The International Journal of Advanced Manufacturing Technology, Volume 55, Numbers 5-8, 2011, Pages 689-697

Optimal inspection policy for 3-state systems monitored by variable sample size control charts

Shaomin Wu

Received: date / Accepted: date

Abstract This paper presents the expected long-run cost per unit time for a system monitored by an adaptive control chart with variable sample sizes (VSS): if the control chart signals that the system is out-of-control, the followed sampling will be conducted with a larger sample size. The system is supposed to have three states: in-control, out-of-control, and failed. Two levels of repair are applied to maintain the system. A minor repair will be conducted if an assignable cause is confirmed by an inspection and a major repair will be performed if the system fails. Both the minor and major repairs are assumed to be perfect. We derive the expected long-run cost per unit time, which can be used to obtain the optimal inspection policy. Numerical examples are conducted to validate the derived cost.

Keywords Quality control · maintenance policy · control chart · repairable system · multi-state system · adaptive control chart

1 Introduction

Condition-based maintenance has nowadays been widely applied to monitoring the performance of important systems for improving their availabilities. Control charts are one of the monitoring tools employed in manufacture for the purpose of removal of assignable causes every time when the process parameter has shifted. As control charts – similar to other monitoring tools – may produce false signals that incorrectly indicate the state of the system, optimally designing the parameters of control charts to minimize the cost incurred by the false signals is a vitally important topic in the research community of statistical process control. Various control charts have been considered by researchers. Some examples are as follows. [1] and [2] separated the $\bar{X}$-chart into several zones and optimized the chart for monitoring a process whose deterioration can be classified into two states, in which one state requires minor repair and the other requires major repair. [3] used the p-chart to derive thresholds for aviation inspection. [4] derived the expected long-run costs per unit time for a system monitored by the cumulative count of conforming chart (CCC chart) where the system is maintained

Cranfield University, School of Applied Sciences; Tel.: +44 -1234-754963; Fax: +44 -1234-751671; E-mail: shaomin.wu@cranfield.ac.uk

with different levels of inspection and maintenance. [5] considered economic design of control charts for optimization of preventive maintenance policies for systems. Other examples of research in this area can also be seen in [6] and [7].

When control charts are used, a general assumption is that the system being monitored has three states: *in-control*, *out-of-control,* and *failure*. the in-control state is the state that the system functions without any problem, the out-of-control state means that the system has been disrupted by the occurrence of events called assignable causes but it still functions, and a failure state is the state that the system stops functioning. The decision variables in designing a control chart can be the sampling interval between consecutive sampling points, the sample size, or the control limits. Typical application areas can be found in continuous manufacturing processes such as electronic item assembly lines.

The parameters in a control chart can be variable, based on which we have two different kinds of control charts: *static* and *adaptive*. A static control chart has fixed parameters such as sample size n, sampling interval h, lower control limit (LCL), and upper control limit (UCL). On the other hand, an adaptive control chart has at least one of its parameters (n, h, LCL and UCL) that is allowed to be changed based on the values of the sample statistics, which provides information about the current state of the process. An introduction to control charts can be found in [8].

An adaptive control chart can utilise the inspection capacity more effectively for better process control ([9–14, 4]). There has been little work in investigating the potential of an adaptive control chart to monitor a system that has different repair levels. However, it is vitally important for industrial practitioners to have tools or formulas that can help them to design maintenance regimes or/and parameters of in-control charts, especially for adaptive controls charts (as a static control chart can be seen as a special case of an adaptive control).

This paper presents the formulas of the expected long-run cost per unit time for a system monitored by an adaptive control chart with variable sample sizes (VSS), which can ultimately be used to optimize the parameters in the control chart. The system is assumed to have three states, in-control, out-of-control, and failed. The adaptive chart has three zones: central, warning, and action zones. If the quality characteristic (for example, the number of the non-defectives in the np control chart or the average of the observations in a subgroup for the $\bar{X}$ control chart) falls in the central zone, no action will be taken and the next sampling interval remains the same as its previous one. If the quality characteristic falls in the warning zones, more products will immediately be sampled. If the quality characteristic in the new sample falls in the central zone, then no action will be taken, otherwise, an inspection will be performed. If the quality characteristic falls in the action zone, then an inspection will immediately be carried out to check the existence of a possible assignable cause. If the assignable cause is confirmed, a minor repair will be conducted to remove the assignable cause. If the system fails, then a major repair will be performed. Both the minor and major repairs are perfect, that is, they can bring the system back to a good-as-new state.

In this paper, we only consider a 3-state situation, which forms a multistate reliability system. Research in multistate systems is another interesting topic in reliability theory and engineering, the reader is referred to [15, 16], and [17] for more information.

This paper does not specify a typical type of control charts. The result can be applied to either attribute control charts (e.g, $\bar{X}$ control charts) or variable control charts (e.g., np control charts). But the numerical example uses an np control chart as an example.

