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Aerofoil theory for swallow tail wings of small aspect ratio

-by-

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SUMMARY

A method is developed for the calculation of the aerodynamic forces acting on a 'swallow tail' wing of small aspect ratio. Lift, induced drag, and aerodynamic centre position of simple swallow tail wings (Fig.1(b)) are computed as an application. For a given incidence, lift and induced drag are, within the limits of the theory, proportional to aspect ratio and independent of speed. The chordwise lift distribution rises linearly from zero at the apex, drops rapidly in the region of the root chord trailing edge, and then decreases gently to zero.

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MEP

1. Introduction and discussion of results.

In a remarkable paper published some years ago (ref.1), R.T. Jones put forward a theory for low aspect ratio pointed wings at all speeds, both below and above the speed of sound. The theory was based on the idea, due to Munk (ref.2), that the induced velocity field set up by a slender moving body in a transonic plane, is essentially two-dimensional. In spite of the simplification involved in this idea, the results of ref.1 were borne out in a striking manner, both by relatively more exact theories and by experiment. The theory was extended by H.S. Ribner (ref.3) to permit the calculation of the stability derivatives of low aspect ratio wings. It has also been used for the calculation of the effect of controls of different types by J. Deyoung (ref.4) and M.D. Hodges (ref.5). G.N. Ward has applied an equivalent method to problems of wing body interference at supersonic speeds in a paper which also includes a rigorous justification of the basic assumption mentioned above for the case under consideration (ref.6).

In the present paper, we are concerned with the calculation of the aerodynamic forces acting on small aspect ratio 'swallow tail' wings such as depicted in Figs.1(a) and 1(b). It is assumed that the outline of the wing planform varies monotonically from a pointed nose to pointed tips (i.e. $\frac{dy}{dx} < 0$ along ABC in Fig.1(a) and $\frac{dy}{dx} > 0$ along AFE). This case is outside the scope of the methods which are given in the papers mentioned above although for one particular planform, a solution for a mathematically equivalent problem is described in ref.7.

Numerical results have been calculated for the case of a 'simple swallow tail' wing (Fig.1(b)). The results depend on two parameters, (i) the aspect ratio, $(\text{span})^2/\text{area}$, or in fact the ratio of any two typical lateral and longitudinal dimensions, and (ii) the ratio c/c_0, where c_0 is the root chord of the wing, and c is the longitudinal coordinate of the tips, measured from the apex (Fig.2). As in the case of the delta wing (ref.1) it is found that the aerodynamic forces acting on the wing, for given air density, speed, area, incidence, and for given ratio c/c_0 are actually proportional to the aspect ratio so that the results can be represented as functions of the ratio c/c_0 only.

Figs. 4(a) and 4(b) show the chordwise lift distribution, i.e. the pressure difference integrated in spanwise direction for a given chord position, for two values of the parameter c/c_0. It will be seen that there is a discontinuity in the

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chordwise lift distribution (and indeed of the pressure) across the coordinate of the wing root trailing edge. Although such discontinuities may occur also under the conditions of ref.1, they are impossible in a real fluid, but they reflect a rapid variation of the pressure in the region under consideration.

Figs. 5, 6 and 7 show the lift curve slope, the position of the aerodynamic centre, and the induced drag coefficient respectively, as functions of the parameter c/c_o. The numerical values used for their construction are given in Table I below. (In that table, A denotes the aspect ratio, and h denotes the distance of the aerodynamic centre from the wing apex in fractions of the chord c).

TABLE I

c/c_o	$\frac{dC_L}{d\alpha}\Big/A$	h	$A\,\frac{C_{D_i}}{C_L^2}$
1.0	1.571	0.667	0.3173
1.5	1.012	0.555	0.3379
2.0	0.739	0.504	0.3836

2. Analysis

We take the origin at the apex of the wing, with the x-axis parallel to the direction of flow, the y-axis pointing to starboard, and the z-axis pointing upwards, so that the coordinate system is right-handed. Let Φ be the total velocity potential and ϕ the induced velocity potential so that $\Phi = Vx + \phi$, where V is the free stream velocity. We denote the induced velocity components by u, v, w, $u = \frac{\partial\phi}{\partial x}$, $v = \frac{\partial\phi}{\partial y}$, $w = \frac{\partial\phi}{\partial z}$, so that the total longitudinal velocity equals V + u. By Bernoulli's equation, the pressure difference at a point of the aerofoil, Δp, is related to the longitudinal induced velocity by

$$(1) \qquad \Delta p = -2\rho V u = -2\rho V \frac{\partial\phi}{\partial x}$$

where ρ is the air density. In accordance with the introduction, the partial differential equation for ϕ is taken as

$$(2) \qquad \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

The boundary condition at the aerofoil is

$$(3) \qquad w = \frac{\partial\phi}{\partial z} = -V\alpha$$

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where α is the local incidence, positive in the nose-up sense. We shall consider only the case of a flat wing, α = const.

We write $y = \pm a = \pm a(x)$ for the spanwise coordinates of the leading edges, $0 \leq x \leq c$, and $y = \pm b = \pm b(x)$ for the spanwise coordinates of the trailing edges, $c_o \leq x \leq c$, so that $a(o) = b(c_o) = 0$, $a(c) = b(c)$. We also introduce the complex variable $\eta = y + iz$, so that ϕ may be regarded as a function of the two variables x and η, $\phi = \phi(x,\eta)$. ϕ may be written as the real part of a complex function of x and η, which is an analytic function of η.

In addition to satisfying Equation (2) and the boundary condition (3), ϕ must be such that Δp, and hence $\frac{\partial\phi}{\partial x}$, vanish at the trailing edges of the aerofoil (Joukowski-condition), while these quantities may become infinite at the leading edge. Finally, since the aerofoil does not penetrate any transverse plane x = const. <0, it follows that ϕ must be constant, and may be assumed to vanish for such x. All these conditions suggest that ϕ can be represented in the form

$$(4) \qquad \phi(x,\eta) = \begin{cases} 0 & \text{for } x<0 \\ \mathcal{R}\displaystyle\int_o^x \frac{A(t)}{\sqrt{[a(t)]^2 - \eta^2}}\,dt & \text{for } 0<x<c_o \\ \mathcal{R}\left[\displaystyle\int_o^{c_o} \frac{A(t)}{\sqrt{[a(t)]^2 - \eta^2}}\,dt + \int_{c_o}^x A(t)\sqrt{\frac{\eta^2 - [b(t)]^2}{[a(t)]^2 - \eta^2}}\,dt\right] & \text{for } c_o<x<c \end{cases}$$

where $\mathcal{R}$ denotes the real part of a complex number as usual, and A is a real function of its argument, which remains to be determined. In fact, differentiating (4) with respect to x and putting $\eta = y$, we see that at the aerofoil

$$(5) \qquad u = \frac{\partial\phi}{\partial x} = \begin{cases} \dfrac{A(x)}{\sqrt{[a(x)]^2 - y^2}} & \text{for } 0<x<c_o \\ A(x)\sqrt{\dfrac{y^2 - [b(x)]^2}{[a(x)]^2 - y^2}} & \text{for } c_o<x<c \end{cases}$$

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This shows that the pressure difference becomes infinite at the leading edges and vanishes at the trailing edges, in accordance with the Joukowski condition.

