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ON THE USE OF APPROXIMATE SOLUTIONS FOR NON LINEAR OSCILLATIONS

by

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CONTENTS

1. Introduction

Without doubt an extremely valuable property of linear systems is that they are predictable, in the sense that once their behaviour when excited in one particular way is known, this information can be used to predict their behaviour when excited in some different fashion. Moreover, the outcome of increasing the strength of the exitation is simply to magnify the response proportionately. The consequence of this last property is that any approximate analytic solution of a linear equation will retain its basic form as the amplitude of the disturbance is varied. That this property is not enjoyed by non linear systems can be demonstrated with the following example. Consider the case of a mass linked to an element (Fig. 1.1) with a restoring force of the form,

$$F(x) = c_1 x + c_3 x^3$$

$$c_1 < 0 \,, \quad c_3 > 0$$

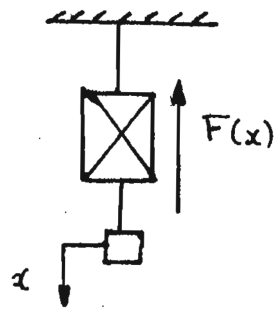

FIG. 1.1

The system has three equilibrium points as shown in the phase plane diagram Fig.1.2.

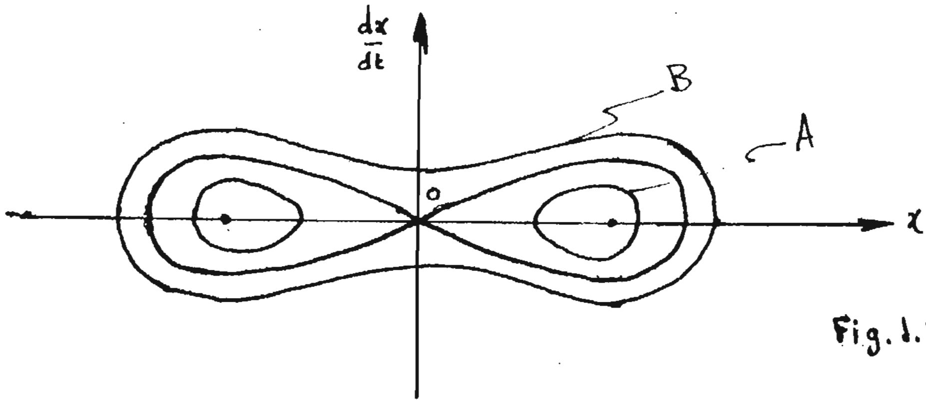

Fig.1.2

The response of the mass when given a small initial displacement with zero initial velocity follows the line marked A: the mass will oscillate about the equilibrium point $x = \sqrt{-\frac{c_1}{c_3}}$.

When the mass is given a small initial velocity it will oscillate about the saddle point O (curve B). The two trajectories are markedly different in character and approximate solutions describing these motions would likewise have to be dissimilar to account for the different types of motion. The point to be made here is that for non linear systems, even after the system equation is known, the form of an approximate analytic solution can depend on the initial trajectory of the system.

In recent years a large number of papers have appeared in the literature concerned with the construction of approximate solutions of non linear oscillation problems. For the most part, these contributions have been aimed at portraying a direct comparison of the approximate solution with 'exact' numerical solutions. This is a somewhat disturbing feature in that many of the contributions seem to end with this comparison which seems to add very little to our understanding of non linear phenomena. The usefulness of an approximate solution would probably be a better guide to the relative merits of different methods than a few comparisons of solution curves.

In what way then, can these approximate analytic forms be used so as to increase our understanding of the non linear problem under investigation? The most obvious use is that in principle they should enable us to (in the case of autonomous systems) calculate the trajectory of the system at any time τ, without having to compute the trajectory for $t<\tau$ as has to be done in numerical integration. However, in practice we often find that although the form of an analytic approximation is known, certain functions involved in the approximation have to be integrated over the preceeding time, and hence the approximation loses the instantaneous nature of the original problem. However, the construction of an approximate solution in terms of known functions is a basically useful exercise in that it should lead to a clearer understanding, by the analyst, of how the solution is affected by changes in the system parameters.

If we are given that a particular equation adequately models a certain phenomenon then it might be possible to estimate the model parameters (coefficients in the system equation) with the use of an approximate analytic solution to the system equation. It is to this use that one section of this note is devoted.

In Section 2 of the present note perturbation methods are applied to a particular class of non linear, one degree of freedom, systems exhibiting oscillatory behaviour. In the cases studies the small parameter in the equations can be interpreted as a damping parameter. The second order differential equations with zero damping have as exact solutions, combinations of the Jacobian Elliptic functions. The two time variable method is used to correct for the non uniformity present in a straight forward expansion scheme. Section 3 is devoted to the estimation technique mentioned above. It is shown in Section 4 how the approximate solutions to three further problems can be constructed.

2. Perturbation Methods applied to non-linear systems with small damping

2.1 Straight forward expansions and the source of non-uniformity

The development of approximate analytic solutions to non linear equations as a straight forward expansion in the powers of a parameter occurring in the equation is usually accompanied by a knowledge of the exact solution of the equation when the parameter is set to zero. This exact solution is often referred to as the zeroth order solution of the expansion and as it stands contains no information regarding the behaviour of the complete solution when the parameter is non zero. It is necessary to determine at least the first order solution of the expansion in order to reveal any new information regarding the complete solution. When the parameter is small this first order correction term can sometimes adequately reveal the manner in which the solution is modified. In many cases the small parameter occurs in a differential equation multiplying a collection of non linear terms and the zeroth order solution is the exact solution of a linear equation. In this note we will be concerned with a class of non linear differential equations which have as zeroth order solutions the Jacobian Elliptic functions or combinations thereof. One particular class of such equations can be written in the general form

$$\frac{d^2x}{dt^2} + f(x) + b\,h(x, \frac{dx}{dt}, t) = 0 \qquad 2.1$$

where b is a small parameter, and $f(x)$ is a polynominal of maximum order $\leqslant 3$. The zeroth order equation,

$$\frac{d^2x_0}{dt^2} + f(x_0) = 0 \qquad 2.2$$

then has an exact solution in terms of Jacobian Elliptic functions (Appendix 1) but is usually solved by inverting the inverse solution $t = t(x_0)$ where $t(x_0)$ is an elliptic integral.

A particularly simple case presents itself when $f(x)$ is an anti-symmetric function of the form

$$f(x) = c_1 x + c_3 x^3$$
$$c_1 > 0 \ , \ c_3 > 0 \qquad 2.3$$

(see Appendix 1 for a study of the cases $c_1 > 0$, $c_3 < 0$; $c_1 < 0$, $c_3 > 0$)

The solution of equation 2.2 is then,

$$x_0(t) = A_0 \, C_n(\omega_0 t + \varphi_0, \mu_0) \tag{2.4}$$

where $C_n(\psi, \mu)$ is the Jacobian Elliptic cosine function of period $4K(\mu)$ ($K(\mu)$ is the complete elliptic integral of the first kind - see appendix 1) and

$$\omega_0^2 = c_1 + c_3 A_0^2$$
$$\mu_0^2 = \frac{\frac{1}{2} c_3 A_0^2}{c_1 + c_3 A_0^2} \tag{2.5}$$

A_0 and φ_0 are determined from the initial conditions of the problem. For the most part this note will be concerned with problems in which $f(x)$ is of the form shown in equation 2.3 and here we include cases in which the general form of $f(x)$ is more complicated but a form such as equation 2.3 is a good approximation in the particular range of $x(t)$ considered.

We begin the description of the approximate method with a popular example in which the function $h(x, \frac{dx}{dt}, t)$ represents a linear 'damping' element of the form

$$h(x, \frac{dx}{dt}, t) = \frac{dx}{dt} \tag{2.6}$$

and $f(x)$ takes the form of equation 2.3.

Equation 2.1 can be written

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c_1 x + c_2 x^3 = 0 \qquad 2.7$$

and we assume an initial trajectory $x(0) = \xi_1$, $\frac{dx}{dt}(0) = \xi_2$

Adopting a direct expansion scheme in powers of b we assume $x(t;b)$ to be analytic and continuous in b and of the form

$$x(t,b) = x_0(t) + b\,x_1(t) + b^2 x_2(t) + O(b^3) \qquad 2.8$$

If we substitute the above form for $x(t;b)$ into equation 2.7 and equate coefficients of b, (When equation 2.8 is substituted into equation 2.7 the resulting equation can be regarded as an identity in b and it follows that the coefficients of the powers of b must all be zero.) we obtain the following set of equations.

$$O(b^0) \qquad \frac{d^2x_0}{dt^2} + c_1 x_0 + c_2 x_0^3 = 0 \qquad 2.9a$$

$$O(b) \qquad \frac{d^2x_1}{dt^2} + (c_1 + 3c_2 x_0^2)\,x_1 = -\,dx_0/dt \qquad 2.9b$$

$$O(b^2) \qquad \frac{d^2x_2}{dt^2} + (c_1 + 3c_2 x_0^2)\,x_2 = -\frac{dx_1}{dt} - 3c_3 x_1^2 x_0 \qquad 2.9c$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

The solution to the zeroth order equation (equation 2.9a) is of the form

$$x_0(t) = A_0 Cn(\omega_0 t + \phi_0, \mu_0) \qquad 2.10$$

The initial conditions for the problem are carried by the function $x_0(t)$ if equation 2.8 is to be valid for all b (including $b=0$).

Thus $\quad x_0(0) = \xi_1 \quad ; \quad \frac{dx_0}{dt}(0) = \xi_2$

and $\quad x_1(0) = 0 \quad ; \quad \frac{dx_1}{dt}(0) = 0$

$$\therefore \quad \omega_0^2 = c_1 + c_3 A_0^2 \quad ; \quad \mu_0^2 = \frac{\frac{1}{2} c_3 A_0^2}{c_1 + c_3 A_0^2} \qquad 2.11$$

and ω_0^2 satisfies the equation,

$$\omega_0^4 - 2 c_3 \xi_2^2 \omega_0^2 - (c_1 + \xi_1^2 c_3)^2 = 0 \qquad 2.12$$

Since $\quad \frac{dx_0}{dt}(0) = - A_0 \omega_0 S_n(\phi_0, \mu_0) D_n(\phi_0, \mu_0)$

The phase angle ϕ_0 is then obtained from the relation,

$$\phi_0 = Cn^{-1}\left(\frac{\xi_1}{A_0}, \mu_0\right) = F\left(\sin^{-1}\sqrt{1-\left(\frac{\xi_1}{A_0}\right)^2}, \mu_0\right) \qquad 2.13$$

where
$$F(u,\mu) = \int_0^u \frac{d\theta}{\sqrt{1-\mu^2\sin^2\theta}} \quad , \quad \mu^2 < 1$$

is an elliptic integral of the first kind.

When the initial velocity is zero, $(\xi_2 = 0)$ the required relationships are simplified,

$$A_0 = \xi_1 \quad \text{and} \quad \phi_0 = 0$$

We continue with a study of this case since the essential features of the non linear oscillation are retained.

To obtain the first order solution we have to solve the inhomogeneous equation in $x_1(t)$ (equation 2.92). This is a linear differential equation with periodic coefficients of period 4K and a periodic forcing term. A certain property of such differential operators can therefore be utilized and instead of actually constructing the first order solution by solving equation 2.9b we can reveal something about the nature of this solution. A most useful result exists concerning the existence of periodic solutions to inhomogeneous equations of the form equation 2.9b. This result is constructed and discussed at length in Appendix 2 of this note and hence we omit any formal analysis at this point. It is shown that if the homogeneous form of equation 2.9b has any periodic solution of period, say, T_ω then the solution to the inhomogeneous equation will be periodic with period T_ω if and only if the given right hand side of equation 2.9b is orthogonal (over T_ω) to every periodic solution of the homogeneous equation. Moreover we can say that if this orthogonality condition is not satisfied then some part of the solution of equation 2.9b will be unbounded as t tends to infinity. This latter effect is often referred to as resonance. To use this result we have to ask if there are any periodic solutions of the homogeneous form of equation 2.9b. There are two fundamental solutions to the homogeneous equation and one of these can be seen to be

$$x_{11}(t) = \frac{dx_0}{dt} = -A_0\omega_0 S_n(\omega_0 t) D_n(\omega_0 t) \qquad 2.14$$

which is in fact periodic, with period 4K. The second fundamental solution can be found from $x_{11}(t)$ by the variation of parameters technique but will not concern us here since a little analysis reveals that this solution is in fact non-periodic (ref 18). The function $x_{11}(t)$ does concern us and the periodicity condition for the solutions of the $O(b)$ equation (2.9b) to be periodic becomes,

$$\int_0^{4K} \left(\frac{dx_0}{dt}\right)^2 dt = 0 \qquad 2.15$$

Equation 2.15 can only be satisfied when $(dx_0/dt) = 0$, a trivial case, and hence it is concluded that there will be no part of the first order solution periodic with period 4K and that this solution will grow unbounded as t grows. This solution cannot be very useful to us since the first order term $(O(b))$ should have been a small correction to the zeroth order solution $x_0(t)$ for small b and the second term in the expansion (equation 2.8) will eventually be of $O(1)$, as t grows, no matter how small b is. In such situations we say the expansion is non-uniform and the region of non-uniformity, manifested in the behaviour of $x_1(t)$, occurs for $t > 1/b$.

As is often the case, for our problem, the straight forward expansion, equation 2.8, fails and if we consider again the zeroth order solution $x_0(t)$ (equation 2.10) the reasons for this become clear. It is apparent that the presence of the term $b\frac{dx}{dt}$ in equation 2.7 when $C_3 = 0$ will result in a gradual decay of the solution as time grows, providing $b > 0$. Physically the motion is damped out. From a knowledge of the phase plane character (Appendix 1) of the solution of equation 2.7 we can deduce that the effect of the non linearity, $C_3 x^3$; $C_3 > 0$, will not change the basic nature of the decaying process. Therefore as t tends to infinity, where the real solution should be disappearing, the zeroth order solution is of the same magnitude as when $t = 0$. To correct for the effects of damping the first order solution would have to be at least as big as the zeroth order solution for large time. It is this accumulative effect of the damping which causes the non-uniformity and to emphasise that the non-linearity is not to blame it is useful to examine the linear case, $C_3 = 0$, in more detail. The zeroth order solution is then simply,

$$x_0(t) = A_0 \cos \omega_0 t \qquad \omega_0 = \sqrt{c_1} \tag{2.16}$$

and the first order equation has periodic solutions (period $2\pi/\omega_0$)

$$x_1(t) = \cos \omega_0 t \; : \; x_{12}(t) = \sin \omega_0 t \tag{2.17}$$

The orthogonality condition is again not satisfied and a simple way of seeing the non-uniformity is to examine the expanded form of the first order equation.

$$\frac{d^2x_1}{dt^2} + e_1 x_1 = A_0 w_0 \sin w_0 t \qquad 2.18$$

The solution of equation 2.18 will involve terms of the form $t \cos w_0 t$ (linear oscillator at resonance) and hence the envelope of the oscillation will grow linearly with time. The exact nature of the unbounded growth of the first order solution in the non linear problem will be more complicated but the nature of the non-uniformity in the two cases is the same in principle.

The failure of the expansion scheme described might suggest that it would be wise to abandon perturbation methods in favour of some other approximate technique. Since one of the main reasons for the failure seemed to be the absence of any decay in the zeroth order approximation, a possible alternative approach would be to artificially introduce the effects of damping into the zeroth order solution by assuming that the constants A_0, w_0 and μ_0 vary with time in a fashion to be determined. For the linear problem we would then write,

$$x(t) = A_0(t) \cos w_0(t) t \qquad 2.19$$

where $A_o(t)$ and $W_o(t)$ are to be determined. This procedure is similar to a scheme developed by Kryloff and Bogoliuboff (Ref 1) although they actually assume a form,

$$x(t) = A_o(t) \cos(W_o t + \phi_o(t)) \qquad 2.20$$

The time derivative of $x(t)$ is then assumed to be of the form,

$$\frac{dx}{dt} = -A_o(t) W_o \sin(W_o t + \phi_o(t)) \qquad 2.21$$

The unknown functions $A_o(t)$ and $\phi_o(t)$ are then determined by constructing two first order differential equations and 'averaging' the equations over one cycle of the oscillation. This basic approach carries the assumption that the damping is small. A similar procedure can be adopted for the non linear problem and was first developed in a series of papers by Barkham and Soudack (Ref. 2 - 5). In a later series of papers Christopher (Ref. 6 - 7) developed an improved method and his results show remarkably good agreement with 'exact' numerical solutions.

But need we have abandoned the perturbation method?

In doing so and resorting to the methods described above we have, it seems, lost some of that valuable quality of approximation methods, that is, rationality. Using a perturbation method it is possible to determine the order of the error of the approximation and to improve the approximation by including terms of higher order. But for the method to be useful the expansion should be uniformly valid for the range of the independant variable considered. For the non linear oscillation problem we would ideally like the solution to have the same order of error for all time i.e. to be uniformly valid over the whole time range. How then do we obtain a uniformly valid solution?

2.2 Obtaining a Uniformly Valid Approximation

The kind of problem we are confronted with occurs in a variety of different fields and many different procedures have been developed to handle the situation. A particular method, well suited to the present problem, will be introduced which will enable us to include in the zeroth order approximation the main features of the damped non linear oscillation. The key to the method lies in the recognition of two different time scales, one fast, one slow, that are involved in the oscillation. On the fast time scale $(O(t))$ the actual oscillation is evolving whilst on the slow time scale $(O(bt))$, the envelope of the oscillation and instantaneous 'frequency' are changing. We begin by defining two new time scales. A slow time scale

$$\tau = bt$$

2.22

and a fast time scale $\eta = g(\tau)/b$

and write the dependent variables as

$$x(t;b) \equiv \tilde{x}(\eta,\tau;b) \qquad 2.23$$

so that

$$\frac{dx}{dt} = \frac{\partial \tilde{x}}{\partial \eta} g'(\tau) + \frac{\partial \tilde{x}}{\partial \tau} b \qquad 2.24$$

where $g'(\tau) = dg/d\tau = d\eta/dt$

and,

$$\frac{d^2x}{dt^2} = g'(\tau)^2 \frac{\partial^2 \tilde{x}}{\partial \eta^2} + 2g'(\tau) b \frac{\partial^2 \tilde{x}}{\partial \eta \partial \tau} + $$
$$+ b g''(\tau) \frac{\partial \tilde{x}}{\partial \eta} + b^2 \frac{\partial^2 \tilde{x}}{\partial \tau^2} \qquad 2.25$$

Substituting equations 2.23 - 2.25 into the original equation 2.7 we obtain the partial differential equation,

$$g'^2 \frac{\partial^2 \tilde{x}}{\partial \eta^2} + b\left(2g'\frac{\partial^2 \tilde{x}}{\partial\eta\,\partial\tau} + g''\frac{\partial\tilde{x}}{\partial\eta} + g'\frac{\partial\tilde{x}}{\partial\eta}\right) + b^2\left(\frac{\partial^2\tilde{x}}{\partial\tau^2} + \frac{\partial\tilde{x}}{\partial\tau}\right) + c_1\tilde{x} + c_2\tilde{x}^3 = 0 \qquad 2.26$$

We assume a perturbation expansion for $\tilde{x}(\eta,\tau;b)$ in terms of the damping parameter b , of the form,

$$\tilde{x}(\eta,\tau;b) = x_0(\eta,\tau) + b x_1(\eta,\tau) + b^2 x_2(\eta,\tau) + O(b^3) \qquad 2.27$$

Substituting equation 2.27 into equation 2.26 and collecting the terms of various orders of b together, we obtain the following set of partial differential equations for the functions $x_i(\eta,\tau)$, $i = 0, 1, \ldots$

$$g'^2 \frac{\partial^2 x_0}{\partial\eta^2} + c_1 x_0 + c_2 x_0^3 = 0 \qquad 2.28$$

$$g'^2 \frac{\partial^2 x_1}{\partial\eta^2} + (c_1 + 3c_2 x_0^2)\, x_1 = -\left(2g'\frac{\partial^2 x_0}{\partial\eta\,\partial\tau} + (g''+g')\frac{\partial x_0}{\partial\eta}\right) \qquad 2.29$$

At this point in the analysis the function $g'(\tau)$ is unknown and the solution to the zeroth order equation (2.28) is also unknown until $g'(\tau)$ has been determined.

Similarly when the periodicity condition is applied to the right hand side of equation 2.29 no result is achieved since $g'(\tau)$ appears here. The whole point of the method now comes into focus: we can actually choose that particular form of $g'(\tau)$ so that the right hand side of equation 2.29 obeys the periodicity condition, i.e. the overall solution of the first order equation is periodic in η and finitely bounded. Since the slow time scale τ does not appear in the differential operation of equations 2.28 and 2.29, we can regard the solution to these equations as being generated by ordinary differential equations in η but having the arbitrary constants now dependent on τ. Thus the zeroth order solution becomes

$$x_0(\eta,\tau) = A_0(\tau)\, C_n(\eta, p_0(\tau)) \qquad 2.30$$

with, as in equation 2.5, a similar relationship between the unknown quantities

$$g'(\tau)^2 = c_1 + c_2 A_0^2 \quad : \quad p_0(\tau)^2 = \frac{\frac{1}{2} c_3 A_0^2}{c_1 + c_2 A_0^2} \qquad 2.31$$

The technique we have introduced is known as the two-time variable expansion method of the general multiple scaling technique described in great detail in Nayfeh (Ref. 8) and Cole (Ref. 9). In an earlier work (Ref. 10) Kusmak presented an analysis for obtaining asymptotic solutions of non linear differential equations of the form

$$\frac{d^2x}{dt^2} + \varepsilon f(x,\tau)\frac{dx}{dt} + F(x,\tau) = 0 \qquad 2.32$$

where ε is a small parameter and $\tau = \varepsilon t$ is a slow time (the class of equation analysed in this note is of course a special case of equation 2.32). Kusmak constructs the solution of equation 2.32 as an asymptotic expansion in powers of ε and introduces a fast time scale akin to $\eta(t)$. The unknown fast time scale is then determined by a method equivalent to the periodicity condition used in this note. Kusmak proceeds to apply the method to an equation of the form

$$\frac{d^2x}{dt^2} + a(\tau)x + b(\tau)x^3 = 0 \qquad 2.33$$

and determines the zeroth order solution in terms of Jacobian Elliptic functions. Equation 2.7 can, of course, be transformed to the form of equation 2.33 with the transformation $x(t) = u(t)\exp(-bt/2)$ or equation 2.7

could be studied directly by Kusmak's method. Whilst retaining the essential features of Kusmak's approach we will proceed along slightly different lines and develop an expression for the unknown function $g'(\tau)$ in a form amenable to computation and the estimation process to be described in a later section.