This paper is structured as follows. The next section briefly introduces the VSS control chart. Section 3 presents assumptions and notation used in the paper. Section 4 formulates the expected long-run cost per unit time for systems monitored by the VSS control chart. Section 5 offers numerical examples to perform sensitivity analysis for various parameter settings. Section 6 concludes the findings of this paper.

2 VSS control chart

A static control chart has two zones (see Figure 2(a)): central zone Z_{f0} and action zones Z_{f1}, whereas an adaptive control chart has three zones (see Figure 2(b)): central zone Z_{a0}, warning zones Z_{a1}, and action zones Z_{a2}. From a comprehensive survey in the developments and the designs of adaptive control charts, the reader is referred to [18].

A VSS control chart uses two different sample sizes alternatively, depending on the quality characteristic of the process. If the quality characteristic is in the central state, then a normal sample size n_0 is employed. Conversely, if the quality characteristic falls in the warning zones (see Figure 2(b)), then a larger number $n_1(> n_0)$ is used as the next sample size to confirm the existence of the possible assignable cause.

3 Assumptions and notation

Consider a system with three states: in-control, out-of-control, and failed, we make the following assumptions.

The first sampling interval is h unit times immediately after the start of the system and n_0 samples are then collected. After that, there are following four situations.

A1. If the quality characteristic of the n_0 samples falls in Z_{a0} (see Figure 2(b)), then the next sample size will remain the same (ie., n_0), and no further action will be taken.

A2. If the quality characteristic of the n_0 samples falls in Z_{a1}, then the next sample size will be n_1 with zero time interval, and an inspection will be carried out to check whether the system is in-control or out-of-control. If the system is confirmed to be out-of-control, then a minor repair is performed, otherwise, no further action will be taken and the next sampling interval will be h and the sample size will be n_0.

A3. If the quality characteristic of the n_0 samples falls in Z_{a2}, then an inspection will be carried out to check the existence of the assignable cause. If the occurrence is confirmed by the inspection, then a minor repair is performed; otherwise, no further action will be taken and the next sampling interval will be h and the sample size will be n_0.

A4. If the system fails, then a major repair will be conducted immediately.

The following assumptions are also held.

- Suppose that the system can shift from the in-control state to the out-of-control state and then to the failure state; but it cannot shift directly from the in-control state to the failure state without going through the out-of-control state, see Figure 1. Neither the failure state nor the out-of-control state can be restored back to the in-control state without any intervention.

- An inspection is assumed to be perfect in that it can reveal whether the system is in-control or out-of-control. During an inspection, the system does not stop and carries on running. Once the system has been confirmed to be in the out-of-control state by the inspection, repairmen will carry out a minor repair which can bring the system back to a good-as-new state. Once the system fails, repairmen will conduct a major repair. The major repair can bring the system back to a good-as-new state.
- For simplicity, times spent on an inspection, a minor or a major repair are so short compared to the sampling interval that can be neglected. But their costs are considered.

We also denote

- X_1, random time from the beginning of the in-control state to the occurrence of an assignable cause;
- $f_1(x_1)$, pdf. of X_1, and $F_1(x_1) = \Pr(X_1 < x_1)$, cdf. of X_1;
- X_2, random time from the beginning of the out-of-control state to failure;
- $f_2(x_2)$, pdf. of X_2, and $F_2(x_2) = \Pr(X_2 < x_2)$, cdf. of X_2;
- n_0, normal sample size;
- n_1, larger sample size;
- h, sampling interval;
- c_s, sampling cost per sample;
- c_i, inspection cost for a possible assignable cause;
- c_{r1}, cost for a minor repair;
- c_{r2}, cost for a major repair;
- α_{ij}, probability that the quality characteristic falls in Z_{aj} $(j = 0, 1, 2)$ when the system is in the in-control state for $i = 0$, or in the out-of-control state for $i = 1$. It is for the situation when a sample size n_0 is applied;
- β_{ij}, the probability that the quality characteristic falls in Z_{aj} $(j = 0, 1, 2)$ when the system is in the in-control state for $i = 0$, or in the out-of-control state for $i = 1$. It is for the situation when a sample size n_1 is applied;
- T_a, renewal cycle length;
- T_{a1}, time to the first minor repair with an assignable cause detected by the control chart in a sampling interval where a longer sample size is used;
- T_{a2}, time to the first minor repair with an assignable cause detected by the out-of-control signal by the control chart when a normal sample size is used;
- T_{a3}, time to failure; and
- C_{a1}, C_{a2}, C_{a3}, costs incurred within times T_{a1}, T_{a2} and T_{a3}, respectively.

In the following, we use the renewal reward theorem, which simply states that the expected long run cost per unit time is the ratio between the expected renewal cycle cost and expected renewal cycle length [19].

