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On the Application of Oblique Coordinates to Problems of Plane Elasticity and Swept-Back Wing Structures

-by-

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SUMMARY

The object of this report is two-fold. On the mathematical side it seeks to illustrate the use of oblique coordinates in applications to Elasticity and Structure Theory. On the practical side it seeks to provide methods by which designers can solve problems of stress distribution and deflection for the case of swept-back wing structures, whose ribs lie parallel to the direction of flight.

The report is divided into three parts. In Part I the mathematical basis is developed. Formulae are derived which express the fundamental concepts and relations of Geometry, Kinetics, Statics and Plane Elasticity in terms of vector components in oblique coordinates. In Part II, the results obtained in Part I are applied to a uniform, symmetrical, rectangular section, swept-back box. A complete theory of stress distribution and deflections is obtained for the case of loading by 'normal' forces and couples* applied to the ends of the box. Some consideration is also given to problems of constraint against warping. In Part III the main results of Part II are generalised to cover the case of a more representative wing structure. This represents an extension of the usual Engineer's Theory of Bending and Torsion to cover the case of swept-back wings with ribs parallel to the flight direction. Practical procedures based upon this extension are laid down for stress distribution and deflection calculations. These will have the same validity for swept-back wings, as the usual design approximations have for the unswept case.

* Forces whose directions and couples where planes are normal to the plane of sweep-back

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Definition of the Symbols Employed

Geometry. Dimensions

$O(x,y,z)$ Main system of oblique Cartesian Coordinates (see Fig.1)

$O(X,Y,z)$ Auxiliary system of oblique coordinates (see Fig.1)

α Angle between the axes Ox, Oy.

(x,y,z) Coordinates of a point referred to axes $O(x,y,z)$

i, j, k, i_1, j_1 Unit vectors in the directions Ox, Oy, Oz, OX, OY respectively.

$\bar{r}$ Position vector

ds Length of vector $d\bar{r}$

θ Angle between $d\bar{r}$ and i

dS Length of the material element $d\bar{r}$ after strain

l Length of a plate or box measured x-wise

c Half width of a plate or box measured y-wise

b Half depth of a box measured z-wise or, in particular, the half depth of the spar $y = c$

b' Half depth of the spar $y = -c$

$\zeta(y)$ Ordinate of the skin line

$\bar{\zeta}$ Mean value of ζ over width $-c \leqslant y \leqslant c$.

t Thickness of the skins

t_W Thickness of the spar webs or in particular, thickness of the web $y = c$

$t_{W'}$ Thickness of the web $y = -c$

t_R Thickness of a diaphragm rib

A Section area of spar flange or in particular, section area of flanges at $y = c$.

A' Section area of flanges at $y = -c$

A_s Section area of stringer

A_R Section area of rib flange

a_s Stringer pitch measured parallel to the ribs

a_R Rib pitch measured parallel to the stringers

$\gamma_R = t_R / a_R$

η Parameter defining the point of action of a certain shear stress distribution in a box. (see equation (143))

η^* Parameter defining a torque axis for use in the calculation of twist (see equation (159))

Kinematics

$\bar{u}$ Displacement vector

(u,v,w) Oblique components of $\bar{u}$

$U = u + v \cos\alpha$

$V = u \cos\alpha + v$

$\bar{p}$ Rotation vector

/(p,q,r) Oblique ....

(p,q,r)	Oblique components of $\overline{p}$ or in particular, components of rib rotation.
C, C_1, C_2	Constants defining a rigid body movement in a plane (see equation 17)
v_R, w_R	Components of rib displacement in the plane of the rib in y and z directions respectively.
w_{Ro}	$= (w_R)_{y=o}$
K_1, K_2	Arbitrary constants occuring in expression for w_{Ro} (equation (65))
u_{W1} w_W, u_W', w_W'	Components of web displacement in directions x and z respectively. Undashed:- $y = c$, dashed:- $y = -c$.
W	Rigid body translation of a rib in z-direction. Defined as $W = w_{Ro}$ in Part II and $W = (w_W+w_W')/2$ in Part III
W_S	'Additional deflection due to shearing' (see § 3.3(5)).
ω	Warping displacement function (eq./118)) or warping displacement itself (equation 131)
ω_1, ω_2	Functions of y occuring in expression for ω (equation 132)
Δ	Section distortion function (equation 118) or section distortion displacement itself. (equation 131)
Δ_1, Δ_2	Functions of y occuring in expression for Δ (equation 132)
p_1, p_2	Constants in expression for p (equation 132)
q_1, q_2	Constants in expression for q (equation 132)
Υ	'Shear Deflection' constant occuring in expression for W (equation 132)
e	Strain in arbitrary direction
e_{xx}, e_{yy}, e_{xy}	Strain components in oblique coordinate system $O(x,y)$.
e_{YY}, e_{xY}	Strain components in rectangular coordinate system $O(x,Y)$
ω	Rotation of an element $\mathbf{d\overline{r}}$.

<u>Statics</u>

$\overline{F}$	Force vector
(X,Y,Z)	Oblique components of $\overline{F}$. Also in Parts II, III, Z is used as resultant z-wire force across a section of a box.
(L,M,N)	Oblique components of a couple - axis $O(x,y,z)$
(L_1, M_1)	Oblique components of a couple - axis $O(X,Y)$. Used also as resultant couple acting across a section of a box.
$T_1, T_2, S(=S_1=S_2)$	Stress resultants in a plate refered to oblique axes $O(x,y)$ (see Fig. 3)
ϕ	Stress function. (see equation 22)

/$\overline{T}_1, \overline{T}_2$, ...

$\overline{T}_1, \overline{T}_2, \overline{S}$ Stress resultants in a plate refered to axes O(x,Y) (see Fig. 4)

$T_1', T_1'', T_2', T_2'', S', S''$ Functions of y occuring in expressions for T_1, T_2, S in equation (37)

S_R Shear per unit length in the ribs, estimated per unit span (x-wise)

S_W, S_W' Shear per unit length in the webs $y = c$, $y = -c$ respectively.

$\overline{L}_1$ Couple component (oblique) about an X-wise axis through a point $y = \eta c$, $z = o$ on a cross-section of a box (equation 145)

L_1^* Ditto about axis through $y = \eta^+ c$, $z = o$ (equation 159).

Elasticity. Influence Coefficients

E Young's Modulus

σ Poisson's Ratio

a_{ij} Matrix relating stress resultants and strains (equation 24)

$(a_{ij})_P$ Part of a_{ij} arising from the plate (equation 27)

$(a_{ij})_R$ Part of a_{ij} arising from the reinforcing members (equation 28)

A_{ij} Matrix inverse to a_{ij} (equation 31)

$\overline{a}_{11}, \overline{a}_{13} = \overline{a}_{31}, \overline{a}_{33}$ Special combinations of a_{ij} (equation 120)

C_{ij} Matrix relating rates of rotation of the ribs with the couple transmitted in a box (see equation 99, 100, 157, 158, 160 and 161)

C_{13} Constant in formula for P_1 (equation 157)

I 'Second Moment of Area' for a swept box (equation 142)

Miscellaneous Parameters and Constants

$A_i (i=0,1,2,3,4)$ constants in expressions for linearly varying stresses in a plate. (see equation (40) and § 2.4)

μ Constant defining the rate of die-away of a special stress system (see equations 44, 47)

μ_i Sequence of values of μc defined by equation (114)

$\lambda_i (i=1,2,3,4)$ Values of λ satisfying equation (46)

$B_i (i=0,1,2,3,4)$ Arbitrary constants in equations 43, 47.

$B_{ij} (i,j = 0,1,2,3,4)$ Coefficients of the linear equations for B_i (See equations 108, 109, 110, 111, 112, 113)

$\beta_j(\ )$ Cofactors of B_{4j} in the determinant $|B_{ij}|$.

C_μ Sequence of arbitrary constants (equations 116, 117)

$P_i, Q_i (i=1,2,3)$ Constants relating rates of rib rotation to Couple transmitted and section warping (eq. 125,126).

D Denominator in expressions for P_i, Q_i (eq. (126))

R_1, R_2, β Constants in the warping equation 127. (see 128)

/Part I. ...

Part I. Generalities and Applications to Problems of Two-Dimensional Elasticity

1.1. Geometry

The frame of reference used in this report is a system of oblique Cartesian coordinates. This system is shown in Fig. 1. The basic axes are $O(x,y,z)$. The angle $x\hat{O}y$ has magnitude α. The axis Oz is at right angles to the plane xOy, and is such that a rotation which brings Ox into the position Oy is right-handed about Oz. Use is also made of auxiliary axes $O(X,Y)$ lying in the plane xOy and such that $O(X,y,z)$ and $O(x,Yz)$ form systems of right-handed rectangular cartesian axes.

It is convenient to introduce unit vectors i,j,k,i_1,j_1 lying in the directions Ox,Oy,Oz,OX,OY respectively. These quantities satisfy, as is easily shown, the following relations.-

$$\left.\begin{aligned} i_1 &= i \operatorname{cosec} \alpha - j \cot \alpha \\ j_1 &= - i \cot \alpha + j \operatorname{cosec} \alpha \end{aligned}\right\} \qquad (1)$$

$$i^2 = j^2 = k^2 = 1, \quad i.j = \cos \alpha, \quad j.k = k.i = 0 \qquad (2)$$

$$\left.\begin{aligned} & ixi = jxj = kxk = 0 \\ & ixj = k \sin \alpha, \quad jxk = i_1, \quad kxi = j_1 \end{aligned}\right\} \qquad (3)$$

The position vector $\bar{r}$ of a point with coordinates (x,y,z) may be written.-

$$\bar{r} = xi + yj + zk \qquad (4)$$

If the length of the differential vector $d\bar{r}$ be denoted by ds, we find from (4) and (2).-

$$ds^2 = d\bar{r}^2 = (dx\, i + dy\, j + dz\, k)^2 = dx^2 + dy^2 + dz^2 + 2dxdy \cos \alpha \qquad (5)$$

The vector $\frac{d\bar{r}}{ds}$ is a unit vector. For the special case in which this vector lies in the plane Oxy (i.e. when $dz/ds = 0$) and is inclined at an angle θ to the axis Ox, we find for the components $\frac{dx}{ds}, \frac{dy}{ds}$ the formulae.-

$$\frac{dx}{ds} = \frac{\sin(\alpha-\theta)}{\sin \alpha}, \quad \frac{dy}{ds} = \frac{\sin \theta}{\sin \alpha} \qquad (6)$$

The relations (6) may be established using (2) and the formulae $i.\frac{d\bar{r}}{ds} = \cos \theta$ and $j.\frac{d\bar{r}}{ds} = \cos(\alpha-\theta)$, or by a simple trigonometrical calculation.

1.2. Kinematics

Any vector may be expressed, as in (4), as a linear combination of i,j,k. The displacement of a point $\bar{u}$ and the rotation about an axis $\bar{p}$ may be written.-

$$\left.\begin{aligned}\bar{u} &= ui + vj + wk \\ \bar{p} &= pi + qj + rk\end{aligned}\right\} \qquad (7)$$

The combinations (u,v,w) and (p,q,r) may be termed the 'components' of the vectors in the axes $O(x,y,z)$, but care must be exercised to avoid applying formulae applicable only to rectangular axes to these quantities. The lengths of vectors are given by formulae like (5). The component u is not the projection of $\bar{u}$ in the direction Ox; this last is given by $u + v \cos\alpha$. If the axis of $\bar{p}$ passes through O, then the displacement $\bar{u}$ induced at a point with position vector $\bar{r}$ is given by.-

$$\bar{u} = \bar{p} \times \bar{r} \qquad (8)$$

Substituting from (4), (7) into (8) and making use of (3), (1) we find.-

$$\left.\begin{aligned}U &= u + v\cos\alpha = (qz - ry)\sin\alpha \\ V &= u\cos\alpha + v = (rx - pz)\sin\alpha \\ w &= (py - qx)\sin\alpha\end{aligned}\right\} \qquad (9)$$

where U, V are the 'projections' of $\bar{u}$ in the directions Ox, Oy respectively.

In the remaining portions of this paragraph we shall restrict our attention to positions and displacements in the plane xOy. Use will be made of our previous notation, with the understanding that z components, such as w, z etc., are taken equal to zero.

If the plane xOy is subjected to a displacement $\bar{u}(x,y)$, a point at $\bar{r}$ will move to $\bar{r} + u$. The length of an element $d\bar{r}$ will change to dS where,

$$ds^2 = d\bar{r}^2, \quad dS^2 = (d\bar{r} + d\bar{u})^2 \qquad (10)$$

Neglecting terms of second order in the displacement we find for the strain e in the element $d\bar{r}$ the formulae.-

$$e = \frac{dS^2 - ds^2}{2ds^2} = \frac{d\bar{r}}{ds}\,\frac{d\bar{u}}{ds} \qquad (11)$$

Substituting from (4), (7) (with $z = w = 0$) and using (2) we find.-

$$e = e_{xx}\left(\frac{dx}{ds}\right)^2 + e_{yy}\left(\frac{dy}{ds}\right)^2 + e_{xy}\left(\frac{dx}{ds}\,\frac{dy}{ds}\right)$$

where

$$e_{xx} = \frac{\partial U}{\partial x}, \quad e_{yy} = \frac{\partial V}{\partial y}, \quad e_{xy} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \qquad (12)$$

and

$$U = u + v\cos\alpha, \quad V = u\cos\alpha + v$$

The quantities e_{xx}, e_{yy} and e_{xy} may be termed 'components of strain', since the complete deformation is defined in terms of them. The formulae in the second line of (12) are familiar, but it must be noticed that U, V are not the true displacement components.

The direct strain e_{YY}, in the direction OY may be obtained from (12), by making use of (6) with $\theta = \pi/2$. We find.-

$$e_{YY} = e_{xx}\cot^2\alpha + e_{yy}\operatorname{cosec}^2\alpha - e_{xy}\cot\alpha\operatorname{cosec}\alpha \qquad (13)$$

/The rotation .....

The rotation ϑ of an element $d\bar{r}$ is given (see Fig. 2) by the formula.-

$$\vartheta = \frac{dv}{ds}\sin(\alpha-\theta) - \frac{du}{ds}\sin\theta \tag{14}$$

Using (14), (12) and (6) we can show that the shear strain e_{xY} associated with the directions Ox, OY is given by.-

$$e_{xY} = (\vartheta)_{\theta=0} - (\vartheta)_{\theta=\pi/2} = -2e_{xx}\cot\alpha + e_{xy}\operatorname{cosec}\alpha \tag{15}$$

When the strain components satisfy a compatibility relation.-

$$\frac{\partial^2 e_{xy}}{\partial x\partial y} = \frac{\partial^2(e_{xx})}{\partial y^2} + \frac{\partial^2(e_{yy})}{\partial x^2} \tag{16}$$

the second line of (12) may be solved for the displacements U, V. The 'complementory function' for this integration is a 'rigid body motion'.-*

$$U = Cy + C_1, \qquad V = -Cx + C_2 \tag{17}$$

where C, C_1, C_2 are arbitary constants. The results (16), (17) are identical with those for rectangular coordinates and the usual proofs apply.