We return to the present problem and to the determination of $g'(\tau)$ by the application of the periodicity condition. If the right hand side of equation 2.29 is orthogonal (over a period in η) to all the periodic solutions of the homogeneous form of equation 2.29 then the complete solution of this equation will also be periodic with period 4K (period of x_0 and $\partial x_0/\partial\eta$). The only periodic solution of the homogeneous equation is $\partial x_0/\partial\eta$ and hence the condition for a periodic (and bounded) solution, becomes

$$\int_0^{4K}\left(2g'(\tau)\frac{\partial^2 x_0}{\partial\eta\,\partial\tau}+\left(g''(\tau)+g'(\tau)\right)\frac{\partial x_0}{\partial\eta}\right)\frac{\partial x_0}{\partial\eta}\,d\eta = 0 \qquad 2.34$$

which can be written as,

$$\frac{\partial}{\partial\tau}\left[g'(\tau)\int_0^{4K}\left(\frac{\partial x_0}{\partial\eta}\right)^2 d\eta\right]+g'(\tau)\int_0^{4K}\left(\frac{\partial x_0}{\partial\eta}\right)^2 d\eta = 0 \qquad 2.35$$

and therefore,

$$g'(\tau)\int_0^{4K}\left(\frac{\partial x_0}{\partial \eta}\right)^2 d\eta = k_0 e^{-\tau} \qquad 2.36$$

where k_0 is a constant.

Now

$$\frac{\partial x_0}{\partial \eta} = -A_0(\tau)\, g'(\tau)\, S_n(\eta, \mu_0)\, D_n(\eta, \mu_0) \qquad 2.37$$

and so equation 2.36 can be written as,

$$\frac{g'(\tau)\, A_0(\tau)^2}{3\mu_0(\tau)^2}\left[(2\mu_0^2 - 1]E(\mu_0) + [1-\mu_0^2]\, K(\mu_0)\right] = k_1 e^{-\tau} \qquad 2.38$$

where $\qquad K(\mu_0) = F(\pi/2, \mu_0)$

and $\qquad E(\mu_0) = E(\pi/2, \mu_0)$

are complete elliptic integrals of the first and second kind respectively. It is worth noting here that a disguised approximation has been used in the application of the periodicity condition (equation 2.35). Equation 2.28 is a partial differential equation and has a solution of the form given in equation 2.30 with period 4K in η .

Since the modulus μ_0 of the Jacobian Elliptic function $C_n(\eta, \mu_0)$ is a function of the slow time scale τ and k is a function of μ_0 then the period of $\chi_0(\eta, \tau)$ and $\partial\chi_0/\partial\eta$ will vary with τ. To correct for the error involved in this approximation one would have to expand the time scale $\eta(t)$ in powers of b in the form,

$$\eta = \frac{g_{-1}(\tau)}{b} + g_0(\tau) + g_1(\tau)b + g_2(\tau)b^2 + O(b^3) \qquad 2.39$$

with each $g_i(\tau)$ being determined by applying the periodicity condition to the right hand side of the $O(b^{i+2})$ equation. By carrying out a higher order analysis it can be shown that $g_0(\tau)$ is actually an arbitrary function and can be set to zero. To determine $g_1(\tau)$ we would have to investigate the periodicity condition for the $O(b^3)$ equation and at this point in the analysis this does not seem to be worthwhile.

The left hand side of equation 2.38 can be written in an approximate form if we expand $E(\mu)$ and $k(\mu)$ in terms of μ and recognise from equation 2.31 that $0 \leqslant \mu^2 \leqslant 0.5$. The expansions of the complete elliptic integrals are,

$$K(\mu) = \frac{\pi}{2}\left[1 + \frac{\mu^2}{4} + \frac{9}{64}\mu^4 + O(\mu^6)\right]$$

$$E(\mu) = \frac{\pi}{2}\left[1 - \frac{\mu^2}{4} - \frac{3}{64}\mu^4 - O(\mu^6)\right] \qquad 2.40$$

Retaining terms of up to $O(\mu^4)$ in $K(\mu)$ and $E(\mu)$, equation 2.38 can be simplified and reduces to,

$$g'(\tau)\,A_0^2(\tau)\,(8 - 3\mu_0^2) = k_2\,e^{-\tau} \qquad 2.41$$

where $k_2 = g'(0)\,A_0^2(0)\,(8 - 3\mu_0^2(0))$, and can be determined from the initial conditions. Using equation 2.31 we can express $A_0(\tau)$ and $\mu_0(\tau)$ in terms of $g'(\tau)$ and substituting these expressions into equation 2.41 we obtain a quartic equation for $g'(\tau)$.

$$(g'(\tau)^2 - c_1)\left(\frac{13}{2}g'(\tau)^2 + \frac{3}{2}c_1\right) = k_2\,c_3\,g'(\tau)\,e^{-\tau} \qquad 2.42$$

Hence $\quad g(\tau) = g(0) + \int_0^\tau g'(\tau)\,d\tau$

and $\quad \eta(t) = g(\tau)/b$

For a given value of t (hence τ) we can solve equation

2.41 for $g'(\tau)$ and obtain $A_0(\tau)$ and $\mu_0(\tau)$ from equation 2.31. It can be seen from equation 2.42 that when $c_2 = 0$ we obtain the real solution, $g'(\tau) = \sqrt{c_1}$, which is, of course, the linear oscillator frequency to $O(b)$. The exact linear frequency is given by $g'(\tau) = \sqrt{c_1 - b^2/4}$ and to obtain the correct result when $c_2 = 0$ we could use $\bar{c}_1 = c_1 - b^2/4$ instead of c_1 in equation 2.42. This type of improvement is not strictly in keeping with the systematic procedure used in the perturbation method but it is hoped that the inclusion of corrections of $O(b)$ will only improve the approximation. Using the technique described some examples have been computed and compared with exact numerical results. The comparisons are shown in Figs. 1 - 9 for various values of c_1, c_2 and b, and are seen to be fairly accurate even for $b = 1.0$. In all the cases a phase error begins to develop and grow with time, which can only be corrected by including the first order correction $x_1(\eta, \tau)$.

Note here that we need not have introduced the new fast time scale of $O(t)$ (i.e. $\eta(t)$): instead we could have simply used t. This would have been equivalent to writing $\eta = g'(\tau)t$ instead of $\eta = \int_0^t g'(\tau)\,dt$ as we actually used. The function $g'(\tau)$ can be considered synonymous with the function $g_{-1}(\nu)$ in the more elaborate expansion of $\eta(t)$ in powers of b given by equation 2.39. Figs. 9a and 9b show the effect on the solution curves of using the simple form $\eta = g'(\nu)t$.

Using the two time variable method, uniformly valid asymptotic solutions of more general equations of the form,

$$\ddot{y} + \varepsilon H_1(y)\dot{y} + M(y)y = 0$$

$$\ddot{y} + \varepsilon H_2(\dot{y})\dot{y} + M(y)y = 0 \qquad 2.43$$

have been obtained by Rasmussen (Refs. 11, 12) and his results show very good agreement with exact numerical results.

At this point in the analysis it is expedient to ask the question - What use is the zeroth order analytic approximation to us? If we are satisfied with seeing the solution curves, what better than the 'exact' numerical ones? The value of the approximation will be accessed when we examine the possible uses to which the analytic forms generated can be put. It is to one of these uses that the next section is devoted.

3. Estimating the characteristics of a Non Linear Oscillation (Ref. 13)

Let us assume that we have a set of displacement response curves of a non linear oscillator, the behaviour of which can be approximately described by the equation,

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c_1 x + c_3 x^3 = 0 \qquad 3.1$$

(b , c_1 and c_3 are unknown)

The response can either be obtained by giving the system an initial displacement or applying an impulse (initial velocity). We propose a method whereby the coefficients b , c_1 and c_3 can be determined from such a set of curves using the analytic approximation described in section 2.

Fig. 3.1 shows a pair of adjacent peaks of a typical oscillation.

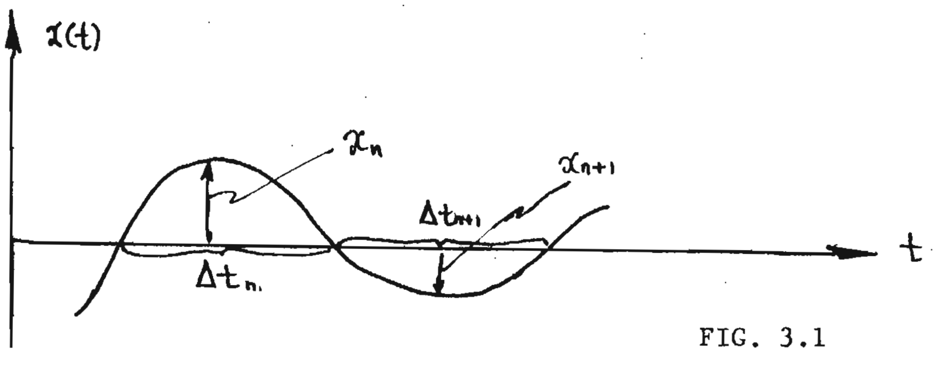

FIG. 3.1

Now the zeroth order approximation is, again,

$$x_0(\eta,\tau) = A_0(\tau)\, C_n(\eta, p_0(\tau))$$

where $\eta = g(\tau)/b$ and $d\eta/dt = g'(\tau)$

$$g'(\tau)^2 = c_1 + c_2 A_0(\tau)^2 \quad \text{and} \quad p_0(\tau)^2 = \frac{\frac{1}{2} c_2 A_0^2}{c_1 + c_2 A_0^2} \qquad 3.2$$

At the peaks of the oscillation (x_n) we can write,

$$x_0(\eta_n, p_n) \simeq A_0(\tau_n) = x_n \;;\; \text{since } C_n(\eta_n, p_n) = 1$$

(the approximate equality is needed since the amplitude envelope does not necessarily touch the oscillation curve exactly at the peak values). Therefore it follows from equation 3.2 that, at the peaks,

$$g'(\tau_n)^2 = c_1 + c_2 x_n^2 \qquad 3.3$$

Now $g'(\tau) = d\eta/dt$ and if we assume that $g'(\tau)$ has the constant value $g'(\tau_n)$ over the internal $\Delta\tau_n$ we can write,

$$g'(\tau_n)\,\Delta t_n = \Delta\eta_n \simeq 2K(p_n) \qquad 3.4$$

and here we assume that p_0 is slowly varying so that over the positive portion of the oscillation $\Delta\eta = 4K(p)/2$.

Combining equation 3.4 with equation 3.3 we can write,

$$\frac{4k^2(\mu_n)}{\Delta t_n^2} = c_1 + c_3 x_n^2 \tag{3.5}$$

Expanding $k(\mu_n)$ in power of μ_n^2 and neglecting terms of $O(\mu_n^4)$ and higher we have that,

$$k(\mu_n) \simeq \frac{\pi}{2}\left(1 + \frac{\mu_n^2}{4}\right) \tag{3.6}$$

so that

$$k(x_n^2) = \frac{\pi}{2}\left(1 + \frac{\frac{1}{8}c_3 x_n^2}{c_1 + c_3 x_n^2}\right) \tag{3.7}$$

According to this relation $k(\mu_n)$ varies between $\frac{9}{8}\frac{\pi}{2}$ when $c_3 x_n^2$ is large and $\pi/2$ when $x_n = 0$. Actually $k(\mu)$ varies between approximately 0.59π and $\pi/2$ as μ^2 varies between $1/2$ and zero. We will make the rather crude approximation that $k(\mu_n)$ assumes the value $\frac{9}{8}\cdot\frac{\pi}{2}$ over the whole range of amplitude of the measured peaks. Equation 3.5 now becomes

$$\left(\frac{k}{\Delta t_n}\right)^2 = c_1 + c_3 x_n^2 \tag{3.8}$$

where $k = \frac{9}{8}\pi$

It is interesting to note that a perturbation method treating $\frac{c_3 x_n^2}{c_1}$ as a small parameter would lead to the approximate relation,

$$\left(\frac{\pi}{\Delta t_n}\right)^2 = c_1 + \frac{3}{4} c_3 x_n^2 + O(c_3^2 x_n^4) \qquad 3.9$$

which would be of the same order of accuracy as equation 3.8 when $\left(\frac{c_3}{c_1} x_n^2\right) \gg 1$. Equation 3.9 would probably be a more uniformly accurate relation to use for estimating c_1 and c_3 . However, the numerical examples shown later are based on the use of equation 3.8. Using measured values of x_n and Δt_n we can now plot the points of $\left(k/\Delta t_n\right)^2$ v. x_n^2 and obtain a least squares linear fit through these points as shown in Fig. 3.2.

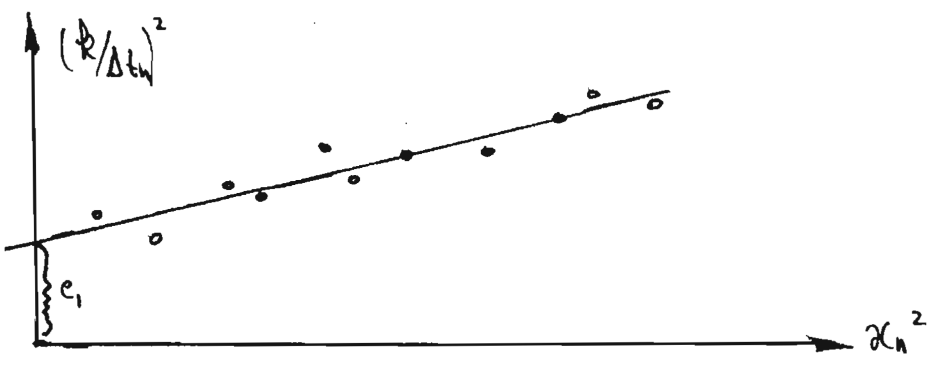

FIG. 3.2

Using a least squares fit to the points the approximate values of C_1 and C_3 are given by

$$C_1 = \frac{k^2}{S_1} \sum_{j=1}^{N} \frac{(S_2 - x_j^2 S_3)}{\Delta t_j^2} \qquad 3.10$$

$$C_3 = \frac{k^2}{S_1} \sum_{j=1}^{N} \frac{(N x_j^2 - S_3)}{\Delta t_j^2} \qquad 3.11$$

where,

$$S_1 = N S_2 - S_3^2$$

$$S_2 = \sum_{i=1}^{N} x_i^4$$

$$S_3 = \sum_{i=1}^{N} x_i^2$$

and N is the total number of amplitude peaks measured. Regarding the approximate relations equations 3.8 and 3.9 a more accurate estimation of C_1 and C_3 would be obtained if we use the mean square peak amplitude defined by

$$x_m^2 = \frac{x_{n+1}^2 + x_n^2}{2}$$

The approximate relations replacing equations 3.8 and 3.9 would then read,

$$\frac{(k\pi)^2}{2}\left[\frac{\Delta t_{n+1}^2 + \Delta t_n^2}{\Delta t_{n+1}^2 \Delta t_n^2}\right] = C_1 + C_3 x_m^2 \; ; \; \frac{\pi^2}{2}\left[\frac{\Delta t_{n+1}^2 + \Delta t_n^2}{\Delta t_{n+1}^2 \Delta t_n^2}\right] = C_1 + \frac{3}{4} C_3 x_m^2$$

We can now use the approximate values of C_1 and C_3 to estimate the damping parameter b . To do this we consider two adjacent peaks (x_n, x_{n+1}) (Fig. 3.1) and define the quantities,

$$\delta = x_{n+1}/x_n \qquad x_m^2 = \frac{x_{n+1}^2 + x_n^2}{2} \qquad \gamma = \frac{C_3 x_m^2}{C_1} \tag{3.12}$$

giving $\quad x_n^2 = \dfrac{2x_m^2}{1+\delta^2} \qquad\qquad x_{n+1}^2 = \dfrac{2\delta^2 x_m^2}{1+\delta^2}$

At the peak $x_n(t_n)$, eqn. 2.41 gives,

$$g'(\tau_n)\, x_n^2 (8 - 3p_n^2) = k_2 e^{-bt_n} \tag{3.13}$$

Substituting for $g'(\tau_n)$, $p^2(\tau_n)$ in terms of x_n^2 (eqn. 2.31) we obtain,

$$x_n^4 \left\{ \frac{\left(C_1 + \frac{13}{16} C_2 x_n^2\right)^2}{C_1 + C_2 x_n^2} \right\} = k_2^2 e^{-2bt_n} \tag{3.14}$$

A similar expression can be obtained for station t_{n+1} and if we divide this by eqn. 3.14 and introduce the quantities δ and γ we obtain,

$$\delta^4 \left\{ \frac{\left(1+\delta^2\left(1+\frac{13}{8}\gamma\right)\right)^2 \left(1+2\gamma+\delta^2\right)}{\left(1+\delta^2(1+2\gamma)\right)\left(1+\frac{13}{8}\gamma+\delta^2\right)^2} \right\} = e^{-2b(t_{n+1}-t_n)} \tag{3.15}$$

Now for a linear system the logarithmic decrement is defined as $\log_e\left|\frac{x_{n+1}}{x_n}\right|$ and is a constant measure of the damping b, independent of amplitude. For a non linear system the log. dec. is still a useful concept although we would expect it to vary with the amplitude of the oscillation. With the intention of introducing the log. dec. into eqn. 3.15 we can expand $\log \delta$ for δ close to unity, retaining only the linear term in δ. Thus

$$\log \delta = \varepsilon \simeq \delta - 1 \qquad 3.16$$

hence we use the approximate forms,

$$\delta \simeq 1+\varepsilon \ , \quad \delta^2 \simeq 1+2\varepsilon \ , \quad \delta^4 \simeq 1+4\varepsilon$$

Substituting these forms into eqn. 3.15 we obtain, after some simplification,

$$\frac{D_1(\gamma) + \varepsilon D_2(\gamma)}{D_1(\gamma) + \varepsilon D_3(\gamma)} = e^{-2b(t_{n+1}-t_n)} \qquad 3.17$$

where
$$D_1(\gamma) = \left(1+\tfrac{13}{16}\gamma\right)(1+\gamma)$$

$$D_2(\gamma) = \left(1+\tfrac{13}{16}\gamma\right)\left[1+4(1+\gamma)\right] + 2(1+\gamma)\left(1+\tfrac{13}{8}\gamma\right)$$

$$D_3(\gamma) = (1+2\gamma)\left(1+\tfrac{13}{16}\gamma\right) + 2(1+\gamma) \qquad 3.18$$

Since b is a small parameter we need also use the first two terms in the expanded form of $e^{-2b(t_{n+1}-t_n)}$. Thus,

$$e^{-2b(t_{n+1}-t_n)} \simeq 1-2b(t_{n+1}-t_n) \qquad 3.19$$

The time internal $(t_{n+1}-t_n)$ can be represented approximately using eqn. 3.8

$$\therefore e^{-2b(t_{n+1}-t_n)} \simeq 1-2b\,D_4(\gamma) \qquad 3.20$$

where $$D_4(\gamma) = \frac{9\pi}{8\sqrt{c_1}}\frac{1}{\sqrt{1+\gamma}} \qquad 3.21$$

Writing $\varepsilon = b\zeta$ eqn. 3.17 becomes,

$$\frac{D_1(\gamma)+b\zeta D_2(\gamma)}{D_1(\gamma)+b\zeta D_2(\gamma)} = 1-2b\,D_4(\gamma) \qquad 3.22$$

$$\therefore b\zeta D_2 = b(\zeta D_3 - 2D_1D_4) + O(b^2) \qquad 3.23$$

hence to $O(b)$ we have,

$$\zeta = \frac{2D_1D_4}{D_3-D_2} = -\frac{9\pi\left(1+\frac{13}{16}\gamma\right)\left(1+\gamma\right)^{1/2}}{4\sqrt{c_1}\left(4+\frac{17}{2}\gamma+\frac{39}{8}\gamma^2\right)} \qquad 3.24$$

We can therefore write,

$$m\varepsilon \simeq \frac{\left(1+\frac{13}{16}\gamma\right)\left(1+\gamma\right)^{1/2}}{\left(1+\frac{17}{8}\gamma+\frac{39}{32}\gamma^2\right)}\, b = G(\gamma)\, b. \qquad 3.25$$

where

$$m = -\frac{16}{9}\frac{\sqrt{c_1}}{\pi}$$

A curve of $m\varepsilon$ v. γ should, according to the approximation, cut the $m\varepsilon$ axis at the value b. A selection of typical measurements are shown by the circles in Fig. 3.3

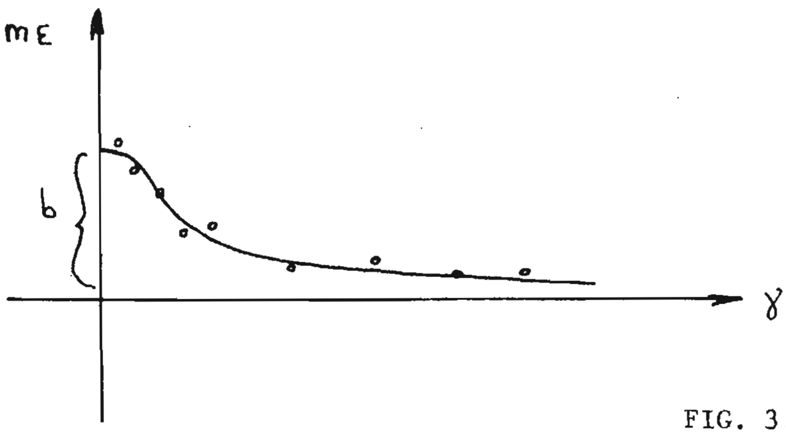

FIG. 3.3

The function $G(\gamma)$ is of course the same for all cases and a least squares fit can again be made on the points to shift the curve representing $G(\gamma)$ to the 'best' position. (full line on Fig. 3.3). According to the least squares fit, the approximate values of b is given by,

$$b \simeq - \frac{\frac{16}{9}\frac{\sqrt{c_1}}{\pi}\sum_{j=1}^{M} G(\gamma_j)\varepsilon_j}{\sum_{j=1}^{M} G(\gamma_j)^2} \tag{3.26}$$

where ε_j are measurements of $\log_e\left|\frac{x_{j+1}}{x_j}\right|$

γ_j are measurements of $\frac{c_3\,(x_{j+1}^2 + x_j^2)}{2c_1}$

Note that when $\gamma \rightarrow 0$

$$\varepsilon \rightarrow -\frac{9}{16}\frac{\pi}{\sqrt{c_1}}b \quad ; \quad \underset{\gamma=0}{\varepsilon_{exact}} = -\frac{b}{2}\frac{\pi}{\sqrt{c_1 - b^2/4}}$$

The curve of $G(\gamma)$ v. γ is shown in Fig. 10a.