4 Expected long-run cost per unit time

From the above assumptions, the system can be renewed by either a minor repair or a major repair, which are listed in Assumptions A2, A3, and A4. As such, these three cases are listed in the following.

Case 1: From Assumption A2, a minor repair is conducted due to an assignable cause that is confirmed by a warning appeared in Z_{a1}. Namely, the system is in the out-of-control state and the quality characteristic falls in Z_{a1}. In this case, the warning is signaled when the sample size n_1 is applied. There might be the following three cases.

- When the system is in the out-of-control state, a warning is signaled during a sampling with the normal sample size n_0. Then an additional sampling with the larger sample size n_1 is immediately conducted, and then an inspection is taken. A minor repair is then conducted. See Figure 3(a).
- The system transits to the out-of-control state when a sampling with the normal sample size n_0 is being conducted. In this case, the signal from this sampling is false and an additional sampling is conducted. See Figure 3(b).
- The system transits to the out-of-control state when a sampling with the normal sample size n_1 is being conducted. See Figure 3(c).

Case 2: Based on Assumption A3, a minor repair is conducted due to an assignable cause that is confirmed by a warning appeared in Z_{a2}. In this case, the warning is signaled when the sample size n_0 is applied.

Case 3: Based on Assumption A4, the system fails, but before the failure, no warning has been signaled.

Below, the expected renewal cycle length of the above three cases are denoted by $E(T_{a1})$, $E(T_{a2})$, and $E(T_{a3})$, respectively.

4.1 *Expected renewal cycle length*

The expected renewal cycle length is $E(T_a) = E(T_{a1}) + E(T_{a2}) + E(T_{a3})$, which is explained as follows.

The expected time between the start and a minor repair triggered by an inspection due to a signal in zone Z_{a1} is given by

$$\begin{aligned} E(T_{a1}) = &\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1}\sum_{k_3=0}^{k_1-k_2}\Bigg\{\alpha_{10}^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3}\alpha_{11}(1-\beta_{10})H_0\int_{H_1-2h}^{H_1-h} f_1(x_1)(1-F_2(H_0-x_1))dx_1 \\ &+\ \alpha_{01}\alpha_{10}^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3+1}(1-\beta_{10})(H_0+2h)\int_{H_1-h}^{H_1} f_1(x_1)(1-F_2(H_0+2h-x_1))dx_1\Bigg\} \\ &+\sum_{k_2=0}^{\infty}\Bigg\{\alpha_{01}(1-\beta_{10})H_2\int_{H_2-h}^{H_2} f_1(x_1)(1-F_2(H_2-x_1))dx_1\Bigg\}, \end{aligned} \tag{1}$$

where $H_0 = k_1h + (\alpha_{01}(k_2-1)+k_3+1)h$, $H_1 = k_2h + \alpha_{01}(k_2-1)h + h$, and $H_2 = k_2h + \alpha_{01}k_2h + 2h$.

Proof The description of three terms in equation (1) is given below.

Denote k_1 as the total number of sampling intervals in both the in-control and out-of-control states, k_2 as the total number of sampling intervals in an in-control state, and k_3 as the number of false signals followed by true ones in the out-of-control state. The number k_2 includes two scenarios: (1) the quality characteristics with the normal sample size n_0 signal warnings that correctly indicate the system in the in-control state; and (2) the quality characteristics with the normal sample size n_0 signal warnings that wrongly indicate that the system is in the out-of-control state, and then further samplings with the larger sample size n_1 are conducted.

There are three scenarios for the system transiting from the in-control state to the out-of-control state. These three states correspond to the following the three terms in equation(1).

Term 1. The system transits from the in-control state to the out-of-control state with a normal sample size n_0. See Figure 3(a). When the system is in the out-of-control state, there might be false signals (with a probability of $\alpha_{10}^{k_1-k_2-k_3}$) with a normal sample size n_0 and the false signals wrongly indicate that the system is in the in-control state, or true signals with a larger sample size n_1 but followed by false signals (with a probability of $(\alpha_{11}\beta_{10})^{k_3}$): the true signals correctly indicate that the system is in the out-of-control state but its following sampling wrongly indicates that the system is in the in-control state. These two scenarios make up an event with a probability of $(\alpha_{10})^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3}$, and take time $(k_1-k_2+k_3)h$. Eventually, a correct signal with a normal sample size is followed by another correct signal with a larger sample size, which has a probability of $\alpha_{11}(1-\beta_{10})$ and a time length of $2h$.
Before the system has transited from the in-control state to the out-of-control state, the time length is $(k_2-1)h+\alpha_{01}(k_2-1)h$. Hence, the total length is $k_1h+(\alpha_{01}(k_2-1)+k_3+1)h=H_0$. The transition occurs in the time interval $((k_2-1)h+\alpha_{01}(k_2-1)h, k_2h+\alpha_{01}(k_2-1)h)$, or (H_1-2h, H_1-h).

