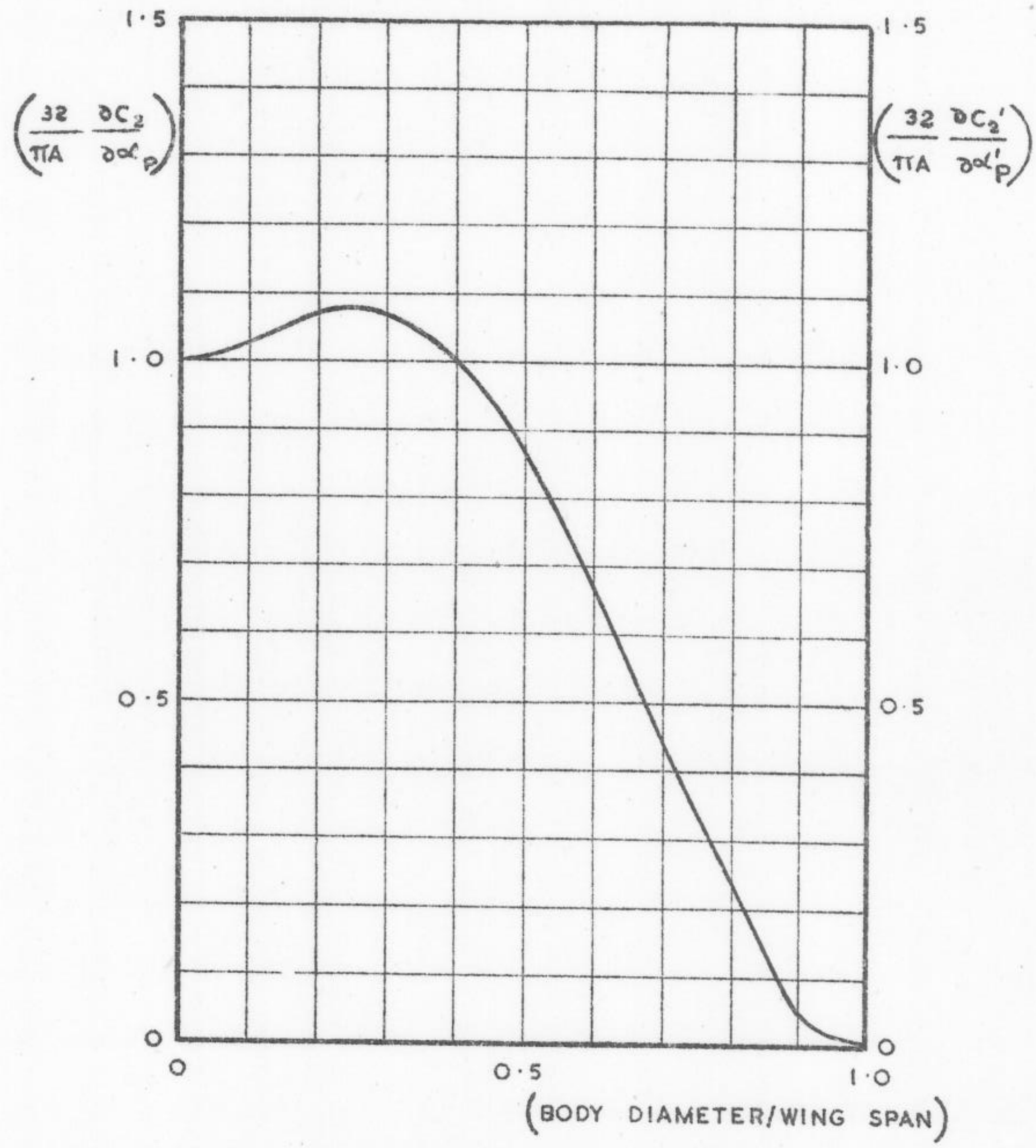


FIG. 11 VARIATION OF ROLLING MOMENT DUE TO ROLL WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY

