REPORT No. 37

May, 1950.

THE COLLEGE OF AERONAUTICS

CRANFIELD.

Wave Reflection near a Wall

-by-

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SUMMARY

The field of flow due to a shock wave or expansion wave undergoes a considerable modification in the neighbourhood of a rigid wall. It has been suggested that the resulting propagation of the disturbance upstream is largely due to the fact that the main flow in the boundary layer is subsonic. Simple models were produced by Howarth, and Tsien and Finston, to test this suggestion, assuming the co-existence of layers of uniform supersonic and subsonic main stream velocities. The analysis developed in the present paper is designed to cope with any arbitrary continuous velocity profile which varies from zero at the wall to a constant supersonic velocity in the main stream. Numerical examples are calculated and it is concluded that a simple inviscid theory is incapable of giving an adequate theoretical account of the phenomenon. The analysis includes a detailed discussion of the process of continuous wave reflection in a supersonic shear layer.

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1. Introduction and Discussion of Results.

It has been known for some time that the field of flow due to a shock wave or expansion wave undergoes a considerable modification in the neighbourhood of a solid wall, in addition to the actual reflection at the wall. A characteristic feature of the process is that as the wave impinges on the wall the disturbance is propagated upstream in the boundary layer (see Ref. 1 for detailed experimental evidence). To construct a simple theoretical model of this effect, Howarth (Ref. 2) considered the propagation of small disturbances in a uniform supersonic stream bounded by a parallel uniform subsonic stream. Since no linear dimension is associated with the main field of flow it is difficult to compare the scale of the effect calculated in this way with the scale of the experimental phenomenon. The model was improved by Tsien and Finston (Ref. 3) who considered the propagation of a disturbance in a uniform supersonic stream bounded on one side by a subsonic stream which in turn is bounded by a rigid wall. In the present paper we attempt to make our basic assumptions even more realistic by assuming that the main stream velocity varies continuously from 0 at the wall up to a supersonic speed at some distance from the wall. The case of a continuously varying main stream velocity profile in a purely supersonic region which is bounded on one side by a wall, has been considered by Liepmann and his associates (Ref. 4). The simplifying assumptions of linearisation etc., made in the present work, are basically the same as in the earlier papers mentioned, more particularly in Ref. 4, except that our method permits us to take into account the wave reflection in the supersonic region completely, whereas the treatment in Ref. 4 is only approximate.

A completely different approach has been used by Lees (Ref. 5). It is based on detailed semi-empirical assumptions on the nature of the flow in the boundary layer.

/Numerical ....

Numerical examples have been calculated for a typical laminar velocity profile. Figs. 3 and 4 show the total reflected disturbance due to a simple incident compression (or expansion) wave at Mach numbers 1·25 and 1·75 respectively. It will be seen that the disturbance is propagated upstream only a few multiples of the thickness of the boundary layer. On the other hand experimental evidence (Ref. 1) shows that the resulting disturbances may be clearly distinguishable at points which are fifty or sixty times the boundary layer thickness upstream of the incoming wave. In trying to account for the discrepancy we note that the basic assumptions of our analysis may be inadequate in three respects, (i) they neglect viscosity, (ii) they neglect vorticity, and (iii) they involve linearisation. By an extension of the present method it may be possible to include viscosity and vorticity while still accepting the linearisation of the problem. There appears to be some justification for putting our result on record, although with some diffidence, since it is at variance with the conclusion reached by Tsien and Finston in Ref. 3.

I am indebted to Mr. A.D. Young for a number of valuable discussions on the subject of the present paper; and to Dr. S. Kirkby and Mr. A.W. Babister for assistance in the calculation of the numerical examples.

/2. Basic ....

2. Basic Analysis.

We consider two-dimensional flow in a semi-infinite expanse of fluid bounded by a wall which is parallel to the x - axis, at $y = y_w$. The main stream, supposed (approximately) parallel to the x - axis, will be assumed to be given by a function $V = V(y)$, $y \geq y_w$, where $V(y)$ is continuous and differentiable, and vanishes at the wall. Also, $V(y)$ shall be constant for sufficiently large y, $V(y) = V_o$ for $y \geq y_o$, say. We write $M(y) = V(y)/a$ for the local main stream Mach number, where $a = a(y)$ is the velocity of sound appropriate to that ordinate, $a(y) = a_o < V_o$ for $y \geq y_o$. It is irrelevant to the subsequent analysis whether or not we assume as a further simplification that a is constant throughout the medium.

Let u,v be the velocity components of a small, steady disturbance imposed on the main stream. With the usual approximations we obtain the linearised equation of continuity,

$$-\lambda(y)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \lambda(y) = [M(y)]^2 - 1. \quad \ldots\ldots.(1)$$

We shall assume that the vorticity associated with the disturbance can be neglected, $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$, (see ref. 4, p.22) so that the motion possesses a velocity potential ϕ, $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$.
By (1), ϕ satisfies the equation

$$-\lambda(y)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad \ldots\ldots\ldots\ldots..(2)$$

Particular solutions of (2) are obtained by assuming that $\phi(x,y)$ is of the form $\phi(x,y) = f(y)\cos(kx + \varepsilon)$, where k and ε are arbitrary. Substitution in (2) yields the following ordinary differential equation for f(y),

$$k^2\lambda(y)f(y) + \frac{d^2 f}{dy^2} = 0. \quad \ldots\ldots\ldots\ldots..(3)$$

When $y \geq y_o$, $\lambda_o = [M(y)]^2 - 1 = M_o^2 - 1$, where $M_o = V_o/a_o$, so that a fundamental set of solutions of (3) is given by $\cos k\beta_o(y-y_o)$, $\sin k\beta_o(y-y_o)$, where $\beta_o = \sqrt{\lambda_o} = \sqrt{M_o^2 - 1}$. More generally we write f(y) as a power series of the parameter k

$$f(y) = \sum_{n=0}^{\infty} f_n(y)k^n \quad \ldots\ldots\ldots\ldots\ldots(4)$$

/where....

where the $f_n(y)$ are independent of k. Substituting this, and the corresponding series for $f''(y)$ in (3), we obtain

$$\sum_{n=0}^{\infty} \lambda(y) f_n(y) k^{n+2} + \sum_{n=0}^{\infty} f_n''(y) k^n = 0. \quad \ldots\ldots\ldots (5)$$

Comparison of coefficients of equal powers of k then yields

$$\left.\begin{aligned} &f_0''(y) = 0 \\ &f_1''(y) = 0 \\ &f_n''(y) + \lambda(y) f_{n-2}(y) = 0, \quad n \geq 2. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

To obtain a continuation of the solution $\cos k\beta_o(y-y_o)$ in the region $y_o \geq y \geq y_w$ (i.e. a solution which passes continuously into $\cos k\beta_o(y-y_o)$, while its first derivative passes continuously into $\left[\cos k\beta_o(y-y_o)\right]' = -k\beta_o \sin k\beta_o(y-y_o)$, we put

$$\left.\begin{aligned} &f_o(y) = 1 \\ &f_1(y) = 0 \\ &f_n(y) = -\int_{y_o}^{y} dy' \int_{y_o}^{y} \lambda(y'') f_{n-2}(y'')\, dy'', \quad n \geq 2. \end{aligned}\right\} \ldots\ldots (7)$$

The set of functions $f_n(y)$ defined in this way, clearly satisfies (6). The corresponding power series $f(y) = \sum_{n=0}^{\infty} f_n(y) k^n$ represents an even func- of k. To discuss its convergence let L be a positive upper bound of $|\lambda(y)|$. Then

$$|f_2(y)| = \left|\int_{y_o}^{y} dy \int_{y_o}^{y'} \lambda(y'') f_o(y'')\, dy''\right| \leq L \left|\int_{y_o}^{y} dy \int_{y_o}^{y'} dy''\right| = L \frac{|y-y_o|^2}{2}$$

$$|f_4(y)| = \left|\int_{y_o}^{y} dy \int_{y_o}^{y'} \lambda(y'') f_2(y'')\, dy''\right| \leq L^2 \left|\int_{y_o}^{y} dy \int_{y_o}^{y'} \frac{|y''-y_o|^2}{2}\, dy''\right| = L^2 \frac{|y-y_o|^2}{4!}$$

and in general

$$|f_{2n}(y)| = \left|\int_{y_o}^{y} dy \int_{y_o}^{y'} \lambda(y'') f_{2(n-1)}(y'')\, dy''\right|$$

$$\leq L^n \left|\int_{y_o}^{y} dy \int_{y_o}^{y'} \frac{|y''-y_o|^{2n-2}}{(2n-2)!}\, dy''\right| = L^n \frac{|y-y_o|^{2n}}{(2n)!}$$