Term 2. See Figure 3(b). The system might also transit from the in-control state to the out-of-control state within a normal sample size after a false signal appears in the in-control state, but a correct signal follows. This event has a probability of $\alpha_{01}(1-\beta_{10})$ and a time length of $h+h$. The time length of the system in the in-control state is $(k_2-1)h+\alpha_{01}(k_2-1)h$, then the transition from the in-control state to the out-of-control state occurs in (H_1-h, H_1). After the system has transited to the out-of-control state, the probability of the appearance of a correct signal is given by $\alpha_{10}^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3+1}\alpha_{01}(1-\beta_{10})$ and has a time length of H_0+2h.

Term 3. When the system is in the in-control state, a false signal appears with a normal sample size. See Figure 3(c). Then a larger sample size is used and the system transits to the out-of-control state in this sampling interval, and then a true signal appears. This event has a probability of $\alpha_{01}(1-\beta_{10})$.

The expected time between the start and a minor repair triggered by an inspection due to a signal in zone Z_{a2} is given by

$$E(T_{a2}) = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1}\sum_{k_3=0}^{k_1-k_2}\left\{\alpha_{10}^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3}\alpha_{12}(H_0-h)\int_{H_1-2h}^{H_1-h} f_1(x_1)(1-F_2(H_0-h-x_1))dx_1\right\}. \tag{2}$$

Proof The proof is similar to that of $E(T_{a1})$, apart from the appearance of the out-of-control signals in a longer interval h in this case.

The expected time between the start and a major repair is given by

$$E(T_{a3}) = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1}\sum_{k_3=0}^{k_1-k_2}\alpha_{10}^{k_1-k_2-k_3}\alpha_{11}^{k_3}\beta_{10}^{k_3}\left\{\int_{H_0-2h}^{H_0-h}\int_{H_1-2h}^{\tau_{k_1k_2k_3}} xf_1(x_1)f_2(x-x_1)dx_1dx\right.$$

$$
\begin{aligned}
&+\ \alpha_{11}\int_{H_0-h}^{H_0}\int_{H_1-2h}^{\tau_{k_1k_2k_3}} xf_1(x_1)f_2(x-x_1)dx_1dx \\
&+\ \int_{H_0-2h}^{H_0-h}\int_{H_1-h}^{\tau'_{k_1k_2k_3}} xf_1(x_1)f_2(x-x_1)dx_1dx \\
&+\ \alpha_{11}\int_{H_0-h}^{H_0}\int_{H_1-h}^{\tau'_{k_1k_2k_3}} xf_1(x_1)f_2(x-x_1)dx_1dx \Bigg\},
\end{aligned}
\tag{3}
$$

where $\tau_{k_1k_2k_3} = \begin{cases} H_1-h & \text{if } k_1-k_2 \neq 0 \\ x & \text{if } k_1-k_2 = 0, \end{cases}$ and $\tau'_{k_1k_2k_3} = \begin{cases} H_1 & \text{if } k_1-k_2 \neq 0 \\ x & \text{if } k_1-k_2 = 0. \end{cases}$

Proof The system might transit from the in-control state to the out-of-control state either in a sampling interval using a normal sample size or in a sampling interval using a larger sample size, and the system can then fail in both sampling intervals, which creates four scenarios. The first two terms in equation (3) correspond to the scenarios when the transition from the in-control state to the out-of-control state occurs in a sampling interval when a normal sample size n_0 is conducted, and they correspond to the scenarios when the transition from the in-control state to the out-of-control state occurs in a sampling interval when a larger sample size n_1 is conducted.

The first term in equation (3) is the scenario when the two transitions (i.e., from the in-control state to the out-of-control state and then fail) occur in longer sampling intervals. The second term means that a correct signal appears in a longer sampling interval h (with a probability β_{11}) followed by a shorter sampling interval h for confirmation, but the system fails within this h. The third term means that the transition from the in-control state to the out-of-control state occurs (with a probability β_{01} followed by a shorter sampling interval). The last term means that the two scenarios occur in short sampling intervals.

4.2 Expected renewal cycle cost

The costs incurred during periods $E(T_{f1})$, $E(T_{f2})$, and $E(T_{f3})$ are derived in the following.