To find the normal induced velocity, we have to differentiate (4) with respect to z. And, since for any analytic function $f(\eta)$,

$$\frac{\partial}{\partial z}\mathcal{R} f(\eta) = \mathcal{R}\frac{\partial f(\eta)}{\partial z} = \mathcal{R}\left[i\frac{\partial f(\eta)}{\partial(iz)}\right] = \mathcal{R}\left[i\frac{\partial f(\eta)}{\partial \eta}\right] = -\mathcal{J} f'(\eta),$$

where $\mathcal{J}$ denotes the (real) coefficient of the imaginary part of a complex number, as usual, we obtain

$$(6)\quad \frac{\partial\phi}{\partial z} = \begin{cases} 0 & \text{for } x<0 \\ -\mathcal{J}\left[\frac{\partial}{\partial\eta}\int_0^x \frac{A(t)}{\sqrt{[a(t)]^2-\eta^2}}\,dt\right] & \text{for } 0<x<c_0 \\ -\mathcal{J}\left[\frac{\partial}{\partial\eta}\int_0^{c_0} \frac{A(t)}{\sqrt{[a(t)]^2-\eta^2}}\,dt + \frac{\partial}{\partial\eta}\int_{c_0}^x A(t)\sqrt{\frac{\eta^2-[b(t)]^2}{[a(t)]^2-\eta^2}}\,dt\right] & \text{for } c_0<x<c \end{cases}$$

We now have to determine $A(t)$ from the condition that

$$(7)\quad \left.\begin{aligned} &\lim_{\eta\to y}\mathcal{J}\frac{\partial}{\partial\eta}\int_0^x \frac{A(t)}{\sqrt{[a(t)]^2-\eta^2}}\,dt \\ &\lim_{\eta\to y}\mathcal{J}\left[\frac{\partial}{\partial\eta}\int_0^{c_0} \frac{A(t)}{\sqrt{[a(t)]^2-\eta^2}}\,dt + \frac{\partial}{\partial\eta}\int_{c_0}^x A(t)\sqrt{\frac{\eta^2-[b(t)]^2}{[a(t)]^2-\eta^2}}\,dt\right] \end{aligned}\right\} = V\alpha \qquad \begin{cases}\text{for } 0<x<c_0 \\ \text{for } c_0<x<c\end{cases}$$

for any point $(x,y,0)$ on the aerofoil. We notice that

$$\lim_{\eta\to y}\mathcal{J}\frac{\partial}{\partial\eta}\frac{1}{\sqrt{[a(x)]^2-\eta^2}} = \lim_{\eta\to y}\mathcal{J}\frac{\partial}{\partial\eta}\sqrt{\frac{\eta^2-[b(x)]^2}{[a(x)]^2-\eta^2}} = 0$$

at all points $(x, y, 0)$ on the aerofoil, so that by (6) the

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normal induced velocity is constant along the chords of the aerofoil for <u>any</u> choice of $A(t)$. Its magnitude is determined entirely by the values of the integrals in (6) ahead of, and in the region of, the leading edge.

It is easy to show that the first equation in (7) is satisfied by

$$(8) \qquad A(t) = - V\alpha \; a(t) \; a'(t) \; , \qquad 0 < t < c_0$$

In fact

$$\int_0^x \left[- V\alpha \; a(t) \; a'(t)\right] \frac{dt}{\sqrt{[a(t)]^2 - \eta^2}} = - V\alpha \left[\sqrt{[a(t)]^2 - \eta^2}\right]_{t=0}^{t=x} = V\alpha \left[i\eta - \sqrt{[a(x)]^2 - \eta^2}\right]$$

and so, for points $(x,y,0)$ on the aerofoil,

$$\lim_{\eta \to y} \mathcal{I} \frac{\partial}{\partial \eta} \int_0^x \left[- V\alpha \; a(t) \; a'(t)\right] \frac{dt}{\sqrt{[a(t)]^2 - \eta^2}} = V\alpha \, \mathcal{I} \left[i - \frac{y}{\sqrt{[a(x)]^2 - y^2}}\right] = V\alpha,$$

as required.

It will be seen that the solution implied by (8) agrees with ref.1. for the particular case considered there.

The left hand side of the second equation in (7) now becomes, with the specified value of $A(t)$ for $0 < t < c_0$,

$$\lim_{\eta \to y} \mathcal{I} \left[\frac{\partial}{\partial \eta} \int_0^x \left[- V\alpha \; a(t) \; a'(t)\right] \frac{dt}{\sqrt{[a(t)]^2 - \eta^2}} + \frac{\partial}{\partial \eta} \int_{c_0}^x A(t) \sqrt{\frac{\eta^2 - [b(t)]^2}{[a(t)]^2 - \eta^2}} \, dt\right]$$

$$= \lim_{\eta \to y} V\alpha \, \mathcal{I} \left[i - \frac{\eta}{\sqrt{[a(c_0)]^2 - \eta^2}}\right] + \lim_{\eta \to y} \mathcal{I} \left[\frac{\partial}{\partial \eta} \int_{c_0}^x A(t) \sqrt{\frac{\eta^2 - [b(t)]^2}{[a(t)]^2 - \eta^2}} \, dt\right]$$

$$= V\alpha + \frac{V\alpha \, y}{\sqrt{y^2 - [a(c_0)]^2}} + \lim_{\eta \to y} \mathcal{I} \left[\frac{\partial}{\partial \eta} \int_{c_0}^x A(t) \sqrt{\frac{\eta^2 - [b(t)]^2}{[a(t)]^2 - \eta^2}} \, dt\right]$$

where we confine ourselves to points of the aerofoil for which $y > a(c_0)$. The condition for $A(t)$, $c_0 < t < c$, now becomes

$$(9) \qquad \lim_{\eta \to y} \left[\mathcal{I} \frac{\partial}{\partial \eta} \int_{c_0}^x A(t) \sqrt{\frac{\eta^2 - [b(t)]^2}{[a(t)]^2 - \eta^2}} \, dt\right] = - V\alpha \frac{y}{\sqrt{y^2 - [a(c_0)]^2}}$$

for points $(x,y,0)$ of the aerofoil such that $x > c_0$, $y > a(c_0)$.