This shows that $\sum_{n=0}^{\infty} f_n(y)k^n$ converges for all (real and complex) k, and therefore represents an integral function of k, for all $y \geq y_w$. This function will be denoted by $C(y;k)$. We have $C(y_o;k) = 1$, from (7), and $C'(y_o;k) = 0$, where the dash indicates differentiation with respect to y. It follows that, for $y \geq y_o$, $C(y;k) = \cos k\beta_o(y-y_o)$, as required. Similarly, if we define the set of functions $f_n(y)$ by

$$\left.\begin{aligned} f_o(y) &= 0 \\ f_1(y) &= \beta_o(y-y_o) \\ f_n(y) &= -\int_{y_o}^{y} dy' \int_{y_o}^{y} \lambda(y'')f_{n-2}(y'')dy'' , \quad n \geq 2 \end{aligned}\right\} \quad \ldots\ldots\ldots (8)$$

then the function

$$S(y;k) = \sum_{n=0}^{\infty} f_n(y)k^n$$

is an odd integral function of k for all $y \geq y_w$ such that $S(y;k) = \sin k\beta_o(y-y_o)$ for $y \geq y_o$. Suitable bounds for the functions $f_n(y)$ now are

$$|f_{2n+1}(y)| \leq \beta_o L^n \frac{|y-y_o|^{2n+1}}{(2n+1)!} ,$$

for n = 0, 1, 2, ... $C(y;k)$ and $S(y;k)$ form a set of independent solutions of the differential equation (3). Particular 'normal' solutions of (2) are given by

$$\left.\begin{matrix} C(y;k) \\ S(y;k) \end{matrix}\right\} \quad \left\{\begin{matrix} \cos kx \\ \sin kx \end{matrix}\right. \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and we note that in the region $y \geq y_o$ these solutions reduce to

$$\left.\begin{matrix} \cos k\beta_o(y-y_o) \\ \sin k\beta_o(y-y_o) \end{matrix}\right\} \quad \left\{\begin{matrix} \cos kx \\ \sin kx \end{matrix}\right. \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

respectively.

In the sequel, $\lambda(y)$, which equals -1 at the wall, and equals $\lambda_o > 0$ for $y > y_o$, will be assumed to be an increasing function of y. There is then just one distinct value of y, $y = y_s$, say, for which $\lambda(y_s) = 0$, corresponding to the sonic line. For convenience we shall refer to the region $y \geq y_o$ briefly as the 'supersonic region' while $y_o \geq y \geq y_s$ and $y_s \geq y \geq y_w$ will be termed 'transonic' and 'subsonic' respectively. We may assume that $y_o = 0$.

/A disturbance .....

A disturbance travelling towards the wall in the supersonic region may be expressed in the form $\phi = F(x+\beta_o y)$. Subject to suitable restrictions on the behaviour of the function for numerically large values of the argument this may be written as a Fourier integral

$$F(x+\beta_o y) = \int_{-\infty}^{\infty} (P(k)+ iQ(k))(\cos k(x+\beta_o y)-i\sin k(x+\beta_o y))dk \qquad \ldots\ldots\ldots\ldots\ldots (11)$$

where $P(k)$ and $Q(k)$ are real functions of k, even and odd, respectively.

Re-writing (11) in the form

$$F(x+\beta_o y) = \int_{-\infty}^{\infty} (P(k)+iQ(k))(\cos k\beta_o y-i\sin k\beta_o y)e^{-ikx}\, dk$$

we see from (9) and (10) that $F(x+\beta_o y)$ is the function to which

$$F^*(x,y) = \int_{-\infty}^{\infty} (P(k)+iQ(k))(C(y;k)-iS(y;k))e^{-ikx}\, dk, \quad y \gtrsim y_w \qquad \ldots\ldots\ldots\ldots\ldots (12)$$

reduces for $y \gtrsim y_o = 0$. $F^*(x,y)$ is real since $C(y;k)$ and $S(y;k)$ are even and odd functions of k, respectively.

Similarly the function

$$G^*(x,y) = \int_{-\infty}^{\infty} (R(k)+iS(k))(C(y;k)+iS(y;k))e^{-ikx} dk \ \ldots (13)$$

$R(k)$ even, $S(k)$ odd,

reduces to

$$G^*(x,y) = \int_{-\infty}^{\infty} (R(k)+iS(k))e^{ik\beta_o y}\, e^{-ikx} dk$$

$$= \int_{-\infty}^{\infty} (R(k) + iS(k)e^{-ik(x-\beta_o y)} dk = G(x-\beta_o y)$$

for $y > y_o = 0$, i.e. in the supersonic region it represents a disturbance travelling away from the wall.

The velocity components in a direction normal to the wall in the two cases are given by

$$\frac{\partial F^*}{\partial y} = \int_{-\infty}^{\infty} (P(k)+iQ(k))(C'(y;k)-iS'(y;k))e^{-ikx} dk \ \ldots\ldots (14)$$

/and ......

and

$$\frac{\partial G^*}{\partial y} = \int_{-\infty}^{\infty} (R(k)+iS(k))(C'(y;k)+iS'(y;k))e^{-ikx} dk \ \ldots\ldots \quad (15)$$

respectively, where the dash denotes differentiation with respect to y.

Let

$$\phi(x,y) = F^*(x,y) + G^*(x,y) \ldots\ldots\ldots\ldots \quad (16)$$

be the velocity potential of a field of flow in the region under consideration. To satisfy the condition of zero normal velocity at the wall we must have

$$\left(\frac{\partial F^*}{\partial y} + \frac{\partial G^*}{\partial y}\right)_{y=y_w} = 0$$

or, by (14) and (15)

$$\int_{-\infty}^{\infty} \Big[(P(k)+iQ(k))(C'(y_w;k)-iS'(y_w;k))+(R(k)+iS(k))(C'(y_w;k)+iS'(y_w;k))\Big] e^{-ikx} dk = 0, \text{ for all } x.$$

This yields the condition

$$(P(k)+iQ(k))(C'(y_w;k)-iS'(y_w;k))+(R(k)+iS(k))(C'(y_w;k)+iS'(y_w;k)) = 0 \ldots\ldots\ldots\ldots \quad (17)$$

Assume now that the incoming wave $F(x+\beta_0 y)$ is specified in the supersonic region. This determines $P(k)$ and $(Q(k)$. It follows that if the potential of the total disturbance can be written as in (16), $R(k)$ and $S(k)$ must satisfy (17) so that these functions are given by

$$R(k)+iS(k) = -\frac{C'(y_w;k)-iS'(y_w;k)}{C'(y_w;k)+iS'(y_w;k)}(P(k)+iQ(k)) \ldots\ldots \quad (18)$$

Since $C(y;k)$ and $S(y;k)$ are independent solutions of the second order differential equation (3), their derivatives cannot vanish simultaneously. It follows that the right-hand side of (18) remains finite for all real k. It will be seen that the real part of the right-hand side of (18) is an even function of k, while its imaginary part is odd, as required.

/The ......

The above formulae enable us to determine a solution corresponding to any incoming wave $F(x+\beta_o y)$ specified in the supersonic region. However, our argument so far has not established that we have obtained the physically correct result expressing the evolution of the incident wave as it travels through the transonic into the subsonic layer and is reflected at the wall. To elucidate this somewhat subtle distinction we may point out that equation (12) which defines our incoming wave (a disturbance travelling towards the wall) in the supersonic region does not in general represent an incoming wave in the transonic region. Thus to ensure that we obtain the physically correct answer we shall trace the gradual evolution of the disturbance on passing through the various layers. In actual fact we shall find that the final result agrees with that obtained by the simple analysis which was given above. However, apart from being necessary as a matter of principle, the following considerations also help to throw light on the physical mechanism of the phenomenon.