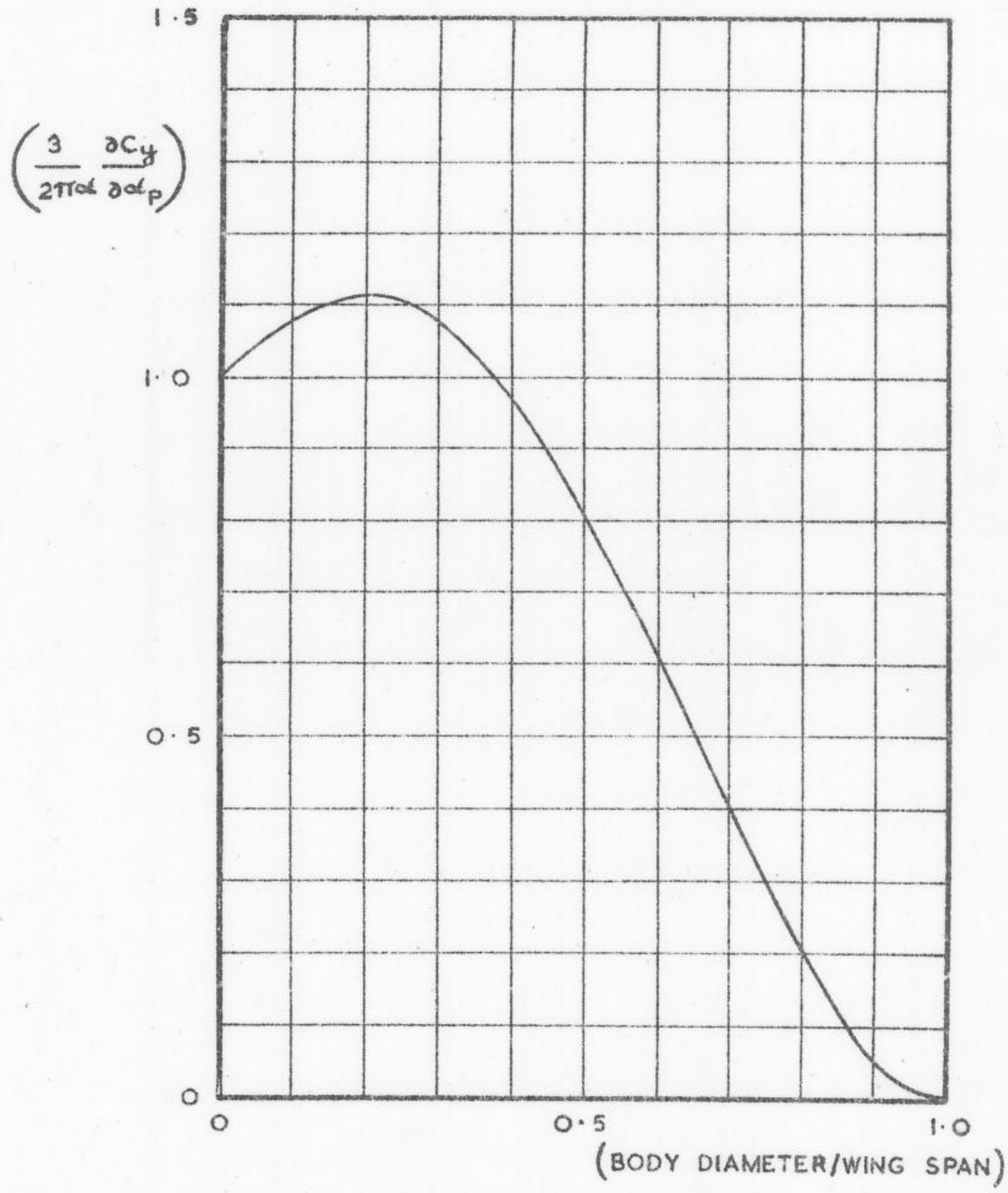


FIG. 12 VARIATION OF SIDE FORCE DUE TO ROLL WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY

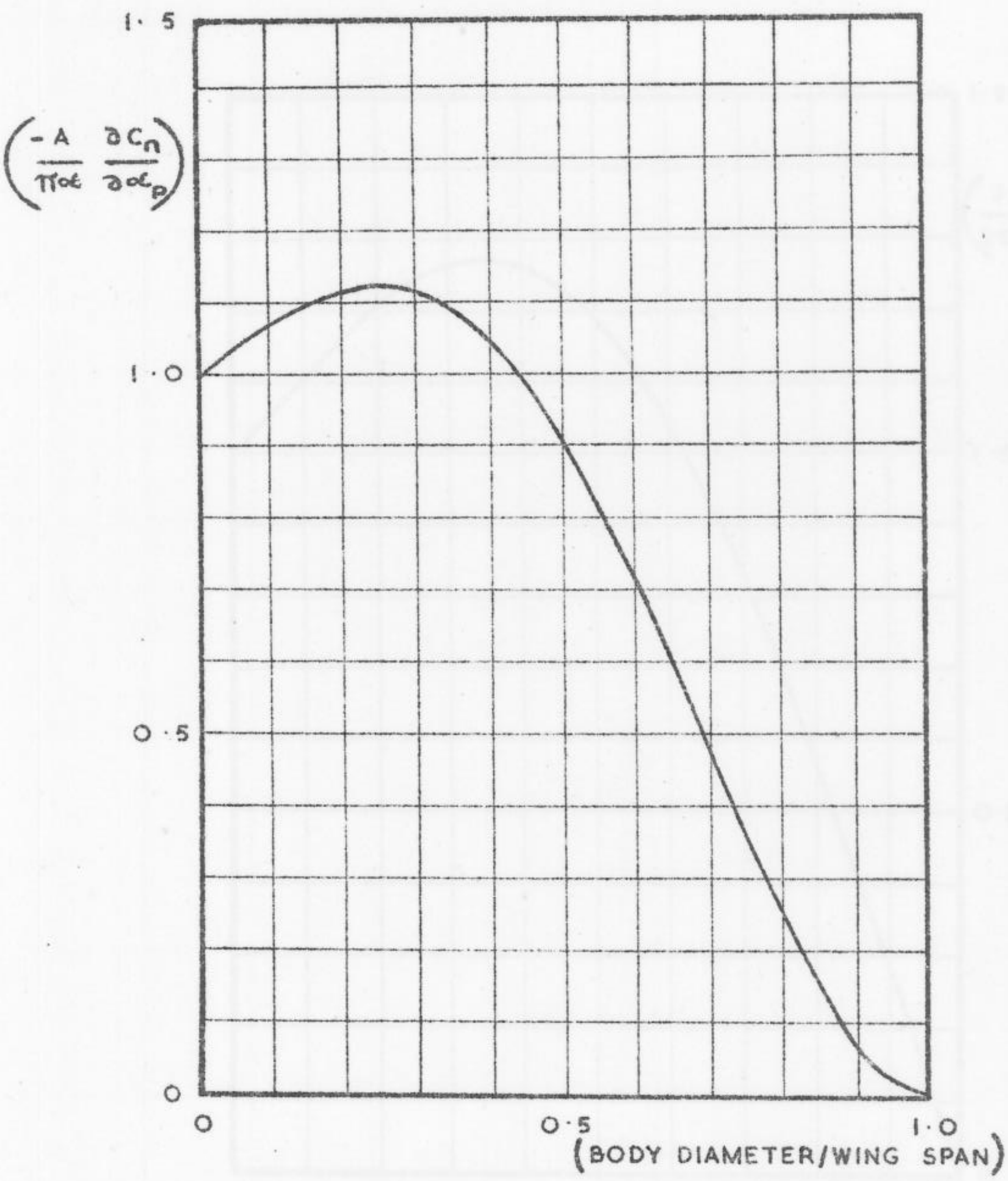


FIG. 13 VARIATION OF YAWING MOMENT DUE TO ROLL WITH BODY DIAMETER FOR A DELTA WING ON A CYLINDRICAL BODY (ORIGIN AT WING APEX ON CENTRE LINE)