We may modify the left-hand side of (9) slightly by replacing $\frac{\partial}{\partial\eta}$ by $\frac{\partial}{\partial y}$, since $\eta = y + iz$. Integrating with respect to y on both sides of (9) we then obtain an alternative condition for $A(t)$, viz.

$$(10)\quad \lim_{\eta\to y} \mathcal{I}\int_{c_o}^{x} A(t)\sqrt{\frac{\eta^2 - [b(t)]^2}{[a(t)]^2 - \eta^2}}\,dt = -V\alpha\sqrt{y^2 - [a(c_o)]^2} + \text{const.}$$

In fact, a solution of (10) for any value of the constant on the right hand side would lead to a solution of (9), but the consideration of the limiting case $x\to c_o$, $y\to a(c_o)$, shows that we must take 0 as the appropriate value of the constant. We may now let η tend to y (i.e. let z tend to 0) on the left hand side of (10) before integration, and then let $(x,y,0)$ tend to the leading edge so that x and y are linked by the relation $y = a(x)$. Then

$$\mathcal{I}\int_{c_o}^{x} A(t)\sqrt{\frac{y^2 - [b(t)]^2}{[a(t)]^2 - y^2}}\,dt = \mathcal{I}\int_{c_o}^{x} A(t)\sqrt{\frac{[a(x)]^2 - [b(t)]^2}{[a(t)]^2 - [a(x)]^2}}\,dt$$

$$= -\int_{c_o}^{x} A(t)\sqrt{\frac{[a(x)]^2 - [b(t)]^2}{[a(x)]^2 - [a(t)]^2}}\,dt$$

so that (10) becomes

$$(11)\quad \int_{c_o}^{x} A(t)\sqrt{\frac{[a(x)]^2 - [b(t)]^2}{[a(x)]^2 - [a(t)]^2}}\,dt = V\alpha\sqrt{[a(x)]^2 - a_o^2}$$

where $a_o = a(c_o)$. This is an integral equation of Volterra's type. The following simple numerical method for its solution avoids any difficulty which might be caused by the fact that the integral on the left hand side of (11) becomes infinite at the upper limit of the integral.

For any x, x', x'', $c_o \leq x' < x'' \leq x \leq c$, we have, by a mean value theorem of the integral calculus

$$(12)\quad \int_{x'}^{x''} A(t)\sqrt{\frac{[a(x)]^2 - [b(t)]^2}{[a(x)]^2 - [a(t)]^2}}\,dt = \int_{x'}^{x''} \frac{A(t)}{a'(t)}\sqrt{[a(x)]^2 - [b(t)]^2}\,\frac{a'(t)\,dt}{\sqrt{[a(x)]^2 - [a(t)]^2}}$$

$$= \frac{A(\xi)}{a'(\xi)}\sqrt{[a(x)]^2 - [b(\xi)]^2}\int_{x'}^{x''}\frac{a'(t)\,dt}{\sqrt{[a(x)]^2 - [a(t)]^2}} = \frac{A(\xi)}{a'(\xi)}\sqrt{[a(x)]^2 - [b(\xi)]^2}\left[\sin^{-1}\frac{a(x'')}{a(x)} - \sin^{-1}\frac{a(x')}{a(x)}\right]$$

where ξ is some intermediate value between x' and x''. We shall accept the approximation that ξ is midway between x' and x'', $\xi = \frac{1}{2}(x' + x'')$.

To solve (11), we divide the interval $\langle c_o, c\rangle$ into m equal sub-intervals, $\langle c_o = x_o, x_1\rangle$, $\langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m = c\rangle$. We then satisfy (11) for $x = x_1, x = x_2, \ldots, x = x_m$, evaluating the integral on the left hand side approximately by applying (12) to all the sub-intervals in $\langle c_o, x\rangle$. In this way we obtain the following triangular system of m linear equations for the unknowns $A_k = A(\xi_k)$, $\xi_k = \frac{1}{2}(x_k + x_{k-1})$, $k = 1,2,\ldots,m$.

$$(13) \quad \sum_{k=1}^{n} \frac{1}{a'(\xi_k)} \left\{ \sqrt{[a(x_n)]^2 - [b(\xi_k)]^2} \left[\sin^{-1} \frac{a(x_k)}{a(x_n)} - \sin^{-1} \frac{a(x_{k-1})}{a(x_n)} \right] \right\} A_k = V\alpha \sqrt{[a(x_n)]^2 - a_o^2}, \quad n = 1,2,\ldots,m$$

For a single swallow-tail wing (Fig.1(b)), $a'(x) = \text{const.} = \frac{a_o}{c_o} = \tan\gamma$, where γ is the semi apex angle of the wing. Hence, putting $B(t) = \frac{A(t)}{V\alpha \tan\gamma}$, $B_k = A_k/V\alpha \tan\gamma$, we obtain the system of equations

$$(14) \quad \sum_{k=1}^{n} \left\{ \sqrt{[a(x_n)]^2 - [b(\xi_x)]^2} \left[\sin^{-1} \frac{a(x_k)}{a(x_n)} - \sin^{-1} \frac{a(x_{k-1})}{a(x_n)} \right] \right\} B_k = \sqrt{[a(x_n)]^2 - a_o^2}$$

$$n = 1,2,\ldots,m$$

For the calculations on which Figs. 4-6 are based, $\langle c_o, c\rangle$ was divided into five sub-intervals, $m = 5$. To obtain an idea of the accuracy of the solution, similar calculations were made for $m = 3$, and $m = 4$, for a value of $\frac{c}{c_o} = 2$. The results are shown in Fig.3, from which it appears that there is good agreement between the quantities obtained for $m = 4$ and $m = 5$, while the calculations based on $m = 3$ are inadequate.