1.3. Statics

A force $\bar{F}$ may be written,

$$\bar{F} = Xi + Yj + Zk \tag{18}$$

If this force acts at the point $\bar{r}$, its moment about the origin O is $\bar{r}\times\bar{F}$. Using (4) (18) (3) and (1) we find,

$$\left.\begin{aligned} &\bar{r}\times\bar{F} = L_1 i_1 + M_1 j_1 + Nk = Li + Mj + Nk \\ \text{where}\quad &L_1 = yZ - zY \quad M_1 = zX - xZ \quad N = (xY-yX)\sin\alpha \\ \text{and}\quad &L = L_1\operatorname{cosec}\alpha - M_1\cot\alpha \quad M = -L_1\cot\alpha + M_1\operatorname{cosec}\alpha \end{aligned}\right\} \tag{19}$$

The conditions for equilibrium of a system of forces are $\Sigma\bar{F} = 0$, $\Sigma\bar{r}\times\bar{F} = 0$. Reference to (18), (19) shows that these may be written.-

$$\left.\begin{aligned} &\Sigma X = \Sigma Y = \Sigma Z = 0 \\ &\Sigma(yZ - zY) = \Sigma(zX - xZ) = \Sigma(xY - yX) = 0 \end{aligned}\right\} \tag{20}$$

These equations have the same form as for rectangular axes.

Turning now to two-dimensional questions, we define the stress resultants T_1, S_1, T_2 and S_2 for a plate. These are the oblique components of forces per unit length, acting across normal sections parallel to axes Ox and Oy, situated in the middle surface of the plate. The sign convention for these forces is shown in Fig. 3. Consider an element of the plate (dx, dy). The forces acting upon it are shown in Fig. 3. The forces on the edges are determined by the stress resultants; the body force is given by $(Xi + Yj)dxdy$. Application of the rules of (20) gives us the following

/differential ...

* A translation $\bar{u} = \operatorname{cosec}^2\alpha\,\{(c_1 - c_2\cos\alpha)i + (c_2 - c_1\cos\alpha)j\}$ and a rotation about O, $\bar{p} = -Ck\operatorname{cosec}\alpha$. (sec(9)).

differential equations of equilibrium.-

$$\left.\begin{aligned} &\frac{\partial T_1}{\partial x} + \frac{\partial S_2}{\partial y} + X = o \\ &\frac{\partial S_1}{\partial x} + \frac{\partial T_2}{\partial y} + Y = o \\ &S_1 = S_2 = S \text{ (say)} \end{aligned}\right\} \qquad (21)$$

The similarity with the equations in rectangular coordinates will be noticed. If $X = Y = 0$ we can satisfy (21) by introducing a stress function $\emptyset$ such that,

$$T_1 = \frac{\partial^2 \emptyset}{\partial y^2}, \quad T_2 = \frac{\partial^2 \emptyset}{\partial x^2}, \quad S = -\frac{\partial^2 \emptyset}{\partial x \partial y} \qquad (22)$$

It is convenient also to introduce stress resultants $\bar{T}_1$, $\bar{T}_2$, $\bar{S}$ refered to axes $O(x,Y)$. The specification of these is shown in Fig. 4. The relations between the un-barred and barred stress resultants may easily be shown to be.-

$$\left.\begin{aligned} T_1 &= \bar{T}_1 \sin\alpha + \bar{T}_2 \cos\alpha \cot\alpha - 2\bar{S}\cos\alpha \\ T_2 &= \bar{T}_2 \operatorname{cosec}\alpha \\ S &= \bar{S} - \bar{T}_2 \cot\alpha \end{aligned}\right\} \qquad (23)$$

1.4. Stress - Strain Relations

In § 1.2 we studied a system of plane strain refered to oblique axes $O(x,y)$. We now interpret these results as refering to the mean strain across the thickness of a uniform plate. Such a state of strain in a plate will give rise to stresses and stress resultants and in § 1.3 we studied the properties of these forces when refered to our oblique axes. If the material of our plate is elastic and obeys the Generalised Hooke's Law, then the stress resultants T_1, T_2 and S will be related to the strain components e_{xx}, e_{yy} and e_{xy} by homogenous linear equations of the form.-

$$\left.\begin{aligned} T_1 &= a_{11}\, e_{xx} + a_{12}\, e_{yy} + a_{13}\, e_{xy} \\ T_2 &= a_{21}\, e_{xx} + a_{22}\, e_{yy} + a_{23}\, e_{xy} \\ S &= a_{31}\, e_{xx} + a_{32}\, e_{yy} + a_{33}\, e_{xy} \end{aligned}\right\} \qquad (24)$$

where as we shall show later,

$$a_{ij} = a_{ji} \qquad (25)$$

For the special case in which the plate is isotropic with thickness t, Young's Modulus E and Poisson's Ratio σ, known theory applied to the rectangular axes $O(x,Y)$ gives.-

$$\left.\begin{aligned} &\bar{T}_1 = \frac{Et}{(1-\sigma^2)}(e_{xx} + \sigma\, e_{YY}), \quad \bar{T}_2 = \frac{Et}{(1-\sigma^2)}(e_{YY} + \sigma\, e_{xx}) \\ &S = \frac{Et}{2(1+\sigma)}\, e_{xY} \end{aligned}\right\} \qquad (26)$$

/Substitution ....

Substitution from (26) in (23) expresses T_1, T_2, S in terms of e_{xx}, e_{yy}, e_{xy}. Use of (13), (15) throws our relations into the form (24) and so determines the a_{ij} for the isotropic plate. Denoting these results by $(a_{ij})_p$ we find.-

$$(a_{ij})_p = \frac{Et}{(1-\sigma^2)}\cdot \operatorname{cosec}^3\alpha. \begin{pmatrix} 1, & \cos^2\alpha+\sigma\sin^2\alpha, & -\cos\alpha \\ \cos^2\alpha+\sigma\sin^2\alpha, & 1 & -\cos\alpha \\ -\cos\alpha, & -\cos\alpha, & \frac{1+\cos^2\alpha-\sigma\sin^2\alpha}{2} \end{pmatrix} \tag{27}$$

In the case where the plate is reinforced by closely spaced stringers of section area A_s at a pitch a_s running parallel to Ox, and by closely spaced ribs of section area A_R at a pitch a_R running parallel to Oy*, then, if the material of the reinforcements has modulus E, loads of magnitudes respectively $EA_s\, e_{xx}$ and $EA_R\, e_{yy}$ will appear in the stringers and ribs. Distributing the stringers and ribs continuously we generate stress resultants $T_1 = EA_s\, e_{xx}/a_s$ and $T_2 = EA_R\, e_{yy}/a_R$ and so for a reinforced plate we must add to (27) the matrix $(a_{ij})_R$ given by.-

$$(a_{ij})_R = \begin{pmatrix} EA_s/a_s & 0 & 0 \\ 0 & EA_R/a_R & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{28}$$

The complete matrix for a plate reinforced in the directions Ox, Oy is thus.-

$$a_{ij} = (a_{ij})_p + (a_{ij})_R \tag{29}$$

The equations (24) may be solved for e_{xx}, e_{yy}, e_{xy} yielding.-

$$\left.\begin{aligned} e_{xx} &= A_{11}T_1 + A_{12}T_2 + A_{13}S \\ e_{yy} &= A_{21}T_1 + A_{22}T_2 + A_{23}S \\ e_{xy} &= A_{31}T_1 + A_{32}T_2 + A_{33}S \end{aligned}\right\} \tag{30}$$

where,

$$A_{ij} = \frac{1}{|a_{ij}|}\begin{pmatrix} a_{22}a_{33}-a_{23}^2, & a_{23}a_{31}-a_{21}a_{33}, & a_{21}a_{32}-a_{31}a_{22} \\ a_{13}a_{32}-a_{12}a_{33}, & a_{11}a_{33}-a_{13}^2, & a_{12}a_{31}-a_{11}a_{32} \\ a_{12}a_{23}-a_{22}a_{13}, & a_{13}a_{21}-a_{11}a_{23}, & a_{11}a_{22}-a_{12}^2 \end{pmatrix} \tag{31}$$

1.5. Compatibility Relation for the Stress Resultants

The strain components must satisfy (16). It follows from (30) that the stress resultants must satisfy.-

$$\left(A_{11}\frac{\partial^2}{\partial y^2} + A_{21}\frac{\partial^2}{\partial x^2} - A_{31}\frac{\partial^2}{\partial x\partial y}\right)T_1$$

$$+\left(A_{12}\frac{\partial^2}{\partial y^2} + A_{22}\frac{\partial^2}{\partial x^2} - A_{32}\frac{\partial^2}{\partial x\partial y}\right)T_2$$

$$+\left(A_{13}\frac{\partial^2}{\partial y^2} + A_{23}\frac{\partial^2}{\partial x^2} - A_{33}\frac{\partial^2}{\partial x\partial y}\right)S = 0 \tag{32}$$

/In the ....

* a_s and a_R are measured parallel to Oy and Ox respectively.

In the case where a stress function ϕ exists we can substitute from (22) into /32) obtaining.-

$$A_{22}\frac{\partial^4\phi}{\partial x^4} - 2A_{23}\frac{\partial^4\phi}{\partial x^3\partial y} + (2A_{12} + A_{33})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2A_{13}\frac{\partial^4\phi}{\partial x\partial y^3} + A_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \tag{33}$$

1.6. Application to Certain Plate Problems

The theories of displacement, strain and stress developed in the previous sections are particularly applicable to plates whose boundaries consist of parallelograms. Let us therefore turn our attention to a plate whose edges lie along the lines $x = o$, $x = l$, $y = \pm c$. (Fig. 5)

We shall not seek here to solve problems with given boundary conditions, but following the 'inverse' method of St. Venant, shall impose certain restrictions on the stress distribution and examine the consequences. However, with an eye on applications to wings, we shall restrict our discussion to solutions which satisfy.-

$$T_2 = 0 \quad \text{when} \quad y = \pm c \tag{34}$$

Let us begin with the simplest of all cases in which the stress resultants are constant.* Equation (34) then implies that $T_2 = 0$ everywhere. The edges $x = o,l$ are loaded by uniform T_1 and S, while the edges $y = \pm c$ are loaded by a uniform S. Writing $T_2 = o$ in (30) we find the following formulae for the constant strain components.-

$$\left.\begin{aligned} e_{xx} &= A_{11}T_1 + A_{13}S \\ e_{yy} &= A_{21}T_1 + A_{23}S \\ e_{xy} &= A_{31}T_1 + A_{33}S \end{aligned}\right\} \tag{35}$$

The displacements follow from (12). The complementory function for this integration is given by (17). We thus find.-

$$\left.\begin{aligned} U &= e_{xx}.x + (e_{xy} + C)y + C_1 \\ V &= e_{yy}.y - Cx + C_2 \end{aligned}\right\} \tag{36}$$

As a second example let us consider another case in which $X = Y = 0$ and assume that the stress resultants vary linearly with x. We write

$$T_1 = x\,T_1' + T_1'', \quad T_2 = x\,T_2' + T_2'', \quad S = x\,S' + S'' \tag{37}$$

where T_1', T_1'', T_2', T_2'', S' and S'' are functions of y. Substituting in (21) with $X = Y = 0$ and using (34) we easily show that,

$$T_1' = -\frac{dS''}{dy} \qquad T_2' = T_2'' = S' = 0 \tag{38}$$

/Substituting from (37) ...

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* This satisfies (34) and implies $X = Y = 0$ by (21).

Substituting from (37) and using (38) we find that.-

$$\left.\begin{aligned} T_1'' &= -\frac{A_{13}}{A_{11}} A_2 y^2 + A_3 y + A_4 \\ S'' &= \tfrac{1}{2} A_2 y^2 + A_1 y + A_o \end{aligned}\right\} \qquad (39)$$

where $A_i\,(i = 0,1,2,3,4)$ are arbitrary constants. Substituting from (38), (39) into (37) we obtain,

$$\left.\begin{aligned} T_1 &= -A_2\left(xy + \frac{A_{13}}{A_{11}} y^2\right) - A_1 x + A_3 y + A_4 \\ T_2 &= 0 \\ S &= \tfrac{1}{2}A_2 y^2 + A_1 y + A_o \end{aligned}\right\} \qquad (40)$$

Substituting in (30) and using (12) we find the following expressions for (U,V).

$$\begin{aligned} U = {} & A_o(A_{13}x + A_{33}y) + A_4(A_{11}x + A_{31}y) \\ & + A_1\left\{-\tfrac{1}{2}A_{11}x^2 + A_{13}xy + \tfrac{1}{2}(A_{21} + A_{33})y^2\right\} \\ & + A_3(A_{11}xy + \tfrac{1}{2}A_{31}y^2) \\ & + A_2\left\{-\tfrac{1}{2}A_{11}x^2y - \tfrac{1}{2}A_{13}xy^2 + (1/3)\left(\tfrac{1}{2}A_{21} - \frac{A_{13}^2}{A_{11}} + \tfrac{1}{2}A_{33}\right)y^3\right\} + (Cy + C_1) \end{aligned} \qquad (41)$$

$$\begin{aligned} V = {} & A_o \cdot A_{23}y + A_4 \cdot A_{21}y + A_1(-A_{13}x^2 - A_{21}xy + \tfrac{1}{2}A_{23}y^2) \\ & + \tfrac{1}{2}A_3(-A_{11}x^2 + A_{21}y^2) \\ & + A_2\left\{(1/6)A_{11}x^3 - \tfrac{1}{2}A_{21}xy^2 + (1/3)\left(\tfrac{1}{2}A_{23} - \frac{A_{21}A_{13}}{A_{11}}\right)y^3\right\} + (-Cx + C_2) \end{aligned} \qquad (42)$$

As a third and last example, let us consider a case where the stress decrease exponentially from the root $x = o$ (i.e. vary as $e^{-\mu x}$, where the real part of μ is positive!) For the sake of possible applications to the box structures of Part II, we introduce a body force.-

$$X = 0, \quad Y = -B_o e^{-\mu x} \qquad (43)$$

where B_o is a constant, which may be a complex number.