We can test the accuracy of the parameter estimation methods described by taking measurements from the exact numerical solution of a particular equation. A variety of cases were studied, and for each case three displacement responses were analysed, (Initial conditions $x(0) = 0.5, 1.0, 1.5$) so that a range of $(x_n, \Delta t_n)$ were obtained. Table 1 below shows some sample results:

	C_1	C_3	b
EXACT	1.0	10.0	1.0
ESTIMATED	0.917	9.475	1.058
EXACT	1.0	20.0	1.0
ESTIMATED	1.091	17.98	1.12
EXACT	1.0	2.0	0.5
ESTIMATED	1.15	1.99	0.512
EXACT	1.0	10.0	0.5
ESTIMATED	0.934	9.54	0.49
EXACT	1.0	20.0	0.5
ESTIMATED	1.184	18.13	0.46

TABLE 1 - COMPARISON OF EXACT AND ESTIMATED PARAMETERS

Accompanying the curve of the function $G(\gamma)$ in Fig. 10 are some measurements of the logarithmic decrement for the cases $b = 0.5$ and 1.0 (Figs. 10b and 10c). Some cases with $b = 0.1$ were analysed but the difficulty the author had in measuring the peak amplitude accurately gave rise to large errors in the estimated parameter. One would however expect good estimations of the parameters when b is small if the peak amplitude could be measured accurately. In the present study the required measurements were read straight from response curves drawn by hand and there is no doubt that this method could be improved.

The author felt that any elaborate improvement of the measurement techniques did not warrant the effort required in the context of the present study.

4. Some Further Examples

In this section we examine the behaviour of three non linear systems and construct zeroth order approximations of the equations of motion. In each case the approximation will be expressed in terms of Jacobian Elliptic functions. The first example exhibits a limit cycle due to non linear damping; in the second we examine the step function response of the non linear oscillator studied in sections 2 and 3; finally we show how some of the techniques described can be applied to a simple two degree of freedom system.

4.1 Equations with Non Linear Damping

This section is mainly concerned with the interaction of two non linearities - a study of eqn. 2.1 where

$$f(x) = c_1 x + c_3 x^3 \quad ; \quad h(x, \frac{dx}{dt}, t) = (\alpha x^2 - 1)\frac{dx}{dt} \qquad 4.1$$

Before we begin to investigate this system however it is illustrative to show that when $C_3 = 0$ (the Van der Pol equation), two quite different approximate techniques generate the same solution. We will apply the Kryloff-Bogliuboff method and the multiple scale method to the Van der Pol equation

$$\frac{d^2x}{dt^2} + b(x^2-1)\frac{dx}{dt} + x = 0 \qquad 4.2$$

In the K.B. technique we apply the transformation,

$$x(t) = a(t)\sin(t + \phi(t))$$

$$\qquad 4.3$$

and $$\frac{dx}{dt} = a(t)\cos(t + \phi(t))$$

Substituting these expressions into eqn. 4.2 and equating to zero the remaining terms in the expression for dx/dt we obtain two first order differential equations for $a(t)$ and $\phi(t)$

$$\frac{da}{dt}\cos(t+\phi) = -b\,h(a\sin(t+\phi),\, a\cos(t+\phi)) \qquad 4.4$$

$$\frac{d\phi}{dt}\sin(t+\phi) = \frac{b}{a}\,h(a\sin(t+\phi),\, a\cos(t+\phi)) \qquad 4.5$$

Multiplying eqn. 4.4 by $\cos(t+\phi)$ and eqn. 4.5 by $\sin(t+\phi)$, integrating from 0 to 2π (assume $a(t)$, da/dt, ϕ, $d\phi/dt$ remain constant over a period of oscillation) we obtain,

$$\frac{da}{dt} = -\frac{b}{2\pi}\int_0^{2\pi} h\cos(t+\phi)\,dt = \frac{b}{2}a\left(1-\frac{a^2}{4}\right) \tag{4.6}$$

$$\frac{d\phi}{dt} = \frac{b}{2\pi a}\int_0^{2\pi} h\sin(t+\phi)\,dt = 0 \tag{4.7}$$

$$\therefore\quad a^2 = \frac{a_0^2 e^{bt}}{1+\frac{1}{4}a_0^2\left(e^{bt}-1\right)} \quad \text{and} \quad \phi = \phi_0\ (\text{const.}) \tag{4.8}$$

The approximate form of the solution (eqn. 4.3) will thus exhibit a limit cycle with an amplitude equal to 2.

To apply the two time scale method to the present example, (this is necessary in a perturbation method since a resonance effect in the first order solution is predicted by a straightforward expansion), we introduce the slow time scale $\tau = bt$ and retain t as the fast time scale. Thus the partial differential equation replacing eqn. 4.1 becomes,

$$\frac{\partial^2\tilde{x}}{\partial t^2} + \tilde{x} + b\left[2\frac{\partial^2\tilde{x}}{\partial t\,\partial\tau} + (\tilde{x}^2-1)\frac{\partial\tilde{x}}{\partial t}\right] + b^2\left[\frac{\partial^2\tilde{x}}{\partial\tau^2} + (\tilde{x}^2-1)\frac{\partial\tilde{x}}{\partial\tau}\right] = 0 \tag{4.9}$$

where $\quad \tilde{x}(t,\tau;b) \equiv x(t;b)$

Expanding $\tilde{x}(t,\tau;b)$ in powers of b ,

$$\tilde{x}(t,\tau;b) = x_0(t,\tau) + b x_1(t,\tau) + O(b^2) \qquad 4.10$$

Substituting into eqn. 4.9 and equating coefficients of b we obtain,

$$O(b^0) : \frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \text{ , hence } x_0(t,\tau) = A_0(\tau)\cos(t+\phi(\tau)) \qquad 4.11$$

$$O(b) : \frac{\partial^2 x_1}{\partial t^2} + x_1 = -2\frac{\partial^2 x_0}{\partial t \partial \tau} + (x^2-1)\frac{\partial x_0}{\partial t} \qquad 4.12$$

Equation 4.12 can thus be written,

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = \left(2\frac{\partial A_0}{\partial \tau} - A_0 + \frac{A_0^3}{4}\right)\sin(t+\phi(\tau)) + 2A_0\frac{\partial \phi}{\partial \tau}\cos(t+\phi(\tau)) + \frac{A_0^3}{4}\sin 3(t+\phi(\tau)) \qquad 4.13$$

The periodicity condition for eqn. 4.13 is simply that the function on the right hand side should contain no component in $\sin t$ or $\cos t$.

$$\therefore \quad 2\frac{\partial A_0}{\partial \tau} - A_0 + \frac{A_0^3}{4} = 0 \quad \text{and} \quad 2A_0\frac{\partial \phi}{\partial \tau} = 0 \qquad 4.14$$

Hence we arrive at the same result as derived by the K.B. method

$$A_0(\tau) = \frac{A_0(0)\, e^{\tau}}{1 + \frac{A_0(0)}{4}(e^{\tau}-1)} \quad \text{and} \quad \phi(\tau) = \phi_0 \ (\text{const.}) \qquad 4.15$$

To investigate the interaction of the two non linearities we now consider the more general forms given by eqn. 4.1. The equation to be studied is,

$$\frac{d^2x}{dt^2} + b(\alpha x^2 - 1)\frac{dx}{dt} + c_1 x + c_2 x^3 = 0 \qquad 4.16$$

$$c_1 > 0 \ , \ c_3 > 0$$

As in the previous example we introduce the slow time scale bt and the scale of $O(t)$, $\eta = g(\tau)/b$. Equation 4.16 becomes the partial differential equation,

$$g'(\tau)^2 \frac{\partial^2 \tilde{x}}{\partial \eta^2} + c_1\tilde{x} + c_3\tilde{x}^3 + b\left[(\alpha\tilde{x}^2 - 1)g'(\tau)\frac{\partial \tilde{x}}{\partial \eta} + 2g'(\tau)\frac{\partial^2 \tilde{x}}{\partial \eta \partial \tau} + g''(\tau)\frac{\partial \tilde{x}}{\partial \eta}\right] + b^2\left[\frac{\partial^2 \tilde{x}}{\partial \tau^2} + (\alpha\tilde{x}^2 - 1)\frac{\partial \tilde{x}}{\partial \tau}\right] = 0 \qquad 4.17$$

We expand the solution to eqn. 4.17 in powers of b , in the form,

$$\tilde{x}(\eta, \tau; b) = x_0(\eta, \tau) + b\, x_1(\eta, \tau) + O(b^2) \qquad 4.18$$

Equating coefficients of b we obtain the zeroth and first order approximate equations,

$$g'(\tau)^2 \frac{\partial^2 x_0}{\partial \eta^2} + c_1 x_0 + c_3 x_0^3 = 0 \qquad 4.19$$

$$g'(\tau)^2 \frac{\partial^2 x_1}{\partial \eta^2} + (c_1 + 3c_3 x_0^2) x_1 = -\left\{ (\alpha x_0^2 - 1)\frac{\partial x_0}{\partial \eta} + 2g'(\tau)\frac{\partial^2 x_0}{\partial \eta \partial \tau} + g''(\tau)\, \partial x_0/\partial \eta \right\} \qquad 4.20$$

The zeroth order approximation thus has the form,

$$x_0(\eta,\tau) = A_0(\tau) C_n(\eta, \mu_0(\tau)) \qquad 4.21$$

$$\text{with } g'(\tau)^2 = c_1 + c_3 A_0(\tau)^2 \;; \quad \mu_0^2 = \frac{\frac{1}{2} c_3 A_0^2}{c_1 + c_3 A_0^2}$$

Once again the function $g'(\tau)$ can be choosen so that the first order solution is periodic with period $4K(\mu_0)$ in η. The periodicity condition reads,

$$\int_0^{4K} \left\{ \left[(\alpha x_0^2 - 1) g'(\tau) + g''(\tau) \right] \frac{\partial x_0}{\partial \eta} + 2g'(\tau) \frac{\partial^2 x_0}{\partial \eta \partial \tau} \right\} \frac{\partial x_0}{\partial \eta} d\eta = 0 \qquad 4.22$$

$$\therefore \frac{\partial}{\partial \tau}\left\{ g'(\tau) \int_0^{4K} \left(\frac{\partial x_0}{\partial \eta}\right)^2 d\eta \right\} - g'(\tau) \int_0^{4K} \left(\frac{\partial x_0}{\partial \eta}\right)^2 d\eta = -g'(\tau) \int_0^{4K} \alpha x_0^2 \left(\frac{\partial x_0}{\partial \eta}\right)^2 d\eta \qquad 4.23$$

with $x_0(\eta, \mu_0(\tau))$ given by equation 4.21

and $$\frac{\partial x_0}{\partial y} = -A_0(\tau)\, S_n(y,\mu_0)\, D_n(y,\mu_0)$$

we have, $$\int_0^{4K} x_0^2\left(\frac{\partial x_0}{\partial y}\right)^2 dy = \frac{A_0(\tau)^4 \pi}{4}$$

$\therefore$ We can write,

$$g'(\tau)\int_0^K \left(\frac{\partial x_0}{\partial y}\right)^2 dy = k_0 e^{\tau} - \frac{\alpha\pi}{16}\int_0^{\tau} e^{(\tau-s)} g'(s) A_0^4(s)\, ds \qquad 4.24$$

where k_0 is a constant.

Using the same approximations as we used to arrive at eqn. 2.41 we obtain,

$$g'(\tau) A_0(\tau)^2 (8-3\mu_0^2) = k_1 e^{\tau} - 2\alpha \int_0^{\tau} e^{(\tau-s)} g'(s) A_0^4(s)\, ds \qquad 4.25$$

or $$(g'(\tau)^2 - c_1)\left(\frac{13}{2} g'(\tau)^2 + \frac{3}{2} c_1\right) = g' k_2 c_3 e^{\tau} - \frac{2\alpha}{c_3} g' \int_0^{\tau} e^{(\tau-s)} g'(g'^2 - c_1)^2 ds \qquad 4.26$$

Equation 4.26 can be solved for $g'(\tau)$ using an iterative method, regarding the integral term as a coefficient of $g'(\tau)$ and iterating until the solution converges. We can see from eqn. 4.25 that when $c_3 = 0$ (i.e. $\mu_0 = 0, g' = \sqrt{c_1}$) the amplitude $A_0(\tau)$ satisfies the equation,

$$\frac{dA_0}{d\tau} - \frac{A_0}{2} + \frac{\alpha}{8} A_0^3 = 0 \qquad 4.27$$

a result which agrees with eqn. 4.14.

For the case when $c_3 \neq 0$ the limit cycle amplitude can be estimated with the aid of eqn. 4.25. If we choose the initial condition so that $x_0(0,0) = A_{0_{LIMIT}}$, $\frac{\partial x_0}{\partial \eta}(0,0) = 0$ then we can write,

$$g'_L A_{0_L}^2 (8 - 3\mu_{0_L}^2)(1 - e^{\tau}) = -2\alpha A_{0_L}^4 g'_L \int_0^{\tau} e^{(\tau - s)} ds \qquad 4.28$$

where the subscript L refers to the limit cycle value of the respective quantity (these will of course be constants).

The limit cycle amplitude is then given by the solution of the bi-quadratic

$$2\alpha c_3 A_{0_L}^4 + (2\alpha c_1 - \frac{13}{2} c_3) A_{0_L}^2 - 8c_1 = 0 \qquad 4.29$$

Any error in the approximate limit cycle amplitude will be reflected in the approximate solution having an incorrect limit cycle frequency and the phase error between approximate and exact solutions in the limit cycle, varying with time. Due to this varying phase error the approximate method can hardly be expected to give a solution curve that is quantitively accurate for large time, no matter how small the damping parameter b . Some sample comparisons of approximate and exact numerical solutions curves are shown in Fig. 11.

We turn now to the use of the approximate method in estimating the parameters c_1 , c_3 , b and α , of the Van der Pol oscillator with non linear stiffness. The parameters C_1 and C_3 can be estimated by a similar method as was used for the problem described in section 3. Once again we use the approximate relation, (eqn. 3.9)

$$\left(\frac{k\pi}{\Delta t_n} \right)^2 \simeq C_1 + C_3 x_n^2 \qquad 4.30$$

and the values of C_1 and C_3 can be obtained by a least squares fit of the measurements as in section 3. Having obtained estimates for C_1 and C_3 the coefficient α can be determined by using the relation given in eqn. 4.29. Use of this equation implies that the limit cycle amplitude A_{0L} has been measured. Estimating the damping b is more complicated than in the case of linear damping since the algebraic equation (eqn. 2.41) has been replaced by the integral equation 4.25.

We can here eliminate k_1 from eqn. 4.25 by considering the equation satisfied at the two points t_n and t_{n+1} (corresponding to the peak values of $x(t)$). We can write,

$$e^{-\tau_n} g'_n x_n^2 (8-3p_n^2) - e^{-\tau_{n+1}} g'_{n+1} x_{n+1}^2 (8-3p_{n+1}^2) = \lambda\alpha \int_{\tau_n}^{\tau_{n+1}} e^{-s} g' A_0^4 \, ds \qquad 4.31$$

The function $e^{-\tau} g'(\tau) A_0^4(\tau)$ can be considered as a slowly varying function in the interval $\tau_n \leqslant \tau \leqslant \tau_{n+1}$ and hence we use the approximation,

$$\int_{\tau_n}^{\tau_{n+1}} f(s)\,ds \simeq \frac{1}{2}(\tau_{n+1}-\tau_n)(f_{n+1}+f_n) \tag{4.32}$$

After some reduction of eqn. 4.31 we can write,

$$e^{2(\tau_{n+1}-\tau_n)} = \frac{g_{n+1}'^{2}\,x_{n+1}^{4}(8-3\mu_{n+1}^2)}{g_n'^{2}\,x_n^{4}(8-3\mu_n^2)}\left\{\frac{(8-3\mu_{n+1}^2)+2\alpha x_{n+1}^2 b(t_{n+1}-t_n)}{(8-3\mu_n^2)-2\alpha x_n^2 b(t_{n+1}-t_n)}\right\} \tag{4.33}$$

Substituting for $g_n'^{2}$, μ_n^2 in terms of x_n^2 and introducing the new variables

$$\varsigma = \frac{x_{n+1}}{x_n} \quad ; \quad \gamma = \frac{c_3 x_m^2}{c_1} \quad ; \quad x_m^2 = \frac{x_{n+1}^2 + x_n^2}{2}$$

and the approximate relation, (cff. eqn. 3.21)

$$t_{n+1}-t_n \simeq k\pi\Big/\sqrt{c_1}\sqrt{1+\gamma}$$

equation 4.33 can be reduced to the form

$$\frac{(1+\varsigma^2+2\gamma)\varsigma^2(1+\varsigma^2+\frac{13}{8}\varsigma^2\gamma)}{(1+\varsigma^2+2\varsigma^2\gamma)(1+\varsigma^2+\frac{13}{8}\gamma)}\left\{\frac{(1+\varsigma^2)(1+\varsigma^2+\frac{13}{8}\varsigma^2\gamma)+\frac{b\alpha}{4}(1+\varsigma^2+2\varsigma^2\gamma)\frac{\varsigma^2 c_1\gamma}{c_3}D_4(\gamma)}{(1+\varsigma^2)(1+\varsigma^2+\frac{13}{8}\gamma)-\frac{b\alpha}{4}(1+\varsigma^2+2\gamma)\frac{c_1\gamma}{c_3}D_4(\gamma)}\right\} = e^{2b(t_{n+1}-t_n)} \tag{4.34}$$

where

$$D_4(\gamma) = k\pi\Big/\sqrt{c_1}\sqrt{1+\gamma}$$

As in section 3 we now introduce an approximate form for the logarithmic decrement,

$$\log_e\left|\frac{x_{n+1}}{x_n}\right| = \varepsilon = b\sigma \simeq \delta - 1$$

and similarly

$$e^{2b(t_{n+1}-t_n)} \simeq 1 + 2b\,D_u(\gamma)$$

Replacing δ by $1+b\sigma$ in eqn. 4.35 and neglecting terms of $O(b^2)$ after some reduction we finally obtain the result,

$$\sigma = \frac{9}{4}\frac{\pi}{\sqrt{c_1}\sqrt{1+\gamma}}\left\{\frac{(1+\gamma)\left(1+\frac{13}{16}\gamma-\frac{\alpha}{4}(1+\gamma)\frac{c_1\gamma}{c_3}\right)}{4+\frac{17}{2}\gamma+\frac{39}{8}\gamma^2}\right\} \tag{4.36}$$

and hence,

$$\log_e\left|\frac{x_{n+1}}{x_n}\right|\frac{16\sqrt{c_1}}{9\pi} \simeq \frac{(1+\gamma)^{1/2}\left(1+\frac{13}{16}\gamma-\frac{\alpha}{4}(1+\gamma)\frac{c_1\gamma}{c_3}\right)}{\left(1+\frac{17}{8}\gamma+\frac{39}{32}\gamma^2\right)}\,b \tag{4.37}$$

$$= b\,G_1\left(\gamma, \frac{\alpha c_1}{4c_3}\right)$$

The function $G_1\left(\gamma, \frac{\alpha_1}{4c_2}\right)$ has the form shown in Fig. 3.4.

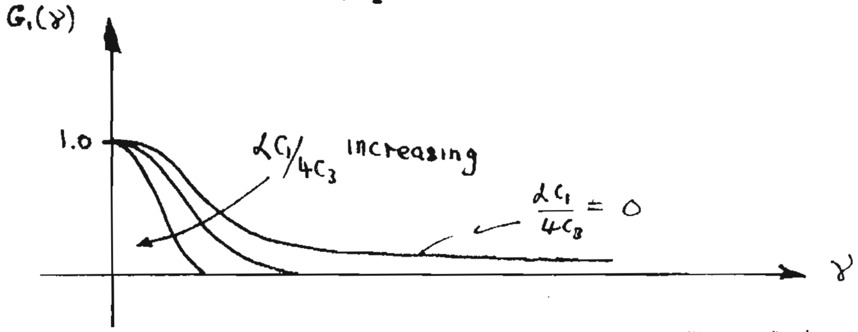

FIG. 3.4

Note that $G_1(\gamma, 0) = G(\gamma)$ (eqn. 3.25)

From measurements of $\log_e \left|\frac{x_{n+1}}{x_n}\right|$ and x_n^2, the curve of $G_1(\gamma)$ can be shifted to give the least squares fit to the data and hence the best estimate for the damping b.

The non linear damping parameter, α, can also be estimated from the curve of $\log_e \left|\frac{x_{n+1}}{x_n}\right| \frac{16}{9}\frac{\sqrt{c_1}}{\pi}$ v. γ, i.e. when $\log_e \left|\frac{x_{n+1}}{x_n}\right| = 0$, $\alpha = \frac{4(1+\frac{13}{16}\gamma)}{(1+\gamma)} \frac{c_3}{c_1\gamma}$.