There are other ways of applying the above mean value theorem. For example, instead of (12), we might have used the following formula -

$$(15) \quad \int_{x_1}^{x''} A(t) \sqrt{\frac{[a(x)]^2 - [b(t)]^2}{[a(x)]^2 - [a(t)]^2}}\, dt = \int_{x'}^{x''} \frac{A(t)}{a(t)a'(t)} \sqrt{[a(x)]^2 - [b(t)]^2}\, \frac{a(t)a'(t)\,dt}{[a(x)]^2 - [a(t)]^2}$$

$$= \frac{A(\xi)}{a(\xi)a'(\xi)} \sqrt{[a(x)]^2 - [b(\xi)]^2} \int_{x'}^{x''} \frac{a(t)a'(t)\,dt}{\sqrt{[a(x)]^2 - [a(t)]^2}}$$

$$= \frac{A(\xi)}{a(\xi)a'(\xi)} \sqrt{[a(x)]^2 - [b(\xi)]^2} \left[\sqrt{[a(x)]^2 - [a(x')]^2} - \sqrt{[a(x)]^2 - [a(x'')]^2} \right]$$

where ξ is again some intermediate value between x' and x'', but of course not necessarily the same as before. (15) can readily be used to find the exact value of $A(c_o + 0)$, i.e. the limiting value of $A(t)$ as t tends to c_o from above. Putting $x' = c_o$ and $x'' = x$ in (15) and substituting the result on the left hand side of (11), we obtain

$$\frac{A(\xi)}{a(\xi)a'(\xi)}\sqrt{[a(x)]^2-[b(\xi)]^2}\sqrt{[a(x)]^2-[a(c_o)]^2} = V\alpha\sqrt{[a(x)]^2-a_o^2}$$

But $a(c_o) = a_o$ and so

$$A(\xi) = V\alpha \frac{a(\xi)a'(\xi)}{\sqrt{[a(x)]^2-[b(\xi)]^2}}$$

Now let x tend to c_o from above. Then ξ tends to c_o also, and so in the limit

(16) $\quad A(c_o + 0) = V\alpha\, a'(c_o)$

The corresponding value for $B(c_o + 0)$ is

(17) $\quad B(c_o + 0) = 1$

Still considering the simple swallow tail wing let $\lambda = \frac{c-c_o}{c_o}$.
It is of interest to determine the limiting form of the integral equation (11) as λ tends to 0. We introduce the non-dimensional variable s by $x = (1 + \lambda s)c_o$.
Then

$$a(x) = a(c_o)\frac{x}{c_o} = a_o(1+\lambda s), \quad b(x) = a_o(1+\lambda)s$$

so that (11) becomes

(18) $$\lambda\int_0^s A^*(\sigma)\sqrt{\frac{(1+\lambda s)^2-(1+\lambda)^2\sigma^2}{(1+\lambda s)^2-(1+\lambda\sigma)^2}}\,d\sigma = V\alpha\tan\gamma\sqrt{(1+\lambda s)^2-1}$$

where we have put $t = (1+\lambda\sigma)c_o$, $A^*(\sigma) = A(t) = A\big((1+\lambda\sigma)c_o\big)$.

For small λ, equation (18) tends to the form

(19) $$\int_0^s A^*(\sigma)\sqrt{\frac{1-\sigma^2}{s-\sigma}}\,d\sigma = 2V\alpha\tan\gamma\sqrt{s},$$

and we may verify directly that this equation is satisfied by

(20) $$A^*(\sigma) = \frac{V\alpha\tan\gamma}{\sqrt{1-\sigma^2}}$$

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On the other hand, for $\lambda \to \infty$, equation (18) becomes

$$(21) \qquad \int_0^s A(\sigma)\,d\sigma = V\alpha \tan\gamma \; s$$

which is solved by

$$(22) \qquad A(\sigma) = V\alpha \tan\gamma$$

3. Calculation of aerodynamic forces

Having computed the function $A(x)$, we are in a position to determine the aerodynamic forces which act on the wing. By (1) and (5), the pressure difference at a point (x, y, o) of the aerofoil is given by

$$(23) \qquad \Delta p = \begin{cases} -2\rho V \dfrac{A(x)}{\sqrt{[a(x)]^2 - y^2}} & \text{for } 0 < x < c_o \\[2ex] -2\rho V A(x) \sqrt{\dfrac{y^2 - [b(x)]^2}{[a(x)]^2 - y^2}} & \text{for } c_o < x < c. \end{cases}$$

The lift per unit chord is given by

$$(24) \qquad \ell(x) = \int_{-a(x)}^{a(x)} \Delta p \, dy \quad ,$$

for $0 < x < c_o$. Thus, in that case

$$\ell(x) = -2\rho V \int_{-a(x)}^{a(x)} \frac{A(x)\,dx}{\sqrt{[a(x)]^2 - y^2}} = -2\rho V A(x) \left[\sin^{-1} \frac{y}{a(x)}\right]_{-a(x)}^{a(x)} = -2\pi\rho V A(x)$$

Or, taking into account (8),

$$(25) \qquad \ell(x) = 2\pi\rho V^2 \alpha \, a(x) \, a'(x) \quad , \qquad 0 < x < c_o \quad .$$

On the other hand, for $c_o < x < c$,

$$\ell(x) = \int_{-a(x)}^{-b(x)} \Delta p \, dy + \int_{b(x)}^{a(x)} \Delta p \, dy = 2 \int_{b(x)}^{a(x)} \Delta p \, dy = -4\rho V A(x) \int_{b(x)}^{a(x)} \sqrt{\frac{y^2 - [b(x)]^2}{[a(x)]^2 - y^2}} \, dy$$

Let $k' = \dfrac{b(x)}{a(x)}$, $k = \sqrt{1 - k'^2}$, and introduce the variable β by

$$y = a(x) \, \mathrm{dn}\,(\beta, k)$$

/where ...

where $dn(\beta,k)$ is the familiar Jacobian elliptic function. Then

$$y^2-[b(x)]^2 = a(x)\ k^2\ cn^2(\beta,k)\ ,\quad [a(x)]^2-y^2 = a(x)\ k^2\ sn^2(\beta,k)$$

and $dy = -\ a(x)\ k^2\ sn(\beta,k)\ cn(\beta,k)$

Hence

$$\int_{b(x)}^{a(x)} \sqrt{\frac{y^2-[b(x)]^2}{[a(x)]^2-y^2}}\ dy = -\ a(x)k^2 \int_0^K cn^2(\beta,k)\ d\beta = -\ a(x)\ [E(k) - k'^2 K(k)]$$

$$= -\ a(x) \left[E\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right) - \left(\frac{b}{a}\right)^2 K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)\right],$$

so that the expression for $\ell(x)$ becomes

(26) $$\ell(x) = 4\rho VA(x)\ a(x) \left[E\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right) - \left(\frac{b}{a}\right)^2 K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)\right]$$

For the case of a simple swallow tail wing we have $A(x) = V\alpha \tan\gamma\ B(x)$, and so