A particular solution of equations (21) and (32) is easily shown to be

$$T_1 = 0, \quad T_2 = \frac{A_{23}}{\mu A_{22}} B_o e^{-\mu x}, \quad S = -\frac{B_o}{\mu} e^{-\mu x} \qquad (44)$$

The displacements corresponding to (44) follow from (30) and (12) We find.-

$$U = -\frac{a_{13}}{A_{22}|a_{ij}|} \frac{B_o}{\mu^2} e^{-\mu x}, \quad V = \frac{a_{11}}{A_{22}|a_{ij}|} \cdot \frac{B_o}{\mu^2} e^{-\mu x} \qquad (45)$$

/where use .....

where use has been made of the algebraic theorem that the cofactors of $|A_{ij}|$ are given by $a_{ij}/|a_{ij}|$. To obtain a complementary function we make use of (33). Assuming that ϕ varies as $e^{\mu(\lambda y-x)}$, we find that

$$A_{11}\lambda^4 + 2A_{13}\lambda^3 + (2A_{12} + A_{33})\lambda^2 + 2A_{33}\lambda + A_{22} = 0 \qquad (46)$$

Denoting the roots of (46) by $\lambda_i (i=1,2,3,4)$ we find a solution of (33) in the form.-

$$\phi = e^{-\mu x}\sum_{i=1}^{4} B_i e^{\mu\lambda_i y} \qquad (47)$$

where B_i are arbitrary constants (complex numbers).

The stress resultants follow from (22).-

$$\left.\begin{aligned} T_1 &= \mu^2 e^{-\mu x}\sum B_i\lambda_i^2 e^{\mu\lambda_i y}, \quad T_2 = \mu^2 e^{-\mu x}\sum B_i e^{\mu\lambda_i y}, \\ S &= \mu^2 e^{-\mu x}\sum B_i\lambda_i e^{\mu\lambda_i y} \end{aligned}\right\} \qquad (48)$$

The corresponding deflections are found to be.-

$$\left.\begin{aligned} U &= -\mu e^{-\mu x}\sum B_i e^{\mu\lambda_i y}(\lambda_i^2 A_{11} + \lambda_i A_{13} + A_{12}) + Cy + C_1 \\ V &= \mu e^{-\mu x}\sum \frac{B_i}{\lambda_i} e^{\mu\lambda_i y}(\lambda_i^2 A_{21} + \lambda_i A_{23} + A_{22}) - Cx + C_2 \end{aligned}\right\} \qquad (49)$$

Imposing the condition (34) upon our complete solution we find.-

$$\sum B_i e^{\mu\lambda_i c} = \sum B_i e^{-\mu\lambda_i c} = -\frac{A_{23}}{\mu^3 A_{22}} B_o \qquad (50$$

which gives two equations for the constants B_i. The imposition of further boundary conditions at $y = \pm c$ would enable the solution to be completed. This development is reserved until the theory of Part II is formulated.

/Part II. .....

Part II Applications to Simple Swept Back Box Structures

2.1. Description of a Simplified Structure. Notation

In part II we shall apply the results developed in Part I to the study of stress distribution and deflection problems for a uniform swept box. Such a simplified structure, while not reproducing all the characteristics of an actual wing structure, will reveal those properties peculiar to sweep back.

The structure to be considered is a uniform rectangular section box swept back through an angle $\pi/2 - \alpha$. (see Fig. 6) Reference axes $O(x,y,z)$, of the kind defined in § 1.1, are so disposed that the faces of the box are given by $y = \pm c$, $z = \pm b$ and the ends by $x = o$, $x = l$. The faces $z = \pm b$ are termed 'skins'. They have thickness t and are reinforced by x-wise closely spaced stringers of section area A_s and y-wise pitch a_s, and by y-wise closely spaced rib booms of section area A_R and x-wise pitch a_R. The faces $y = \pm c$ are termed 'spar webs'. They have thickness t_w and are assumed to carry only shear stresses. Such direct load carrying capacity as they may possess will be assumed integrated with the 'spar flanges', which run along the four edges of the box and have a cross sectional area A. The corresponding rib booms on the skins $z = \pm b$ are joined by 'rib webs' thickness t_R, which are assumed to carry only shear stresses. These rib webs are of course rigidly attached to the spar webs. The materials of all the components are assumed to have Young's Modulus E and Poisson's Ratio σ.

2.2. Theory of the Simplified Structure

We shall limit ourselves in what follows to cases in which the displacements occuring in the skins $z = \pm b$ are equal and opposite to one another. The notation applied to plates in Part I will here be applied to the 'skin' $z = b$. Corresponding values of displacements and stresses for $z = -b$, can then be obtained by reversal of sign.

Let us begin by considering the rib webs. These are to be treated as continuously distributed in the x direction. The 'thickness' of ribs within an element dx will thus be $\gamma_R dx$ where,

$$\gamma_R = t_R/a_R \tag{51}$$

The shear per unit length carried by the rib web, within dx will be written $S_R dx$ where S_R is a function of x only. The y and z components of displacement in the plane of the rib webs will be denoted by $\bar{v}_R$ and $\bar{w}_R$ respectively. These definitions are illustrated in Fig. 7. The relation between S_R and the displacements is clearly.-

$$S_R = \frac{E\gamma_R}{2(1+\sigma)}\left(\frac{\partial v_R}{\partial z} + \frac{\partial w_R}{\partial y}\right) \tag{52}$$

The kinematics of a 'pure shear carrying plate' are not well defined. We shall therefore in the interests of simplicity, assume that w_R is independent of z, thus attributing a limited rigidity to the ribs. Experience with the theory of unswept

/boxes suggests .....

boxes suggests that this restriction is not of any real significance. Differentiation of (52) with respect to z then shows that v_R is a linear function of z. and so, remembering that rib displacements must conform with those in the skins at $z = \pm b$, we find.-

$$v_R = Vz/b \tag{53}$$

Equations (52) and (53) then yield.-

$$w_R = \frac{2(1+\sigma)S_R}{E\gamma_R} y - \frac{1}{b}\int_0^y V\,dy + w_{Ro} \tag{54}$$

where $w_{Ro} = (w_R)_{y=o}$ is a function of x.

We turn now to the spar webs considering first of all the surface y = c. The x and z wise displacement components in the plane of the spar will be written u_W and w_W respectively. Conformity with the rib displacements implies.-

$$w_W = (w_R)_{y=c} \tag{55}$$

The component w_W is thus independent of z and so, just as in the case of the rib webs, we deduce that u_W is linear in z and thus is given by.-

$$u_W = (U)_{y=c}\cdot \frac{z}{b} \tag{56}$$

since spar web displacements must agree with those in the skins at $z = \pm b$. The shear per unit length in the spar web will be written S_W and is related to u_W and w_W by.-

$$S_W = \frac{Et_W}{2(1+\sigma)}\left(\frac{\partial u_W}{\partial z} + \frac{\partial w_W}{\partial x}\right) \tag{57}$$

The notation for the spar web is illustrated in Fig. 8. S_W is a function of x only and its variation is brought about by the shear S_R applied by the ribs. Equilibrium of an element dz dx yields the equation.-

$$\frac{dS_W}{dx} - S_R = 0 \tag{58}$$

Substituting from (54) into (55) and from (55), (56) into (57) into (58) we find.-

$$\frac{d^2S_R}{dx^2} - \frac{\gamma_R S_R}{ct_W} + \frac{E\gamma_R}{2c(1+\sigma)}\left[\frac{1}{b}\left(\frac{\partial U}{\partial x}\right)_{y=c} - \frac{1}{b}\int_0^c \frac{\partial^2 V}{\partial x^2}dy + \frac{d^2 w_{Ro}}{dx^2}\right] = 0 \tag{59}$$

We shall denote corresponding quantities for the surface y = -c, by the same symbols as for y = c, but with a dash added (i.e. u_W', w_W' and S_W'). The equations corresponding to (55) - (59) are.-

$$w_W' = (w_R)_{y=-c} \tag{60}$$

$$u_W' = (U)_{y=-c}\cdot \frac{z}{b} \tag{61}$$

/S_W' = ...

$$S'_W = \frac{Et_W}{2(1+\sigma)}\left(\frac{\partial u'_W}{\partial z} + \frac{\partial w'_W}{\partial x}\right) \tag{62}$$

$$\frac{dS'_W}{dx} + S_R = 0 \tag{63}$$

$$-\frac{d^2S_R}{dx^2} + \frac{\tau_R}{ct_W}S_R + \frac{E\tau_R}{2c(1+\sigma)}\left[\frac{1}{b}\left(\frac{\partial U}{\partial x}\right)_{y=-c} - \frac{1}{b}\int_0^{-c}\frac{\partial^2 V}{\partial x^2}dy + \frac{d^2 w_{Ro}}{dx^2}\right] = 0 \tag{64}$$

Transforming (59) and (64) we obtain the following equations for w_{Ro} and S_R in terms of the displacements in the skin $z = b$.-

$$w_{Ro} = \frac{1}{2b}\left(\int_0^c Vdy + \int_0^{-c} Vdy\right) - \frac{1}{2b}\int_0^x\left\{(U)_{y=c} + (U)_{y=-c}\right\}dx + K_1x + K_2 \tag{65}$$

$$\frac{d^2S_R}{dx^2} - \frac{\tau_R}{ct_W}S_R = \frac{E\tau_R}{4bc(1+\sigma)}\left[\int_{-c}^{c}\frac{\partial^2 V}{\partial x^2}dy - \left\{\left(\frac{\partial u}{\partial x}\right)_{y=c} - \left(\frac{\partial u}{\partial x}\right)_{y=-c}\right\}\right] \tag{66}$$

where K_1, K_2 are arbitrary constants.

The equations governing the behaviour of the skin $z = b$ have already been developed in Part I. The external force (X,Y) arises in this case from shear flows S_R applied by the ribs. We have in fact.-

$$X = 0, \qquad Y = -S_R \tag{67}$$

The boundary conditions at the edges $y = \pm c$ can be obtained by considering the equilibrium of elements dx of the spar flanges. The balance of y - components gives.-

$$(T_2)_{y=\pm c} = 0 \tag{68}$$

The x-wise balance of forces is shown in Fig. 9. We thus find.-

$$S_W + (S)_{y=c} = EA\left(\frac{\partial e_{xx}}{\partial x}\right)_{y=c} \tag{69}$$

$$S'_W - (S)_{y=-c} = EA\left(\frac{\partial e_{xx}}{\partial x}\right)_{y=-c} \tag{70}$$

Formulae for S_W, S'_W in terms of U,V,S_R and w_{Ro} were obtained implicitly during the derivation of (59) (64). These may be expressed as

$$S_W + S'_W = \frac{Et_WK_1}{2(1+\sigma)} \tag{71}$$

/$S_W - S'_W = \ldots\ldots$

$$S_W - S_W' = \frac{Et_W}{2(1+\sigma)}\left[\frac{1}{b}\left((U)_{y=c} - (U)_{y=-c}\right) + \frac{4(1+\sigma)c}{E\gamma_R}\frac{dS_R}{dx} - \frac{1}{b}\int_{-c}^{c}\frac{\partial V}{\partial x}\,dy\right] \quad (72)$$

where use has been made of (65). Our boundary conditions (69) and (70) can then be written.-

$$(S)_{y=c} - (S)_{y=-c} = EA\frac{d}{dx}\left\{(e_{xx})_{y=c} + (e_{xx})_{y=-c}\right\} - \frac{Et_W K_1}{2(1+\sigma)} \quad (73)$$

$$(S)_{y=c} + (S)_{y=-c} = EA\frac{d}{dx}\left\{(e_{xx})_{y=c} - (e_{xx})_{y=-c}\right\}$$

$$- \frac{Et_W}{2(1+\sigma)}\left[\frac{(U)_{y=c} - (U)_{y=-c}}{b} + \frac{4(1+\sigma)c}{E\gamma_R}\frac{dS_R}{dx} - \frac{1}{b}\int_{-c}^{c}\frac{\partial V}{\partial x}\,dy\right] \quad (74)$$

The mathematical problem presented by our swept box is thus reduced to a plate problem of the type studied in Part I where the 'body force' $Y = -S_R$ is given by equation (66) and the boundary conditions at the edges $y = \pm c$ are given by (68) (73) and (74).

Finally let us write formulae for the static resultant of the forces acting across a section with coordinate x. These reduce to a force $Z.k$ at the centre of the section (x,o,o) and a couple $L_1.i_1 \wedge M_1.j_1$ where.-

$$Z = 2b(S_W + S_W') = \frac{Ebt_W K_1}{(1+\sigma)} \quad (75)$$

and,

$$\left.\begin{aligned} L_1 &= 2bc(S_W - S_W') - 2b\int_{-c}^{c} S\,dy \\ M_1 &= 2bEA\left\{(e_{xx})_{y=c} + (e_{xx})_{y=-c}\right\} + 2b\int_{-c}^{c} T_1\,dy \end{aligned}\right\} \quad (76)$$

It is to be remarked that we have found it convenient to use the oblique axes OX, OY for defining the couple. If it is desired to write the couple Li + Mj using the axes Ox, Oy, then the necessary transformation is given in (19).

2.3. Simple Loading Conditions.- (1) Constant Couple.

We now apply the results of the first example in plate theory of § 1.6 to a problem of swept boxes. The constant stresses T_1 and S of this example will be assumed to be acting in the skin $z = b$. The corresponding strains and deflections are given in equations (35) and (36). The body force $Y = -S_R$ is zero in this case. Substituting $S_R = 0$ and the values of U, V given in (36) into (66), we find this equation identically satisfied. Since S and e_{xx} are constant equation (73) shows that $K_1 = 0$ and so by (75) that $Z = 0$. Equation (74) shows that,

$$C = -\frac{1}{2}e_{xy} - \frac{b(1+\sigma)}{Et_W c}S \quad (77)$$

Equations (69) and (70) shows that,

/S_W = ...

$$S_W = - S, \qquad S'_W = S \tag{78}$$

Assuming for simplicity that $U = V = 0$ when $x = y = o$ and that $w_{Ro} = o$, when $x = o$ we find from (65) that,

$$w_{Ro} = - \frac{e_{xx}}{2b} . x^2 \tag{79}$$

Using (53), (54), (55), (56), (60) and (61) we find,

$$\left.\begin{aligned} v_R &= (e_{yy}.y - Cx)z/b \\ w_R &= - \frac{e_{xx}}{2b}.x^2 + \frac{C}{b}xy - \frac{e_{yy}}{2b}.y^2 \end{aligned}\right\} \tag{80}$$

$$\left.\begin{matrix} u_W \\ u'_W \end{matrix}\right\} = \left\{ e_{xx}.x \pm (e_{xy} + C)c \right\} z/b \tag{81}$$

$$\left.\begin{matrix} w_W \\ w'_W \end{matrix}\right\} = - \frac{e_{xx}}{2b} x^2 \pm \frac{Cc}{b} x - \frac{e_{yy}c^2}{2b} \tag{82}$$

The magnitudes of the stress resultants T_1 and S follows from (76). We find.-

$$\left.\begin{aligned} T_1 &= \frac{M_1 + \frac{EAA_{13}}{2c} L_1}{4bc(1 + \frac{EA\ A_{11}}{c})} \\ S &= - \frac{L_1}{8bc} \end{aligned}\right\} \tag{83}$$

The formulae developed in this paragraph together with (35), (36) solve the stress distribution and deflection problems for the case where our simplified swept box is loaded by constant couples.