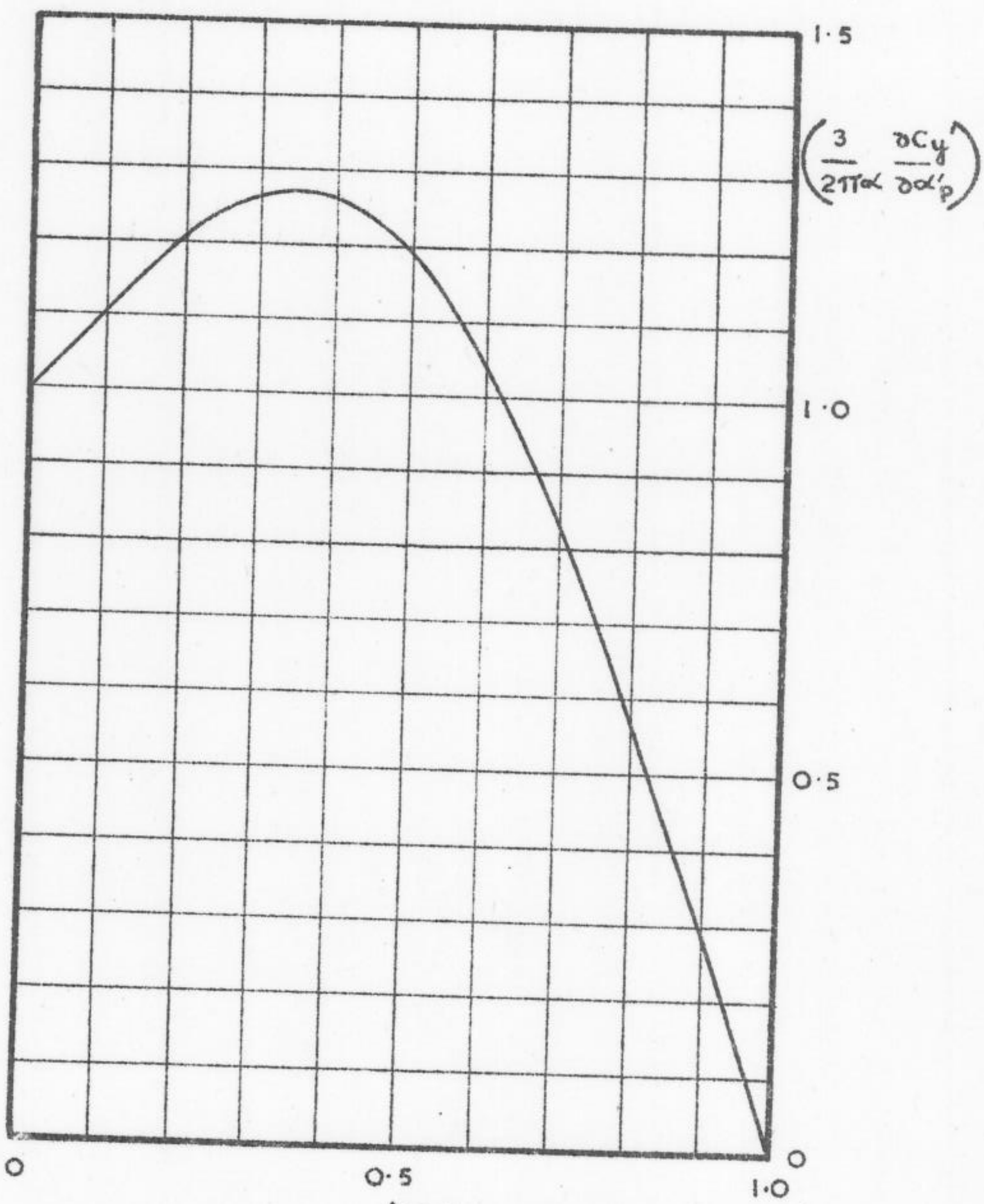


FIG. 14 VARIATION OF SIDE FORCE DUE TO ROLL WITH BODY DIAMETER FOR DELTA WING OR CYLINDRICAL BODY (WIND AXES)