(27) $$\ell(x) = 4\rho V^2\alpha\ a(x)\ \tan\gamma\ B(x) \left[E\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right) - \left(\frac{b}{a}\right)^2 K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)\right]$$

For $x = \xi_k$, we take B to be given by $B(\xi_k) = B_k$, as determined from (14). Thus

(28) $$\ell(\xi_k) = 4\rho V^2\alpha\ a(x)\ \tan\gamma\ B_k \left[E\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right) - \left(\frac{b}{a}\right)^2 K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)\right]$$

The total lift, L, is given by

(29) $$L = \int_0^c \ell(x)\,dx = L_f + L_r, \text{ say, where}$$

$$L_f = \int_0^{c_0} \ell(x)\,dx\ ,\qquad L_r = \int_{c_0}^{c} \ell(x)\,dx$$

Integrating (25), we obtain immediately

(30) $$L_f = \pi\rho V^2\alpha\ [a(c_0)]^2 = \pi\rho V^2\ a_0^2\alpha$$

in agreement with ref. 1. On the other hand, L_f must be obtained by numerical integration, and in view of the preceeding analysis the use of the following simple formula seems appropriate.

$$(31) \qquad L_r = \frac{c-c_o}{m} \sum_{k=1}^{m} \ell(\xi_k) = \frac{4\rho V^2 \alpha \tan\gamma\ (c-c_o)}{m} \sum_{k=1}^{m} a(\xi_k)\, B_k \left\{ E\left(\sqrt{1 - \left[\frac{b(\xi_k)}{a(\xi_k)}\right]^2}\right) - \left[\frac{b(\xi_k)}{a(\xi_k)}\right]^2 K\left(\sqrt{1 - \left[\frac{b(\xi_k)}{a(\xi_k)}\right]^2}\right) \right\}$$

The pitching moment round the apex, M, is given by

$$(32) \qquad M = \int_o^e x\, \ell(x)\, dx = M_f + M_r$$

$$M_f = \int_o^{e_o} x\, \ell(x)\, dx \qquad M_r = \int_{c_o}^{c} x\, \ell(x)\, dx$$

Thus, for the case of a simple swallow tail wing,

$$(33) \qquad M_f = \int_o^{c_o} x.\ 2\pi\rho V^2 \alpha \tan^2\gamma\ x\, dx = \frac{2\pi}{3}\, \rho V^2 \alpha \tan^2\gamma$$

while a numerical formula for M_r is

$$(34) \qquad M_r = \frac{c-c_o}{m} \sum_{k=1}^{m} \xi_k\, \ell(\xi_k)$$

$$= \frac{4\rho V^2 \alpha \tan\gamma\ (c-c_o)}{m} \sum_{k=1}^{m} \xi_k\, a(\xi_k)\, B_k \left\{ E\left(\sqrt{1-\left[\frac{b(\xi_k)}{a(\xi_k)}\right]^2}\right) - \left[\frac{b(\xi_k)}{a(\xi_k)}\right]^2 K\left(\sqrt{1-\left[\frac{b(\xi_k)}{a(\xi_k)}\right]^2}\right) \right\}$$

The distance of the aerodynamic centre from the apex of the aerofoil is then given by $d = M/L$.

Finally we establish a formula for the induced drag of the aerofoil, D_i. D_i is the difference between the surface pressure drag $D_p = L\alpha$ and the forward suction force D_s exerted on the leading edges of the wing.

$$(35) \qquad D_i = D_p - D_s = L\alpha - D_s$$

To calculate D_s we surround the leading edges of the wing by small cylindrical surfaces S as given by the equation

$$(36) \qquad \underline{r} = \underline{r}\,(\xi, \theta) = \xi\, \underline{i} + (\pm\, a(\xi) + \varepsilon \cos\theta)\, \underline{j} + \varepsilon \sin\theta)\ \underline{k}$$

/where ...

$$+ w \sin \theta)\, \varepsilon \, d\theta \, d\xi$$

To continue, we require formulae which express the infinitesimal behaviour of the velocity components in the neighbourhood of the leading edges. Confining ourselves to starboard we see that equation (4) yields, for fixed $x = \xi$ and small ε,

$$u = \mathcal{R} \frac{A(\xi)}{\sqrt{[a(\xi)]^2 - y^2}} = \mathcal{R} \frac{A(\xi)}{\sqrt{[a(\xi)]^2 - [a(\xi) + \varepsilon \exp (i\theta)]^2}} = \mathcal{R} \frac{A(\xi)}{\sqrt{-2a(\xi)\varepsilon \exp (i\theta)}} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

or

(40) $$u = - \frac{A(\xi)}{\sqrt{2a(\xi)}} \frac{1}{\sqrt{\varepsilon}} \sin \frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{ for } 0 < \xi < c_o$$

and similarly

(41) $$u = - \frac{A(\xi)}{\sqrt{2a(\xi)}} \sqrt{[a(\xi)]^2 - [b(\xi)]^2} \frac{1}{\sqrt{\varepsilon}} \sin \frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{ for } c_o < \xi < c$$

Again, the unbounded components of v and w

/depend ...

where ε is a small positive quantity, and the limits of variation of ξ and θ are $0 \leq \xi \leq c$, $0 \leq \theta \leq 2\pi$. By the momentum theorem the force exerted by the air on the portions of the wing which are inside S, $\underline{F}(\varepsilon)$ say, is given by

$$(37) \qquad \underline{F}(\varepsilon) = -\int_S p\, \underline{dS} - \rho \int_S \underline{q}\, (\underline{q}\, \underline{dS})$$

where p is the pressure, $\underline{q}$ is the velocity vector, $\underline{q} = (V+u)\underline{i} + v\underline{j} + w\,\underline{k}$, and $\underline{dS}$ is the directed surface element pointing outwards from the surface. Thus $\underline{dS} = (\underline{r}_\theta \wedge \underline{r}_\xi)\, d\theta\, d\xi$, so that (37) may be replaced by

$$(38) \quad \underline{F}(\varepsilon) = -\int_S p(\underline{r}_\theta \wedge \underline{r}_\xi)\, d\theta\, d\xi - \rho \int_S \underline{q}\Big(\underline{q}\ (\underline{r}_\theta \wedge \underline{r}_\xi)\Big) d\theta\, d\xi$$

where + or − are to be taken in the expression for $\underline{r}_\xi$, as given by (36), on starboard and port respectively. We have $\underline{r}_\theta \wedge \underline{r}_\xi = (-\varepsilon \sin\theta \underline{j} + \varepsilon \cos\theta\, \underline{k}) \wedge (\underline{i} \pm a'(\xi)\underline{j}) = \pm a'(\xi)\varepsilon \cos\theta \underline{i} + \varepsilon \sin\theta\, \underline{k}$