2.4. Simple Loading Conditions.- (2) Bending by a z-wire force.

We now apply the results of our second example of § 1.6 to our swept box. The stress resultants for the face $z = b$ are assumed given by equation (40). The deflections for this face are then given by (41) and (42). Since $Y = 0$ for this solution we have $S_R = 0$ as in § 2.3. Substituting from (41) and (42) into (66) we find that $S_R = 0$ implies.-

$$A_2 = 0, \quad A_3 = - \frac{3}{2} \frac{A_{13}}{A_{11}} A_1 \tag{84}$$

Substituting from (30), (40) into (73) and recalling (75) we find.-

$$A_1 = -\frac{Z}{4bc\left\{1 + \frac{EAA_{11}}{c}\right\}} \tag{85}$$

Substituting from (30), (40), (41), (42) into (74) and recalling (84) we find.-

$$C = -\left(\frac{b(1+\sigma)}{Et_w c} + \frac{A_{33}}{2}\right) A_o - \frac{A_{31}}{2} A_4 \tag{86}$$

Equations (69) (70) and (75) give.-

$$\left.\begin{matrix} S_w \\ S'_w \end{matrix}\right\} = \frac{Z}{4b} \mp A_o \tag{87}$$

If we assume that our force Z is located along the line $x = l$, $y = o$ i.e. applied centrally at the top rib, we find by (19) that .-

$$L_1 = 0, \quad M_1 = -Z(l-x) \tag{88}$$

Substituting in (76) and using (88), (85) we find that .-

$$A_o = 0, \quad A_4 = A_1 \ell \tag{89}$$

Substituting from (84), (85), (86), (89) in (40), (41), (42) and (87) we find.-

$$\left.\begin{aligned} T_1 &= -\frac{Z\left(l-x-\frac{3}{2}\frac{A_{13}}{A_{11}}y\right)}{4bc\left\{1+\frac{EAA_{11}}{c}\right\}} = \frac{M_1}{4bc\left\{1+\frac{EAA_{11}}{c}\right\}} - \frac{3}{2}\frac{A_{13}}{A_{11}}S \\ S &= -\frac{Zy}{4bc\left\{1+\frac{EAA_{11}}{c}\right\}} \end{aligned}\right\} \tag{90}$$

$$\left.\begin{aligned} U &= -\frac{Z}{4bc\left\{1+\frac{EAA_{11}}{c}\right\}}\left[A_{11}lx+\tfrac{1}{2}A_{31}ly - \tfrac{1}{2}A_{11}x^2-\tfrac{1}{2}A_{13}xy + \tfrac{1}{2}\left(A_{21}+A_{33}-\tfrac{3}{2}\frac{A_{13}^2}{A_{11}}\right)y^2\right] \\ V &= -\frac{Z}{4bc\left\{1+\frac{EAA_{11}}{c}\right\}}\left[\tfrac{1}{2}A_{31}lx + A_{21}ly - \tfrac{1}{4}A_{13}x^2 - A_{21}xy + \tfrac{1}{2}\left(A_{23}-\tfrac{3}{2}\frac{A_{21}A_{13}}{A_{11}}\right)y^2\right] \end{aligned}\right\} \tag{91}$$

$$S_w = S'_w = Z/4b \tag{92}$$

/Where in (91) .....

Where in (91) we have assumed $U = V = 0$ when $x = y = o$. The equations (90) show that the conditions at the tip $x = 1$ are not exactly those corresponding to 'freedom', even from direct stress. For our solution to be valid equal and opposing couples must be applied to the faces $z = \pm b$ by loads normal to the rib $x = 1$, not to mention linearly varying shear loads applied parallel to this rib. However, the effects of this self-equilibrating system will die away as one proceeds along the span and so our solution may be considered practically valid at (say) a distance 2c from the end.

Substituting in (65) we find assuming $w_{RO} = 0$ when $x = o$,

$$w_{Ro} = \frac{Z}{4bc\left\{1+\frac{EAA_{11}}{c}\right\}}\left[\frac{A_{11}}{2b}x^2\left(1-\tfrac{1}{3}x\right)+\left\{\frac{4c(1+\sigma)\left(1+\frac{EAA_{11}}{c}\right)}{Et_w}\right.\right.$$

$$\left.\left.+\frac{c^2}{2b}\left(2A_{21}+A_{33}-\frac{3}{2}\frac{A_{13}^2}{A_{11}}\right)\right\}x\right] \qquad (93)$$

The remaining deflections can be written down using (53), (54), (55), (56), (60) and (61), but since the formulae are lengthy we shall not give them here.

2.5. Analysis of the Deflections for the Simple Loading Conditions

The deflections at any plane section (coordinate x) of our box may be analysed into the sum of a translation, a rotation, a warping from the plane and a distortion in the plane of section. Let us consider a translation Wk and a rotation $pi + qj$, where W, p and q are functions of x. These will produce displacements at our section given by.-

$$U = qz \sin\alpha, \quad V = -pz\sin\alpha, \quad w = W + py\sin\alpha \qquad (94)$$

Where use has been made of (9) and the rotation has been located at (x,o,o). For this one equation U,V have a 'general' significance as in (9) and are not confined to $z = b$. Comparison of the first of (94) with (56) and (61) suggests the identification

$$q = \frac{(U)_{y=c} + (U)_{y=-c}}{2b\sin\alpha} \qquad (95)$$

Comparison of the second of (94) with (53) suggests.-

$$p = -\frac{(\text{Terms of } V \text{ independent of } y)}{b\sin\alpha} \qquad (96)$$

Comparison of the third of (94) with (54) gives.-

$$W = w_{RO} \qquad (97)$$

and (96) again. The term in (54) containing S_R does not

/occur in (53) ....

does not occur in (53) and gives a shear strain not a rotation. We shall adopt the definitions (95), (96) and (97) for p, q and W. Other definitions are possible, but the differences are bound up with questions of 'shear deflection' and 'root conditions', with which we are not particularly concerned here.

Let us now apply our formulae to the case of loading by a couple analysed in § 2.3. Substituting from (36) with $C_1 = C_2 = 0$ and (79) we find,

$$p = \frac{Cx \operatorname{cosec} \alpha}{b}, \quad q = \frac{e_{xx} \cdot x \operatorname{cosec} \alpha}{b}, \quad W = -\frac{e_{xx}}{2b} x^2 \qquad (98)$$

Substituting from (77), (35) and finally (83) we find the following relations.-

$$\left.\begin{aligned} \frac{dp}{dx} &= C_{11}L_1 + C_{12}M_1 \\ -\frac{d^2W}{dx^2} \operatorname{cosec} \alpha &= \frac{dq}{dx} = C_{21}L_1 + C_{22}M_1 \end{aligned}\right\} \qquad (99)$$

where,

$$\left.\begin{aligned} C_{11} &= \frac{\operatorname{cosec} \alpha}{8bc}\left\{\frac{(1+\sigma)}{Et_w c} + \frac{A_{33}}{2b} - \frac{EAA_{13}^2}{2bc\left(1+\dfrac{EAA_{11}}{c}\right)}\right\} \\ C_{12} = C_{21} &= -\frac{A_{13} \operatorname{cosec} \alpha}{8b^2 c\left(1+\dfrac{EAA_{11}}{c}\right)} \\ C_{22} &= \frac{A_{11} \operatorname{cosec} \alpha}{4b^2 c\left(1+\dfrac{EAA_{11}}{c}\right)} \end{aligned}\right\} \qquad (100)$$

The relations (99) generalise the usual curvature - bending moment and twist-torque relations valid for an unswept box (beam).

The remaining terms in the deflection formulae can be analysed into firstly a 'linear warping'.-

$$\left.\begin{aligned} U &= (e_{xy} + C)y, \quad V = 0 \\ \left.\begin{aligned} u_W \\ u'_W \end{aligned}\right\} &= \pm (e_{xy} + C)cz/b \end{aligned}\right\} \qquad (101)$$

and a 'cross sectional distortion'.-

$$\left.\begin{aligned} U &= 0, \quad V = e_{yy} \cdot y \\ v_R &= e_{yy} \cdot yz/b, \quad w_R = -e_{yy} \cdot y^2/2b \end{aligned}\right\} \qquad (102)$$

The warping, which consists of spanwise displacement, depends upon both L_1 and M_1. The cross-sectional distortion consists of an 'antielastic' bending of the ribs.

We turn now to the analysis of the deflections for the case of z-wise loading at the tip, dealt with in § 2.4.

/Substituting from ....

Substituting from (91) into (95), (96) recalling (93), (97) and (100) we find,

$$\frac{dp}{dx} = - C_{12}Z(1-x), \quad - \frac{d^2W}{dx^2} \operatorname{cosec} \alpha = \frac{dq}{dx} = - C_{22}Z(1-x) \tag{103}$$

Recalling (88) we see that the relations (99) are valid for this case as well. The remaining displacement terms can be analysed into firstly a 'linear warping'.-

$$U = \frac{A_{31}M_1 y}{8bc(1+\frac{EAA_{11}}{c})}, \quad V = o \tag{104}$$

secondly, a 'parabolic warping'.-

$$\left.\begin{aligned} U &= \frac{Z}{8bc(1+\frac{EAA_{11}}{c})}\left(A_{21} + A_{33} - \frac{3}{2}\frac{A_{13}^2}{A_{11}}\right)(c^2 - y^2) \\ V &= o \end{aligned}\right\} \tag{105}$$

and finally a cross sectional distortion.-

$$U = o, \; V = \frac{A_{21}M_1 y}{4bc(1+\frac{EAA_{11}}{c})} - \frac{Z\left\{A_{23} - \frac{3}{2}\frac{A_{21}A_{13}}{A_{11}}\right\}}{8bc(1+\frac{EAA_{11}}{c})}y^2 \tag{106}$$

The formula (101) when expressed in terms of M_1 (with $L_1 = o$) agrees with (104). Similarly (102) agrees with the first term of (106). The warping of (105) is analogous to that occuring in unswept boxes and will give rise to a theory of 'shear lag', just as the linear warping will give rise to a theory of 'end constraint' similar to that arising in the case of the torsion of unswept boxes.

2.6. Internal Systems of Stress

The third example of § 1.6 may be used to construct systems of stress for which the static resultant on a cross section is zero. We take as displacements in the surface $z = b$ the sum of the expressions given in equations (45) and (49), where the constants B_j, which occur in these, are limited by the relations (50). Equations (43) and (67) show that

$$S_R = B_o e^{-\mu x} \tag{107}$$

Our assumed solution must satisfy (66), (73) (with $K_1 = o$ by (75) and (74). Making the necessary substitutions, we find, incidentally, that the constant C of (49) is zero. The three remaining equations together with (50) form a homogeneous set of linear equations in the five constants B_j (j=o,1,2,3,4). These equations may be written

$$\sum_{j=o}^{4} B_{ij}B_j = 0 \tag{108}$$

/where the .....

where the equations for $i = o,1$, are obtained from (50) by addition and subtraction, the equation for $i = 2$ is from (66), that for $i = 3$ from (73) and that for $i = 4$ from (74). The constants B_{ij} are given by .-

$$B_{oo} = 0, \quad B_{oj} = \sinh \mu\lambda_j c \quad (j=1,2,3,4) \tag{109}$$

$$B_{1o} = \frac{A_{23}}{\mu^3 A_{22}}, \quad B_{1j} = \cosh \mu\lambda_j c \quad (j=1,2,3,4) \tag{110}$$

$$\left.\begin{aligned} B_{2o} &= \frac{a_{11}c}{A_{22}\left|a_{ij}\right|} + \frac{2bc(1+\sigma)}{E\gamma_R}\left(\frac{\gamma_R}{ct_w} - \mu^2\right) \\ B_{2j} &= \mu^2\left(\frac{A_{22}}{\lambda_j^2} + \frac{A_{23}}{\lambda_j} - A_{13}\lambda_j - A_{11}\lambda^2{}_j\right)\sinh \mu\lambda_j c \quad (j=1,2,3,4) \end{aligned}\right\} \tag{111}$$

$$\left.\begin{aligned} B_{30} &= \frac{a_{13}}{A_{22}\left|a_{ij}\right|\mu} \\ B_{3j} &= \frac{\mu}{EA}\lambda_j \sinh \mu\lambda_j c + \mu^2(\lambda_j^2 A_{11} + \lambda_j A_{13} + A_{12})\cosh \mu\lambda_j c \\ &\qquad (j=1,2,3,4) \end{aligned}\right\} \tag{112}$$

$$\left.\begin{aligned} B_{4o} &= \frac{\mu t_w c}{\gamma_R} + \frac{1}{\mu} - \frac{Et_w c a_{11}}{2(1+\sigma)bA_{22}\left|a_{ij}\right|\mu} \\ B_{4j} &= -\mu^2\lambda_j \cosh \mu\lambda_j c \\ &+ \frac{Et_w\mu}{2b(1+\sigma)}\left[\lambda_j^2 A_{11} + \lambda_j A_{13} - \frac{1}{\lambda_j}A_{23} - \frac{1}{\lambda_j^2}A_{22} - \right. \\ &\left. - \frac{2\mu^2 Ab(1+\sigma-}{t_w}(\lambda_j^2 A_{11} + \lambda_j A_{13} + A_{12})\right] \quad \sinh \mu\lambda_j c \end{aligned}\right\} \tag{113}$$

Equations (108) are satisfied by non-zero B_j if .-

$$\left|B_{ij}\right| = 0 \tag{114}$$

Equation (114) is a transcendental equation for μ. It is very complex as inspection of (109) - (113) shows. The mathematical examination of its roots is therefore out of the question, but physical intuition, based upon experience with unswept boxes, suggests the existence of an infinite sequence of roots with positive real parts, which may be written.-

$$\mu = \frac{1}{c}(\mu_1, \mu_2, \mu_3, \ldots\ldots.) \tag{115}$$

/They can of .....

They can of course be calculated numerically in a special case. The solution of the first four equations of (108) gives the ratio between the B_j.
We may write

$$B_j = C_\mu \beta_j(\mu) \tag{116}$$

where C_μ is an arbitary complex constant and $\beta_j(\mu)$ are the cofactors of B_{4j} in the determinant of (114).