4.2 The Step Function Response of a Damped Non-Linear Oscillator

We outline here the formal analysis for obtaining the zeroth order solution, corrected by the two time scale method for the effects of damping (non uniformity), of a damped non linear oscillator excited by a step function. The analysis is terminated when the form of the zeroth order approximation is obtained. The equation to be solved is,

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c_1 x + c_3 x^3 = f_0 1_+(t) \qquad 4.38$$

Introducing the transformation,

$$\xi(t) = x(t) - x_m(t) \qquad 4.39$$

where $x_m(t)$, the steady state response is given by,

$$c_1 x_m + c_3 x_m^3 = f_0 \qquad 4.40$$

$$(\dot{x}_m = \ddot{x}_m = 0,\ t > -\infty)$$

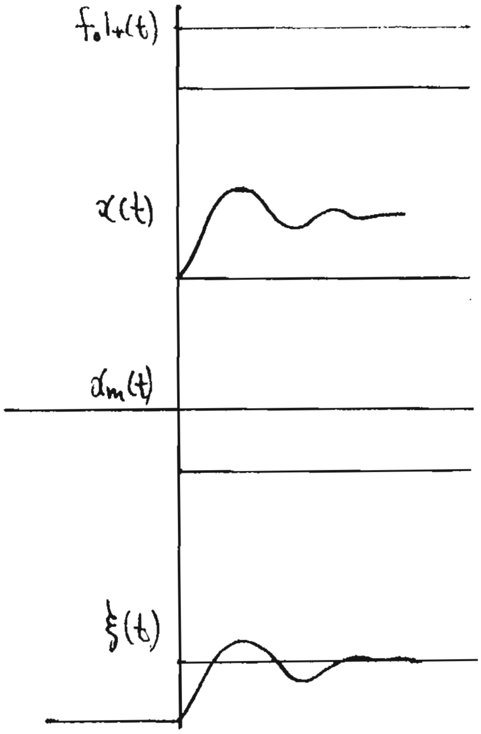

FIG. 4.1

we obtain the equation in terms of $\xi(t)$,

$$\frac{d^2\xi}{dt^2} + b\frac{d\xi}{dt} + (c_1 + 3c_3 x_m^2)\xi + 3c_3 x_m \xi^2 + c_3\xi^3 = -f_0 1_-(t) \qquad 4.41$$

Fig. 4.1 portrays the behaviour of the various functions in use. We are interested in $\xi(t)$ for $t>0$ and eqn. 4.41 can thus be written,

$$\frac{d^2\xi}{dt^2} + b\frac{d\xi}{dt} + \bar{c}_1\xi + c_2\xi^2 + c_3\xi^3 = 0 \qquad 4.42$$

with $\xi(0) = -x_m$, $\dot{\xi}(0) = 0$

By introducing the transformation eqn. 4.39 the indicial response problem has been transformed into an initial value problem. However, a term in $\xi(t)^2$ has appeared in the non linear spring force which will give rise to an asymmetric oscillation in $\xi(t)$, as is to be expected. We will proceed to solve eqn. 4.42 by a perturbation method using two time scales to include the dissipation effects.

Since it is required to state the problem in terms of the original parameters b, c_1, c_3, and f_0, it would be useful to formulate a relation between the old and new parameters (b, $\bar{c}_1$, c_2, x_m). These relations are,

$$c_1 x_m + c_3 x_m^3 = f_0 \; ; \; \bar{c}_1 = c_1 + 3c_3 x_m^2 \; ; \; c_2 = 3c_3 x_m . \qquad 4.43$$

However, it is more convenient to convert these relations into the more useful parametric form,

$$\frac{c_2}{3\sqrt{c_1 c_3}}\left[1 + \frac{c_2^2}{9c_1 c_3}\right] = \frac{f_0\, c_3^{1/2}}{c_1^{3/2}} \qquad 4.44$$

So that with f_0, c_1, c_3 specified, c_2 can be obtained from the curve Fig. 4.2 and x_m and $\bar{c}_1$ are then given by,

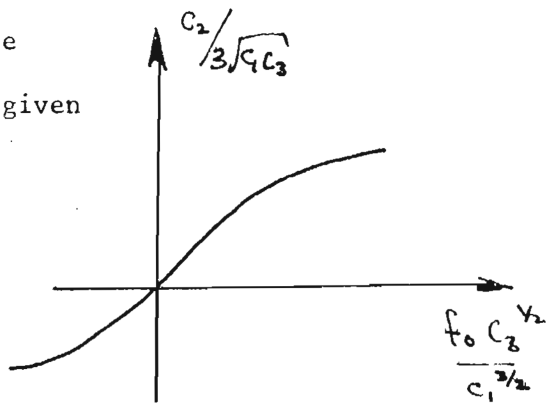

FIG. 4.2

$$\begin{aligned} x_m &= c_2/3c_3 \\ \bar{c}_1 &= c_1 + 3c_2 x_m^2 \end{aligned} \qquad 4.45$$

The relation between c_2 and f_0 can be further explored by examining the limiting cases.

(1) f_0 small $\qquad c_2 \simeq 3f_0 c_3/c_1$

4.46

(2) f_0 Large $\qquad c_2 \simeq 3\,(f_0 c_3^2)^{1/3}$

It follows that c_2 is not necessarily small when f_0 is small. Returning to eqn. 4.34 we introduce the two time scales $\eta(t)$ and τ defined as before,

$$\eta(t) = g(\tau)/b \quad ; \quad \tau = bt \qquad 4.47$$

Hence the partial differential equation replacing eqn. 4.42 becomes,

$$g'^2 \frac{\partial^2 \bar{\xi}}{\partial \eta^2} + \bar{c}_1\bar{\xi} + c_2\bar{\xi}^2 + c_3\bar{\xi}^3 + b\left[2g'\frac{\partial^2\bar{\xi}}{\partial\eta\partial\tau} + (g'+g'')\frac{\partial\bar{\xi}}{\partial\eta}\right]$$
$$+ b^2\left[\frac{\partial^2\bar{\xi}}{\partial\tau^2} + \frac{\partial\bar{\xi}}{\partial\tau}\right] = 0 \qquad 4.48$$

where

$$\bar{\xi}(\eta,\tau;b) \equiv \xi(t;b)$$

We now expand $\bar{\xi}(\eta,\tau;b)$ in the form,

$$\bar{\xi}(\eta,\tau;b) = \xi_0(\eta,\tau) + b\,\xi_1(\eta,\tau) + O(b^2) \qquad 4.49$$

Substituting the expansion (eqn. 4.49) into eqn. 4.48 and equating the powers of b we obtain,

$$O(b^0):\quad g'^2\frac{\partial^2\xi_0}{\partial\eta^2} + \bar{c}_1\xi_0 + c_2\xi_0^2 + c_3\xi_0^3 = 0 \qquad 4.50$$

$$\xi_0(0) = -x_m \quad ; \quad \partial\xi_0/\partial\eta\,(0) = 0$$

$$O(b):\quad g'^2\frac{\partial^2\xi_1}{\partial\eta^2} + (\bar{c}_1 + 2c_2\xi_0 + 3c_3\xi_0^2)\xi_1 = -\left\{2g'\frac{\partial^2\xi_0}{\partial\eta\partial\tau} + (g'+g'')\frac{\partial\xi_0}{\partial\eta}\right\} \qquad 4.51$$

with $\xi_1(0) = 0 \quad ; \quad \partial\xi_1/\partial\eta\,(0) = 0$

Because of the asymetric term $c_2\xi_0^2$ in eqn. 4.50 the solution of these equations cannot be written as a single elliptic function and it is not immediately obvious what form the solution does take. The solution can however, be written in an inverse form as a Jacobian Elliptic integral.

Let

$$\bar{C}_1^* = \bar{C}_1/g'(H)^2 \; ; \; C_2^* = C_2/g'(H)^2 \; ; \; C_3^* = C_3/g'(H)^2$$

Then equation 4.50 can be written in the alternative form,

$$\left(\frac{\partial x_0}{\partial \eta}\right)^2 = \left\{\bar{C}_1^*(\xi_m^2-\xi_0^2) + \frac{2C_2^*}{3}(\xi_m^3-\xi_0^3) + \frac{C_3^*}{2}(\xi_m^4-\xi_0^4)\right\} \qquad 4.52$$

where $\xi_m = \xi_m(\tau)$ is the amplitude envelope of the negative portion of oscillation and $\xi_m(0) = \xi_0(0) = -x_m$.

Eqn. 4.52 can be written in the form,

$$\eta = \int_{\xi_m(\tau)}^{\xi_0(\eta,\tau)} \frac{d\xi_0}{\sqrt{\bar{C}_1^*(\xi_m^2-\xi_0^2) + \frac{2}{3}C_2^*(\xi_m^3-\xi_0^3) + \frac{C_3^*}{2}(\xi_m^4-\xi_0^4)}} \qquad 4.53$$

The integral in eqn. 4.53 is an elliptic integral and in order to use the standard tables (Ref. 14) we require to factorise the denominator of the integrand. Although $\xi_m(\tau)$ is a root of equation,

$$\bar{C}_1^*(\xi_m^2-\xi_0^2) + \frac{2}{3}C_2^*(\xi_m^3-\xi_0^3) + \frac{C_2^*}{2}(\xi_m^4-\xi_0^4) = 0 \qquad 4.54$$

for $C_2^* \neq 0$, we find that $-\xi_m(\tau)$ is not a second root. The second real root we require is exactly that value of $\xi_0(\eta,\tau)$ which occurs when $\frac{\partial \xi_0}{\partial \eta} = 0$ (say $\xi_n(\tau)$) as shown in Fig. 4.3.

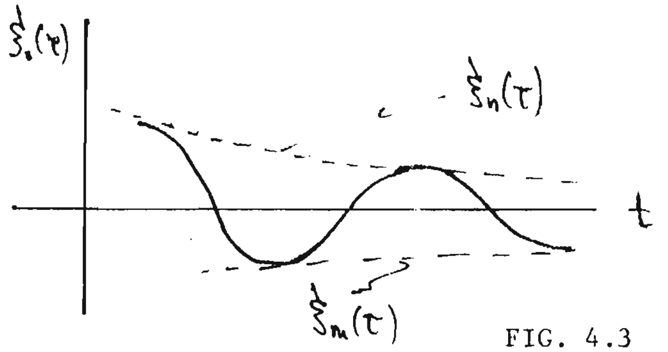

FIG. 4.3

Extracting the root $\xi_0 = \xi_m$ from eqn. 4.54 we obtain,

$$\frac{\bar{c}_3^*}{2}(\xi_m-\xi_0)\left\{\frac{2\bar{c}_1}{c_3}\xi_m + \frac{4}{3}\frac{c_2}{c_3}\xi_m^2 + \xi_m^3 + \xi_0\left(\frac{2\bar{c}_1}{c_3}+\frac{4}{3}\frac{c_2}{c_3}\xi_m+\xi_m^2\right) + \xi_0^2\left(\frac{4}{2}\frac{c_2}{c_3}+\xi_m\right)+\xi_0^3\right\} = 0 \qquad 4.55$$

The remaining cubic will have the one real root $\xi_n(\tau)$ and for the case $c_1 > 0,\ c_3 > 0$, the other two roots will be complex conjugates. (To see this, compare with symmetric oscillator, $c_2 = 0$). For c_2 small we can obtain an approximation to the root $\xi_n(\tau)$ by a perturbation analysis.

i.e. let

$$\xi_n(\tau) = \xi_m(\tau) + c_2\phi(\tau) \qquad 4.56$$

and we find that,

$$\phi(\tau) = \frac{-\frac{2}{3}\xi_m(\tau)^2}{\bar{c}_1 + c_3\xi_m(\tau)^2} \qquad 4.57$$

The elliptic integral in eqn. 4.53 has the general form,

$$I = \int_b^y \frac{ds}{\sqrt{(a-s)(s-b)(t-c)(t-\bar{c})}}$$

c , $\bar{c}$ are complex conjugate roots.

From Byrd and Freidman (Ref. 14) p. 133, we can write

$$I = h\int_0^{u_1} du = hu_1 = h\,cn^{-1}(\cos\varphi, \mu) = h\,F(\varphi, \mu) \qquad 4.58$$

where

$$\cos\varphi = \left[\frac{(a-s)A-(s-b)B}{(a-s)B+(s-b)A}\right] \qquad 4.59$$

and $A^2 = (a-b_1)^2 + a_1^2$; $B^2 = (b-b_1)^2 + a_1^2$

$$b_1 = \frac{c+\bar{c}}{2} \quad ; \quad a_1^2 = -\frac{(c-\bar{c})^2}{4} \qquad 4.60$$

$$\mu^2 = \frac{(a-b)^2 - (A-B)^2}{4AB} \qquad 4.61$$

$$h = 1/\sqrt{AB} \tag{4.62}$$

From eqn's. 4.53 and 4.58 we can therefore write,

$$\eta = \sqrt{\frac{2}{C_3^*}}\, h(\tau)\, C_n^{-1}(\cos\psi(\tau), \mu(\tau)) \tag{4.63}$$

Now, $a = \xi_n(\tau)$, $b = \xi_m(\tau)$, $c = e(\tau)$

$A = A(\tau)$, $B = B(\tau)$

$$\therefore \quad \cos\psi(\tau) = C_n\left(\sqrt{\frac{C_3^*}{2}}\frac{\eta}{h(\tau)}, \mu(\tau)\right) \tag{4.64}$$

and finally from eqn. 4.59,

$$\xi_0(\eta,\tau) = \frac{aB + bA - (aB - bA)C_n\left(\sqrt{\frac{C_3^*}{2}}\frac{\eta}{h(\tau)}, \mu(\tau)\right)}{A + B - (A - B)C_n\left(\sqrt{\frac{C_3^*}{2}}\frac{\eta}{h(\tau)}, \mu(\tau)\right)} \tag{4.65}$$

Although the expressions for the coefficient in eqn. 4.65 are likely to be of a simpler form than they appear, especially for the case $C_2 \ll 1$, in order to obtain these expressions is closed form the cubic factor in eqn. 4.55

has itself to be factorised. Due to such complexities it was not considered worthwhile, by the author, at this stage to continue the analysis beyond this point.

The present example serves to illustrate the construction of a solution in terms of Jacobian elliptic functions through the inversion of the more primitive elliptic integral.

4.3 The Transient Response of a Simple Two Degree of Freedom System.

Consider the two mass system shown in Fig. 4.4.

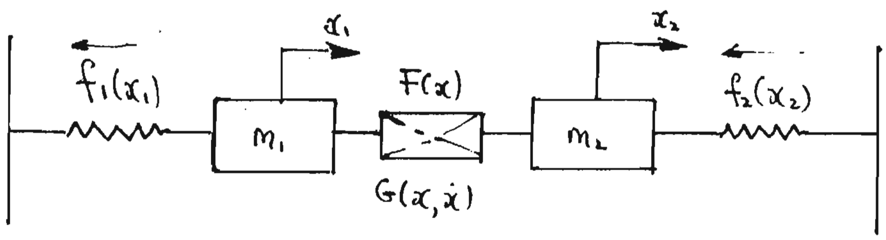

FIG. 4.4

The two masses m_1, m_2 are connected to each other by a unit having stiffness and damping characteristics defined by the functions $F(x)$ and $G(x,\dot{x})$ respectively, and to the supporting structure by pure stiffness units having characteristics $f_1(x_1)$ and $f_2(x_2)$ as shown.

The equations of motion of the masses are,

$$m_1\ddot{x}_1 = -f_1(x_1) - F(x_1 - x_2) - G(x_1 - x_2, \dot{x}_1 - \dot{x}_2) \qquad 4.66$$

$$m_2\ddot{x}_2 = -f_2(x_2) + F(x_1 - x_2) + G(x_1 - x_2, \dot{x}_1 - \dot{x}_2) \qquad 4.67$$

with initial conditions , $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ 4.68

($\dot{x}_1$ denotes differentiation of x_1 w.r.t. time)

To illustrate how we might apply the techniques used previously for one degree of freedom systems, we simplify the model and consider the particular case defined by,

$$m_1 = m_2 = m \quad : \quad f_1(x_1) = f_2(x_2) = k(x)$$

4.69

$$F(x) = c_1 x + c_3 x^3 \quad : \quad G(x, \dot{x}) = b\, dx/dt$$

The equations of motion thus take the form,

$$m_1\ddot{x}_1 + kx_1 + c_1(x_1 - x_2) + c_3(x_1 - x_2)^3 + b(\dot{x}_1 - \dot{x}_2) = 0 \qquad 4.70a$$

$$m_2\ddot{x}_2 + kx_2 - c_1(x_1 - x_2) - c_3(x_1 - x_2)^3 - b(\dot{x}_1 - \dot{x}_2) = 0 \qquad 4.70b$$

Introducing the new coordinate system defined by,

$$y_1 = x_1 - x_2 \quad ; \quad y_2 = x_1 + x_2$$

we can subtract and add eqn's 4.70a, 4.70b to obtain the equations of motion in the 'new' coordinate system. It is exactly this transformation which in the linear case $(c_3 = 0)$ decouples the system equation and defines the normal modes of the system. A similar decoupling occurs for the nonlinear systems with the particular characteristics choosen. The new equations read,

$$\ddot{y}_1 + 2b\dot{y}_1 + (k + 2c_1)y_1 + 2c_3 y_1^3 = 0 \qquad 4.71a$$

$$\ddot{y}_2 + k y_2 = 0 \qquad 4.71b$$

with initial conditions

$$y_1(0) = y_1^0 = x_1^0 - x_2^0$$
$$y_2(0) = y_2^0 = x_1^0 + x_2^0 \qquad 4.72$$

It can be seen that for the present case it is possible to regard the complete motion of each mass as linear combinations of two independent motions, $y_1(t)$ described by the non linear equation (4.71a) and $y_2(t)$ described by the linear equation (4.71b)

i.e. $$x_1(t) = \frac{y_1(t) + y_2(t)}{2} \; ; \; x_2(t) = \frac{y_2(t) - y_1(t)}{2} \qquad 4.73$$

Eqn. 4.71a has exactly the same form as eqn. 2.7 and hence provided $(k+2c_1) > 0,\ c_3 > 0$, the zeroth order, uniformly valid solution (for small b) is given by,

$$y_{10}(\eta,\tau) = A_{10}(\tau)\, C_n(\eta, p_{10}(\tau)) \qquad 4.74$$

where $$\eta = g(\tau)/b \quad , g'(\tau)^2 = (k+2c_1) + 2c_3 A_{10}(\tau)^2$$

$$p_{10}(\tau)^2 = c_3 A_{10}^2 / g'^2 \qquad 4.75$$

and $g'(\tau)$ is obtained from the equation, $g' A_{10}^2 (8 - 3p_{10}^2) = k_1 e^{-\tau}$

The solution of eqn. 4.71b is simply,

$$y_2(t) = A_2 \cos \sqrt{k}\, t \qquad 4.76$$

so that,

$$x_1(t) = \frac{A_{10}(t)\, Cn(\gamma, \mu_{10}) + A_2 \cos\sqrt{k_2}\, t}{2}$$

$$x_2(t) = \frac{A_2 \cos\sqrt{k}\, t - A_{10}(t)\, Cn(\gamma, \tau)}{2} \qquad 4.77$$

Any asymmetry in the outer stiffnesses i.e. $f_1(x) \neq f_2(x)$, will give rise to coupling terms in eqn's 4.71, as will the case of unequal masses. Also, if the outer stiffness are non linear functions of displacement e.g. $f(x) = k_1 x + k_2 x^3$, then even in the symmetric case the equation of motion in (y_1, y_2) will be coupled.

LIST OF SYMBOLS

(in common use)

$x(t)$	Displacement of Non-Linear Oscillator (Eqn. 2.1)
$f(x)$	Spring Force (Eqn. 2.1)
ε	A small parameter (Eqn. 2.1)
c_1, c_3	Coefficients of linear and cubic terms in $f(x)$
$Cn(\eta,\mu)$ $Sn(\eta,\mu)$ $Dn(\eta,\mu)$	Jacobian elliptic functions of argument η and modulus μ
b	damping parameter (Eqn. 2.6)
$F(\nu,\mu)$	Elliptic integral of the first kind
$E(\nu,\mu)$	Elliptic integral of the second kind
$K(\mu), E(\mu)$	Complete Elliptic integrals of 1st and 2nd kind respectively.
$x_i(t)$	i_{TH} order approximation in a straightforward asymptotic expansion. (Eqn. 2.8)
τ, η	Time scales of order bt and t respectively (Eqn. 2.22)
$A_0(\tau)$	Amplitude function (Eqn. 2.30)
$g'(\tau)$	'Frequency' of oscillation in time scale η (eqn. 2.30)
x_n, x_{n+1}	Adjacent peaks in an oscillation
δ	Ratio of x_{n+1} to x_n
x_m	A mean peak value defined as $x_m^2 = \dfrac{x_{n+1}^2 + x_n^2}{2}$

γ	$C_2 x_m^2 / C_1$
$G(\gamma)$	An approximate form for the log. dec.
α	Coefficient in Non-Linear Damping term (Eqn. 4.1)
ϕ	Phase angle (Eqn. 4.3)
A_{0L}	Limit Cycle amplitude
f_0	Step function amplitude (Eqn. 4.40)
x_m	Steady state response of non linear oscillator to a step function
C_2	$3 C_3 x_m$
$\xi(t)$	transformed displacement given by $\xi(t) = x(t) - x_m(t)$
$x_1(t), x_2(t)$	displacement of masses m_1 and m_2 in two degree of freedom system (Fig. 4.4)
$y_1(t), y_2(t)$	defined as $y_1 = x_1 - x_2$, $y_2 = x_1 + x_2$
k	linear stiffness (Eqn. 4.69)

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Appendix 1

Elliptic integrals, functions and the solution of the equation $\ddot{x} + c_1 x + c_3 x^3 = 0$

The character of the solution of the conservative system described by the equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0 \qquad \text{A1.1}$$

depends markedly on the values of the parameters c_1, c_3 and the initial trajectory. Here we examine the range of parameter values for which the solution has an oscillatory character.

The initial trajectory is defined by the relations,

$$x(0) = \eta_1 \quad ; \quad \frac{dx}{dt}(0) = \eta_2 \qquad \text{A1.2}$$

Equation A1.1 can be integrated once to give the relation,

$$\left(\frac{dx}{dt}\right)^2 = \eta_2^{\,2} + c_1(\eta_1^{\,2} - x^2) + \frac{c_3}{2}(\eta_1^{\,4} - x^4) \qquad \text{A1.3}$$

and this equation enables us to portray the trajectory in phase space $\left(dx/dt \ v.\ x \right)$. Before we discuss these portraits in more detail we see that eqn. A1.3 can be integrated once more to give,

$$t = \int_{\eta_1}^{x} \frac{dx}{\sqrt{\eta_2^2 + c_1(\eta_1^2 - x^2) + \frac{c_3}{2}(\eta_1^4 - x^4)}} \qquad \text{A1.4}$$

The result given by eqn A1.4 is quite a common feature of non linear oscillation problems i.e. the solution is given in the inverse form $t = t(x)$. The integral in eqn. A1.4 is an elliptic integral and in order to proceed with the inversion process it is necessary to reduce the denomenator to standard form. Since all possible oscillatory phase plane trajectories cross the x axis at some time, we do not lose any generality by setting $\eta_2 = 0$. In this case the standard form of eqn. A1.4 reads

$$t = \left(\frac{2}{c_3}\right)^2 \int_{\eta_1}^{x} \frac{dx}{\sqrt{(\eta_1^2 - x^2)(x^2 + \eta_1^2 + \frac{2c_1}{c_3})}} \qquad \text{A1.5}$$

With the help of Ref. 14 (Handbook of Elliptic Integrals - Bryd and Freidman) we can now invert eqn A1.5 for the various ranges of values of η_1 , c_1 and c_3 . Before we do this it is expedient to examine more closely the phase plane portraits of the system for the various cases. There are three cases of interest.