3. Reflection in the Transonic Region.

In a region of uniform supersonic flow, i.e. in our case for $y > y_o = 0$, any perturbation velocity potential can be written in the form $F(x+\beta_o y) + G(x-\beta_o y)$, so that the incoming and outgoing disturbances which constitute the field of flow correspond simply to the first and second term of that expression, respectively. The position in a region of variable supersonic stream velocity is less simple. Writing $\beta(y) = \sqrt{\lambda(y)}$ in (2) for $\lambda(y) > 0$, the equation becomes

$$-\beta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

The characteristic curves (Mach lines) of this equation are given by

$$\beta^2\, dy^2 - dx^2 = 0,$$

or Fig. 1)

$$\frac{dx}{dy} = \pm\, \beta(y) \quad ; \quad x = \pm \int \beta(y)dy + \text{const.} \ldots (20)$$

/An ......

An 'incoming wave' now is a disturbance whose fronts, or potential lines of discontinuity are the Mach lines.

$$x = -\int \beta(y)dy + \text{const.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

e.g. the line NPQ in Fig. 1, while an outgoing wave is a disturbance whose potential lines of discontinuity are the lines

$$x = \int \beta(y)dy + \text{const.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

More precisely, the analytical expression for an incoming wave must be such that a disturbance, or modification of initial values at a point R downstream of one of the Mach lines (21) does not affect conditions at a point upstream of PQ, e.g. the point S. Since β is now variable we are no longer in a position to represent the two types of waves simply by functions $F(x+\beta y)$ and $G(x-\beta y)$ respectively. Nevertheless it is still possible to associate distinct disturbances with the two families of Mach lines in the following way.

Given any solution $\phi(x,y)$ of the differential equation (19) we put

$$\left.\begin{aligned} f_1(x,y) &= \tfrac{1}{2}\left(\beta\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) \\ f_2(x,y) &= \tfrac{1}{2}\left(\beta\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y}\right) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

so that

$$\frac{\partial\phi}{\partial x} = \frac{1}{\beta}(f_1 + f_2)$$

$$\frac{\partial\phi}{\partial y} = f_1 - f_2.$$

Then

$$\frac{\partial f_1}{\partial x} - \frac{1}{\beta}\frac{\partial f_1}{\partial y} = \tfrac{1}{2}\left(\beta\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y} - \frac{1}{\beta}\frac{d\beta}{dy}\frac{\partial\phi}{\partial x} - \frac{1}{\beta}\frac{\partial^2\phi}{\partial y^2}\right)$$

$$= -\frac{1}{2\beta}\frac{d\beta}{dy}\frac{\partial\phi}{\partial x},$$

or

$$\frac{\partial f_1}{\partial x} - \frac{1}{\beta}\frac{\partial f_1}{\partial y} = -\frac{1}{2\beta^2}\frac{d\beta}{dy}(f_1+f_2) \ldots\ldots\ldots\ldots (24)$$

/and ....

and similarly

$$\frac{\partial f_2}{\partial x} + \frac{1}{\beta}\frac{\partial f_2}{\partial y} = \frac{1}{2\beta^2}\frac{d\beta}{dy}(f_1 + f_2)$$

Denoting by D_1/Dx, D_2/Dx differentiation along the families of Mach lines (21) and (22) respectively, we have

$$\frac{D_1}{Dx} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - \frac{1}{\beta}\frac{\partial}{\partial y}$$

and similarly

$$\frac{D_2}{Dx} = \frac{\partial}{\partial x} + \frac{1}{\beta}\frac{\partial}{\partial y}$$

(24) may then be written as

$$\frac{D_1 f_1}{Dx} = -\frac{1}{2\beta^2}\frac{d\beta}{dy}(f_1 + f_2) \quad \text{................} \quad (25)$$

$$\frac{D_2 f_2}{Dx} = \frac{1}{2\beta^2}\frac{d\beta}{dy}(f_1 + f_2)$$

An equivalent set of equations is

$$\frac{D_1 f_1}{Dy} = \frac{1}{2\beta}\frac{d\beta}{dy}(f_1 + f_2) \quad \text{.................} \quad (26)$$

$$\frac{D_2 f_2}{Dy} = \frac{1}{2\beta}\frac{d\beta}{dy}(f_1 + f_2).$$

If β is a constant, then the right-hand sides of (25) and (26) vanish so that the functions $f_1(x,y)$, and $f_2(x,y)$ are constant along their respective Mach lines. Also in that case $\phi(x,y)=F(x+\beta y)+G(x-\beta y)$ so that

$$f_1 = \tfrac{1}{2}\left(\beta\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) = \beta F'(x+\beta y) \quad \text{.............} \quad (27)$$

$$f_2 = \tfrac{1}{2}\left(\beta\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y}\right) = \beta G'(x-\beta y)$$

Thus, $f_1(x,y)$ and $f_2(x,y)$ are associated with the incoming and outgoing waves respectively, and the two disturbances do not interact. More generally, if β is variable we may still regard $f_1(x,y)$ and $f_2(x,y)$ as the incoming and outgoing waves respectively, but there is now a gradual interaction between the two disturbances as indicated by the right-hand sides of (25) and (26).

/Thus........

Thus, assume that an incoming disturbance $f_1(x,y)$ is specified for some $y=y_1$, $f_1(x,y)=f_o(x)$, say, and that we wish to calculate its variation as it travels towards $y=y_2$, $y_2 < y_1$, as well as the variation of the disturbance $f_2(x,y)$ which is built up from $f_1(x,y)$ by gradual reflection within the layer $y_1 \geq y \geq y_2$. It follows from this definition of $f_2(x,y)$ that it does not include any disturbance which penetrates into the layer across $y=y_2$ along the second family of characteristics, (22), so that $f_2(x,y_2) = 0$. The two boundary conditions

$$f_1(x,y_1) = f_o(x), \quad f_2(x,y_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(28)}$$

together with the set of differential equations (24) then determine $f_2(x,y)$ and $f_2(x,y)$ for the region $-\infty < x < \infty$, $y_1 \geq y \geq y_2$.

However, the above analysis is inconclusive because we have not shown as yet that the function $f_1(x,y)$ obtained in this way satisfies the crucial test for an 'incoming wave': we have not established that a modification of $f_o(x)$ at a point R downstream of any given Mach line PQ of the family (21) does not affect conditions at any point S upstream of PQ (Fig. 1). The rigorous proof of this fact is adjourned to section 5.

Coming back to the particular case under consideration, we identify y_1 with $y_o = 0$, and y_2 - in the limit - with the sonic line y. We assume that the incoming wave is given by (11) in the supersonic region and we write the velocity potential of the motion in the transonic region which is due to the joint effect of the incoming wave and of the gradually developing reflected wave in the form

$$\Phi_1(x,y) = \int_{-\infty}^{\infty} \Big[(P_1(k)\ C(y;k) + P_2(k)\ S(y;k)) + i\,(Q_1(k)C(y;k) + Q_2(k)S(y;k)) \Big] e^{-ikx}\, dk \quad \ldots\ldots\ldots\ldots \text{(29)}$$

$\Phi_1(x,y)$ does not include the disturbances originating from the evolution of the wave in the subsonic region or from the reflection at the wall. The first boundary condition in (28) yields, taking into account (11)

/$f_1(x,0)$.......

$$f_1(x,0) = \tfrac{1}{2}\left[\beta \frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_1}{\partial y}\right]_{y=0} = \beta_o F'(x) \quad \ldots\ldots\ldots\ldots\ldots (30)$$

or

$$\tfrac{1}{2}\int_{-\infty}^{\infty}\left[-ik\beta_o(P_1(k)+iQ_1(k))+k\beta_o(P_2(k)+iQ_2(k))\right]e^{-ikx}dk$$

$$= \beta_o\int_{-\infty}^{\infty} -ik(P(k)+iQ(k))e^{-ikx}dk$$

since $\beta(y) = \beta_o$, $C(0;k) = 1$, $C'(0;k) = 0$, $S(0;k) = 0$, $S'(0;k) = k\beta_o$.