A 'general' internal system may be obtained by summation of our results with respect to μ over the sequence (115). The resulting displacements U, V for the surface z = b may be written.-

$$\left.\begin{aligned} U &= \sum_\mu C_\mu e^{-\mu x}\left[-\frac{a_{13}\beta_o(\mu)}{A_{22}\left|a_{ij}\right|\mu^2} - \mu\sum_{j=1}^{4}\beta_j(\mu)e^{\mu\lambda_j y}.\left(\lambda_j^2 A_{11}+\lambda_j A_{13}+A_{12}\right)\right] + c_1 \\ V &= \sum_\mu C_\mu e^{-\mu x}\left[\frac{a_{11}\beta_o(\mu)}{A_{22}\left|a_{ij}\right|\mu^2} + \mu\sum_{j=1}^{4}\frac{\beta_j(\mu)}{\lambda_j}e^{\mu\lambda_j y}\left(\lambda_j^2 A_{21}+\lambda_j A_{33}+A_{22}\right)\right] + c_2 \end{aligned}\right\} \tag{117}$$

It must be understood in (117) that the real parts of the expressions given are to be taken.

The solution (117) could be used to remove the 'warping' and 'section distortion', from the simple solutions analysed in § 2.5, at one particular section (say) x = o. However another difficulty arises here, because the constants C_μ cannot be obtained by the usual harmonic analysis. Multiplication of (117) by $e^{-\mu\lambda_j y}$ and operating $\int_{-c}^{c}(\quad)dy$ yields an infinite set of equations for the C_μ! An alternative process might begin by limiting the expansions (117) to a <u>finite</u> number of terms and then proceed by choosing the C_μ to remove the warping at a <u>finite</u> number of points on the section.

The processes sketched above are very complex and hardly practicable. Recourse must doubtless be made to approximate methods of calculation to handle problems of constraint against warping for swept back wings.

2.7. Approximate Calculation of Root Constraint for the Case of Loading by a Constant Couple

The general methods of § 2.6. are hardly feasible for design calculations. However, an approximate calculation is possible if certain restrictions are made as to the deformation possibilities. We assume that the section of the box can only warp and distort in its plane according to the pattern defined in equations (101) and (102), that is, in the same way as occurs when a constant couple is transmitted, with no restraint at the ends. Other modes of deformation of the section cannot occur, in particular the rib webs are rigid in shear ($t_R \to \infty$). The deformation of the skins and spar webs is then given by.-

/U = qb......

$$\left.\begin{aligned} U &= qb \sin\alpha + \omega\, y/c \\ V &= -\, pb \sin\alpha + \Delta y \\ \left.\begin{matrix} u_w \\ u'_w \end{matrix}\right\} &= qz \sin\alpha \pm \omega z/b \\ \left.\begin{matrix} w_w \\ w'_w \end{matrix}\right\} &= W \pm pc \sin\alpha - \Delta c^2/2b \end{aligned}\right\} \qquad (118)$$

where, p, q, W, ω, Δ are functions of x. Making the supposition that $T_2 = o$ the stress resultants follow from (118). We find

$$\left.\begin{aligned} T_1 &= \left\{\left(-\bar{a}_{13}\frac{dp}{dx} + \bar{a}_{11}\frac{dq}{dx}\right)b \sin\alpha + \frac{\bar{a}_{13}\omega}{c}\right\} + \left\{\frac{\bar{a}_{11}}{c}\frac{d\omega}{dx} + a_{13}\frac{d\Delta}{dx}\right\} y \\ T_2 &= 0 \\ S &= \left\{\left(-\bar{a}_{33}\frac{dp}{dx} + \bar{a}_{31}\frac{dq}{dx}\right)b \sin\alpha + \frac{\bar{a}_{33}\omega}{c}\right\} + \left\{\frac{\bar{a}_{31}}{c}\frac{d\omega}{dx} + \bar{a}_{33}\frac{d\Delta}{dx}\right\} y \end{aligned}\right\} \qquad (119)$$

where

$$\bar{a}_{11} = a_{11} - \frac{a_{12}^2}{a_{22}} \qquad \bar{a}_{13} = \bar{a}_{31} = a_{13} - \frac{a_{12}a_{23}}{a_{22}} \qquad \bar{a}_{33} = a_{33} - \frac{a_{23}^2}{a_{22}} \qquad (120$$

and

$$\left.\begin{matrix} S_w \\ S'_w \end{matrix}\right\} = \frac{Et_w}{2(1+\sigma)}\left\{\pm\frac{dp}{dx} c \sin\alpha + q \sin\alpha + \frac{dW}{dx} \pm \frac{\omega}{b} - \frac{d\Delta}{dx}\frac{c^2}{2b}\right\} \qquad (121)$$

Equations (12), (24), (57), (62) and (118) have been used in the derivation of (119) (120) and (121). Writing $Z = 0$ in (75) we find from (121).-

$$q \sin\alpha + \frac{dW}{dx} - \frac{d\Delta}{dx}\frac{c^2}{2b} = 0 \qquad (122)$$

Substituting in (76) we find.-

$$\left.\begin{aligned} L_1 &= 4bc \sin\alpha . \left\{\frac{Et_w c}{2(1+\sigma)} + b\bar{a}_{33}\right\}\frac{dp}{dx} - 4b^2 c\bar{a}_{31} \sin\alpha . \frac{dq}{dx} \\ &\qquad + 4\left\{\frac{Et_w c}{2(1+\sigma)} - b\bar{a}_{33}\right\}\omega \\ M_1 &= 4b^2 \sin\alpha . (c\bar{a}_{11} + EA)\frac{dq}{dx} - 4b^2 c\bar{a}_{13} \sin\alpha . \frac{dp}{dx} \\ &\qquad + 4b\bar{a}_{13}\omega \end{aligned}\right\} \qquad (123)$$

Substituting in (69) and (70) we find.-

$$\left.\begin{aligned} &\bar{a}_{31}\frac{d\omega}{dx} + \bar{a}_{33}c\frac{d\Delta}{dx} = EAb \sin\alpha . \frac{d^2q}{dx^2} \\ &\left\{\frac{Et_w c}{2(1+\sigma)} - b\bar{a}_{33}\right\}\sin\alpha . \frac{dp}{dx} + b\bar{a}_{31} \sin\alpha . \frac{dq}{dx} \\ &\qquad + \left\{\frac{Et_w}{2b(1+\sigma)} + \frac{\bar{a}_{33}}{c}\right\}\omega = EA\frac{d^2\omega}{dx^2} \end{aligned}\right\} \qquad (124)$$

/Solution .....

Solution of (123) for $\frac{dp}{dx}$, $\frac{dq}{dx}$ yields.-

$$\left.\begin{aligned} \frac{dp}{dx} &= P_1L_1 + P_2M_1 + P_3\,\omega \\ \frac{dq}{dx} &= Q_1L_1 + Q_2M_1 + Q_3\omega \end{aligned}\right\} \qquad (125)$$

where,

$$P_1 = \frac{4b^2\sin\alpha\,(c\bar{a}_{11} + EA)}{D}$$

$$P_2 = \frac{4b^2c\,\bar{a}_{31}\,\sin\alpha}{D} = Q_1$$

$$P_3 = -\frac{16b^2\sin\alpha}{D}\left[bc\bar{a}_{13}^2\left\{\frac{Et_wc}{2(1+\sigma)} - b\bar{a}_{33}\right\}(c\bar{a}_{11}+EA)\right]$$

$$Q_2 = \frac{4bc\,\sin\alpha}{D}\left\{\frac{Et_wc}{2(1+\sigma)} + b\bar{a}_{33}\right\} \qquad (126)$$

$$Q_3 = -\frac{16Et_wb^2c^2\bar{a}_{13}\,\sin\alpha}{D(1+\sigma)}$$

$$D = 16b^3c\,\sin^2\alpha\left[\left\{\frac{Et_wc}{2(1+\sigma)} + b\bar{a}_{33}\right\}(c\bar{a}_{11}+EA) - bca_{13}^2\right]$$

Substitution from (125) into the second of (124) yields.-

$$\frac{d^2\omega}{dx^2} - \beta^2\omega = R_1L_1 + R_2M_1 \qquad (127)$$

where,

$$R_1 = \frac{\sin\alpha}{EA}\left[\left\{\frac{Et_wc}{2(1+\sigma)} - b\bar{a}_{33}\right\}P_1 + b\bar{a}_{31}Q_1\right]$$

$$R_2 = \frac{\sin\alpha}{EA}\left[\left\{\frac{Et_wc}{2(1+\sigma)} - b\bar{a}_{33}\right\}P_2 + b\bar{a}_{31}Q_2\right] \qquad (128)$$

$$\beta^2 = \frac{\sin\alpha}{EA}\left[\left\{\frac{Et_wc}{2(1+\sigma)} - b\bar{a}_{33}\right\}P_3 + b\bar{a}_{31}Q_3\right] + \frac{1}{EA}\left\{\frac{Et_w}{2b(1+\sigma)} + \frac{\bar{a}_{33}}{c}\right\}$$

A solution of (127) which vanishes at $x = o$ and remains finite as $x \to \infty$ is.-

$$\omega = -\frac{(R_1L_1 + R_2M_1)}{\beta^2}(1 - e^{-\beta x}) \qquad (129)$$

The first of (124) gives assuming $\Delta = o$ for $x = o$.-

$$\Delta = \frac{\left\{EAbQ_3\sin\alpha - \bar{a}_{31}\right\}}{\bar{a}_{33}c}\,\omega \qquad (130)$$

The remaining unknowns are easily found. p.q. follow from (125), W from (122) and the stress resultants from (119) and (121) The solution found solves the problem of 'root constraint' for a 'long' swept box loaded by any couple at the tip. It may be applied with the usual approximation to other cases of loading. The method used here may be extended to deal with the parabolic warping of (105) and so yield an approximate solution of the shear lag problem for the swept box.

/Part III. ...

Part III Applications to Swept Back Wing Structures

3.1. Generalisation of the Engineering Theory of Bending and Torsion to Include the Case of Swept Back Wing Structures

The intention of the present section is to generalise the solutions obtained in Part II for the simple cases of loading (§ 2.3, § 2.4, § 2.5), to cover the case of a uniform swept box, whose section bears a closer resemblance to an actual wing structure, than that considered previously. The box to which we shall now devote attention is shown in Fig. 10. The section has unequal spars, that at $y = c$ has thickness t_w and depth $2b$, while that at $y = -c$ has corresponding dimensions t_w' and $2b'$. The skins are identical in both geometry and elasticity and so the section is symmetrical about the y-axis. The skins may be curved, but the development below is restricted to the case where $d\zeta/dy$ is small,* where $\zeta(y)$ is the ordinate. This will ensure that the angle α between the stringers and the rib-skin intersections may be treated as constant over the skin surfaces. The flanges of the spars $y = \pm c$ will have section areas A and A' respectively.

The notation for displacements, strains, stress resultants etc. will be the same as in Part II. However, in the case of the curved skins, displacements etc. will be treated as occuring 'in the surface'. For example V will represent a displacement parallel to the tangent of the curve of cross section.

We make the following assumptions with regard to displacements.-

1. Each section $x = x$ moves as a rigid body with displacement Wk and rotation $pi + qj$. W, p, q are functions of x, the last two being quadratic and the first cubic.

2. The section is warped from the plane by a displacement which is linear in x. In the skins we have $U = \omega_1(y).x + \omega_2(y)$ and the warping in the spar webs is linear in z. By a suitable definition of q we may assume that the rotation of linear elements of the two spar webs to be equal and opposite.

3. The section is distorted in the plane in such a way that $S_R = 0$ and that $V = \Delta_1(y).x + \Delta_2(y)$.

Reference to § 2.5, in particular to equations (104), (105) and (106) shows that our assumptions are sufficiently general to deal with the loading cases and the simple box treated there. Putting our assumptions into mathematical form, we can write.-

/$U = q\ S \sin\alpha + \ldots$

---

* This implies that $|b - b'|/2c$ is small

$$\left.\begin{aligned}
U &= q\,\zeta \sin\alpha + \omega \\
V &= -\,p\left(\zeta - y\frac{d\zeta}{dy}\right)\sin\alpha + W\frac{d\zeta}{dy} + \Delta \\
u_W &= qz\sin\alpha + (\omega)_{y=c}\cdot z/b \\
w_W &= W + pc\sin\alpha \\
u'_W &= qz\sin\alpha + (\omega)_{y=-c}\cdot z/b' \\
w'_W &= W - pc\sin\alpha
\end{aligned}\right\} \qquad (131)$$

where,

$$\left.\begin{aligned}
p &= p_1x + \tfrac{1}{2}p_2x^2 \\
q &= q_1x + \tfrac{1}{2}q_2x^2 \\
W &= \gamma x - \tfrac{1}{2}x^2(q_1 + \tfrac{1}{3}q_2x)\sin\alpha \\
\omega &= \omega_1x + \omega_2 \\
\Delta &= \Delta_1x + \Delta_2
\end{aligned}\right\} \qquad (132)$$

The quantities p_1, p_2, q_1, q_2, γ are constants, while ω_1, ω_2, Δ_1, Δ_2 are functions of y. The terms in (131) involving p, q are obtained by an application of (9). Those involving $\frac{d\zeta}{dy}$ in the formula for V represent the tangential component of those parts of w_R which express rigid body motions (see Fig. 11). The remaining component of the portions of w_R are included in Δ. The definition of W in (131) is $(w_W + w'_W)/2$, which will differ from that used in § 2.5 equation (97), by a term which depends upon the cross sectional distortion and so will be linear in x. This difference will therefore not affect the relation between W and q given in (99) and (103). This relation has been adopted here and used to derive the formula for W in (132). From equations (12) and (131) we find for the strains in the skins.-

$$\begin{aligned}
e_{xx} &= (q_1 + q_2x)\,\zeta\sin\alpha + \omega_1 \\
e_{xy} &= -(p_1 + p_2x)\left(\zeta - y\frac{d\zeta}{dy}\right)\sin\alpha + \gamma\frac{d\zeta}{dy} + \Delta_1 + \frac{d\omega_1}{dy}x + \frac{d\omega_2}{dy}
\end{aligned} \qquad (133)$$

It follows that the stress resultants T_1 and S are linear in x and so assuming in accordance with the findings of Part II that $T_2 = o$ and $S_R = o$ we find from (21) writing $X = Y = T_2 = 0$ that.-

$$\begin{aligned}
T_1 &= -\,x\frac{dS}{dy} + (T_1)_{x=o} \\
T_2 &= 0 \\
S &= S(y)
\end{aligned}$$

/Equation (30) then .....