$$
\begin{aligned}
E(C_{a1}) = &\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1}\sum_{k_3=0}^{k_1-k_2}\Bigg\{\alpha_{10}^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3}\alpha_{11}(1-\beta_{10})\int_{H_1-2h}^{H_1-h} C_0 f_1(x_1)(1-F_2(H_0-x_1))dx_1 \\
&+\ \alpha_{01}\alpha_{10}^{k_1-k_2-k_3}\alpha_{11}^{k_3+1}\beta_{10}^{k_3+1}(1-\beta_{10})\int_{H_1-h}^{H_1} C_1 f_1(x_1)(1-F_2(H_0+2h-x_1))dx_1\Bigg\} \\
&+\ \sum_{k_2=0}^{\infty}\Bigg\{\alpha_{01}(1-\beta_{10})\int_{H_2-h}^{H_2} C_2 f_1(x_1)(1-F_2(H_2-x_1))dx_1\Bigg\},
\end{aligned}
\tag{4}
$$

where $C_0 = k_1n_0c_s+(\alpha_{01}(k_2-1)+k_3+1)n_1c_s+((\alpha_{02}+\alpha_{01}(1-\beta_{00}))(k_2-1)+1)c_i+c_{r1}$, $C_1 = C_0+(n_0+n_1)c_s$, and $C_2 = k_2n_0c_s+\alpha_{01}k_2n_1c_s+(n_0+n_1)c_s+(\alpha_{02}+\alpha_{01}(1-\beta_{00}))k_2c_i+c_i+c_{r1}$.

Proof After the system transited from the in-control state to the out-of-control state in a sampling interval when a normal sample size n_0 is conducted, there will be two possible scenarios before two warning signals appear consecutively in two sampling intervals with a normal sample size n_0 and a larger sample size n_1, respectively. The first scenario is that incorrect signals (with a probability of β_{10}) appears, the second scenario is that a correct signal followed by an incorrect signal (with a probability of $\beta_{11}\beta_{10}$). These two scenarios make up an event with a probability of $(\beta_{10})^{k_1-k_2-k_3}(\beta_{11}\beta_{10})^{k_3}\beta_{11}(1-\beta_{10})$, and the event incurs sampling cost $(k_1-k_2+1)nc_s+(k_3+1)nc_s$. Before the system has transited from the in-control state to the out-of-control state, the sampling cost is $k_2nc_s+\beta_{01}k_2nc_s$, inspection cost $(\beta_{02}+\beta_{01}(1-\beta_{00})k_2+1)c_i$, and cost c_{r1} on minor repair. Hence, the sub-total cost is C_0.

The system might also transit from the in-control state to the out-of-control state within a interval when a larger sample n_1 is conducted after a false signal appear in the in-control state. This event incurs cost $nc_s+c_i+c_{r1}$. The cost incurred before the transition is $k_2nc_s+\beta_{01}k_2nc_s+\beta_{01}\beta_{01}k_2c_i$. The sub-total cost is C_1.

A similar explanation to the third term in equation (4) can be given.

Similarly,

$$E(C_{a2})=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1}\sum_{k_3=0}^{k_1-k_2}\left\{\alpha_{10}^{k_1-k_2-k_3}(\alpha_{11}\beta_{10})^{k_3}\alpha_{12}\int_{H_1-2h}^{H_1-h}C_3f_1(x_1)(1-F_2(H_0-h-x_1))dx_1\right\}, \tag{5}$$

where $C_3=C_0-n_1c_s$.

And finally, we have

$$\begin{aligned}E(C_{a3})=&\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1}\sum_{k_3=0}^{k_1-k_2}\alpha_{10}^{k_1-k_2-k_3}\alpha_{11}^{k_3}\beta_{10}^{k_3}\left\{\int_{H_0-2h}^{H_0-h}\int_{H_1-2h}^{\tau_{k_1k_2k_3}}C_4f_1(x_1)f_2(x-x_1)dx_1dx\right.\\&+\alpha_{11}\int_{H_0-h}^{H_0}\int_{H_1-2h}^{\tau_{k_1k_2k_3}}C_5f_1(x_1)f_2(x-x_1)dx_1dx\\&+\int_{H_0-2h}^{H_0-h}\int_{H_1-h}^{\tau'_{k_1k_2k_3}}C_6f_1(x_1)f_2(x-x_1)dx_1dx\\&\left.+\alpha_{11}\int_{H_0-h}^{H_0}\int_{H_1-h}^{\tau'_{k_1k_2k_3}}C_7f_1(x_1)f_2(x-x_1)dx_1dx\right\},\end{aligned} \tag{6}$$

where $C_4=k_1n_0c_s+(\alpha_{01}(k_2-1)+k_3)n_1c_s-n_0c_s+((\alpha_{02}+\alpha_{01}(1-\beta_{00}))(k_2-1))c_i+c_{r2}$, $C_5=C_4+n_0c_s+c_i$, $C_6=C_4$, and $C_7=C_5$.

Hence, the expected long-run cost per unit time is given by

$$E_a(T,C)=\frac{E(C_{a1})+E(C_{a2})+E(C_{a3})}{E(T_{a1})+E(T_{a2})+E(T_{a3})}. \tag{7}$$

$E_a(T,C)$ in equation (7) can be minimized to obtain the optimal parameters such as α_{ij} and β_{ij}, which is equivalent to optimize inspection policy for 3-state systems monitored by the adaptive control charts.