1. $c_1 > 0$, $c_3 > 0$: (the hard spring)

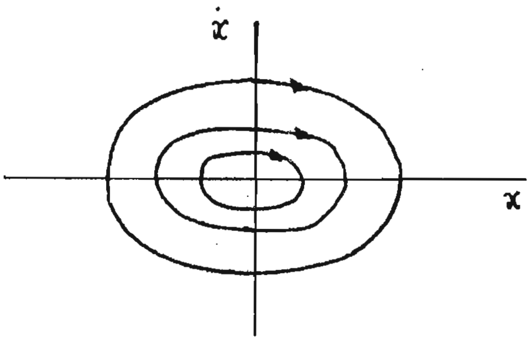


FIG. A.1

In this case there is one equilibrium point $x = 0$ and this is a centre.

In this case the inversion of eqn. A1.5 can be carried out using relation 213.00 in Ref. 14 and is simply

$$x(t) = A\,cn(w_0 t, \mu_0) \qquad \text{A1.6}$$

where $w_0^2 = c_1 + c_3 A_0^2$ and $\mu_0^2 = \dfrac{\frac{1}{2} c_3 A_0^2}{c_1 + c_3 A_0^2}$. This is the A1.7
elliptic cosine function with period $4K(\mu)$: where $k(\mu)$ is the complete elliptic integral of the first kind given by,

$$k(\mu_0) = F(\pi/2, \mu_0) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \mu_0^2 \sin^2\theta}} \qquad \text{A1.8}$$

The period in t is given by,

$$T_t = 4k/w_0 \qquad \text{A1.9}$$

hence

$$\overline{T_t} = \frac{4(1-2\mu_0^2)^{1/2}}{c_1^{1/2}} K(\mu_0) \qquad \text{A1.10}$$

Fig. A.2 below shows the behaviour of μ_0^2 and $\overline{T_t}$ as the amplitude of oscillation varies (Ref. 15)

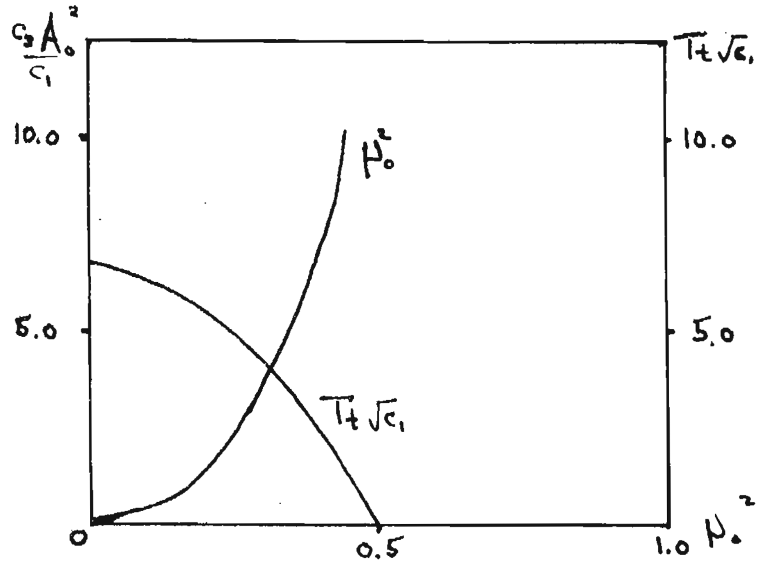


FIG. A.2.

We note that for the present case $c_1 > 0$, $c_3 > 0$, $0 \leq \mu_0^2 \leq 1/2$. For c_1 or c_3 negative there is no guarantee that μ_0 will remain positive and if we choose to retain the form Eqn. A1.6 for the complete range of c_1, c_3 we would be forced to consider values of $\mu > 1$ and complex. Values of μ are usually tabulated for $0 \leq \mu \leq 1$ and hence we look for different forms of the solution for other ranges of c_1 and c_3.

2. $c_1 > 0$, $c_3 < 0$ (soft spring)

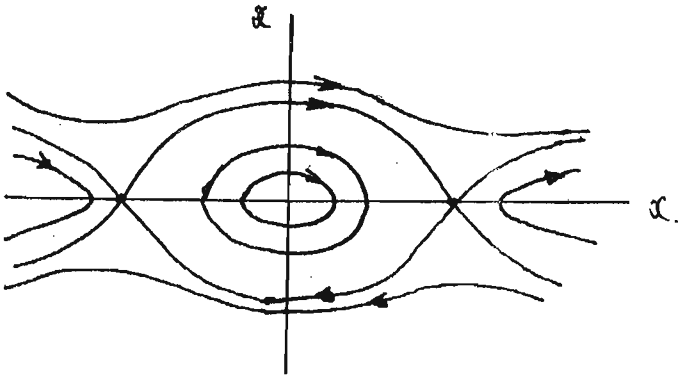


FIG. A.3

In this case there are 3 equilibrium points $x = 0$,
$x = \pm\sqrt{\frac{-c_1}{c_2}}$: the first being a centre and the second two being saddles. We can see that the trajectory will only lie inside the separatrix if $|x_0| < \sqrt{\frac{-c_1}{c_3}}$ and for this case the inverse of eqn A1.5 is accomplished by ex. 219.00 of Ref. 14.

$$x(t) = A_1 S_n(\omega_1 t + \phi_1, \mu_1) \qquad \text{A1.11}$$

where $\omega_1^2 = c_1 + \frac{1}{2}c_3 A_1^2$ and $\mu_1^2 = \dfrac{-\frac{1}{2}c_3 A_1^2}{c_1 + \frac{1}{2}c_3 A_1^2}$ A1.12

$S_n(\omega_1 t, \mu_1)$ is the elliptic sine function with period 4K. The period in t is given by

$$T_t = 4K/\omega_1 = 4\sqrt{\frac{1+\mu_1^2}{c_1}}\,K(\mu_1) \qquad \text{A1.13}$$

and Fig. A.4 shows the behaviour of μ_1^2 and T_t as A_1^2 is varied.

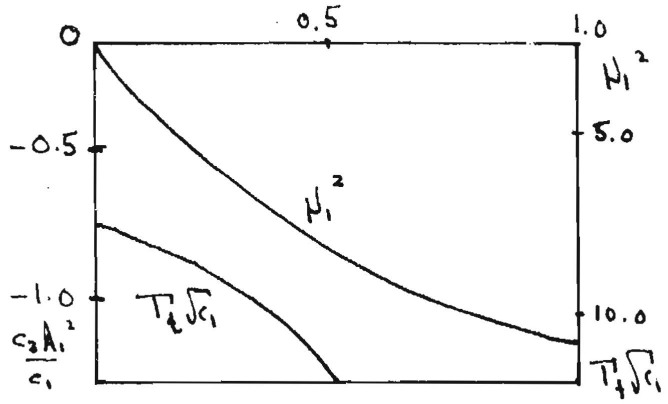


FIG. A.4.

When the initial trajectory lies outside the separatrix $\eta_1 > \sqrt{\frac{-c_1}{c_3}}$ the total spring force at $t = 0$ is negative and the solution will diverge - i.e. no oscillation.

In this case

$$x(t) = A_2 \, ns(\omega_2 t, \mu_2) \qquad \text{A1.14}$$

Where $ns(u) = 1/sn(u)$

$$\omega_2^2 = -\frac{c_3 A_2^2}{2} \; ; \; \mu_2^2 = 1/\mu_1^2$$

3. $c_1 < 0$, $c_3 > 0$. This third case provides the most interesting portrait (Fig. A.5)

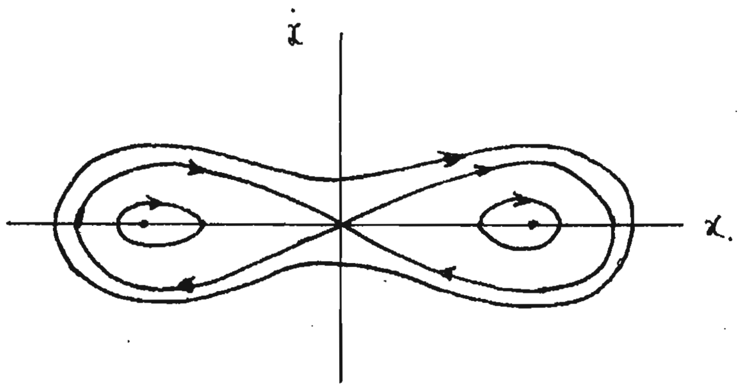


FIG. A.5.

We again have three equilibrium points $x = 0$, $x = \pm\sqrt{-\frac{c_1}{c_3}}$ but this time the origin is the saddle and the other two equilibrium points are centres. This case is particularly interesting in that two types of oscillation are possible, depending on the initial condition. It can be shown that the separatrix cuts the x axis at the points $x = 0$, $x = \pm\sqrt{-\frac{2c_1}{c_3}}$ and if the initial displacement η_1 lies inside these latter two points, (with initial velocity zero) the solution curve will oscillate about the non zero equilibrium point (asymetric oscillation). For $|\eta_1| > \sqrt{-\frac{2c_1}{c_3}}$ the trajectory oscillates about the origin in the manner shown by the outer curve in Fig. 5. Note that any non zero initial velocity (with initial displacement zero) will also cause the trajectory to follow this curve. Let us examine the two cases separately.

a) $|\eta_1| < \sqrt{-\frac{2c_1}{c_3}}$. In order that μ remains in the range 0 to 1 we have to use ex. 217.00 (Ref. 14) if $|\eta_1| < \sqrt{\frac{-c_1}{c_3}}$ and ex. 218.00 (Ref. 14) if $|\eta_1| > \sqrt{\frac{-c_1}{c_3}}$.

For $|\eta_1| < \sqrt{\frac{-2c_1}{c_3}}$ we have,

$$x(t) = A_3 \, nd(w_3 t, \mu_3) \qquad \text{A1.15}$$

where $$nd(u,\mu) = 1/dn(u,\mu)$$

and $$\mu_3^2 = \frac{c_1 + c_3 A_3^2}{c_1 + \frac{1}{2} c_3 A_3^2}, \quad w_3^2 = c_1 + \frac{c_3 A_3^2}{2} \qquad \text{A1.16}$$

For $|\eta_1| > \sqrt{\frac{-c_1}{c_3}}$ we use,

$$x(t) = A_4 \, dn(w_4 t, \mu_4) \qquad \text{A1.17}$$

where $$w_4^2 = \frac{1}{2} c_3 A_4^2 \; ; \; \mu_4^2 = \frac{c_1 + c_3 A_4^2}{\frac{1}{2} c_3 A_4^2}$$

note that,

$$dn(u+2k) = dn(u)$$
$$dn(k) = \sqrt{1-\mu^2}$$
$$dn(k/2) = \left(\sqrt{1-\mu^2}\right)^{1/2}$$
$$dn(u+k) = \sqrt{1-\mu^2}\, nd(u)$$

A1.18

$$dn(u, i\mu) = nd\left(u\sqrt{1+\mu^2}, \mu_1\right), \quad \mu_1 = \mu/\sqrt{1+\mu^2}$$

These relations give the properties of the asymetric oscillation and ensure that the two solutions match when $|\eta_1| = \sqrt{\frac{-c_1}{c_3}}$.

b) $|\eta_1| > \sqrt{\frac{-2c_1}{c_3}}$. For this case the solution is once again constructed with the aid of ex. 213.00 (Ref. 14) i.e.

$$x(t) = A_5 C_n(\omega_5 t, \mu_5) \qquad \text{A1.19}$$

where $\omega_5^2 = c_1 + c_3 A_5^2$, $\mu_5^2 = \frac{\frac{1}{2} c_3 A_5^2}{c_1 + c_3 A_5^2}$

We note that the trajectory does not have the elliptic shape as in case 1. This result is explained by considering the behaviour of the velocity dx/dt ,

$$\frac{dx}{dt} = -A_5 \omega_5 S_n(\omega_5 t, \mu_5) D_n(\omega_5 t, \mu_5) \qquad \text{A1.20}$$

This function has a maximum or minimum when $d^2x/dt^2 = 0$.

$$\text{i.e.} \quad -A_5 \omega_5^2 C_n \left[D_n^2 - \mu_5^2 S_n^2\right] = 0 \qquad \text{A1.21}$$

eqn A1.21 has the solution $C_n(u) = 0$ (i.e. $x(t) = 0$, $t = 0, 2k, 4k$ etc)

and also $\frac{D_n^2}{S_n^2} = \mu_5^2$ or $C_n^2 = \frac{2\mu_5^2 - 1}{2\mu_5^2} = \frac{-c_1}{c_3 A_5^2}$

In the present case $C_1 < 0$ and hence in any one period $4K$ the velocity has an extra four stationary values, occuring when $x(t)$ attains its equilibrium value, $x(t) = \pm\sqrt{\frac{-C_1}{C_3}}$.
The velocity response will look something like that shown in Fig. A.6.

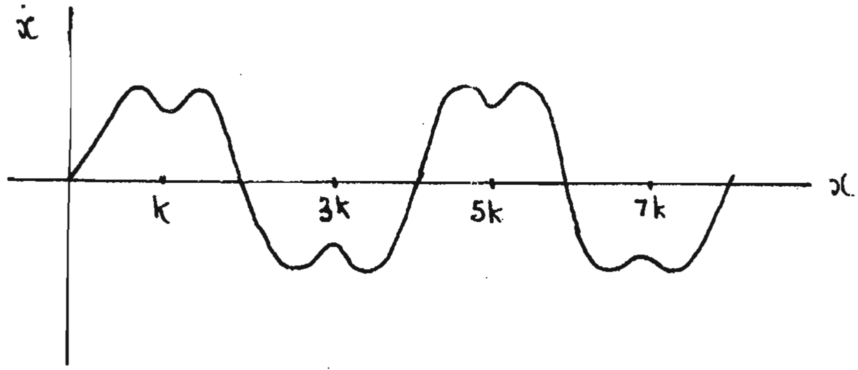


FIG. A.6

This case completes the range of C_1 and C_3 giving the solution an oscillatory character.

Appendix 2

Condition for the existence of periodic solutions of periodic linear differential equations

In section 2, whilst discussing the behaviour of the first order approximation generated by a direct perturbation expansion, we introduced an important property of linear periodic differential equations. For periodic solutions to exist the right hand side of eqn. 2.9b was restricted to obey the condition given in eqn. 2.15. Using the two time variable method a uniformly valid solution was obtained by utilizing the above condition, in that the first order approximation was forced to have periodic solutions i.e. the right hand side of eqn. 2.29 was required to obey the so called periodicity condition (eqn. 2.34). We outline here the proof of this property as given by Hahn (Ref. 16 p. 359 - 264).

The equations of the first order approximations considered in this note belong to the general class of equations written in the form

$$\frac{d\underset{\sim}{u}}{dt} - A(t)\underset{\sim}{u} = \underset{\sim}{\chi}(t) \qquad \text{A2.1}$$

where $A(t)$ is an $(n \times n)$ matrix with elements periodic such that $A(t) = A(t+T_w)$, $\underset{\sim}{z}(t)$ is a periodic n -vector, $\underset{\sim}{z}(t) = \underset{\sim}{z}(t+T_w)$ and $\underset{\sim}{U}(t)$ is the unknown state n -vector. We wish to determine the conditions for the existence of periodic solutions of eqn. A2.1 such that

$$\underset{\sim}{U}_p(t) = \underset{\sim}{U}_p(t+T_w) \qquad \text{A2.2}$$

The solution of eqn. A2.1 can be written in the general form

$$\underset{\sim}{U}(t) = K(t,t_0)\underset{\sim}{U}_0 + \int_0^t K(t,\tau)\underset{\sim}{z}(\tau)\,d\tau \qquad \text{A2.3}$$

where the matrix $K(t,t_0)$, satisfies the equation,

$$\frac{dK(t,t_0)}{dt} = A(t)\,K(t,t_0) \qquad \text{A2.4}$$

and has the following properties (Ref. 16).

$$\begin{aligned} K(t,t_0) &= K(t,t_1)\,K(t_1,t_0) \\ K(t,t_0) &= K(t_0,t)^{I} \\ K(t_0,t_0) &= E \end{aligned} \qquad \text{A2.5}$$

(subscript I denotes matrix inverse)

$\underset{\sim}{w}_0$ is the initial vector of the trajectory $\underset{\sim}{w}(t)$.

Without loss of generality we can choose $t = 0$ and write eqn. A2.3 at $t = T_w$ as,

$$\underset{\sim}{w}(T_w) = K(T_w, 0)\,\underset{\sim}{w}_0 + \int_0^{T_w} K(T_w, \tau)\,\underset{\sim}{z}(\tau)\,d\tau \qquad \text{A2.6}$$

From the properties, eqn. A2.5, we can write,

$$K(T_w, \tau) = K(T_w, 0)\,K(0, \tau) \quad \text{and} \quad K(T_w, 0) = K(0, T_w)^I$$

Hence eqn. A2.6 can be written,

$$\left[K(0, T_w) - E\right]\underset{\sim}{w}_0 = \int_0^{T_w} K(0, \tau)\,\underset{\sim}{z}(\tau)\,d\tau \qquad \text{A2.7}$$

For the case when the homogeneous equation,

$$\frac{d\underset{\sim}{w}}{dt} - A\underset{\sim}{w} = \underset{\sim}{0}$$

has no periodic solution we have, $\det\left[K(0, T_w) - E\right] \neq 0$ i.e. the matrix $K(0, T_w)$ has no eigenvalue equal to unity. In this case the initial vector takes the form

$$\underset{\sim}{w}_0 = \left[K(0, T_w) - E\right]^I \int_0^{T_w} K(0, \tau)\,z(\tau)\,d\tau \qquad \text{A2.8}$$

and hence the complete periodic solution of eqn. A2.1 can be written in the form

$$\underset{\sim}{w}p(t) = k(t,0)\left[k(0,T_w)-E\right]^{I}\int_0^{T_w} k(0,\tau)\,\underset{\sim}{z}(\tau)\,d\tau + \int_0^{t} k(t,\tau)\,\underset{\sim}{z}(\tau)\,d\tau \qquad \text{A2.9}$$

When the homogeneous form of eqn. A1.1 has periodic solutions the situation is not so straightforward, for in this case,

$$\det\left[k(0,T_w)-E\right] = 0 \qquad \text{A2.10}$$

Equation A2.7 can be regarded as a linear equation of the form,

$$B\underset{\sim}{q} = \underset{\sim}{p} \qquad \text{A2.11}$$

and eqn. A2.10 becomes $\det B = 0$. The set of equations A2.11 are only compatable (solvable) when the vector $\underset{\sim}{p}$ lies in the range space of the matrix B and this implies that for every vector which satisfies the adjoint equation,

$$B^{T}\underset{\sim}{t} = 0 \quad \text{, we must have,} \quad \underset{\sim}{t}^{T}\underset{\sim}{p} = 0 \qquad \text{A2.12}$$

i.e. the vector $\underline{p}$ must be orthogonal to the null space of the matrix B^T. (See Lanczos, Ref. 17 Sections 3.6 - 3.9).

Returning to eqn. A2.7, the condition given by eqn. A2.12 can be written

$$\tilde{V}_0^T \int_0^{T_w} k(0,\tau)\, \tilde{z}(\tau)\, d\tau = 0 \qquad \text{A2.13}$$

for every $\tilde{V}_0$ satisfying the equation,

$$[k(0,T_w) - E]^T \tilde{V}_0 = \tilde{0} \qquad \text{A2.14}$$

or

$$k(0,T_w)^T \tilde{V}_0 = \tilde{V}_0 \qquad \text{A2.15}$$

Eqn. A2.15 implies that the vector

$$\tilde{V}(t) = k(0,t)^T \tilde{V}_0$$

is a periodic solution of the adjoint equation,

$$\frac{d\tilde{V}}{dt} - A(t)^T \tilde{V} = \tilde{0} \qquad \text{A2.16}$$

and hence eqn. A2.13 can be written as,

$$\int_0^{T_\omega} \underset{\sim}{V}^T(\tau)\, \underset{\sim}{z}(\tau)\, d\tau = 0 \qquad \text{A2.17}$$

for all periodic $\underset{\sim}{V}(t)$ satisfying eqn. A2.16. This is exactly the result we require and if eqn. A2.17 is satisfied then eqn. A2.1 has a periodic solution.

As shown by Hahn (Ref. 16 - p 361) one can go further to show that if eqn. A2.17 is not satisfied for every periodic $\underset{\sim}{V}(t)$ then the solution of eqn. A2.1 will grow unbounded as t increases (Resonance). If we write,

$$\underset{\sim}{h} = \int_0^{T_\omega} k(T_\omega, \tau)\, z(\tau)\, d\tau$$

then,

$$\underset{\sim}{U}(kT_\omega) = K(T_\omega, 0)^k \underset{\sim}{U}_0 + \sum_{j=0}^{k-1} K(T_\omega, 0)^j \underset{\sim}{h} \qquad \text{A2.18}$$

and the sum on the right grows without limit as k increases. Finally, if the solvability conditions are satisfied for all periodic $\underset{\sim}{V}(t)$, $\underset{\sim}{V}^{(\ell)}(t)$, $\ell = 1, 2, \cdots m$, then equation A2.1 will have a periodic solution of the form,

$$\underset{\sim}{U}_p(t) = \underset{\sim}{\phi}(t) + \sum_{\ell=1}^{m} C_\ell\, \underset{\sim}{U}^{(\ell)}(t) \qquad \text{A2.19}$$

where $\underset{\sim}{\phi}(t)$ is a particular periodic solution and $\underset{\sim}{U}^{(\ell)}(t)$ are the linearly independent periodic solutions of the homogeneous form of eqn. A2.1.

GUIDE TO FIGURES

Figs. 1 - 9 show comparisons of approximate and exact numerical solutions of the equation

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = 0$$

for various combinations of the parameters c_1, c_3 and b. The full line in all cases represents the numerical solution and the circles represent the approximation.

Fig. 10 shows how the logarithmic decrement varies with amplitude for a non-linear system. Some estimates of the damping parameter b are also shown.

Fig. 11 shows a comparison of approximate and exact numerical solutions of the equation.

$$\ddot{x} + b(\alpha x^2 - 1)\dot{x} + c_1 x + c_3 x^3 = 0$$

Again, the full line represents the numerical solution.