Equation (30) then gives.-

$$\left.\begin{aligned} e_{xx} &= A_{11}(-x\frac{dS}{dy} + (T_1)_{x=o}) + A_{13}S \\ e_{xy} &= A_{31}(-x\frac{dS}{dy} + (T_1)_{x=o}) + A_{33}S \end{aligned}\right\} \qquad (135)$$

Comparing (133) and (135) we deduce using (134).-

$$\left.\begin{aligned} T_1 &= \frac{(q_1 + q_2x)\zeta \sin\alpha}{A_{11}} + \frac{(\omega_1)_{y=-c}}{A_{11}} - \frac{A_{13}}{A_{11}}(S)_{y=-c} - \frac{p_2\sin\alpha}{A_{11}}(y\zeta + cb') \\ &\quad + 2(p_2 + \frac{A_{31}}{A_{11}}q_2)\frac{\sin\alpha}{A_{11}}\int_{-c}^{y}\zeta\, dy \\ S &= (S)_{y=-c} - \frac{q_2\sin\alpha}{A_{11}}\int_{-c}^{y}\zeta\, dy \\ \omega_1 &= (\omega_1)_{y=-c} + (2p_2 + \frac{A_{31}}{A_{11}}q_2)\sin\alpha\int_{-c}^{y}\zeta\, dy - p_2\sin\alpha(y\zeta + cb') \\ \Delta_1 + \frac{d\omega_2}{d} &= A_{31}(T_1)_{x=o} + A_{33}S + p_1(\zeta - y\frac{d\zeta}{dy})\sin\alpha - \gamma\frac{d\zeta}{dy} \end{aligned}\right\} \qquad (136)$$

Substituting from (133) and (136) in (69) and (70) (with A' written for A) we find.-

$$\left.\begin{aligned} S_\omega &= -(S)_{y=-c} + q_2\sin\alpha\,(\frac{2c\bar{\zeta}}{A_{11}} + EAb) \\ S'_\omega &= (S)_{y=-c} + EA'b'q_2\sin\alpha \end{aligned}\right\} \qquad (137)$$

where,

$$\bar{\zeta} = \frac{1}{2c}\int_{-c}^{c}\zeta\, dy \qquad (138)$$

Substituting from (131), (132) in (57) and (62) (with t'_w written for t_w) we find expressions for S_w, S'_w which may be compared with (137) yielding.-

$$\left.\begin{aligned} \frac{(\omega_1)_{y=c}}{b} &= -\frac{(\omega_1)_{y=-c}}{b'} = -p_2c\sin\alpha \\ \gamma &= -\frac{(1+\sigma)}{E}(\frac{1}{t_w} - \frac{1}{t'_w})(S)_{y=-c} + \frac{(1+\sigma)q_2\sin\alpha}{E}\left\{\frac{EAb}{t_w} + \frac{EA'b'}{t'_w} + \frac{2c\bar{\zeta}}{A_{11}t_w}\right\} \\ \frac{(\omega_2)_{y=c}}{b} &= -\frac{(\omega_2)_{y=-c}}{b'} = -p_1c\sin\alpha - \frac{(1+\sigma)}{E}(\frac{1}{tw} + \frac{1}{t'_w})(S)_{y=-c} \\ &\quad + \frac{(1+\sigma)}{E}q_2\sin\alpha\left\{\frac{EAb}{t_w} - \frac{EA'b'}{t'_w} + \frac{2c\bar{\zeta}}{A_{11}t_w}\right\} \end{aligned}\right\} \qquad (139)$$

/where in the .....

where in the last equation we have made use of the anti-symmetric nature of the warping in the spar webs (cf. assumption (2) given above). Substituting now from the third of (136) into the first of (139) we find.-

$$p_2 = - \frac{A_{31}}{2A_{11}} q_2 \tag{140}$$

The formulae for T_1, S, S_W, S_W' ((136), (137)) obtained above contain the unknown constants q_1, q_2, p_2 $(\omega_1)_{y=-c}$ and $(S)_{y=-c}$. Equations (139), (140) show that two of them are expressible in terms of the remaining three. These last three and hence the stresses can be determined by use of equations of overall equilibrium like (75) and (76). However, these require modification for the present structure. We find easily that.-

$$\left.\begin{aligned} Z &= 2bS_W + 2b'S_W' + 2\int_{-c}^{c} S\frac{d\zeta}{dy}\,dy \\ L_1 &= 2bcS_W - 2b'cS_W' + 2\int_{-c}^{c} yS\frac{d\zeta}{dy} - 2\int_{-c}^{c} \zeta S dy \\ M_1 &= 2bEA(e_{xx})_{y=c} + 2b'EA'(e_{xx})_{y=-c} + 2\int_{-c}^{c}\zeta T_1 dy \end{aligned}\right\} \tag{141}$$

where allowance has been made for the z-wise components of skin shear $S\frac{d\zeta}{dy}$.

Substituting from (136) and (137) into (141) and making use of (139) and (140) we find after some transformation.-

$$\left.\begin{aligned} q_2 &= \frac{Z}{EI\sin\alpha} \\ \text{where}\quad I &= 2(Ab^2 + A'b'^2 + \frac{1}{EA_{11}}\int_{-c}^{c}\zeta^2 dy) \end{aligned}\right\} \tag{142}$$

$$\left.\begin{aligned} (S)_{y=-c} &= -\frac{L_1}{8c\bar{\zeta}} + \frac{\eta Z}{8\bar{\zeta}} + \frac{Z}{EIA_{11}}\int_{-c}^{0}\zeta\,dy \\ \text{where}\quad & \\ y &= \frac{2E(Ab^2 - A'b'^2) + \frac{2}{cA_{11}}\int_{-c}^{c} y\zeta^2 dy + \frac{4}{cA_{11}}\int_{-c}^{c}\int_{0}^{y}\zeta(y).\zeta(y_1)dy_1 dy}{EI} \end{aligned}\right\} \tag{143}$$

$$q_1 = - \frac{\left(Zl + \frac{A_{31}}{2A_{11}}L_1\right)}{EI\sin\alpha} \tag{144}$$

It is to be remarked that Z and L_1 are constant in our solution, whereas M_1 is linear in x. It is assumed in (144) that Z is applied at $x = l$ and hence that M_1 is given by the expression in (88).

Formulae for the stress resultants can now be

/obtained. .....

obtained. Substituting from (142), (143), (144), (139) and (140) in the formulae of (136) we find.-

$$\left.\begin{aligned} S &= -\frac{\overline{L}_1}{8c\overline{\zeta}} - \frac{Z}{EIA_{11}}\int_0^y \zeta\, dy \\ \text{where}\quad \overline{L}_1 &= L_1 - \eta c Z \end{aligned}\right\} \tag{145}$$

and

$$\left.\begin{aligned} T_1 &= \frac{e_{xx}}{A_{11}} - \frac{A_{13}}{A_{11}} S \\ \text{where}\quad e_{xx} &= \frac{\left\{M_1 + \frac{A_{31}}{2A_{11}}(Zy - L_1)\right\}\zeta}{EI} \end{aligned}\right\} \tag{146}$$

Substituting in (137) we find.-

$$\left.\begin{aligned} S_w &= \frac{\overline{L}_1}{8c\overline{\zeta}} + \frac{Z}{EIA_{11}}\int_0^c \zeta\, dy + \frac{AbZ}{I} \\ S'_w &= -\frac{\overline{L}_1}{8c\overline{\zeta}} + \frac{Z}{EIA_{11}}\int_{-c}^0 \zeta\, dy + \frac{A'b'Z}{I} \end{aligned}\right\} \tag{147}$$

The point $y = \eta c$, $z = o$ on a section $x = x$ may be termed the 'shear centre' at the section. It may be remarked that $\eta = o$ when there is symmetry about the z axis. The torque L_1 about an axis through the shear centre may be termed the 'Batho Torque' and is seen to be reacted by a uniform shear flow given by the usual Batho formula.

We turn now to the calculation of the deflections. Combination of (137) with the second of (139) gives.-

$$\gamma = \frac{(1+\sigma)}{E}\left(\frac{S_w}{t_w} + \frac{S'_w}{t'_w}\right) \tag{148}$$

The quantity γ can then be obtained using (147). It is equal to the mean shear strain in the two spar webs and so the term in W(eq. (138))- 'γx' is the 'shear deflection'. The calculation of the rotations requires a knowledge of p_1, which we have not found as yet. To determine p_1 we must consider the deformations of the ribs. The rib displacements are calculated upon the supposition that $S_R = o$ and that w_R is a function of y only (cf. § 2.2). We find by (52) that

$$v_R = -z\frac{dw_R}{dy}, \quad \omega_R = \omega_R(y) \tag{149}$$

The displacement V at the skin is given by.-

$$V = (v_R)_{z=\zeta} + \omega_R\frac{d\zeta}{dy} = -\zeta^2\frac{d}{dy}\left(\frac{w_R}{\zeta}\right) \tag{150}$$

Recalling (60) we find.-

$$\frac{W_R}{\zeta} - \frac{\omega'_w}{b'} + \int_{-c}^{y}\frac{V}{\zeta^2}\, dy = 0 \tag{151}$$

/Substituting .....

Substituting from (131) in (151 we find.-

$$w_R = py \sin\alpha + W - \zeta \int_{-c}^{y} \frac{\Delta}{\zeta^2}\, dy \tag{152}$$

and

$$\int_{-c}^{c} \frac{\Delta_1}{\zeta^2}\, d\eta = \int_{-c}^{c} \frac{2}{\zeta^2}\, d\eta = 0 \tag{153}$$

Now the strain e_{yy} in the curved skin can be calculated in two ways. Firstly from the displacements V and $_R$* by a well known formula and secondly from equation (30). We thus find.-

$$e_{yy} = \frac{\partial V}{\partial y} - w_R \frac{d^2\zeta}{dy^2} = A_{21}T_1 + A_{23}S \tag{154}$$

Substituting from (131), (152), (145) and (146) and equating coefficients of x in the resulting formulae we find.-

$$\frac{\partial\Delta_1}{\partial y} + \zeta \frac{d^2\zeta}{dy^2} \int_{-c}^{y} \frac{\Delta_1}{\zeta^2}\, dy = \frac{A_{21}}{A_{11}} \frac{Z\zeta}{EI} \tag{155}$$

The remaining terms of our identity give an equation for Δ_2, which we do not write here. The solution of (155) which satisfies (153) is,

$$\Delta_1 = \frac{A_{21}Z}{A_{11}EI}\left\{\zeta y - \tfrac{1}{2}(y^2 - c^2)\frac{d\zeta}{dy}\right\} \tag{156}$$

The quantity p_1 can now be calculated. Opeaating on (136) with $\int_{-c}^{c}(\quad)\,dy$ and using (156), (145) and (146) we find an expression for $\left\{(\omega_2)_{y=c} - (\omega_2)_{y=-c} + \Upsilon(b-b')\right\}$. This quantity can also be obtained from (139) using (142) and (145). Equating the two results we find for p_1.-

$$\left.\begin{aligned}
p_1 &= C_{11}L_1 - C_{12}Z\ell + C_{13}Z \qquad \text{, where} \\
C_{11} &= \frac{\operatorname{cosec}\alpha}{8\bar{\zeta}c}\left\{\frac{(1+\sigma)(b/t_w + b'/t'_w)}{2E\bar{\zeta}c} + \frac{(A_{33} - A_{13}^2/A_{11})}{2\bar{\zeta}} + \frac{2A_{31}^2\bar{\zeta}c}{EIA_{11}^2}\right\} \\
C_{12} &= -\frac{A_{31}\operatorname{cosec}\alpha}{2A_{11}EI} \\
C_{13} &= -\frac{\operatorname{cosec}\alpha}{16\bar{\zeta}^2}\left\{\frac{(1+\sigma)}{Ec}\left(\frac{b}{t_w} + \frac{b'}{t'_w}\right) + (A_{33} - A_{13}^2/A_{11})\right\} \\
&\qquad / + \frac{\operatorname{cosec}\alpha}{2\bar{\zeta}A_{11}EI} \quad \ldots\ldots
\end{aligned}\right\} \tag{157}$$

---

* In all strictness $w_R - v_R \frac{d\zeta}{dy}$, but the inclusion of the second term only introduces terms of the order neglected here.

$$+\frac{\operatorname{cosec}\alpha}{2\zeta A_{11}EI}\left[\frac{(1+\sigma)}{Ec}\left\{\frac{b}{t_w}\int_0^c \zeta\,dy - \frac{b'}{t'_w}\int_{-c}^0 \zeta\,dy + EA_{11}\left(\frac{Ab^2}{t_w} - \frac{A'b'^2}{t'_w}\right)\right\} + \frac{(A_{33} - A_{13}^2/A_{11})}{2c}\left(c\int_0^c \zeta\,dy - c\int_{-c}^0 \zeta\,dy - \int_{-c}^c y\,\zeta\,dy\right) + \frac{(A_{21} - A_{31}^2/4A_{11})}{c}\int_{-c}^c y\,\zeta\,dy\right] \qquad (157)$$

Using (140), (142) and (157) we then find.-

$$\frac{dp}{dx} = C_{11}\,L_1^* + C_{12}\,M_1 \qquad (158)$$

where $\quad L_1^* = L_1 - \eta^* c\,Z$

and $\quad \eta^* = -\,C_{13}/cC_{11} \qquad (159)$

Using (144), (142) and (132) we find.-

$$-\frac{d^2W}{dx^2}\operatorname{cosec}\alpha = \frac{dq}{dx} = C_{21}L_1 + C_{22}M_1 \qquad (160)$$

where, $\quad C_{21} = C_{12}$

and $\quad C_{22} = \operatorname{cosec}\alpha/EI \qquad (161)$

The formular(158) and (160) have the same form as (99) and it can be shown that the constants C_{ij} of (157) and (161) reduce to the forms given in (100) when the proper specialisation is introduced. The difference in the new formulae lies in the introduction of L_1^* in (158). L_1^* is the moment about a line $y = \eta^* c$. The intersection of this line with a rib wise section (coordinate x) may be termed the 'centre for twist' at that section.

The aim set at the beginning of this section has now been accomplished. Formulae for stresses and deflections have been obtained for the case of a uniform swept wing structure loaded by 'normal' forces and couples at the ends. This represents a generalisation of the usual Bending-cum-Batho formulæ which are used by aircraft engineers to obtain a first approximation to the behaviour of unswept wings.

3.2. Procedure for Practical Stress Analysis

Consider now an actual swept back wing structure having two straight spars, skins reinforced by stringers and ribs parallel to the 'direction of flight'. (See Fig. 12) The wing possesses a small amount of taper and the dimensions of the structure vary in a gradual manner along the span. The existance of a plane of symmetry intersecting the spar webs will be assumed. If no such plane exists in reality, then the actual top and bottom surfaces should be replaced by fictitious

/surfaces having .....

surfaces having ordinates and geometry which are mean values of the real quantities for the two surfaces. This plane of symmetry will be taken as the x,y plane of a coordinate system. The y-axis will be taken parallel to the ribs the x-axis will intersect the traces of the ribs on the x,y plane at their mid points and the z-axis will be normal to the x,y plane. Attention will be directed in what follows to a single rib-wise section with coordinate x. The geometry of this section and of the various structural elements at this section, will be described by the symbols used in § 3.1. and illustrated in Fig. 10. It will be assumed that the wing is loaded by forces acting in a z-wise direction and by couples whose axes lie in the x,y plane.