5 A data example

In this section, we conduct use one numerical data example to investigate th impacts of the cost parameters, assuming $F_1(x_1) = 1 - \exp(-(\frac{x_1}{300})^{2.5})$, and $F_2(x_2) = 1 - \exp(-(\frac{x_2}{200})^4)$.

We also assume the parameter values in Table 1 for the numerical example where an np chart is used.

Table 2 indicates the results of the minimum expected long-run cost per unit time. For example, the optimum n_1 is 144 when $n_0 = 80$. This ensures the expected long-run cost per unit time to be minimal, or $E_a(T, C) = 5.87$. Comparing all of the costs $E_a(T, C)$, it can be found that the expected long-run cost per unit time reaches the minimal $E_a(T, C) = 2.276$ when $n_0 = 130$ and $n_1 = 131$. We also notice that the ratio $\frac{n_1}{n_0}$ becomes smaller when n_0 increases.

5.1 Sampling cost c_s

If c_s changes from 0.1 to 9.1 with step 1, the optimal n_0 and n_1 will change from 125 and 137 to 95 and 104 as shown in Table 3. It can be seen that $E_a(T, C)$ changes from 2.285 to 11.761 when c_s changes from 0.1 to 9.1.

5.2 Inspection cost c_i

If c_i changes from 10 to 460, the optimal n_0 and n_1 will change from 140 and 154 to 120 and 132 as shown in Table 4. It is noticed that the sample sizes n_0 and n_1 remain unchanged when c_i changes in intervals (40,90), (100,110), or (160,460).

5.3 Minor repair cost c_{r1}

If c_{r1} changes from 50 to 4500, the optimal n_0 and n_1 changes as shown in Table 5.

It is noticed that the optimum samples n_0 and n_1 do not change dramatically when c_{r1} changes from 50 to 4500: the optimum n_0 and n_1 change from 130 and 143 to 110 and 121, respectively. This suggests that parameter c_{r1} is not sensitive to $E_a(T, C)$ when c_{r1} is in intervals (50,1500) or (2000,4000). We also notice that $E_a(T, C)$ has a large change, from 0.867 to 14.862 when c_{r1} changes from 50 to 4500.

It is noticed that in the above three situations, optimum sample sizes are moving in an opposite direction to that of the changes of costs, c_s, c_i, and c_{r1}: the optimum sample sizes become smaller when those costs become larger.

5.4 Major repair cost c_{r2}

If c_{r2} changes from 500 to 10000, the optimal n_0 and n_1 change as shown in Table 6.

When the major repair c_{r2} increases, the optimum sample sizes increase. It is noticed that the optimum sample sizes n_0 and n_1 remain their respective values, 130 and 143, unchanged, when c_{r2} changes from 4000 to 10000. The optimum sample sizes n_0 and n_1 change when c_{r2} changes from 500 to 1000. This suggests that the parameter

c_{r2} is not sensitive to the cost $E_a(T,C)$ when c_{r2} is in the interval (4000,1000), but it is sensitive to the cost $E_a(T,C)$ when c_{r2} changes from 500 to 1000. In other words, parameter c_{r2} is not sensitive to cost $E_a(T,C)$ when c_{r2} is bigger, whereas c_{r2} is sensitive to the cost $E_a(T,C)$ when it is smaller. It is also noticed that $E_a(T,C)$ has only a slight change, from 1.986 to 2.516, when c_{r2} conducts a big change, from 500 to 10000.

5.5 Discussion

From the above analysis, one can see that in some cases, the optimum sample sizes n_0 and n_1 remain unchanged although cost may change.

It is also noticed that the sampling cost is the most sensitive one impacting $E_a(T,C)$. For cost c_{r2}, it is interesting to notice that the cost $E_a(T,C)$ changes in different directions from the above three costs: c_i, c_s and c_{r1}: the optimum sample sizes increases when cost c_{r2} on major repair increases, and the optimum sample sizes decreases when cost c_{r2} on major repair increases.

6 Concluding remarks

In this paper, the expected long-run cost per unit time is derived for the situation where adaptive control charts with variable sample size are applied to monitor a system with three states: in-control, out-of-control and failure states. We have also conducted sensitivity analysis to investigate the impact of each cost to the expected long-run cost per unit time. It is found that the sample sizes become smaller when any of the individual cost (including sampling cost, inspection cost, and cost on minor repair) increases. However, the sample sizes become larger when cost on major repair increases. Among the four costs, sampling cost is the most sensitive one impacting the expected long-run cost per unit time.

In practice, it is often found that estimating real costs incurred by sampling, inspection or repair is not easy. The sensitivity analysis on the parameters suggests that practitioners can obtain optimum solutions although costs estimated may fall in intervals, instead of precise values.

Our further work will be focused on investigating the scenario when different types of maintenance models (see [20], for example) are considered.

Acknowledgment

We are grateful to Professor Wenbin Wang for helpful discussion. We would like to thank the reviewers for the helpful comments that lead to some improvements.