We therefore obtain as a first condition for the four coefficients $P_1(k)$, $P_2(k)$, $Q_1(k)$, $Q_2(k)$

$$(P_1(k)-Q_2(k))+i(P_2(k)+Q_1(k))=2(P(k)+iQ(k)) \quad \ldots\ldots\ldots\ldots (31)$$

The second boundary condition in (28) is, in terms of the function $\Phi_1(x,y)$

$$f_2(x,y_2) = \tfrac{1}{2}\left[\beta \frac{\partial\Phi_1}{\partial x} - \frac{\partial\Phi_1}{\partial y}\right]_{y=y_2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (32)$$

In the limit as y_2 tends to y_s, $\beta(y)$ tends to 0, and so

$$\left(\frac{\partial\Phi_1}{\partial y}\right)_{y=y_s} = 0. \quad \ldots\ldots\ldots\ldots\ldots (33)$$

This is the same boundary conditions as would be obtained at a wall, so that if we could place a rigid wall along the sonic line, $\Phi_1(x,y)$ would represent the total perturbation potential.

The Fourier integral expression for (33) is

$$\int_{-\infty}^{\infty}\left[(P_1(k)C'(y_s;k)+P_2(k)S'(y_s;k))+ i(Q_1(k)C'(y_s;k) + Q_2(k)S'(y_s;k))\right]e^{-ikx}dk = 0,$$

so that

$$(P_1(k)C'(y_s;k)+P_2(k)S'(y_s;k))+i(Q_1(k)C'(y_s;k) + Q_2(k)S'(y_s;k)) = 0. \quad \ldots\ldots\ldots\ldots\ldots (34)$$

We may re-write (31) and (34) in the form

$$(P_1 + iQ_1) + (P_2 + iQ_2) = 2(P(k) + iQ(k))$$

$$C'(y_s;k)(P_1+iQ_1) + S'(y_s;k)(P_2+iQ_2) = 0 \quad \ldots\ldots\ldots\ldots (35)$$

/The.......

The determinant of this system is $S'(y_s;k) - iC'(y_s;k)$, which cannot vanish for real k. Hence

$$P_1(k)+iQ_1(k) = 2\,\frac{S'(y_s;k)}{S'(y_s;k)-iC'(y_s;k)}(P(k)+iQ(k))$$

$$P_2(k)+iQ_2(k) = -2\,\frac{C'(y_s;k)}{S'(y_s;k)-iC'(y_s;k)}(P(k)+iQ(k)) \qquad \ldots\ldots(36)$$

Since $P_1(k)$, $P_2(k)$, $Q_1(k)$, $Q_2(k)$ are real functions of k, they are determined by (36). However, $\Phi_1(x,y)$ can also be expressed directly in terms of P_1+iQ_1 and P_2+iQ_2,

$$\Phi_1(x,y) = \int_{-\infty}^{\infty}\left[(P_1(k)+iQ_1(k)C(y;k)+(P_2(k)+iQ_2(k)S(y;k))\right]e^{-ikx}dk$$

Hence

$$\Phi_1(x,y) = 2\int_{-\infty}^{\infty}\frac{S'(y_s;k)C(y;k)-C'(y_s;k)S(y;k)}{S'(y_s;k)-iC'(y_s;k)}(P(k)+iQ(k))e^{-ikx}dk \qquad \ldots\ldots\ldots\ldots\ldots\ldots(37)$$

It follows from the construction of (37), that if the incoming wave has a sharp front, e.g. if it vanishes upstream of the Mach line NP in the supersonic region (Fig. 1) then $\Phi_1(x,y)$ vanishes upstream of the Mach line PQ which is the continuation of NP in the transonic region.

In the supersonic region, $\Phi_1(x,y)$ as given by (37) should represent the incoming wave (11) together with the outgoing wave which is built up through gradual reflection in the transonic layer. It may serve as a check on our calculations to show that the difference between $\Phi_1(x,y)$ and $F(x+\beta_o y)$ does in fact represent an outgoing wave for $y>0$.

We have, for $y>0$

$$\Phi_1(x,y)-F(x+\beta_o y) = \int_{-\infty}^{\infty}\left[\left(2\,\frac{S'(y_s;k)\cos k\beta_o y-C'(y_s;k)\sin k\beta_o y}{S'(y_s;k)-iC'(y_s;k)}\right.\right.$$

$$\left.\left. -(\cos k\beta_o y-i\sin k\beta_o y)\right)\right](P(k)+iQ(k))\,e^{-ikx}dk$$

$$= \int_{-\infty}^{\infty}\frac{S'(y_s;k)+iC'(y_s;k)}{S'(y_s;k)-iC'(y_s;k)}(P(k)+iQ(k))(\cos k\beta_o y+i\sin k\beta_o y)e^{-ikx}dk$$

/Comparison .....

Comparison with (13) shows that this is an outgoing wave, as required.

4. Reflection at the Wall.

We now come to the consideration of the subsonic region, $y_s \geq y \geq y_w$. The characteristic curves being now complex, there are no physical lines along which a disturbance is propagated. Thus, we only have to ensure that the potential, together with its derivatives is continuous across the sonic line, and that the normal velocity vanishes at the wall. For this purpose we write

$$\Phi(x,y) = \Phi_1(x,y) + \Phi_2(x,y)$$

where $\Phi_1(x,y)$ is given by (29) and (37), and $\Phi_2(x,y)$ is a solution of (2) which, when continued through the transonic into the supersonic region, finally yields an outgoing wave. It follows that $\Phi_2(x,y)$ is of the form

$$\Phi_2(x,y) = \int_{-\infty}^{\infty} (R(k)+iS(k))(C(y;k)+iS(y;k))e^{-ikx}dk \quad \ldots\ldots (38)$$

The boundary condition at the wall is $\frac{\partial}{\partial y}(\Phi_1+\Phi_2) = 0$, or

$$\int_{-\infty}^{\infty} (P_1(k)+iQ_1(k)C'(y_w;k)+(P_2(k)+iQ_2(k))S'(y_w;k)+(R(k)+iS(k))$$

$$C'(y_w;k)+iS'(y_w;k))e^{-ikx}dk = 0,$$

so that $R(k)+iS(k)$ is given by

$$(P_1(k)+iQ_1(k))C'(y_w;k)+(P_2(k)+iQ_2(k))S'(y_w;k)+(R(k)+iS(k))$$

$$(C'(y_w;k)+iS'(y_w;k)) = 0,$$

or

$$R(k)+iS(k) = 2i\frac{\left[S'(y_s;k)C'(y_w;k)-C'(y_s;k)S'(y_w;k)\right]\left[P(k)+iQ(k)\right]}{\left[S'(y_s;k)-iC'(y_s;k)\right]\left[S'(y_w;k)-iC'(y_w;k)\right]}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Hence

$$\Phi(x,y) = 2\int_{-\infty}^{\infty}\left\{\frac{S'(y_s;k)C(y;k)-C'(y_s;k)S(y;k)}{S'(y_s;k)-iC'(y_s;k)}\right.$$

$$+ i \frac{\left[S'(y_s;k)C'(y_w;k)-C'(y_s;k)S'(y_w;k\right]\left[C(y;k)+iS(y;k)\right]}{\left[S'(y_s;k)-iC'(y_s;k\right]\left[S'(y_w;k)-iC'(y_w;k)\right]}\Bigg\}$$

$$(P(k)+iQ(k))e^{-ikx}dk$$

$$= 2\int_{-\infty}^{\infty}\frac{S'(y_w;k)C(y;k)-C'(y_w;k)S(y;k)}{S'(y_w;k)-iC'(y_w;k)}(P(k)+iQ(k))e^{-ikx}dk$$

$$= \int_{-\infty}^{\infty}(P(k)+iQ(k))(C(y;k)-iS(y;k))e^{-ikx}dk -$$

$$-\int_{-\infty}^{\infty}\frac{C'(y_w;k)-iS'(y_w;k)}{C'(y_w;k)+iS'(y_w;k)}(P(k)+iQ(k))(C(y;k)+iS(y;k))e^{-ikx}dk$$

This shows that $\Phi(x,y)$ can be written as the sum of the two functions $F^*(x,y)$ and $G^*(x,y)$ as given by (12), (13) and (18). The same formula for the total perturbation potential still applies in the transonic region and in the supersonic region, propagation upstream being no longer inadmissible owing to the effect of the subsonic layer. More particularly, the disturbance produced by reflection at the wall is given by (38) also in the transonic and supersonic regions. It will be seen that (38) does not in general represent a pure outgoing wave in the transonic region, since it accounts also for the 'incoming waves' which are obtained from the wave reflected at the wall by subsequent reflection in the transonic layer. However, a further analysis of this process will not be necessary.