The procedure for estimation of the stresses at section x may be outlined as follows:-

1. Tabulation of the values of the following quantities at this section:-

$\alpha, c, b, b', \zeta(y), t, t_W, t_W', t_R, A, A', A_S, A_R, a_S, a_R, E, \sigma$

If any of these, apart from ζ, vary across the section, then mean values should be taken. Allowance for the bending stiffness of the spar and rib webs should be made by augmenting the areas A, A' and A_R.

2. Calculation of sundry constants for the section:-

$\dfrac{A_S}{a_S}$, $\dfrac{A_R}{a_R}$, $(a_{ij})_P$ (equation (27)), $(a_{ij})_R$ (equation (28)),

a_{ij} (equation (29)), the determinant $|a_{ij}|$, A_{ij} (equation (131)), $\bar{\zeta}$ (equation (138)), $\int_{-c}^{c} \zeta^2 dy$, $\int_{-c}^{c} y\zeta^2 dy$, $\int_{-c}^{c}\int_{0}^{y} \zeta(y).\zeta(y_1) dy_1 dy$,

I (equation (142)), η (equation (143)).

3. Calculation of the resultant static action across the section:-

Z sum of z-wise forces acting at points outboard of section. This acts at the centre of the section (y=o).

L_1, M_1 Oblique components, refered to axes O(X,Y) (see Fig. 1) of the sum of the moments, about the centre of the section, of all forces and couples acting at places out board of the section. These may be calculated using the formulae of equation (19). If the external forces are denoted by Z_i and act at (x_i, y_i) we may write:-

$$L_1 = \sum_{x}^{1} y_i Z_i \qquad M_i = - \sum_{x}^{1} (x_i - x) Z_i$$

where the summation $\sum_{x}^{1}$ is with respect to i over all the points x_i such that $x < x_i \leq 1$ (where x = 1 is the tip). Any 'couples' must be replaced by forces before inclusion in these formulae.

/L_1 .....

$\bar{L}_1$ X-wise component of moment about $y = \eta c$ (see equation (145)).

4. Calculation of the Stress Resultants:-

S Shear per unit length (oblique component) in skins (see equation (145)).

T_1 Tension per unit length (oblique component) in skins. (see equation (146)).

The remaining component T_2 is zero.

S_W, S_W' Shear per unit length in spar webs (see equation (142))

The shear per unit length in the rib webs S_R is zero, except of course for effects due to local loads applied to the ribs.

5. Calculation of the stresses in the various components:-

$E\,e_{xx}$ Stress in the stringers. e_{xx} has already been found in the calculation of T_1 (equation (146)).

$E(e_{xx})_{y=\pm c}$ Stresses in the spar flanges.

The loads in the spar flanges and the tensions in the stringer-skin combination (T_1 per unit length) will have normal shear components if the wing structure is tapered. Corrections of the type, usually introduced in the stress analysis of unswept wings, can be introduced here to allow for the 'shear carried by end load', if it is felt to be worthwhile.

$E\,e_{yy}$ Stress in the rib flanges. This is given by (30). We find, $e_{yy} = A_{21}T_1 + A_{23}S$.

e_{xx}, e_{yy}, e_{xy} Strain components in the skin (oblique axes). e_{xx}, e_{yy} have already been found. e_{xy} follows from equation (30):- $e_{xy} = A_{31}T_1 + A_{33}S$

e_{xx}, e_{YY}, e_{xY} Strain components in the skin (rectangular axes $O(x,Y)$. e_{xx} has been calculated. e_{YY}, e_{xY} follow using equations (13) and (15).

$\bar{T}_1/t, \bar{T}_2/t, \bar{S}/t$ Stress components in the skin (rectangular axes $O(x,Y)$.

These follow from equation (26).

$S_W/t_W, S_W'/t_W'$ Shear stresses in the spar webs.

This completes the analysis of the stresses at a section of the wing. For a complete stress analysis these calculations must, of course, be repeated at a number of sections. The solution given will be in error near the tip, near large concentrated loads and at the root, but these errors are present in the customary application of the beam theory to unswept wings. A sufficiently accurate estimate of these

/errors ....

errors may be obtained by idealising the wing structure and treating it as a uniform doubly symmetric rectangular section box and applying the methods developed in § 2.7. The warping equation (127) found there is so similar to that for an unswept box that the outline given in § 2.7 should be an adequate basis for application.

3.3. Procedure for Deflection Calculations

The procedure given here for the calculation of deflections will be based upon the same assumptions with regard to the wing structure as the procedure for stress analysis of § 3.2. The calculations described must be carried out at a reasonable number of sections of the wing so that numerical integrations to obtain actual deflections and rotations can be carried out.

1. Calculation of Section Constants supplementary to those of § 3.2. (2)

$\int_0^c \zeta\, dy$, $\int_{-c}^{0} \zeta\, dy$, $\int_{-c}^{c} y\, \zeta\, dy$.

C_{11}, $C_{12} = C_{21}$, C_{13}, C_{22}. Formulae for these constants are given in equations (157), (161).

η^* (see equation (159).

2. Calculation of a special couple component supplementary to § 3.2 (3).

L_1^* X-wise component of moment about $y = \eta^* c$ (see equation (159)).

3. Calculation of Rates of Section Rotation.

$\frac{dp}{dx}$, $\frac{dq}{dx} = -\frac{d^2W}{dx^2}$ cosec α These quantities follow by equations (158) and (160).

4. Calculation of the Deflections and the Rotations

p, q These follow by integration of the expressions found in (3). This rotation is about an axis passing through the centre of the section ($y = z = o$)

$p \sin \alpha$ Decrease in 'incidence' of a rib section.

W This follows by integration of an expression found in (3). If the root is 'fixed' we may write $W = \frac{dW}{dx} = o$ at the root. However see (5) below in this connection.

5. Calculation of the 'Deflection due to Shear'.

γ This is the 'rate of shear deflection' and is given by equation (148). In § 3.1. it was a constant and equal to $\left(\frac{dW}{dx}\right)_{x=o}$ (see (132)). In general it will be variable.

/W_s The .....

W_s The 'additional deflection due to shearing'. This is obtained by integrating.-

$$\frac{dW_s}{dx} = \gamma.$$

W_s must be added to W to obtain the 'total' mean spar deflection. This prodedure will give the correct root conditions for the total deflection $W + W_s$.

This completes our analysis of deflections. No account has been given of the calculation of section warping and distortion, since this is of little practical importance. Rough estimates of these effects can however be made using the simplified structure of Part II (see equations (101), (102) (104),(105) and (106)).

FIG. 1

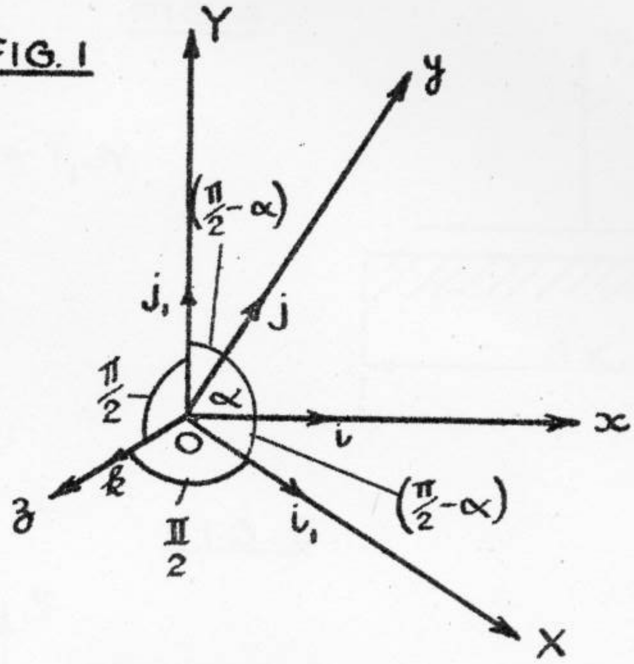

FIG. 2.

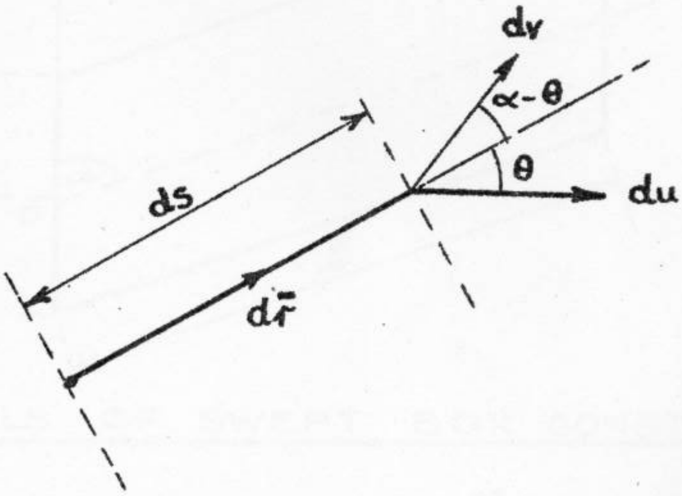

FIG. 3

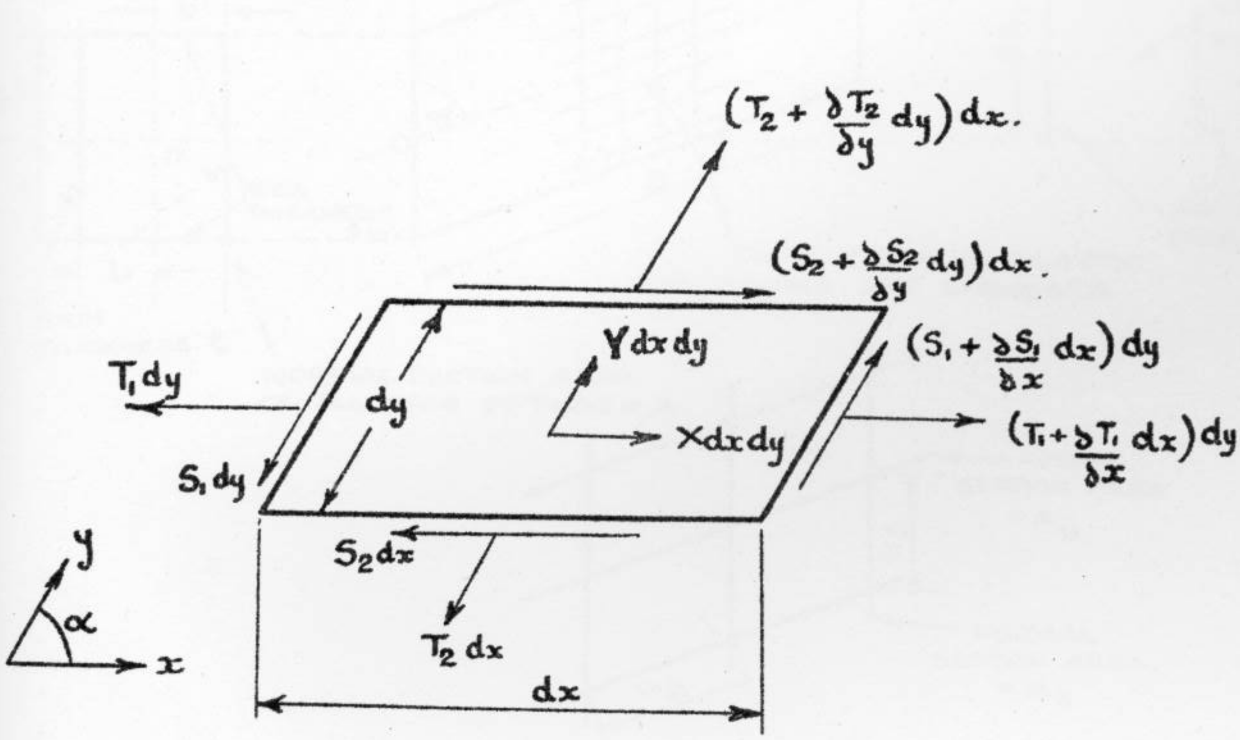

FIG. 4.

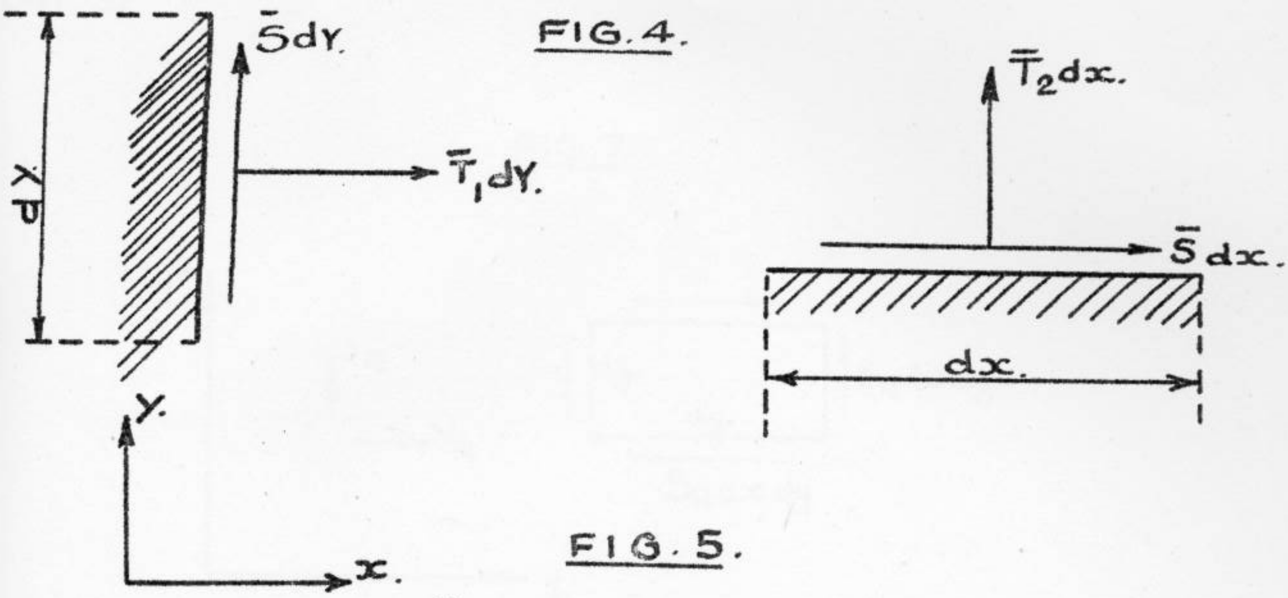

FIG. 5.