Thus, we obtain for the longitudinal component $D(\varepsilon)$ of $\underline{F}(\varepsilon)$, in which we are chiefly interested,

$$(39)\ D(\varepsilon) = \mp \int_S p\, a'(\xi) \cos\theta\, \varepsilon\, d\theta\, d\xi - \rho \int_S u(\pm u a'(\xi) \cos\theta + v \cos\theta + w \sin\theta)\, \varepsilon\, d\theta\, d\xi$$

To continue, we require formulae which express the infinitesimal behaviour of the velocity components in the neighbourhood of the leading edges. Confining ourselves to starboard we see that equation (4) yields, for fixed $x = \xi$ and small ε,

$$u = \mathcal{R} \frac{A(\xi)}{\sqrt{[a(\xi)]^2 - y^2}} = \mathcal{R} \frac{A(\xi)}{\sqrt{[a(\xi)]^2 - [a(\xi) + \varepsilon \exp(i\theta)]^2}} = \mathcal{R} \frac{A(\xi)}{\sqrt{-2a(\xi)\varepsilon \exp(i\theta)}} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

or

$$(40) \qquad u = -\frac{A(\xi)}{\sqrt{2a(\xi)}} \frac{1}{\sqrt{\varepsilon}} \sin\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{ for } 0 < \xi < c_o$$

and similarly

$$(41) \qquad u = -\frac{A(\xi)}{\sqrt{2a(\xi)}} \sqrt{[a(\xi)]^2 - [b(\xi)]^2}\ \frac{1}{\sqrt{\varepsilon}} \sin\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{for } c_o < \xi < c$$

Again, the unbounded components of v and w

/depend ...

depend only on the value of the integrands near the upper limit $x = \xi$ of the integrals in (4). Thus, we may replace these integrals by

$$\mathcal{R}\int^{\xi} \frac{A(\xi)}{a(\xi)a'(\xi)} \frac{a(t)a'(t)}{\sqrt{[a(t)]^2-\eta^2}}\,dt \quad \text{and} \quad \mathcal{R}\int^{\xi} \frac{A(\xi)\sqrt{[a(\xi)]^2-[b(\xi)]^2}}{a(\xi)\,a'(\xi)} \frac{a(t)a'(t)}{\sqrt{[a(t)]^2-\eta^2}}\,dt$$

for $0<\xi<c_o$ and $c_o<\xi<c$, respectively. Integrating with respect to t, and then differentiating with respect to η, we obtain the following expressions for v and w

$$(42)\quad v = -\frac{A(\xi)}{a(\xi)a'(\xi)}\,\mathcal{R}\frac{\eta}{\sqrt{[a(\xi)]^2-\eta^2}} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) = \frac{A(\xi)}{a'(\xi)\sqrt{2a(\xi)}}\frac{1}{\sqrt{\varepsilon}}\sin\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

for $0<\xi<c_o$, since $\eta = a(\xi) + \varepsilon e^{i\theta}$, as before. For the same case,

$$(43)\quad w = \frac{A(\xi)}{a(\xi)a'(\xi)}\,\mathcal{J}\frac{\eta}{\sqrt{[a(\xi)]^2-\eta^2}} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) = -\frac{A(\xi)}{a'(\xi)\sqrt{2a(\xi)}}\frac{1}{\sqrt{\varepsilon}}\cos\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

Similarly, for $c_o<\xi<c$

$$(44)\quad v = \frac{A(\xi)}{a'(\xi)\sqrt{2a(\xi)}}\sqrt{[a(\xi)]^2-[b(\xi)]^2}\,\frac{1}{\sqrt{\varepsilon}}\sin\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

$$w = -\frac{A(\xi)}{a'(\xi)\sqrt{2a(\xi)}}\sqrt{[a(\xi)]^2-[b(\xi)]^2}\,\frac{1}{\sqrt{\varepsilon}}\cos\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

We may summarise (40) - (44) in the following formulae

$$(45)\quad u = -G(\xi)\frac{1}{\sqrt{\varepsilon}}\sin\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right) \qquad v = \frac{G(\xi)}{a'(\xi)}\frac{1}{\sqrt{\varepsilon}}\sin\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

$$w = -\frac{G(\xi)}{a'(\xi)}\frac{1}{\sqrt{\varepsilon}}\cos\frac{\theta}{2} + o\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

where

$$(46)\quad G(\xi) = \begin{cases} \dfrac{A(\xi)}{\sqrt{2a(\xi)}} & \text{for } 0<\xi<c_o \\[2ex] \dfrac{A(\xi)}{\sqrt{2a(\xi)}}\sqrt{[a(\xi)]^2-[b(\xi)]^2} & \text{for } c_o<\xi<c \end{cases}$$

By Bernoulli's equation

$$p = P + \tfrac{1}{2}\rho\left[V^2-(V+u)^2 - v^2 - w^2\right]$$

where P is the pressure at infinity, and so

/(47) ...

(47) $$p = -\frac{1}{2}\rho\left[G(\xi)\right]^2\left(\sin^2\frac{\theta}{2} + \frac{1}{\left[a'(\xi)\right]^2}\right)\frac{1}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)$$

Substituting the values of u, v, w, and p from (45) and (47) in (39), and taking into account the contributions from both port and starboard, we obtain

$$D(\varepsilon) = 2\rho\int_0^c \left[G(\xi)\right]^2 d\xi \int_{-\pi}^{\pi} \frac{1}{2}\left(\sin^2\frac{\theta}{2} + \frac{1}{\left[a'(\xi)\right]^2}\right) a'(\xi)\cos\theta$$

$$+ \sin\frac{\theta}{2}\left(-a'(\xi)\sin\frac{\theta}{2}\cos\theta + \frac{1}{a'(\xi)}\sin\frac{\theta}{2}\cos\theta - \frac{1}{a'(\xi)}\cos\frac{\theta}{2}\sin\theta\right)d\theta + o(1)$$

$$= 2\rho\int_0^c \left[G(\xi)\right]^2 a'(\xi)\,d\xi \int_{-\pi}^{\pi}\left(-\frac{1}{2}\sin^2\frac{\theta}{2}\cos\theta - \frac{1}{\left[a'(\xi)\right]^2}\sin^2\frac{\theta}{2}\right)d\theta + o(1)$$

Integrating

(48) $$D(\varepsilon) = 2\pi\rho\int_0^c \frac{\left[G(\xi)\right]^2}{a'(\xi)}\left(\left[\frac{a'(\xi)}{2}\right]^2 - 1\right)d\xi + o(1)$$

The suction force D_s as defined above equals $\left(-\lim_{\varepsilon\to o} D(\varepsilon)\right)$.