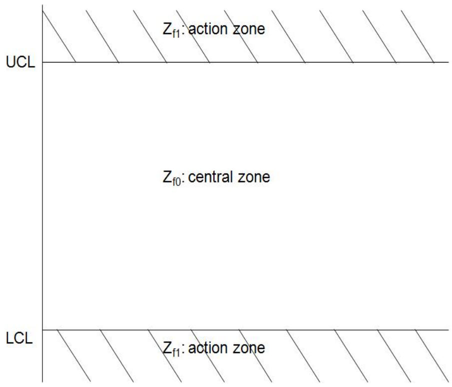
References

1. G. Tagaras, H. Lee, Economic design of control charts with different control limits for different assignable causes, Management Science 34 (11) (1988) 1347–1366.
2. G. Tagaras, H. Lee, Approximate semi-economic design of control charts with multiple control limits, Naval Research Logistics 36 (1989) 337–353.

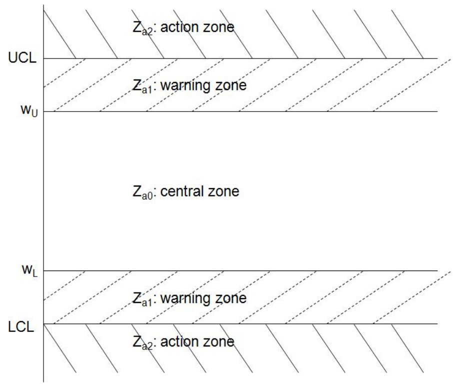
3. A. Cheng, R. Liu, J. Luxhj, Thresholds for safety inspection measurements based on control charts, International Journal of Reliability, Quality and Safety Engineering 42 (1997) 205–225.
4. L. Chan, S. Wu, Optimal design for inspection and maintenance policy based on the ccc chart, Computers and Industrial Engineering 57 (3) (2009) 667–676.
5. R. Cassady, R. Bowden, L. Liew, E. Pohl, Combining preventive maintenance and statistical process control: a preliminary investigation, IIE Transactions (Institute of Industrial Engineers) 32 (6) (2000) 471–478.
6. T. Goh, Some practical issues in the assessment of nonconforming rates in a manufacturing process, International Journal of Production Economics 33 (1-3) (1994) 81–88.
7. M. Xie, T. Goh, P. Ranjan, Some effective control chart procedures for reliability monitoring, Reliability Engineering and System Safety 77 (2) (2002) 143–150.
8. D. Montgomery, Introduction to statistical quality control, John Wiley and Son, 2001.
9. H. Ohta, M. Rahim, A dynamic economic model for an $\bar{X}$-control chart design, IIE Transactions (Institute of Industrial Engineers) 29 (6) (1997) 481–486.
10. B. Parkhideh, K. Case, Economic design of a dynamic X-control chart, IIE Transactions (Institute of Industrial Engineers) 21 (4) (1989) 313–323.
11. M. De Magalhes, E. Epprecht, A. Costa, Economic design of a vp $\bar{X}$ chart, International Journal of Production Economics 74 (1-3) (2001) 191–200.
12. Y. Chen, K. Hsieh, C. Chang, Economic design of the VSSI $\bar{X}$ control charts for correlated data, International Journal of Production Economics 107 (2) (2007) 528–539.
13. S. Zhang, Z. Wu, A CUSUM scheme with variable sample sizes for monitoring process shifts, International Journal of Advanced Manufacturing Technology 33 (9-10) (2007) 977–987.
14. J. Y. Liu, M. Xie, T. N. Goh, Q. H. Liu, Z. H. Yang, Cumulative count of conforming chart with variable sampling intervals, International Journal of Production Economics 101 (2) (2006) 286–297.
15. B. Natvig, Two suggestions of how to define a multistate coherent system, Advances in Applied Probability 14 (2) (1982) 434–455.
16. S. Wu, L. Chan, Performance utility-analysis of multi-state systems, IEEE Transactions on Reliability 52 (1) (2003) 14–21.
17. S. Wu, Joint importance of multistate systems, Computers and Industrial Engineering 49 (1) (2005) 63–75.
18. G. Tagaras, A survey of recent developments in the design of adaptive control charts, Journal of Quality Technology 30 (3) (1998) 212–231.
19. S. Ross, The exponential distribution and the Poisson process, Introduction to probability models. 9th ed. (2007) 302–364.
20. S. Wu, M.J. Zuo, Linear and nonlinear preventive maintenance models, IEEE Transactions on Reliability 59 (1) (2010) 242-249.

$$\begin{tikzcd}
\mathrm{In-control\ state} \arrow[r] & \mathrm{Out-of-control\ state} \arrow[r] \arrow[l, dashed, bend right, "minor\ repair"'] & \mathrm{Failed\ state} \arrow[ll, dashed, bend left, "major\ repair"]
\end{tikzcd}$$

Fig. 1 Transitions between the states of the system (where the dash line represents repair type and the solid line represents transition).