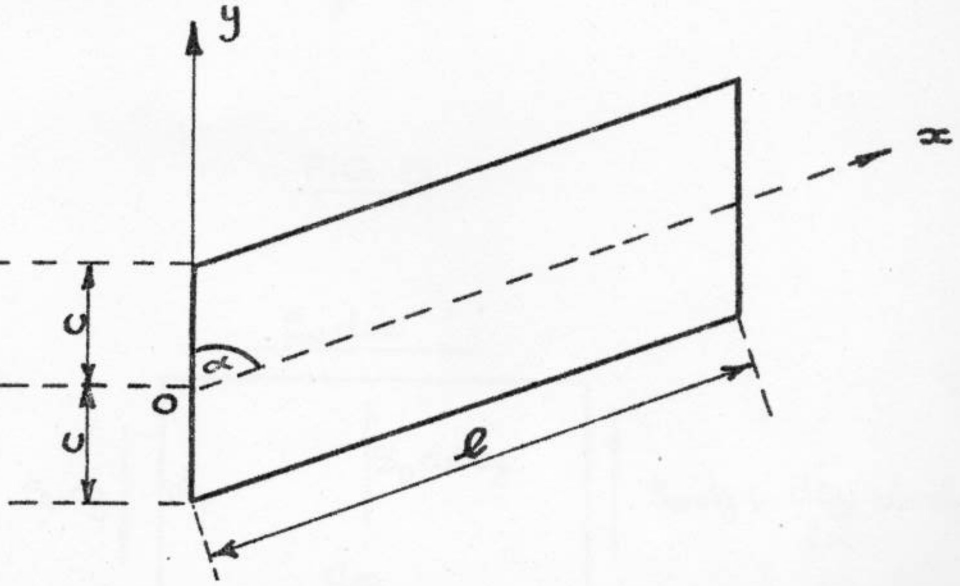

FIG. 6 DETAILS OF SWEPT BOX CONSTRUCTION

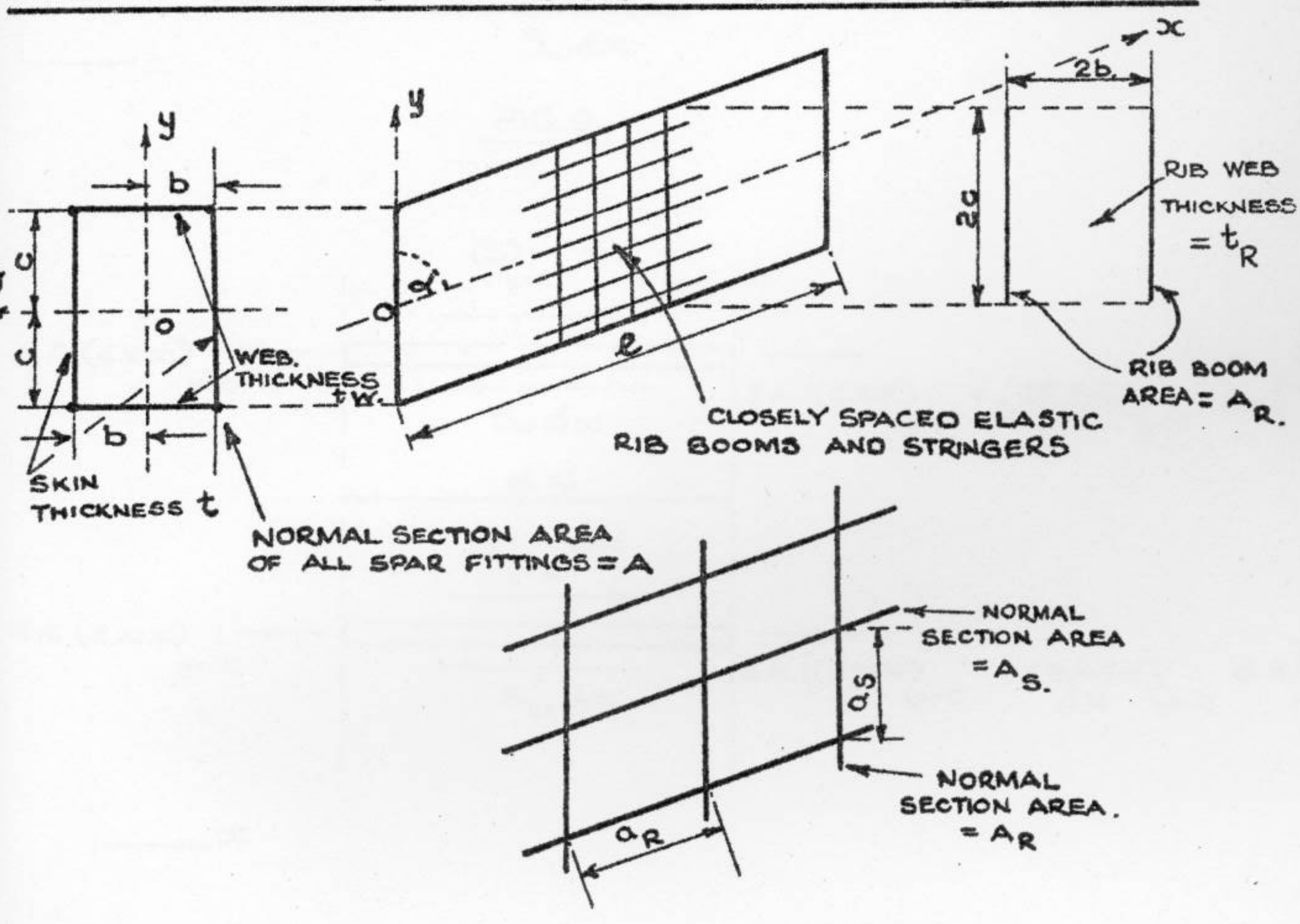

FIG. 7.

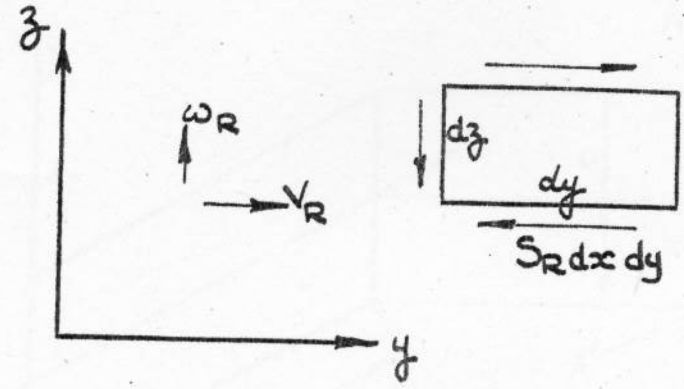

FIG. 8.

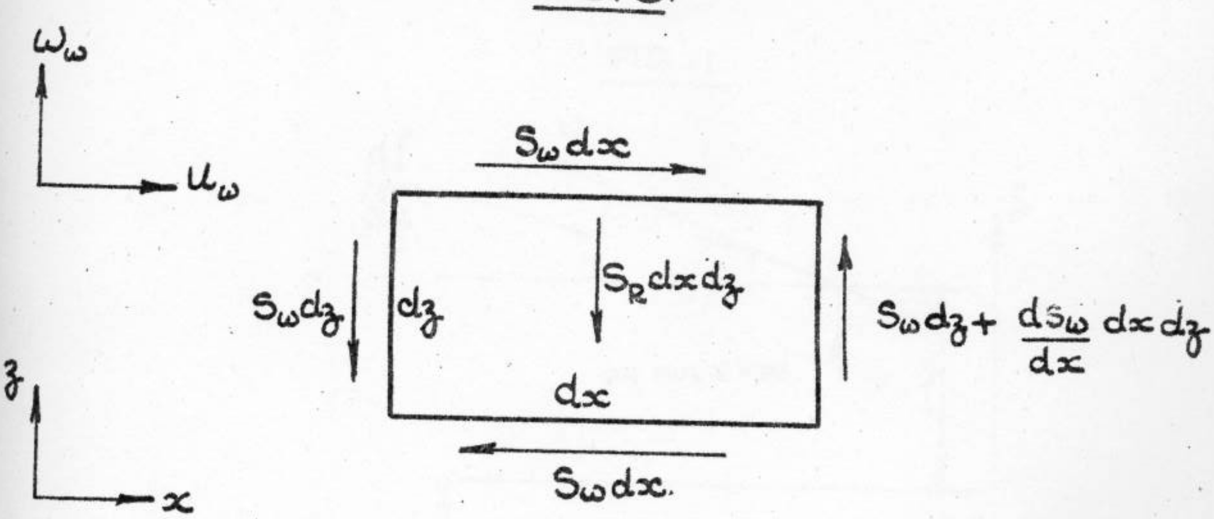

FIG. 9.

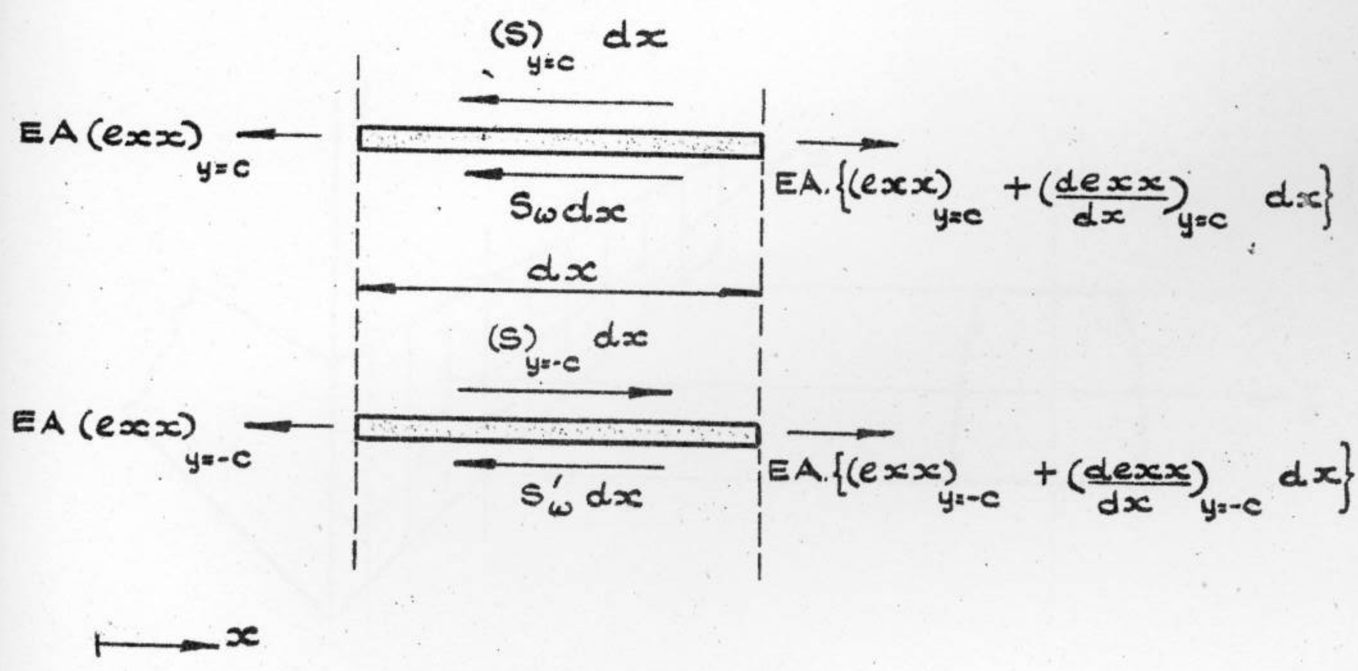

FIG.10.

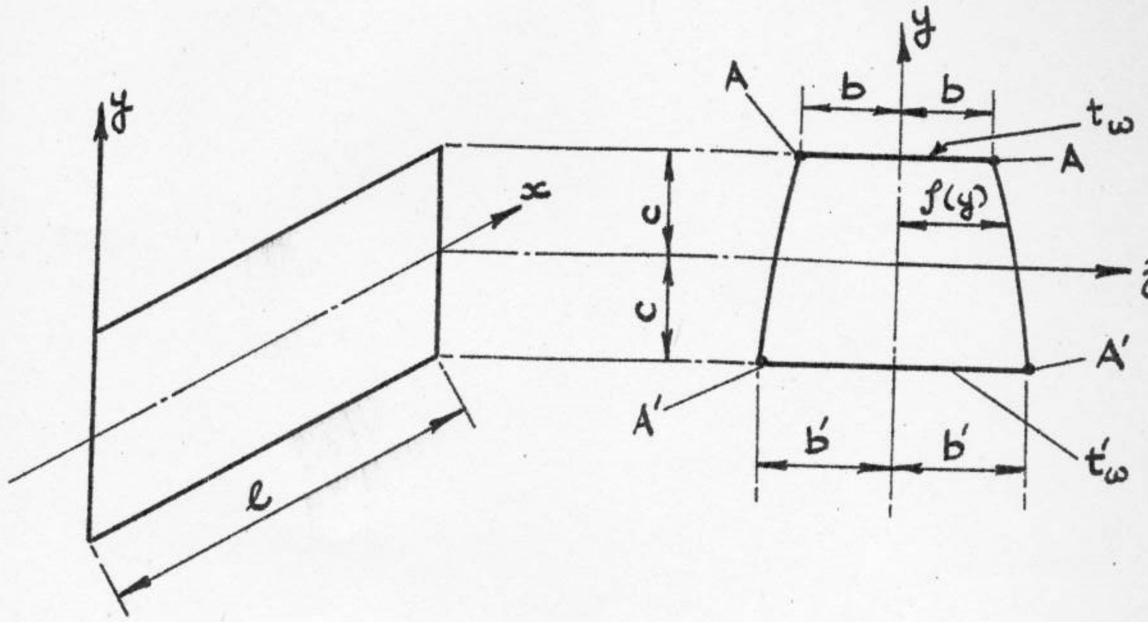

FIG.11.

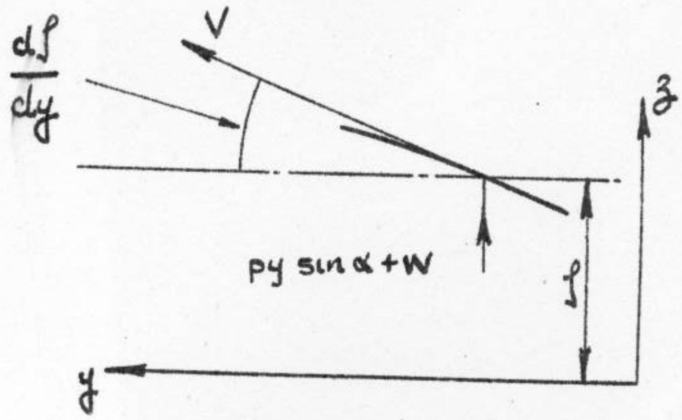

FIG.12.