Also, since we are dealing with the limiting case of wings of small aspect ratio, we may assume that $\left[a'(\xi)\right]^2$ is small compared with 1. (For a simple swallow tail wing $a'(\xi) = \tan\gamma$ is proportional to the aspect ratio, for given c/c_o.)

Hence

(49) $$D_s = 2\pi\rho\int_0^c \frac{\left[G(\xi)\right]^2}{a'(\xi)}\,d\xi$$

For a simple swallow tail wing, the formula becomes

(50) $$D_s = \frac{1}{2}\pi\rho V^2 a_o^2\alpha^2 + \pi\rho V^2\alpha^2\tan\gamma\int_{c_o}^{c}\left[B(\xi)\right]^2\frac{\left[a(\xi)\right]^2 - \left[b(\xi)\right]^2}{a(\xi)}\,d\xi$$

For the special case of a delta wing, the value of the induced drag obtained with the aid of this formula agrees with the result given in ref.1. For the general case, the second integral in (50) can be evaluated numerically, as before.

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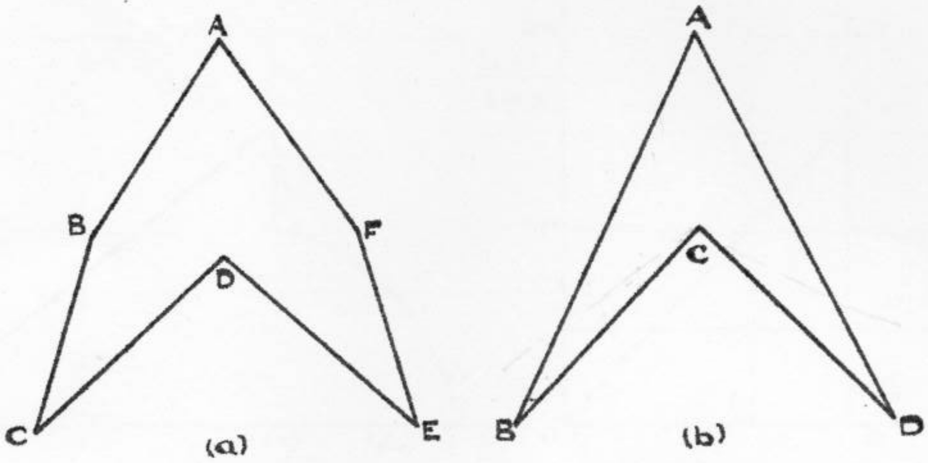


FIG 1

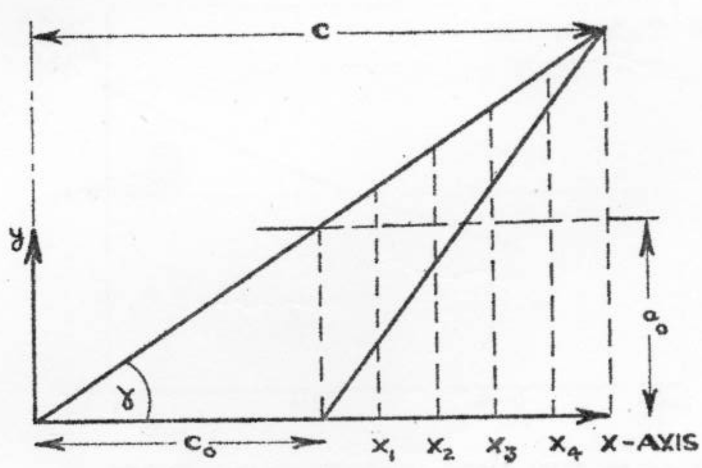


FIG 2 DIMENSIONS OF STARBOARD WING

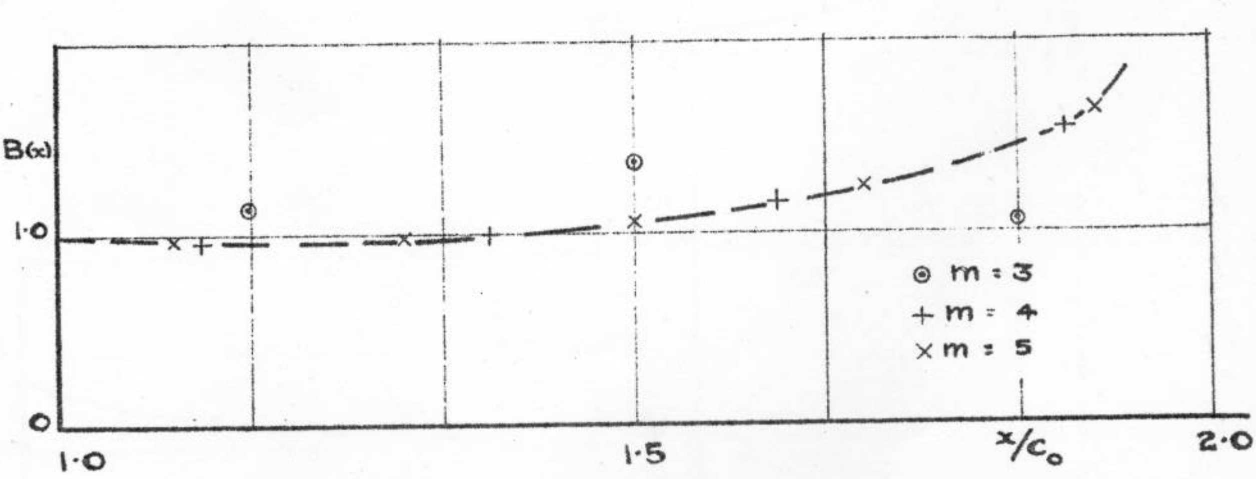
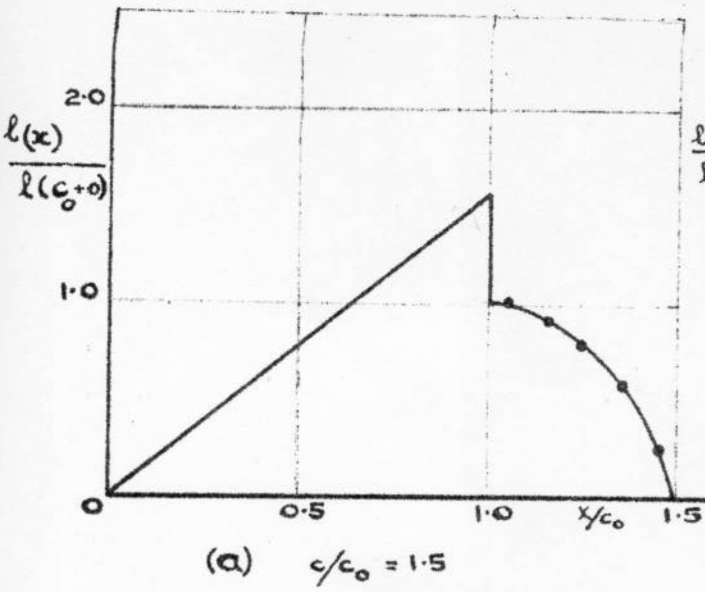