We have obtained the same final result as provided by the simple analysis of section 2. The same formulae still apply if we place the wall in the (uniform) supersonic region. But the fact that (12), (13) and (18) do not provide the correct answer when the wall is placed in the transonic region may serve as a sufficient indication that a more detailed analysis was not out of place.

5. A Two-line Boundary Value Problem.

We still have to settle the question mentioned towards the end of section 3. For this purpose we shall study the following problem from a purely mathematical point of view.

/Consider.....

Consider the system of differential equations

$$\left.\begin{aligned}\frac{\partial f_1}{\partial x} + c_1(y)\frac{\partial f_1}{\partial y} &= a_{11}(y)f_1(x,y)+a_{12}(y)f_2(x,y)\\ \frac{\partial f_2}{\partial x} + c_2(y)\frac{\partial f_2}{\partial y} &= a_{21}(y)f_1(x,y)+a_{22}(y)f_2(x,y)\end{aligned}\right\}\quad\ldots\ldots\ldots (40)$$

where $c_1(y)$, $c_2(y)$, $a_{11}(y)$, $a_{12}(y)$, $a_{21}(y)$, $a_{22}(y)$, are continuous functions of y in an interval $y_1 \geq y \geq y_2$, such that $c_1(y) < 0$, $c_2(y) > 0$ throughout. The characteristic curves of the system are

$$\frac{dy}{dx} = c_1(y) \qquad x = \int \frac{dy}{c_1(y)} + \text{const}\ldots\ldots\ldots\ldots\ldots (41)$$

and

$$\frac{dy}{dx} = c_2(y) \qquad x = \int \frac{dy}{c_2(y)} + \text{const}\ldots\ldots\ldots\ldots\ldots (42)$$

Assume that $f_1(x,y)$ is specified for $y=y_1$, $f_1(x,y_1)=F(x)$, and that $f_2(x,y)$ is specified for $y=y_2$, $f_2(x,y_2)=G(x)$, $-\infty < x < \infty$, such that both $F(x)$ and $G(x)$ vanish for sufficiently small x, $F(x) = G(x) = 0$ for $x < x_0$, say.

Let P be any point in the region R defined by $-\infty < x < \infty$, $y_1 \geq y \geq y_2$. Through P draw the two characteristics ℓ_1 and ℓ_2 belonging to the families (41) and (42), respectively (Fig. 2). These characteristics meet the straight lines $y=y_1$ and $y=y_2$ in points $x=x_1$ and $x=x_2$, respectively. We propose to show that subject to certain conditions of regularity the given problem possesses a solution $f_1(x,y)$, $f_2(x,y)$. Furthermore, we shall show that the value of $f_1(x,y)$ and $f_2(x,y)$ at P is determined completely and uniquely by the values of $F(x)$ for $x \leq x_1$ and of $G(x)$ for $x \leq x_2$.

We may replace (40) by

$$\left.\begin{aligned}\frac{D_1 f_1}{Dx} &= a_{11}f_1 + a_{12}f_2\\ \frac{D_2 f_2}{Dx} &= a_{21}f_1 + a_{22}f_2\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots (43)$$

where D_1/Dx and D_2/Dx denote differentiation along

/the two.......

the two characteristics through any given point, similarly as in equation (25). These equations in turn are equivalent to

$$\left.\begin{aligned} f_1(x,y) &= f_1(x_1,y_1)+\int_{x_1}^{x}\left[a_{11}(\eta)f_1(\xi,\eta)+a_{12}(\eta)f_2(\xi,\eta)\right]d\xi \\ f_2(x,y) &= f_2(x_2,y_2)+\int_{x_2}^{x}\left[a_{21}(\eta)f_1(\xi,\eta)+a_{22}(\eta)f_2(\xi,\eta)\right]d\xi \end{aligned}\right\}\ldots(44)$$

where the integrals on the right-hand side are taken along the characteristics of the first and second family, (41) and (42) respectively. x and y are co-ordinates of any point P in R, and x_1 and x_2 are defined as in Fig. 2.

(44) suggests a method of successive approximation. We define $f_1(x,y)$ and $f_2(x,y)$ in the region R as any set of functions which satisfy the specified boundary conditions, e.g.

$$f_{1,0}(x,y) = F(x_1,y_1) \qquad f_{2,0}(x,y) = G(x_2,y_2)\ldots\ldots\ldots(45)$$

so that $f_{1,0}(x,y)=f_{2,0}(x,y)=0$ for $x\leq x_0$. Next we define sequences of functions $f_{1,n}(x,y)$, $f_{2,n}(x,y)$, $n=1,2.....$ successively by

$$\left.\begin{aligned} f_{1,n}(x,y) &= f_{1,n-1}(x_1,y_1)+\int_{x_1}^{x}\left[a_{11}(\eta)f_{1,n-1}(\xi,\eta)+a_{12}(\eta)\cdot f_{2,n-1}(\xi,\eta)\right]d\xi \\ f_{2,n}(x,y) &= f_{2,n-1}(x_2,y_2)+\int_{x_2}^{x}\left[a_{21}(\eta)f_{1,n-1}(\xi,\eta)+a_{22}(\eta)f_{2,n-1}(\xi,\eta)\right]d\xi \end{aligned}\right\}\ldots(46)$$

where the integrals are again taken along (41) and (42) respectively.

We propose to show that the sequences $f_{1,n}(x,y)$, $f_{2,n}(x,y)$ converge at all points of R.

Let

$$A = \max_{y_1\leq y\leq y_2}\left(|a_{11}(y)|,|a_{12}(y)|,|a_{21}(y)|,|a_{22}(y)|\right),$$

and let M be a common bound to $|F(\xi)|$ and $|G(\xi)|$ for ξ smaller than some arbitrarily large value X,

/$\xi\leq X$........

$\xi \leq X$. Denoting by $d_{1,n}(x,y)$, $d_{2,n}(x,y)$ the differences

$$d_{1,n}(x,y) = f_{1,n}(x,y) - f_{1,n-1}(x,y)$$

$$d_{2,n}(x,y) = f_{2,n}(x,y) - f_{2,n-1}(x,y)$$

we then obtain

$$\left.\begin{aligned} d_{1,1}(x,y) &= \int_{x_1}^{x} \left[a_{11}(\eta)f_{1,0}(\xi,\eta)+a_{12}(\eta)f_{2,0}(\xi,\eta)\right] d\xi \\ d_{2,1}(x,y) &= \int_{x_2}^{x} \left[a_{21}(\eta)f_{1,0}(\xi,\eta)+a_{22}(\eta)f_{2,0}(\xi,\eta)\right] d\xi \end{aligned}\right\} \ldots\ldots (47)$$

where the integral on the right-hand side is taken along the respective characteristics, as before. But $|f_{1,0}(x,y)| = |F(x_1)| \leq M$ for $x \leq X$, since $x_1 \leq x$, and similarly $|f_{2,0}(x,y)| = |G(x_2)| \leq M$.
Hence

$$|d_{1,1}(x,y)| = \int_{x_1}^{x} \left[a_{11}(\eta)f_{1,0}(\xi,\eta)+a_{12}(\eta)f_{2,0}(\xi,\eta)\right] d\xi$$

$$\leq 2AM\int_{x_1}^{x} dx \leq 2AM(x-x_o),$$

and similarly

$$|d_{2,1}(x,y)| \leq 2AM(x-x_o).$$

Also for $n \geq 2$,

$$d_{1,n}(x,y) = \int_{x_1}^{x} \left[a_{11}(\eta)d_{1,n-1}(\xi,\eta)+a_{12}(\eta)d_{2,n-1}(\xi,\eta)\right] d\xi$$

$$d_{2,n}(x,y) = \int_{x_1}^{x} \left[a_{21}(\eta)d_{1,n-1}(\xi,\eta)+a_{22}(\eta)d_{2,n-1}(\xi,\eta)\right] d\xi$$

and so we obtain successively

$$|d_{1,2}(x,y)| \leq (2A)^2M\frac{|x-x_o|^2}{2}, \quad |d_{2,2}(x,y)| \leq (2A)^2M\frac{|x-x_o|^2}{2}$$

. . . .

$$|d_{1,n}(x,y)| \leq (2A)^nM\frac{|x-x_o|^n}{n!}, \quad |d_{2,n}(x,y)| \leq (2A)^nM\frac{|x-x_o|^n}{n!}$$

/We ......