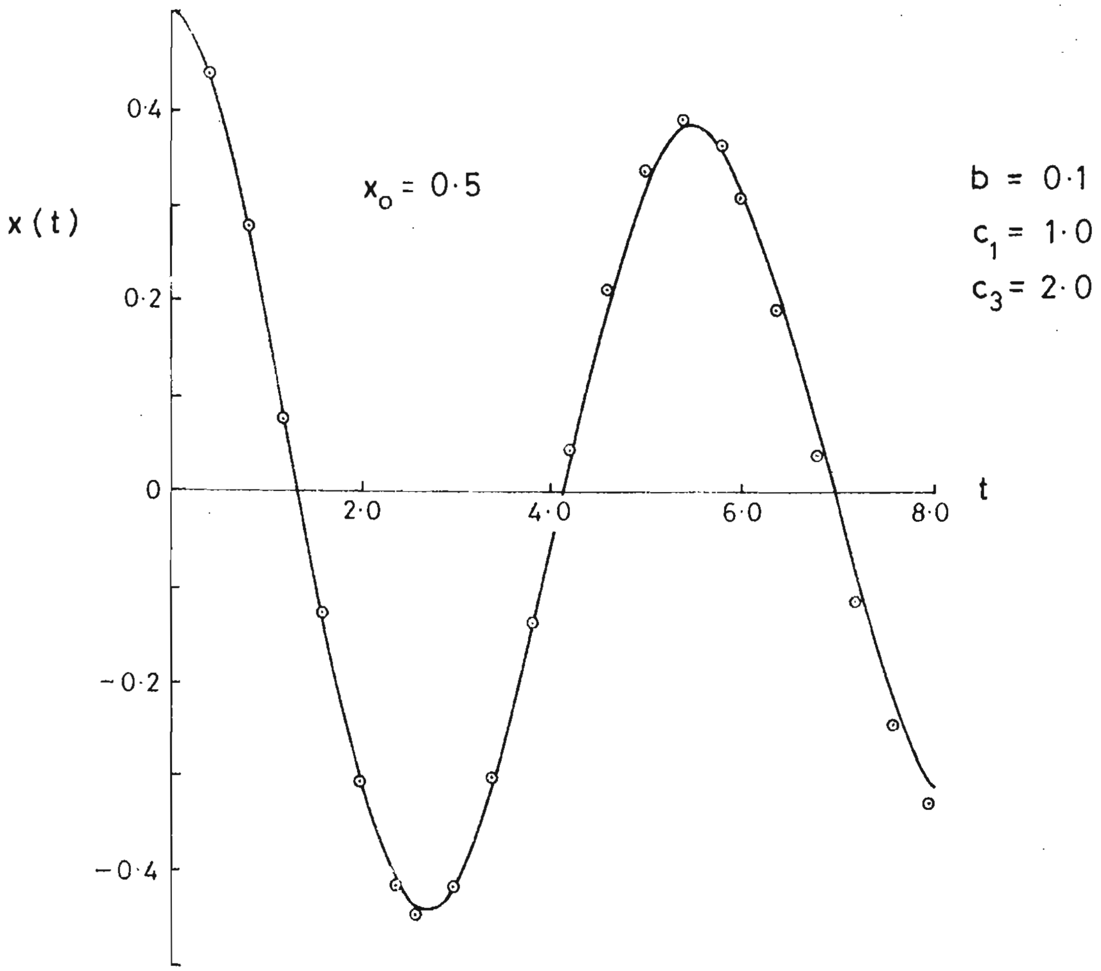

0·4
0·2
0
−0·2
−0·4
x (t)
$x_o = 0·5$
2·0
4·0
6·0
8·0
t
b = 0·1
c_1 = 1·0
c_3 = 2·0

Fig. 1a.

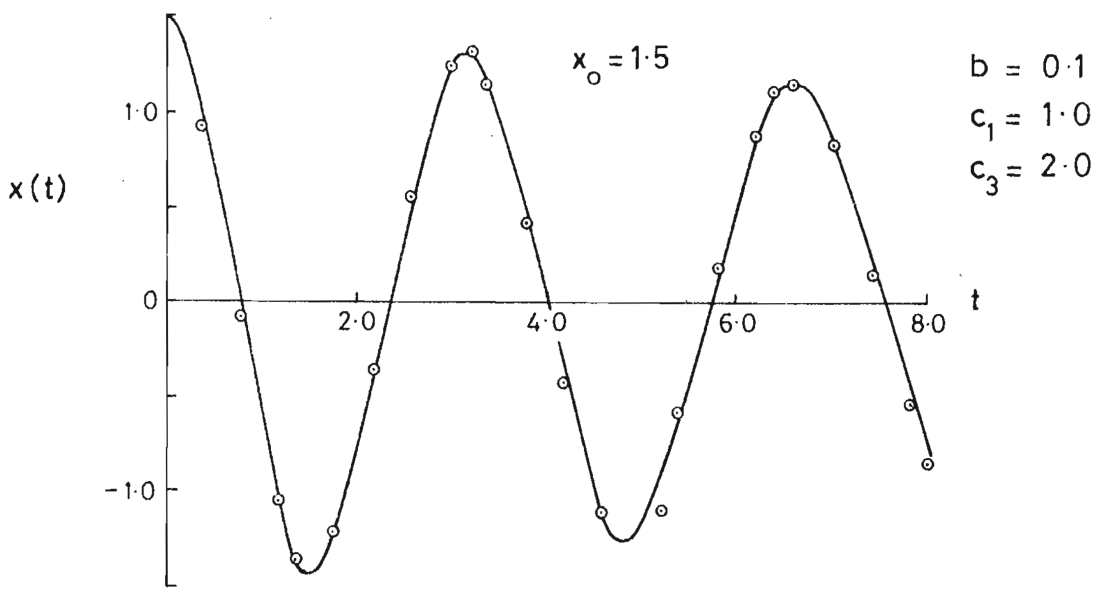
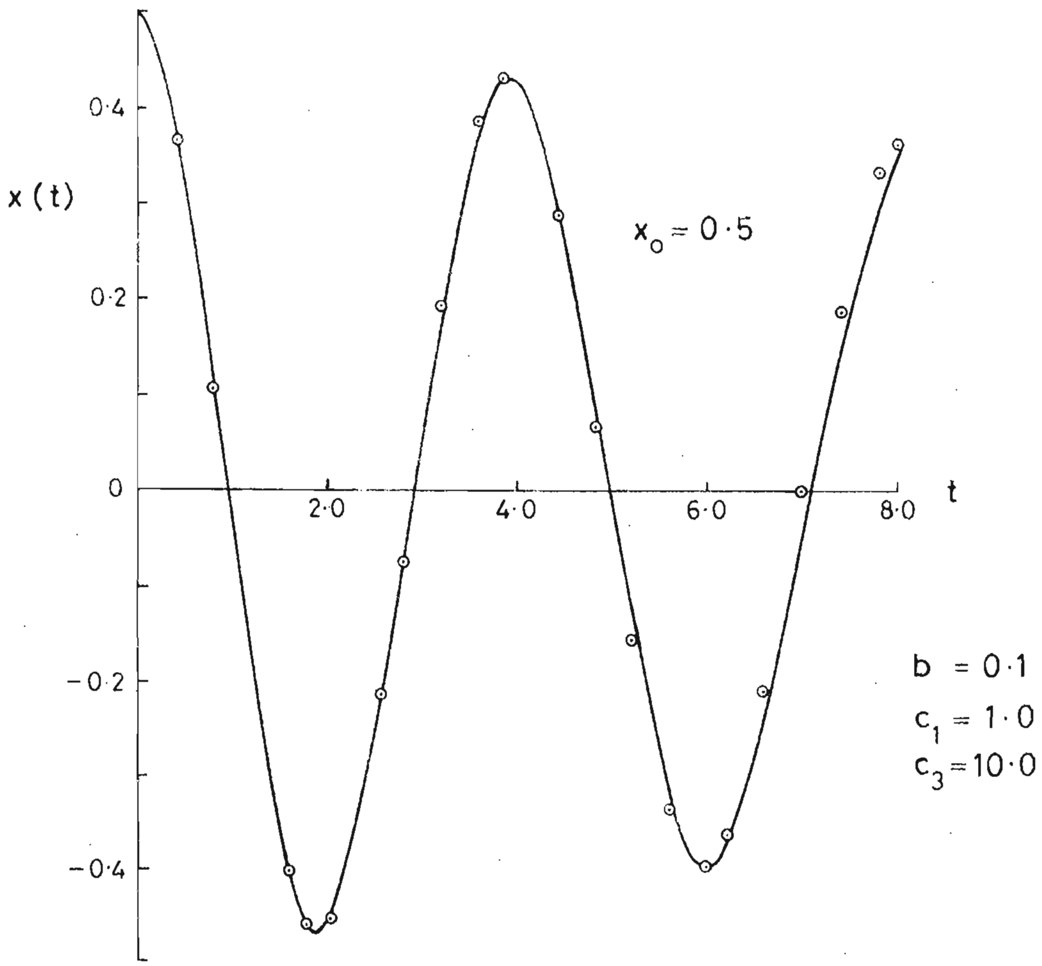
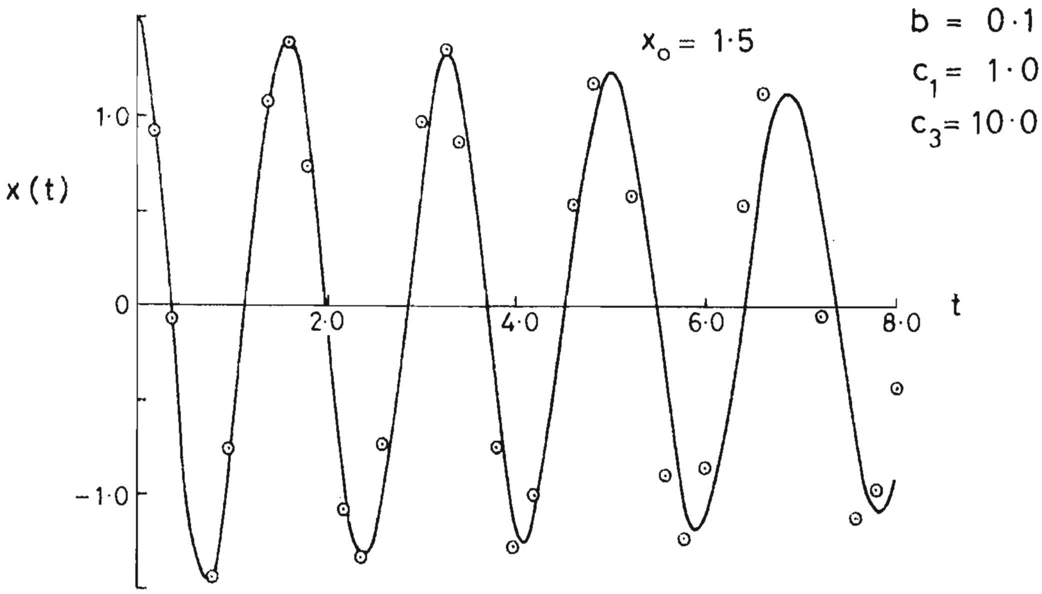

1·0
0
-1·0
x(t)
$x_o = 1·5$
2·0
4·0
6·0
8·0
t
b = 0·1
c_1 = 1·0
c_3 = 2·0

Fig.1b.



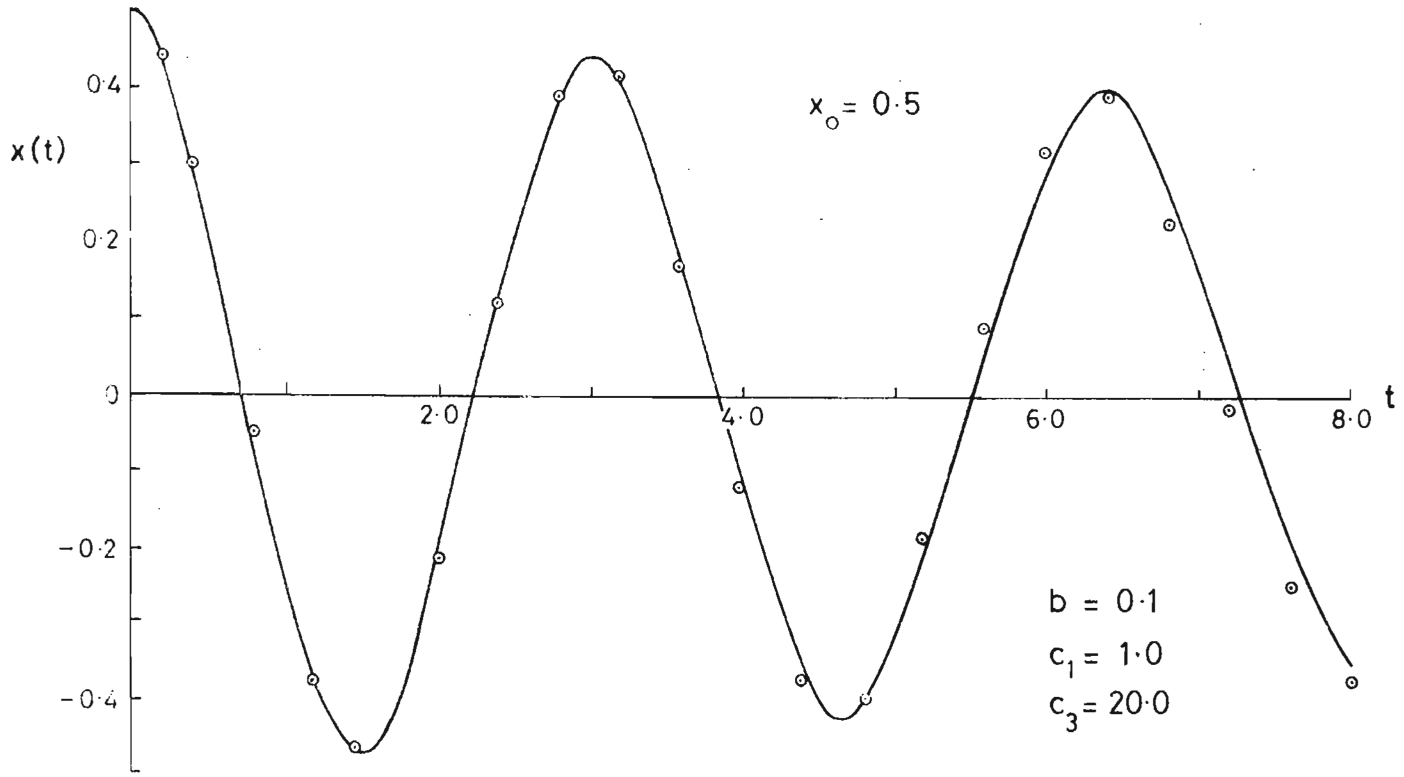
0·4
0·2
0
−0·2
−0·4
x (t)
$x_o = 0·5$
2·0
4·0
6·0
8·0
t
b = 0·1
c_1 = 1·0
c_3 = 10·0

Fig. 2a.



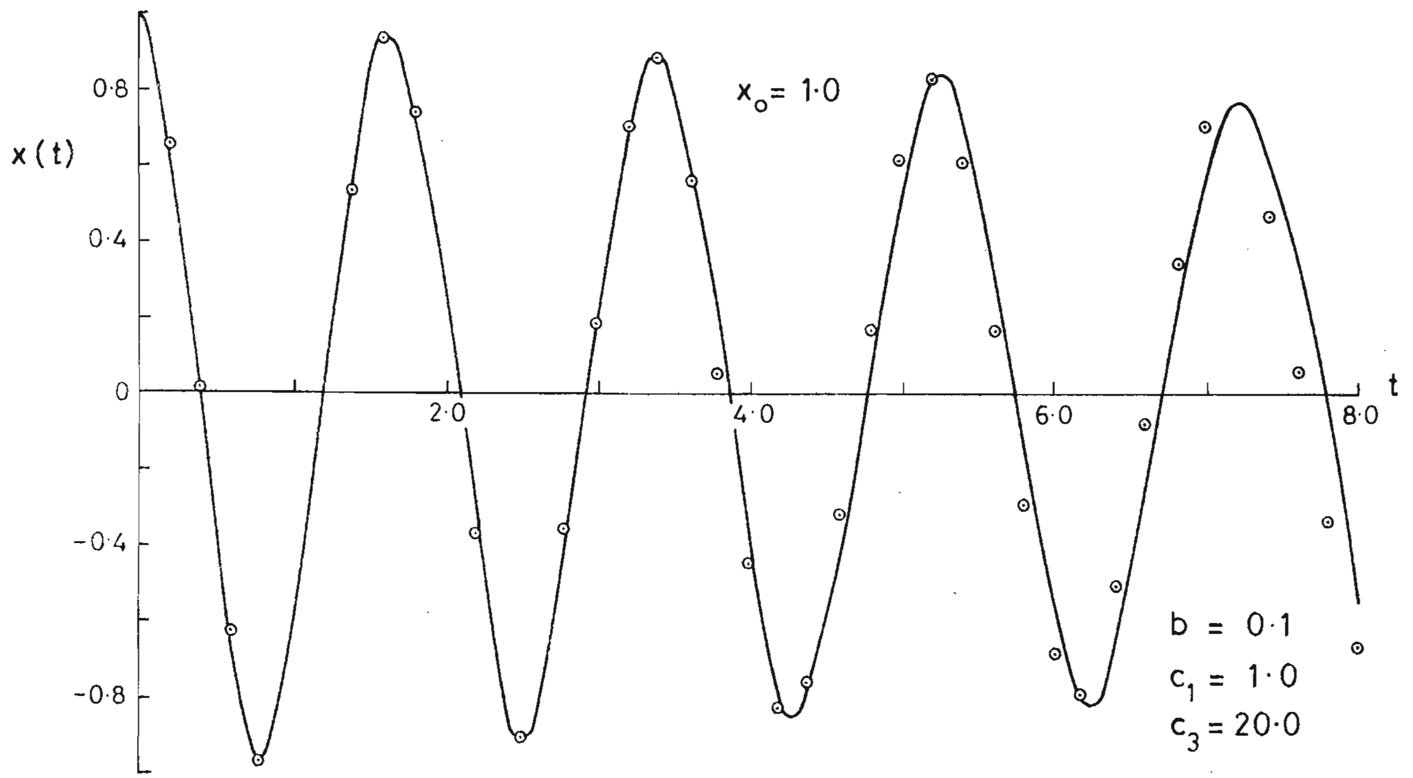
1·0
0
−1·0
x (t)
x_o = 1·5
2·0
4·0
6·0
8·0
t
b = 0·1
c_1 = 1·0
c_3 = 10·0

Fig. 2b.



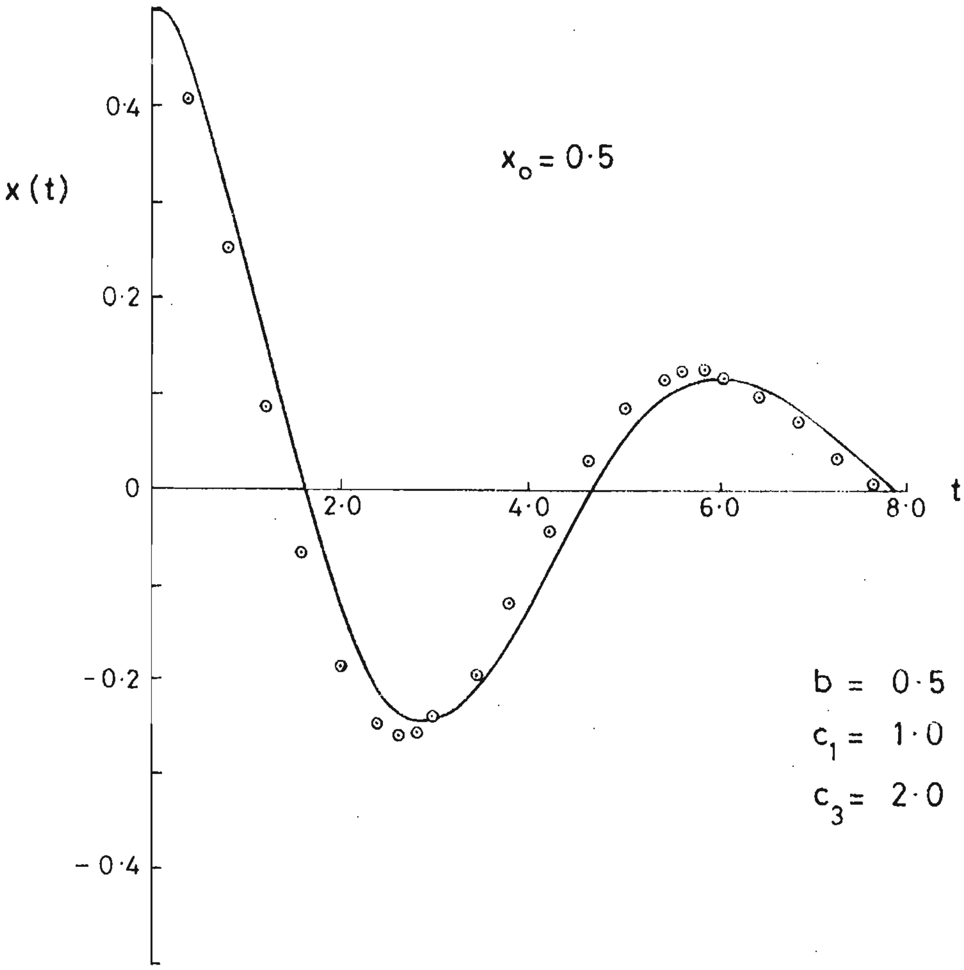
x(t)
0·4
0·2
0
-0·2
-0·4
2·0
4·0
6·0
8·0
t
$x_o = 0·5$
b = 0·1
$c_1 = 1·0$
$c_3 = 20·0$

Fig. 3a.