FIG 3 CONVERGENCE OF NUMERICAL APPROXIMATIONS



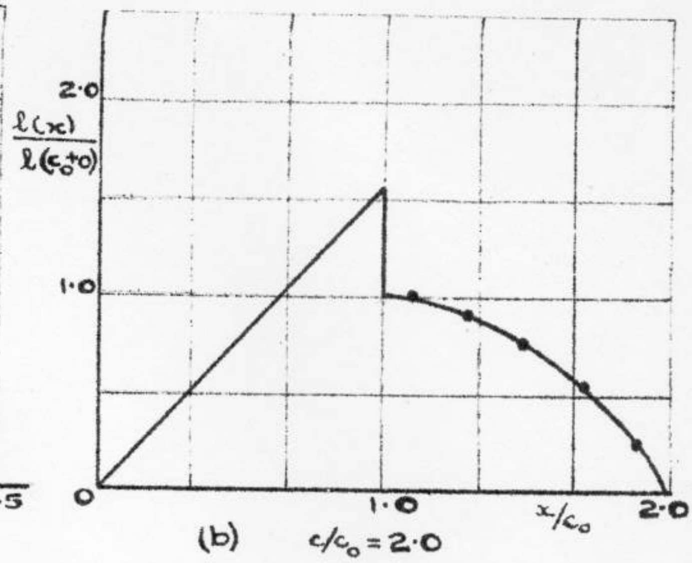


FIG 4 CHORDWISE LIFT DISTRIBUTION

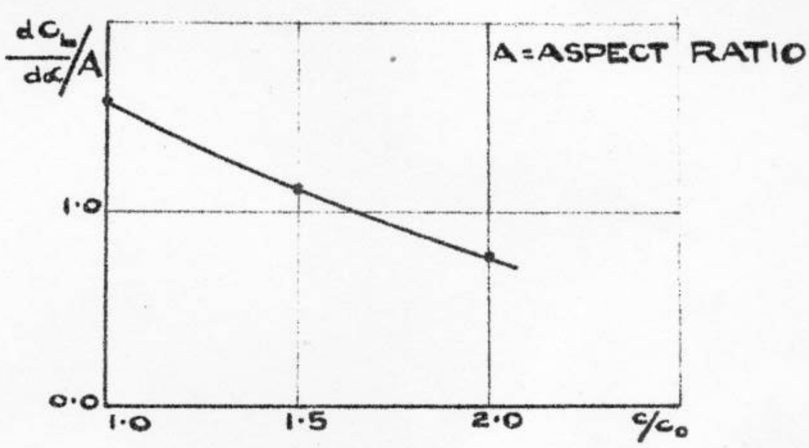


FIG 5 VARIATION OF LIFT CURVE SLOPE

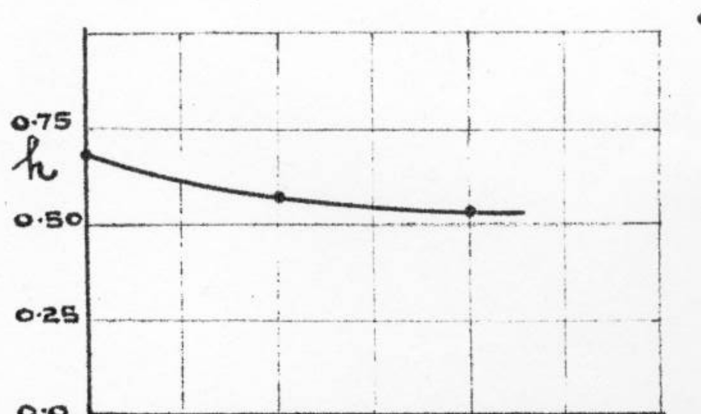


FIG 6 POSITION OF AERODYNAMICS CENTRE AFT OF WING APEX, h (FRACTIONS OF c)

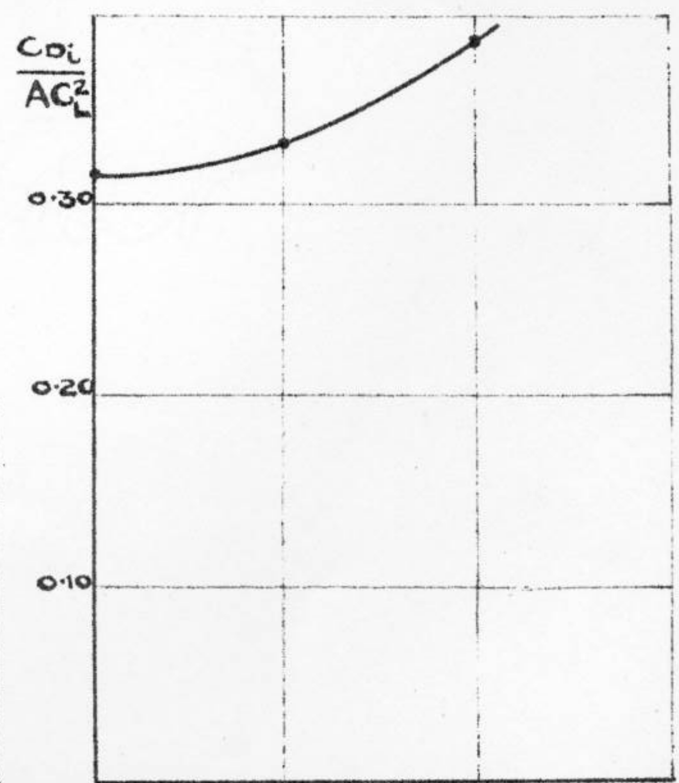


FIG 7 VARIATION OF INDUCED DRAG COEFFICIENT