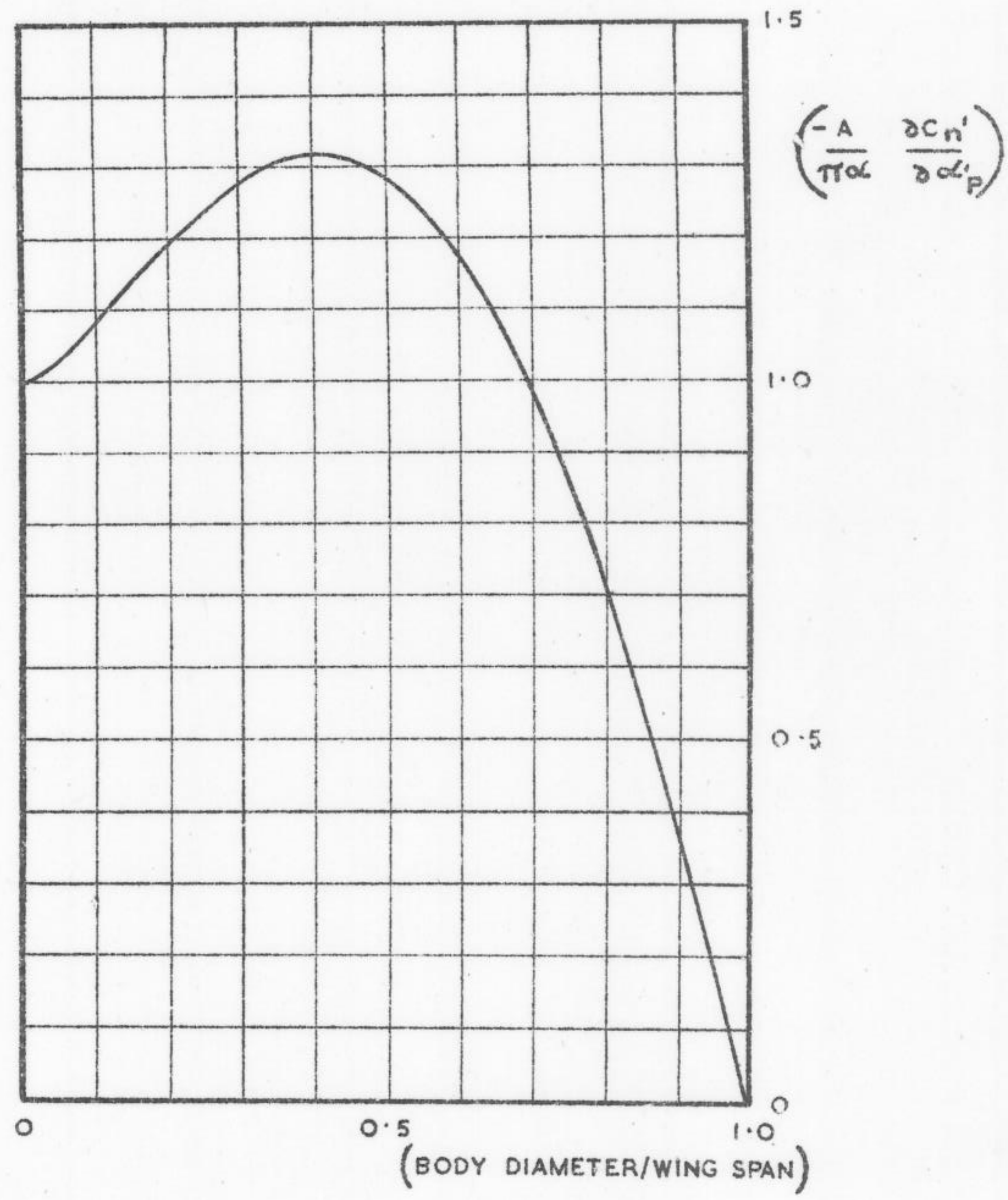


FIG. 15 VARIATION OF YAWING MOMENT DUE TO ROLL WITH BODY DIAMETER FOR A DELTA WING ON A CYLINDRICAL BODY (WIND AXES WITH ORIGIN AT WING APEX)