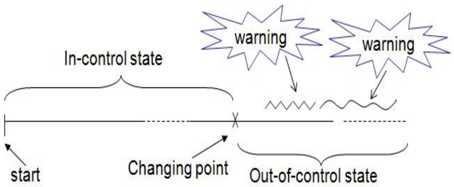


(a) Static control chart

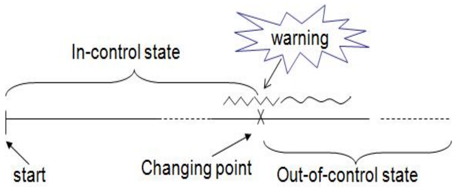


(b) Adaptive control chart

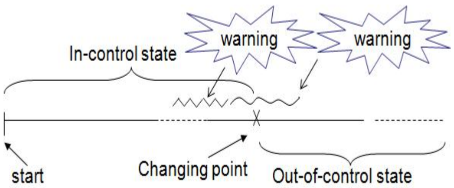
Fig. 2 Control zones in the control charts.



(a) Scenario of Item 1 in T_{a1}



(b) Scenario of Item 2 in T_{a1}.



(c) Scenario of Item 3 in T_{a1}.

Fig. 3 Three scenarios in $E(T_{a1})$ (where the zigzag lines represent a sampling with the normal sample size n_0 and the wave lines represent a sampling with the larger sample size n_1)

Table 1 Parameters used in the numerical example.

α_0	α_1	β_{00}	β_{01}	β_{02}	β_{10}	β_{11}	β_{12}	c_s	c_i	c_{r1}	c_{r2}	n
0.98	0.1	0.833	0.147	0.02	0.02	0.08	0.9	1	100	500	5000	100

Table 2 $E_a(T, C)$ with values of n_0 and n_1.

n_0	n_1	$E_a(T, C)$	n_0	n_1	$E_a(T, C)$
80	144	5.870	120	121	2.293
85	140	4.592	125	126	2.278
90	139	3.679	130	131	2.276
95	133	3.088	135	136	2.283
100	130	2.731	140	141	2.295
105	121	2.521	145	146	2.314
110	119	2.399	145	147	2.315
115	116	2.329	150	151	2.337

Table 3 The expected long-run cost per unit time with c_s, n_0 and n_1.

c_s	n_0	n_1	$E_a(T, C)$	c_s	n_0	n_1	$E_a(T, C)$
0.1	125	137	2.285	5.1	100	110	7.818
1.1	115	126	3.497	6.1	100	110	8.832
2.1	110	121	4.635	7.1	100	110	9.846
3.1	105	115	5.720	8.1	95	104	10.805
4.1	105	115	6.786	9.1	95	104	11.761

Table 4 The expected long-run cost per unit time with c_i, n_0 and n_1.

c_i	n_0	n_1	$E_a(T, C)$	c_i	n_0	n_1	$E_a(T, C)$
10	140	154	1.964	100	125	137	2.286
20	135	148	2.002	110	125	137	2.319
30	135	148	2.039	160	120	132	2.495
40	135	148	2.075	210	120	132	2.658
50	130	143	2.111	260	120	132	2.822
60	130	143	2.146	310	120	132	2.985
70	130	143	2.181	360	120	132	3.149
80	130	143	2.216	410	120	132	3.312
90	130	143	2.251	460	120	132	3.476

Table 5 The expected long-run cost per unit time with c_{r1}, n_0 and n_1.

c_{r1}	n_0	n_1	$E_a(T, C)$	c_{r1}	n_0	n_1	$E_a(T, C)$
50	130	143	0.867	900	130	143	3.547
100	130	143	1.025	1000	130	143	3.862
200	130	143	1.340	1500	130	143	5.438
300	130	143	1.656	2000	120	132	7.014
400	130	143	1.971	2500	120	132	8.585
500	130	143	2.286	3000	120	132	10.157
600	130	143	2.601	3500	120	132	11.729
700	130	143	2.916	4000	120	132	13.301
800	130	143	3.232	4500	110	121	14.862

Table 6 The expected long-run cost per unit time with c_{r2}, n_0 and n_1.

c_{r2}	n_0	n_1	$E_a(T,C)$	c_{r2}	n_0	n_1	$E_a(T,C)$
500	90	125	1.986	5500	130	143	2.309
1000	110	121	2.064	6000	130	143	2.332
1500	120	132	2.104	6500	130	143	2.355
2000	120	132	2.132	7000	130	143	2.378
2500	120	132	2.160	7500	130	143	2.401
3000	120	132	2.187	8000	130	143	2.424
3500	120	132	2.215	8500	130	143	2.447
4000	130	143	2.241	9000	130	143	2.470
4500	130	143	2.264	9500	130	143	2.493
5000	130	143	2.286	10000	130	143	2.516