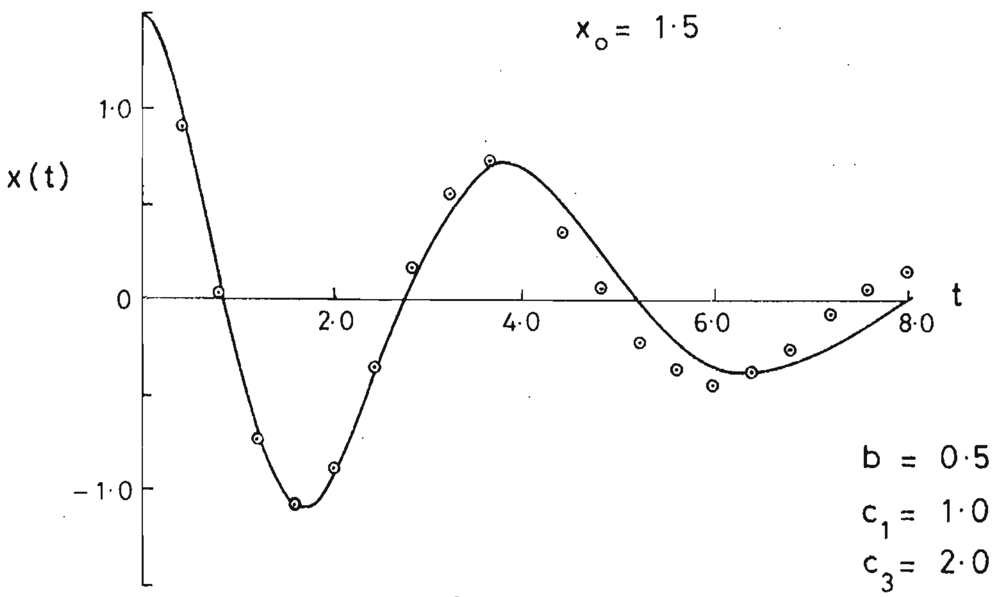
x(t)
0·8
0·4
0
-0·4
-0·8
2·0
4·0
6·0
8·0
t
$x_o = 1·0$
b = 0·1
$c_1 = 1·0$
$c_3 = 20·0$

Fig. 3b.



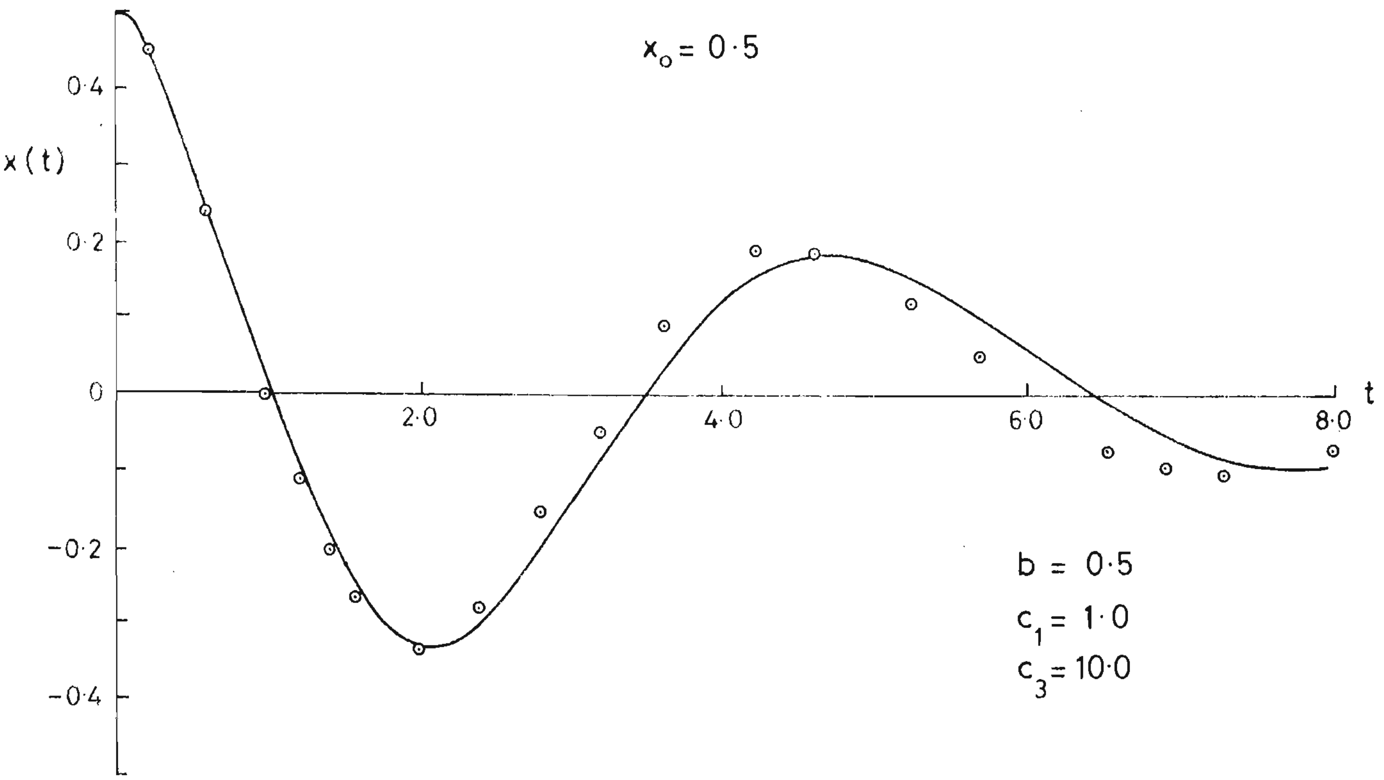
0·4
0·2
0
-0·2
-0·4
x(t)
$x_o = 0·5$
2·0
4·0
6·0
8·0
t
b = 0·5
c_1 = 1·0
c_3 = 2·0

Fig. 4a.



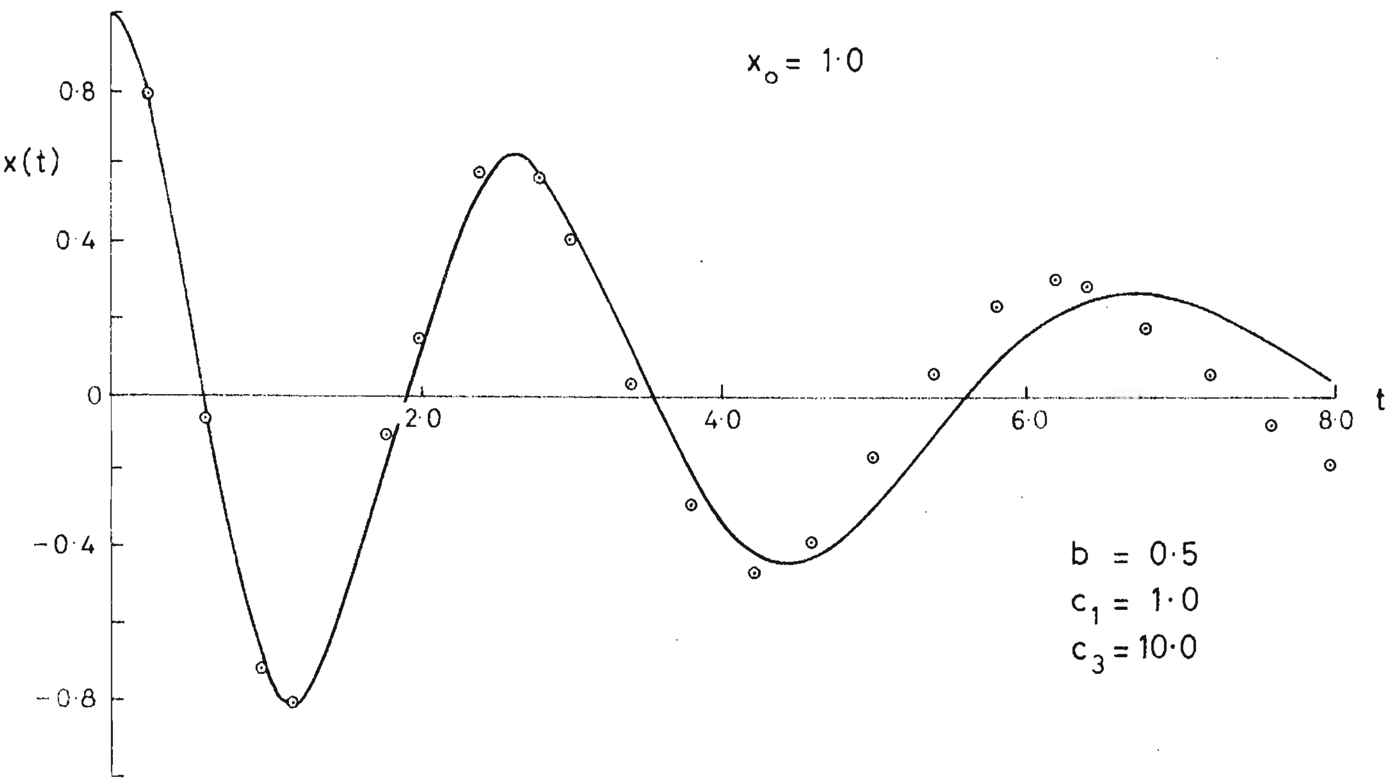
x_o = 1·5
1·0
0
-1·0
x(t)
2·0
4·0
6·0
8·0
t
b = 0·5
c_1 = 1·0
c_3 = 2·0

Fig. 4b.



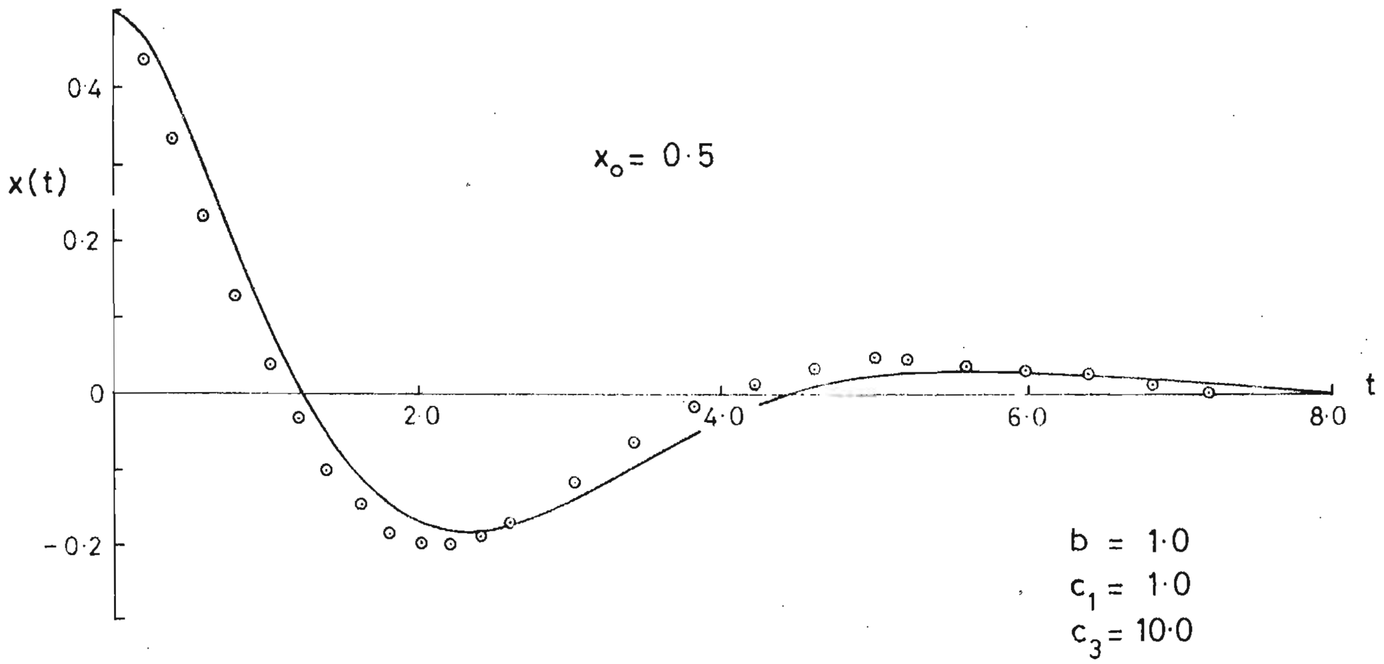
$x_o = 0\cdot5$
0·4
x(t)
0·2
0
−0·2
−0·4
2·0
4·0
6·0
8·0
t
$b = 0\cdot5$
$c_1 = 1\cdot0$
$c_3 = 10\cdot0$

Fig. 5a.



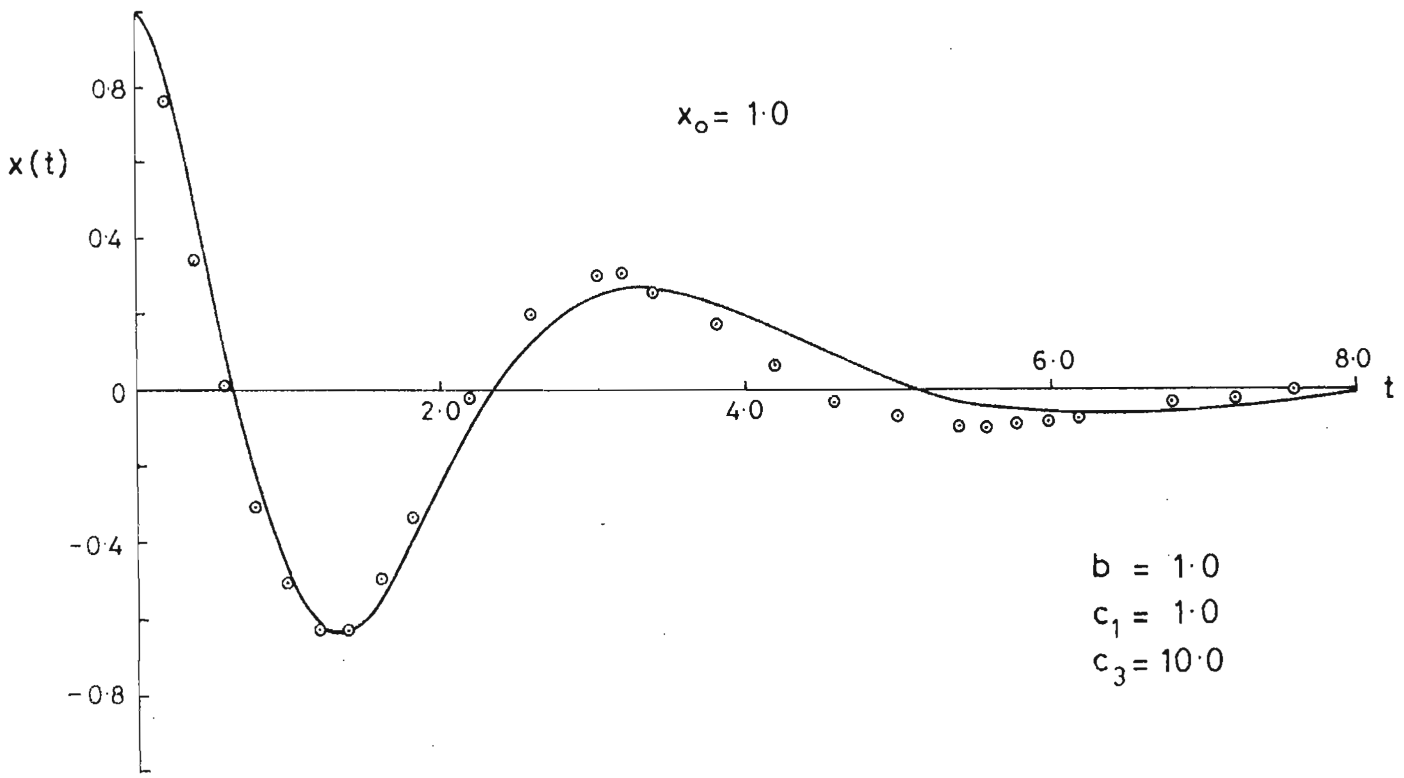
$x_o = 1\cdot0$
0·8
x(t)
0·4
0
−0·4
−0·8
2·0
4·0
6·0
8·0
t
$b = 0\cdot5$
$c_1 = 1\cdot0$
$c_3 = 10\cdot0$

Fig. 5b.



0·4
0·2
0
-0·2
x(t)
x_o = 0·5
2·0
4·0
6·0
8·0
t
b = 1·0
c_1 = 1·0
c_3 = 10·0

Fig. 6a.



0·8
0·4
0
-0·4
-0·8
x(t)
x_o = 1·0
2·0
4·0
6·0
8·0
t
b = 1·0
c_1 = 1·0
c_3 = 10·0

Fig. 6b.

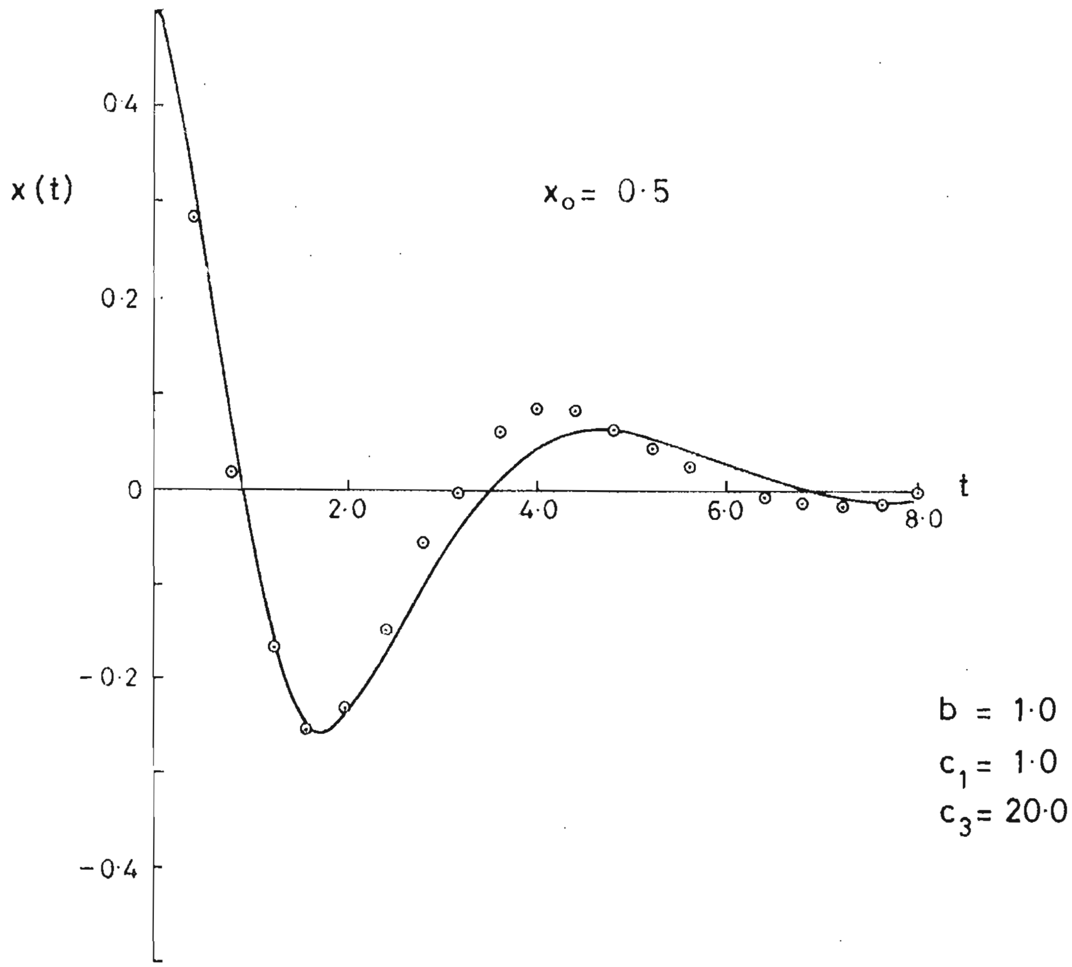

0·4
0·2
0
-0·2
-0·4
x (t)
x_o = 0·5
t
2·0
4·0
6·0
8·0
b = 1·0
c_1 = 1·0
c_3 = 20·0

Fig. 7a.