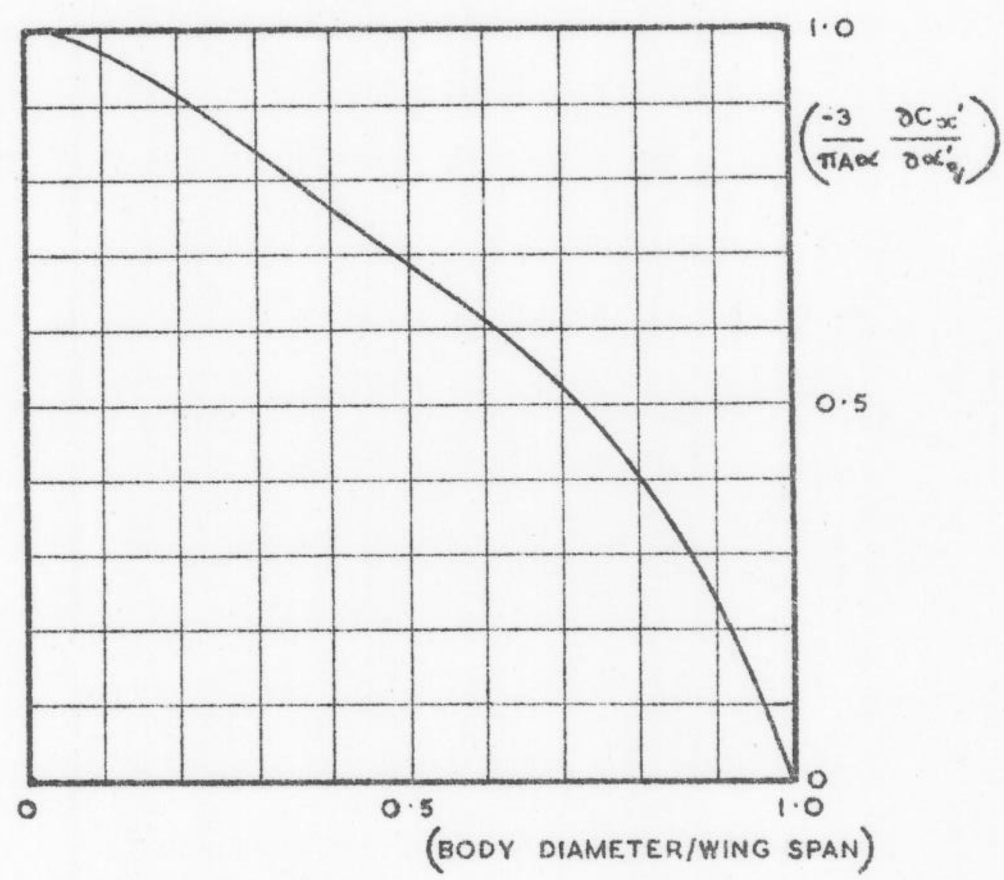


FIG. 16. VARIATION OF LONGITUDINAL FORCE DUE TO PITCH WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY (WIND AXES WITH ORIGIN AT WING APEX.)