We conclude in the usual way (compare Refs. 6,7) that the series $\sum d_{1,n}$, $\sum d_{2,n}$ converge uniformly in any bounded sub-region of R, and hence that the sequences $f_{1,n}(x,y)$ and $f_{2,n}(x,y)$ converge to functions $f_1(x,y)$ and $f_2(x,y)$ which satisfy (43) as well as the boundary conditions $f_1(x,y_1) = F(x)$, $f_2(x,y_2) = G(x)$. If we assume moreover that $F(x)$ and $G(x)$ are differentiable, then we may show that the functions $\partial f_1/\partial x$, $\partial f_2/\partial x$, $\partial f_1/\partial y$, $\partial f_2/\partial y$ exist so that D_1f_1/Dx and D_2f_2/Dx may be replaced by

$$\frac{\partial f_1}{\partial x} + c_1(y)\frac{\partial f_1}{\partial y} \quad \text{and} \quad \frac{\partial f_2}{\partial x} + c_2(y)\frac{\partial f_2}{\partial y}$$

respectively. It follows that $f_1(x,y)$ and $f_2(x,y)$ satisfy (40). We may also show, by the procedure adopted for ordinary differential equations, that the solution is unique. It follows from the construction that the values of $f_1(x,y)$ and $f_2(x,y)$ at any given point P involve only the values of $F(x)$ for $x \leq x_1$, and of $G(x)$ for $x \leq x_2$.

The above theory can be applied directly to the case discussed in Section 3 only so long as $y_2 > y_s$, since the coefficients of (24) become infinite for $y = y_s$. However, the boundary condition remains determinate as $y_2 \to y_s$ and we therefore conclude, subject to the limitations of linearised theory, that (33) is the correct boundary condition for $y_2 = y_s$.

6. General Properties of the Solution.

The incoming wave in the supersonic region is given in the form of a function of one variable, $F(x+\beta_0 y) = F(z)$, say, while the outgoing wave is expressed as $G(x-\beta_0 y) = G(z)$. And by (11), (13) and (18) if

$$F(z) = \int_{-\infty}^{\infty} (P(k) + iQ(k))e^{-ikz}\, dk \quad \ldots\ldots\ldots\ldots\ldots (48)$$

then

$$G(z) = -\int_{-\infty}^{\infty} (P(k)+iQ(k))\frac{C'(y_w;k)-iS'(y_w;k)}{C'(y_w;k)+iS'(y_w;k)}e^{-ikz}\, dk \ldots (49)$$

/We may.....

We may use the linearised version of Bernoulli's equation

$$p = V \frac{\partial \Phi}{\partial x} \quad \text{........................} (50)$$

to find the pressure increment p due to any perturbation potential Φ in the supersonic region. Its applicability in the region of variable main stream velocity is doubtful. The form of (50) shows that if the pressure increment p_i due to incoming wave is given by $p_i/\rho_o V_o = F(x-\beta_o y)$ where $F(z)$ is defined by (48) then the pressure increment p_r due to the resulting outgoing wave is given by $p_r/\rho_o V_o = G(x-\beta_o y)$, where $G(z)$ is defined by (49).

It was mentioned earlier that our analysis should still yield the (theoretically) correct answer if the wall is placed in the supersonic region so that β is constant, $\beta = \beta_o$, for all $y \geq y_w$. It may serve as a check on our calculations to show that this is indeed the case.

We now have $C(y;k) = \cos k\beta_o y$, $S(y;k) = \sin k\beta_o y$, and so

$$C'(y_w;k) \pm iS'(y_w;k) = -k\beta_o(\sin k\beta_o y_w \mp i\cos k\beta_o y_w)$$

$$= \pm ik\beta_o(\cos k\beta_o y_w \pm i\sin k\beta_o y_w).$$

Hence

$$\frac{C'(y_w;k)-iS'(y_w;k)}{C'(y_w;k)+iS'(y_w;k)} = -e^{-2ik\beta_o y_w},$$

so that (49) becomes, for a given function F(z)

$$G(z) = \int_{-\infty}^{\infty} (P(k)+iQ(k))e^{-2ik\beta_o y_w} e^{-ikz} dk$$

$$= \int_{-\infty}^{\infty} (P(k)+iQ(k))e^{-ik(z+2\beta_o y_w)} dk$$

$$= F(z + 2\beta_o y_w)$$

and this is the correct answer.

/Coming ......

Coming back to the general case we may use Fourier's integral theorem to express G(z) directly in terms of F(z). Inverting (48) we obtain

$$P(k) + iQ(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{ikt}\,dt \quad \text{.....................} (51)$$

Substitution in (49) then yields

$$G(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C'(y_W;k)-iS'(y_W;k)}{C'(y_W;k)+iS'(y_W;k)}dk \int_{-\infty}^{\infty} F(t)e^{ik(t-z)}dt \text{....} (52)$$

We shall now deduce the following general principle.

If G(z) is the total reflected wave for a specified incoming wave F(z) as given by (49), then $G^*(z) = F(-z)$ is the reflected wave corresponding to an incoming wave $F^*(z) = G(-z)$.

In fact by (52)

$$G^*(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C'(y_W;k)-iS'(y_W;k)}{C'(y_W;k)+iS'(y_W;k)}dk \int_{-\infty}^{\infty} F^*(t)e^{ik(t-z)}dt$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C'(y_W;k)-iS'(y_W;k)}{C'(y_W;k)+iS'(y_W;k)}dk \int_{-\infty}^{\infty} G(-t)e^{ik(t-z)}dt.$$

Putting $\tau = -t$, $\zeta = -z$, $\ell = -k$, we obtain

$$G^*(-\zeta) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C'(y_W;\ell)-iS'(y_W;\ell)}{C'(y_W;-\ell)+iS'(y_W;\ell)}d\ell \int_{-\infty}^{\infty} G(\tau)e^{-i\ell(\zeta-\tau)}d\tau$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C'(y_W;\ell)+iS'(y_W;\ell)}{C'(y_W;\ell)-iS'(y_W;\ell)}d\ell \int_{-\infty}^{\infty} G(\tau)e^{i\ell(\tau-\zeta)}d\tau$$

.... (53)

On the other hand, by the inversion of (49)

$$(P(k)+iQ(k))\frac{C'(y_W;k)-iS'(y_W;k)}{C'(y_W;k)+iS'(y_W;k)} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} G(t)e^{ikt}dt$$

/and so ....

and so, by (48)

$$F(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{C'(y_W;k)+iS'(y_W;k)}{C'(y_W;k)-iS'(y_W;k)}dk\int_{-\infty}^{\infty}G(t)e^{ik(t-z)}dt \quad\ldots\ldots(54)$$

Comparing (53) and (54) we find $G^{*}(z) = F(-z)$, as asserted. The above principle shows that while a sharp front, such as presented by a pure compression is smoothed out by the subsonic layer, the opposite effect is equally likely - at least in theory. In reality the conclusion is modified by the intervention of viscosity effects etc., which are neglected in our theory.

Another general relation between the incoming and the outgoing wave is

$$\int_{-\infty}^{\infty}\left[F(z)\right]^2 dz = \int_{-\infty}^{\infty}\left[G(z)\right]^2 dz \quad\ldots\ldots\ldots\ldots\ldots\ldots(55)$$

In fact, by applying Parseval's formula to (48) we obtain

$$\int_{-\infty}^{\infty}\left[F(z)\right]^2 dz = \int_{-\infty}^{\infty}\left|P(k)+iQ(k)\right|^2 dk.$$

Also, by applying the same formula to (49) -

$$\int_{-\infty}^{\infty}\left[G(z)\right]^2 dz = \int_{-\infty}^{\infty}\left|(P(k)+iQ(k))\frac{C'(y_W;k)-iS'(y_W;k)}{C'(y_W;k)+iS'(y_W;k)}\right|^2 dk$$

$$= \int_{-\infty}^{\infty}\left|P(k)+iQ(k)\right|^2 dk,$$

since $\left|\dfrac{C'(y_W;k)-iS'(y_W;k)}{C'(y_W;k)+iS'(y_W;k)}\right| = 1$, for all real k.