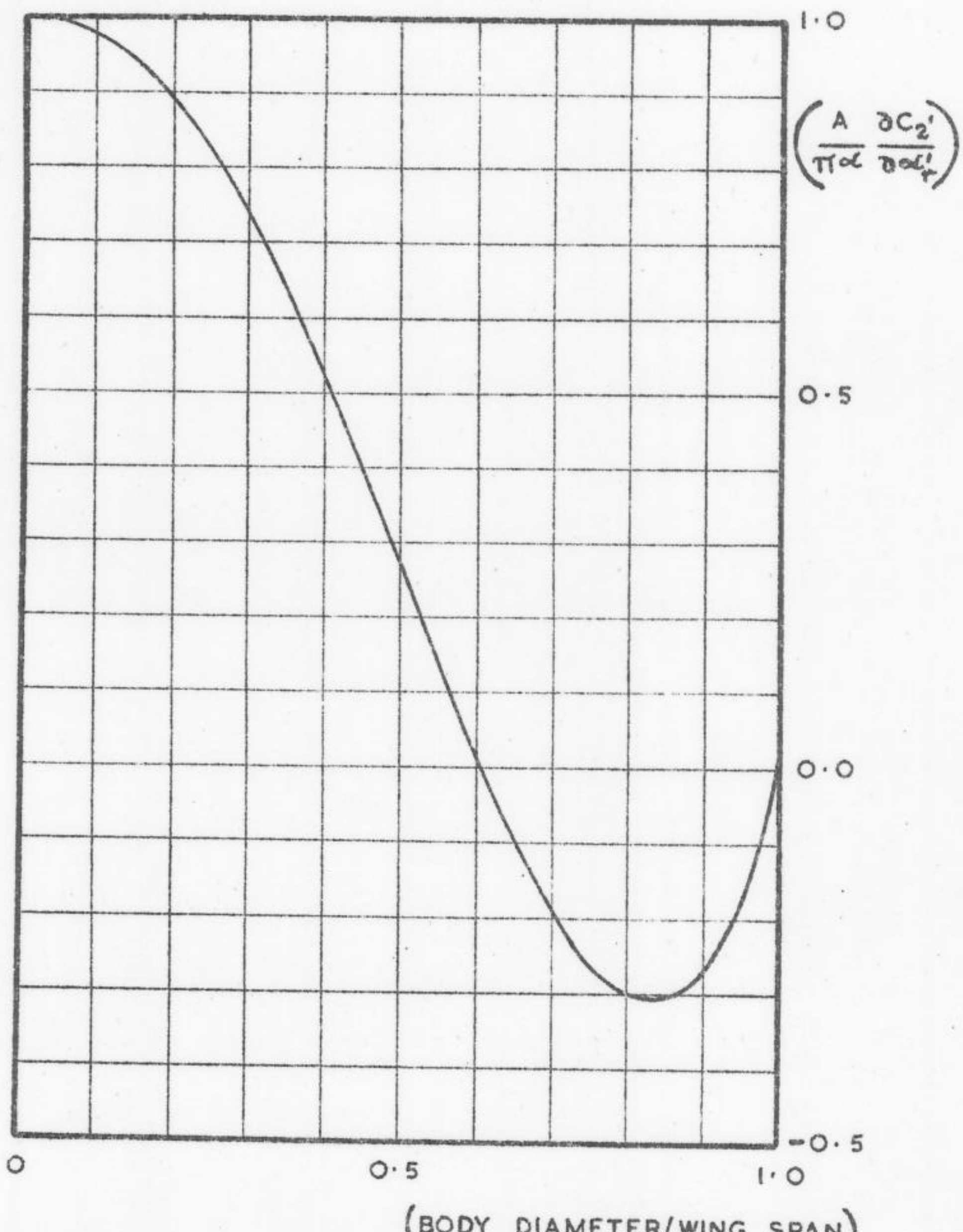


FIG. 17. VARIATION OF ROLLING MOMENT DUE TO YAW WITH BODY DIAMETER FOR DELTA WING ON A CYLINDRICAL BODY (WIND AXES WITH ORIGIN AT WING APEX)