This proves our assertion.

We conclude this discussion of some of the mathematical features with the calculation of the variation of a discontinuity in the incoming wave along the Mach lines of the transonic region.

Thus assume that the function $f_1(x,y) = \beta F'(x + \beta y)$, which represents the incoming wave in the supersonic region, possesses a discontinuity of magnitude $j = j_0$ along the Mach line NP. Let PQ be the continuation of

/NP across......

NP across the transonic region. Writing down the first equation of (26) at points just downstream of NPQ, and at points just upstream of NPQ and subtracting, we obtain the following equation for the variation of the discontinuity j along NPQ.

$$\frac{D_1 j}{Dy} = \frac{1}{2\beta}\frac{d\beta}{dy}\, j \qquad \text{.....................(56)}$$

Regarding j, which is defined on NPQ, as a function of y we may integrate with respect to y and obtain

$$j(y) = \text{const.}\ \sqrt{\beta}(y).$$

Since $j = j_o$ in the supersonic region where $\beta = \beta_o$, we therefore have

$$\frac{j(y)}{j_o} = \sqrt{\frac{\beta(y)}{\beta_o}} \qquad \text{.....................(57)}$$

Equation (57) shows that any initial discontinuity is diffused completely by the time it reaches the sonic line (where $\beta(y) = 0$). This is consistent with the results provided by more exact theories.

7. Numerical Examples.

The exact evaluation of (49) or (52) for given F(z) appears to be, generally speaking, impossible. However, it can be shown that subject to the condition

$$\int_{-\infty}^{\infty} |P(k) + iQ(k)|\, dk < \infty \qquad \text{.....................(58)}$$

legitimate approximations to (49) are obtained by replacing the functions C(y;k), S(y;k) on the right-hand side by their partial sums of specified order, as given by (4), (7) and (8). If moreover P(k)+iQ(k) is a rational function of k then the resulting integral can be evaluated by the calculus of residues. A function F(z) such that P(k)+iQ(k) satisfies all the requisite conditions is given by

$$F(z) = \begin{cases} 0 & \text{for } z < z_o \\ e^{-\gamma(z-z_o)} & \text{for } z_o < z < z_1 \\ e^{-\gamma(z-z_o)} - e^{-\gamma(z-z_1)} & \text{for } z > z_1 \end{cases} \qquad \text{.....................(59)}$$

/where.....

where $z_0 < z_1$, and $\gamma > 0$. If we let z_1 tend to ∞ while keeping z_0 constant, we obtain an incoming wave of the type considered by Tsien and Finston

$$F(z) = \begin{cases} 0 & \text{for } z < z_0 \\ e^{-\gamma(z-z_0)} & \text{for } z > z_0, \end{cases}$$

and if furthermore we let γ tend to 0, then we obtain a step function

$$F(z) = \begin{cases} 0 & \text{for } z < z_0 \\ 1 & \text{for } z > z_0, \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots (60)$$

which corresponds to a simple compression or expansion wave.

Numerical calculations were carried out for the case of a simple incoming wave as given by (60) and for a typical laminar velocity profile specified by

$$\left.\begin{aligned} V(y) &= V_0 && \text{for } y > 0 \\ V(y) &= V_0 \cos \frac{\pi y}{2y_w} && \text{for } 0 > y > y_w \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (61)$$

The functions $C(y;k)$ and $S(y;k)$ were approximated by their seventh partial sums. Figs. 3 and 4 show the total pressure increment Δp at the 'outer edge' of the boundary layer, i.e. for $y_0 = 0$, at a distance $|y_w|$ from the wall. The abscissae are measured in multiples of $|y_w|$.

The effect of an incoming wave concentrated at the origin can be calculated by differentiating the curves in Figs. 3 and 4, and approximate numerical results for other incoming waves may then be obtained by integration. However all the numerical results appear to be inadequate as shown by the discussion in Section 1 above.

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No.	Author	Title, etc.
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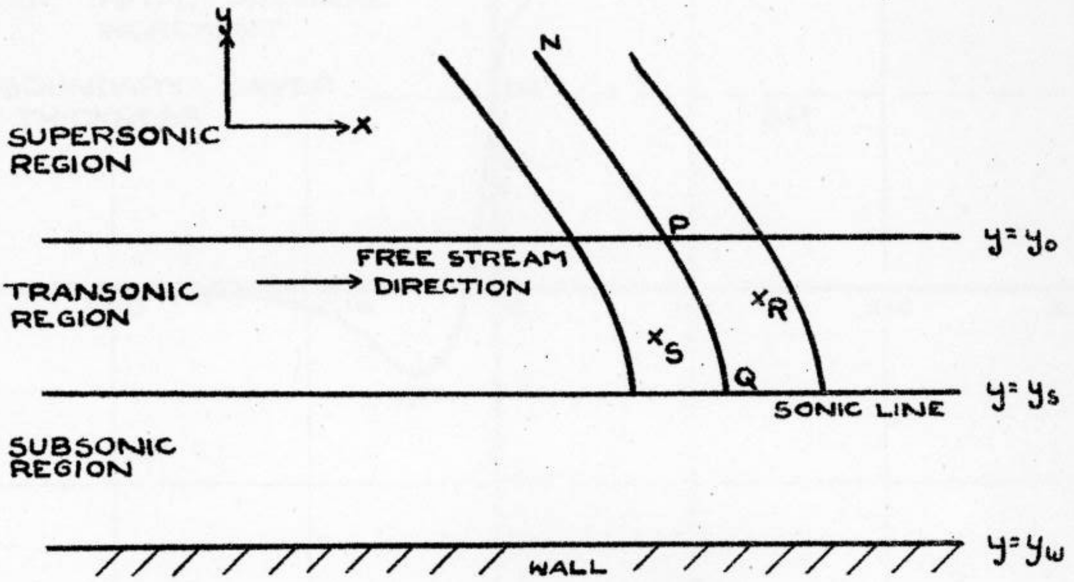
2.	L. Howarth	The propagation of steady disturbances in a supersonic stream bounded on one side by a parallel subsonic stream. Proc. Cambridge Phil. Soc., Vol. 4, pp. 380-390, 1948.
3.	H.S. Tsien and M. Finston	Interaction between parallel streams of supersonic velocities. Journal Aero. Sci. Vol. 16, pp. 515-528, 1949.
4.	Transonic Research Group	Problems in shock reflection. GALCIT Report, 1949 (Contract W 33-038 ac-1717, 11592).
5.	L. Lees	Interaction between the laminar boundary layer over a plane surface and an incident oblique shock wave. Princeton University, Aero. Eng. Lab. Report No. 143, Jan. 1949.
6.	E. Picard	Traité d'Analyse, Vol. 2, pp. 340-351, 1904.
7.	E.L. Ince	Ordinary differential equations, pp. 63-75. U.S. edition, 1944.



(ONLY MACH LINES OF INCOMING WAVES ARE SHOWN e.g. N.P.Q,

FIG 1

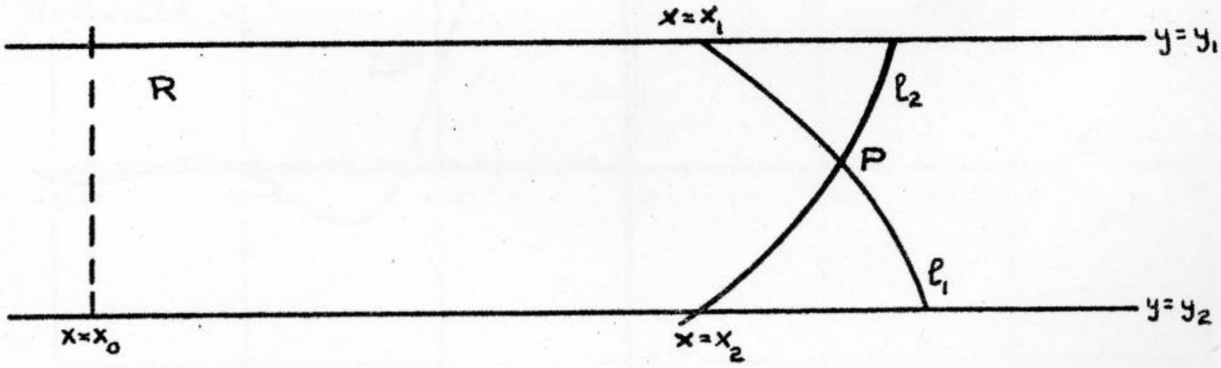
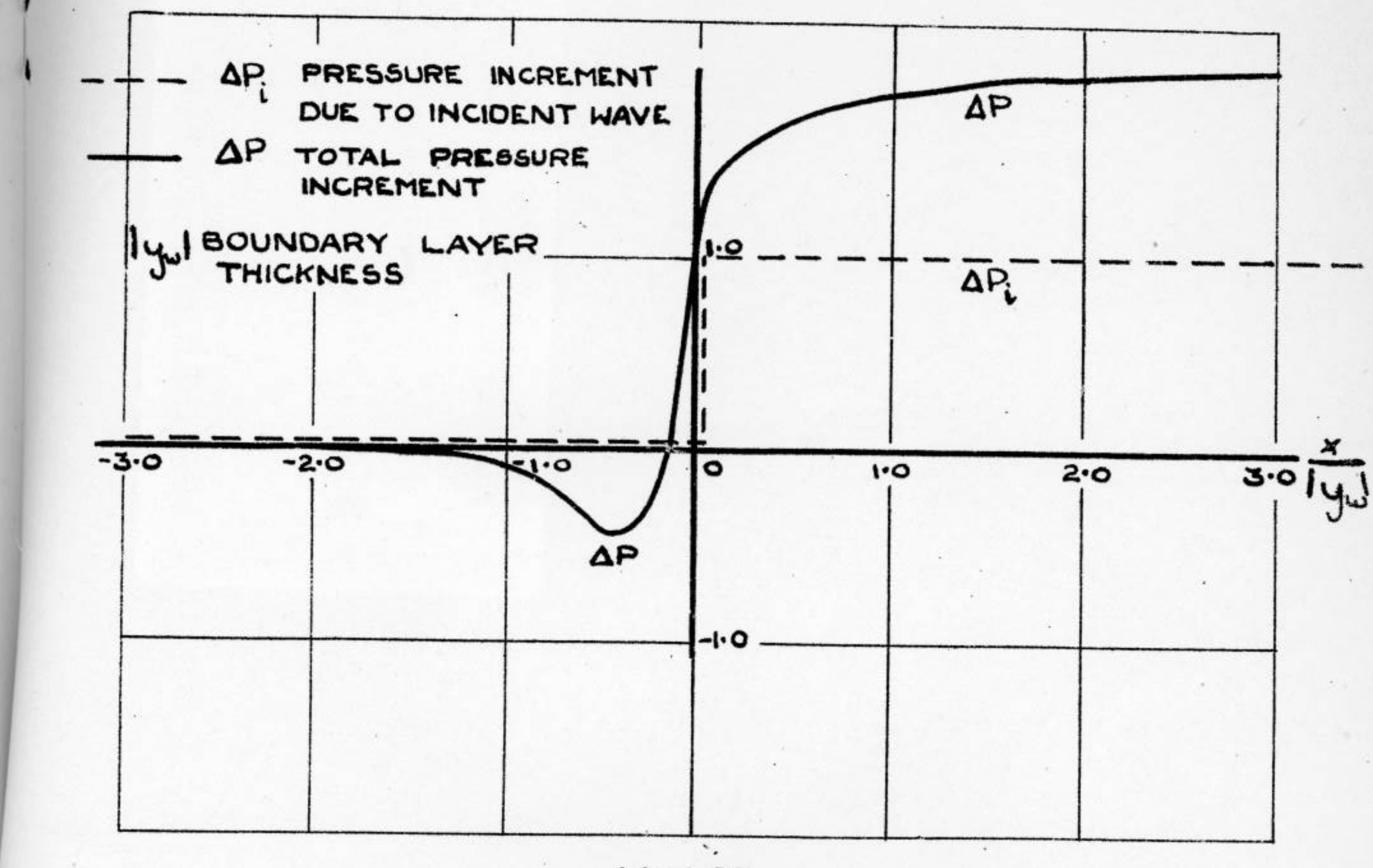
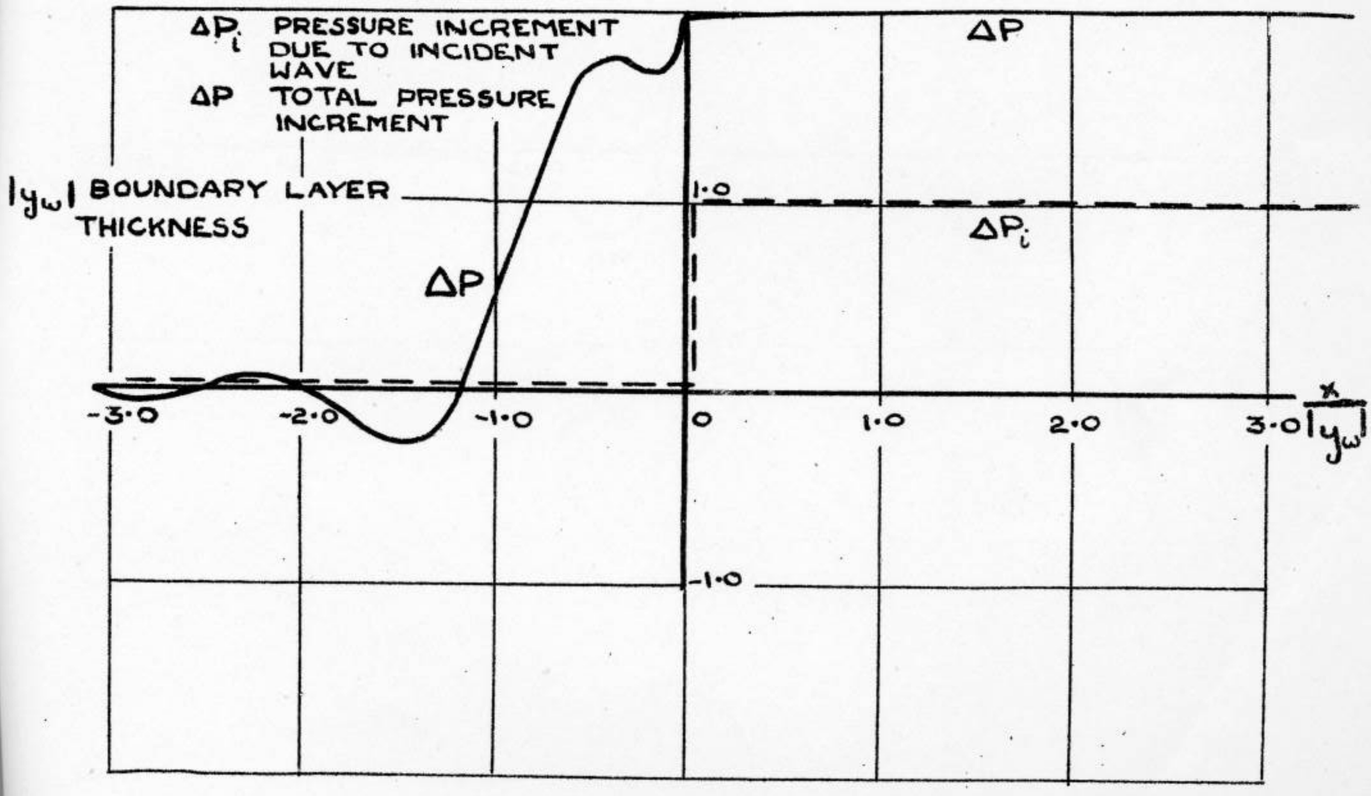


FIG 2



M = 1·25

FIG 3



M = 1·75

FIG 4