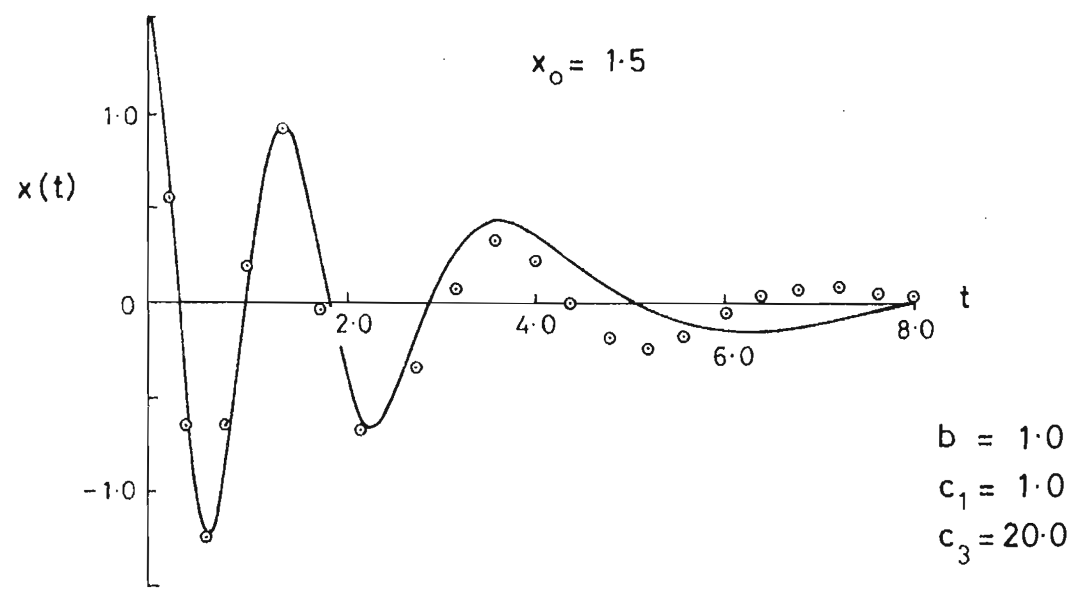
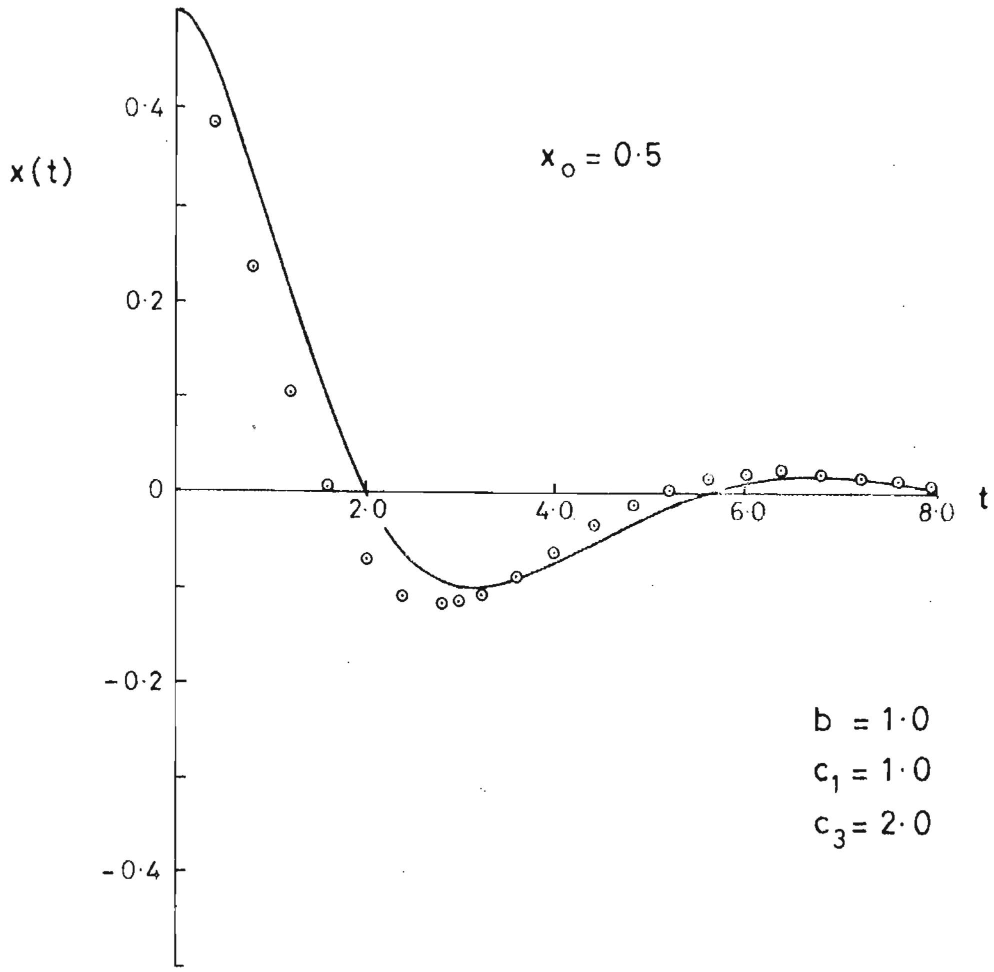
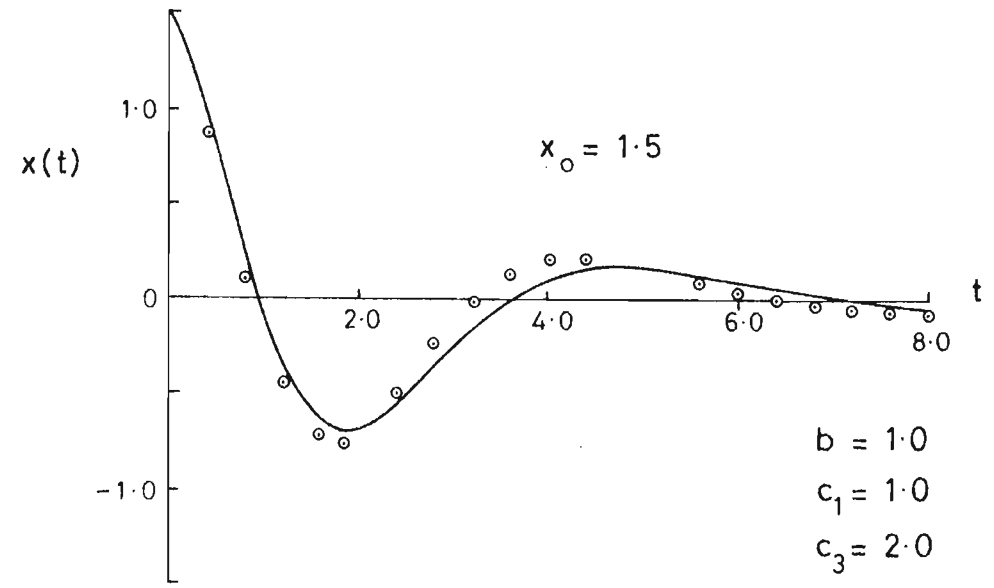

1·0
0
-1·0
x (t)
x_o = 1·5
t
2·0
4·0
6·0
8·0
b = 1·0
c_1 = 1·0
c_3 = 20·0

Fig. 7b.



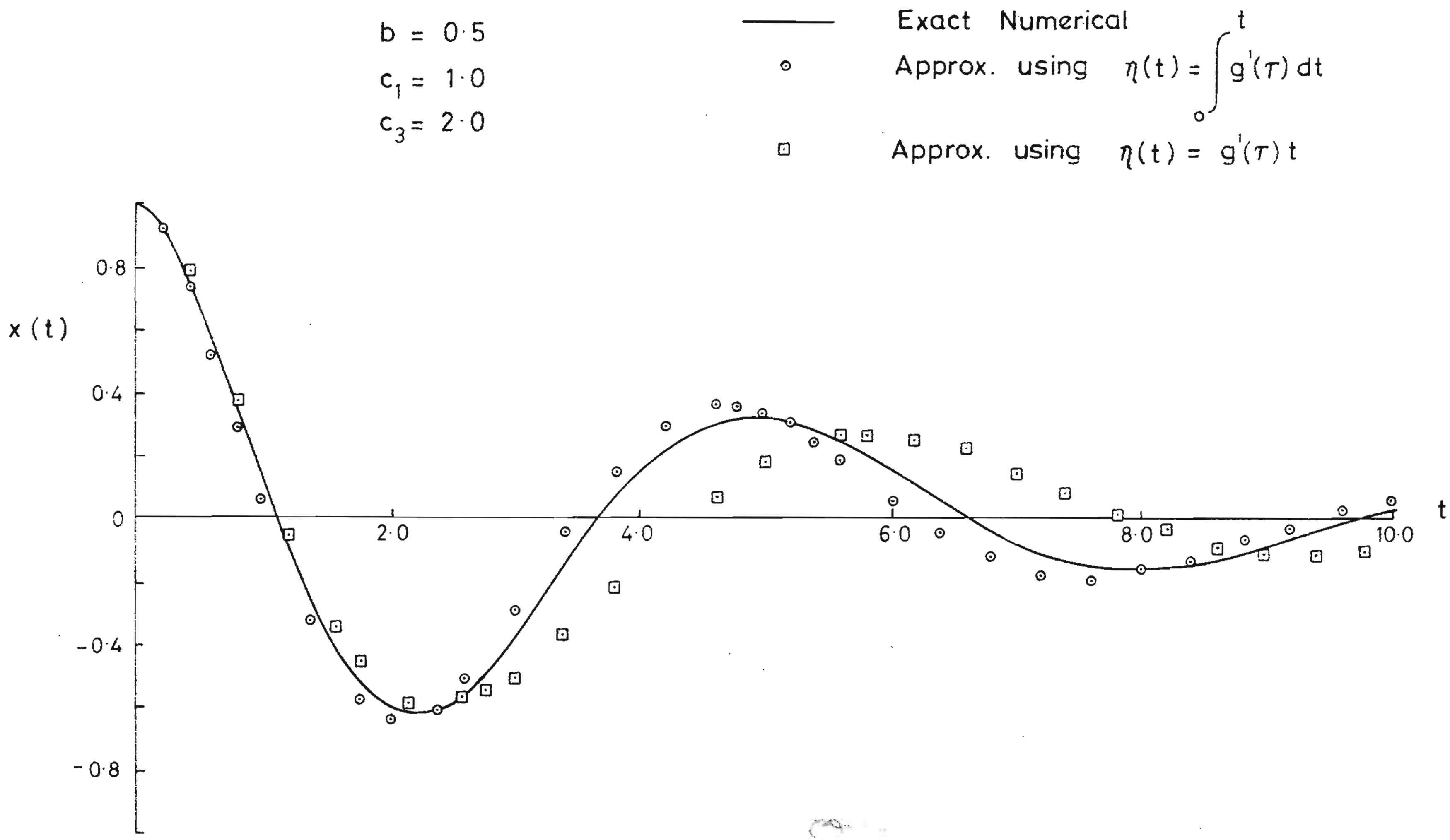
x(t)
0·4
0·2
0
-0·2
-0·4
$x_o = 0·5$
2·0
4·0
6·0
8·0
t
$b = 1·0$
$c_1 = 1·0$
$c_3 = 2·0$

Fig. 8a.



x(t)
1·0
0
-1·0
$x_o = 1·5$
2·0
4·0
6·0
8·0
t
$b = 1·0$
$c_1 = 1·0$
$c_3 = 2·0$

Fig. 8b.



b = 0·5
c1 = 1·0
c3 = 2·0
Exact Numerical
Approx. using η(t) = ∫ g'(τ) dt
Approx. using η(t) = g'(τ) t
x (t)
0·8
0·4
0
-0·4
-0·8
2·0
4·0
6·0
8·0
10·0
t

Fig. 9a.

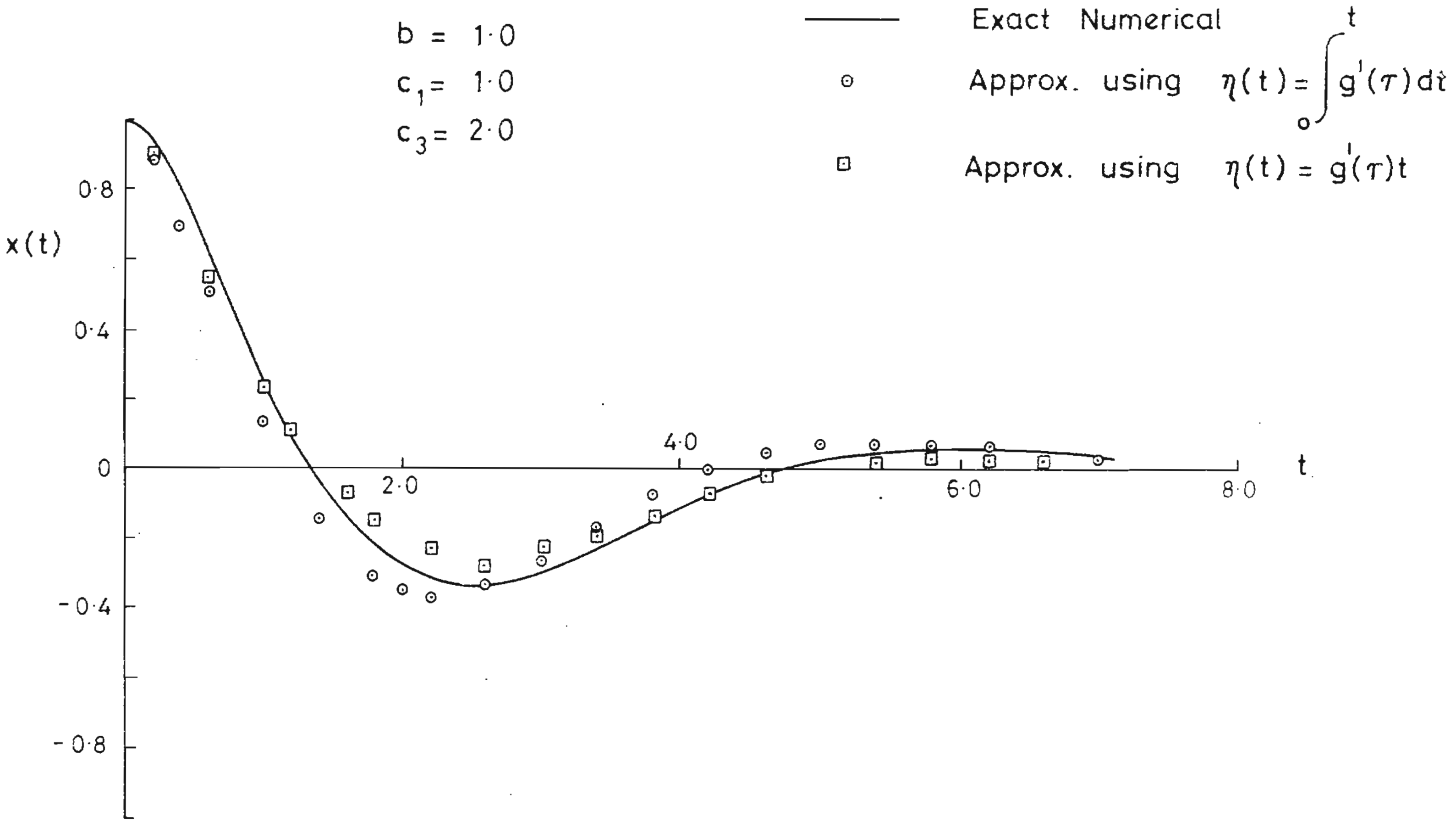

b = 1·0
c₁= 1·0
c₃= 2·0
Exact Numerical
Approx. using $\eta(t) = \int_0^t g'(\tau)d\tau$
Approx. using $\eta(t) = g'(\tau)t$
x(t)
0·8
0·4
0
-0·4
-0·8
2·0
4·0
6·0
8·0
t

Fig. 9b.

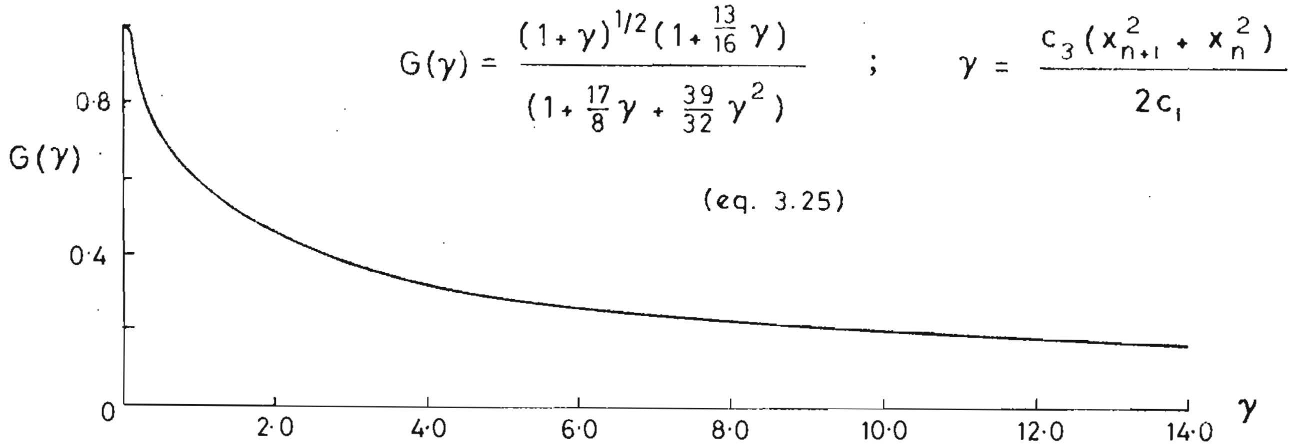


FIG. 10 a.

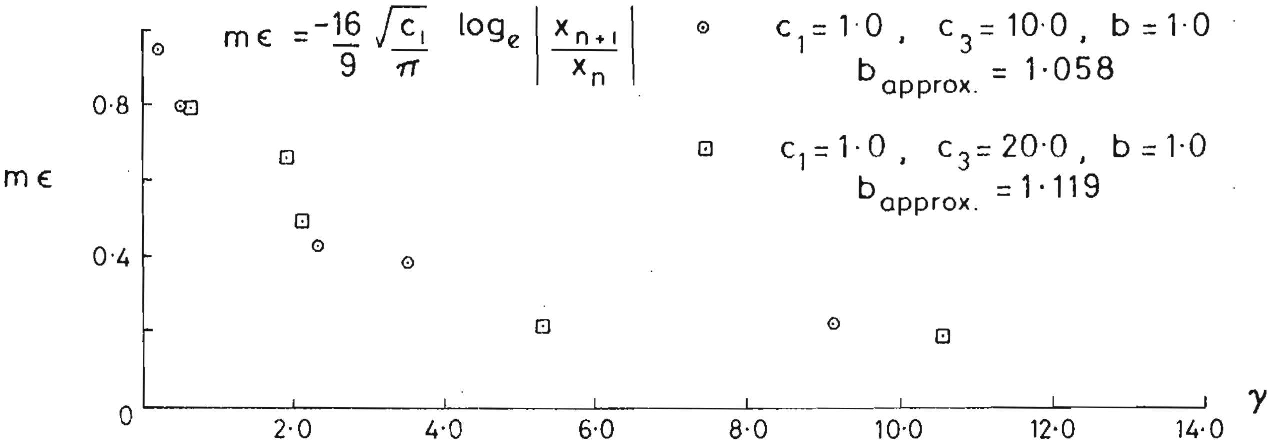


FIG. 10b.

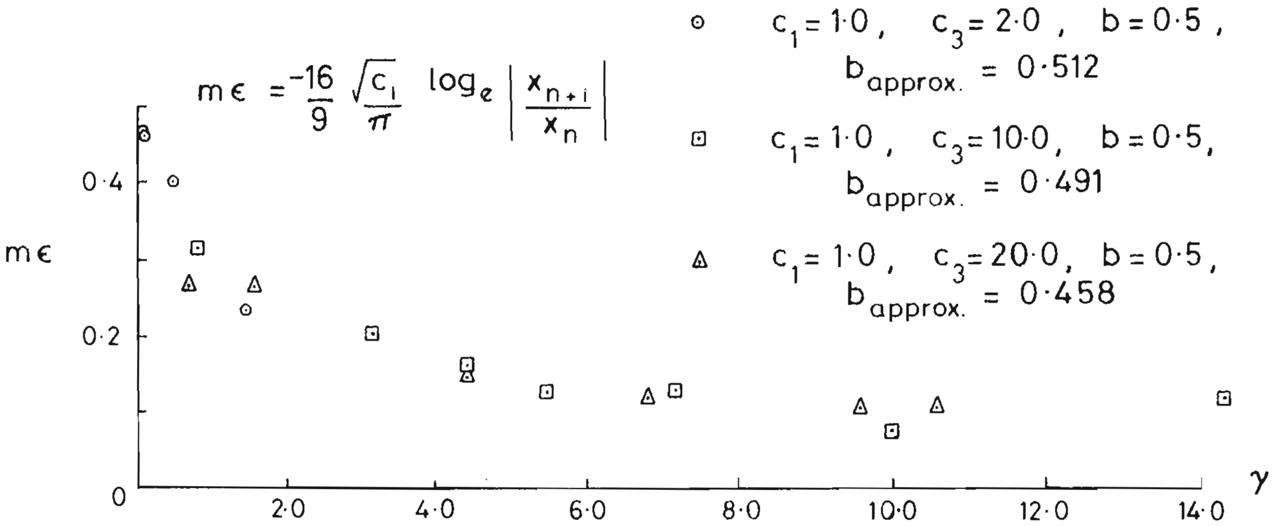


FIG. 10c.

VARIATION OF LOGARITHMIC DECREMENT WITH γ FOR NON-LINEAR OSCILLATIONS

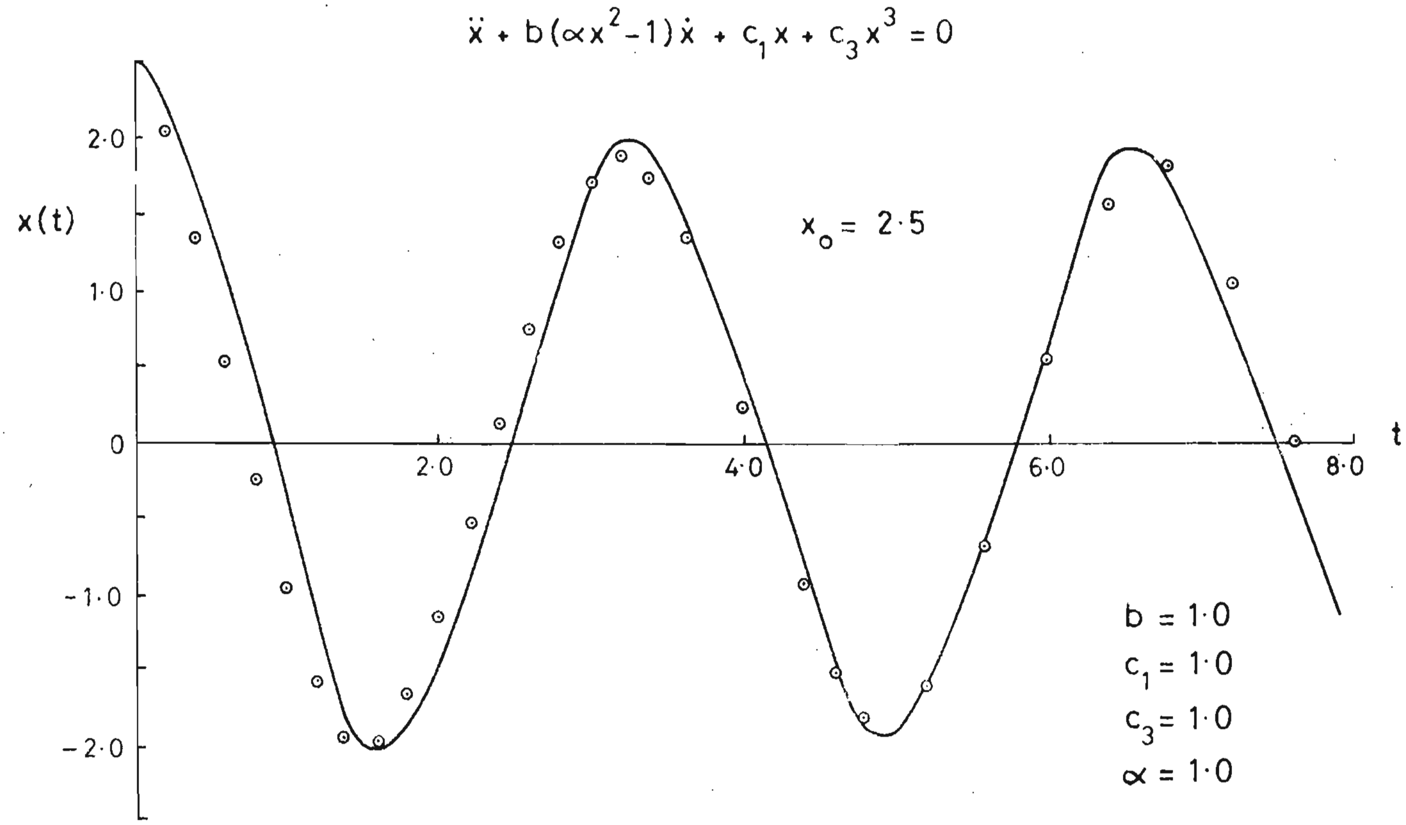


Fig. 11b.

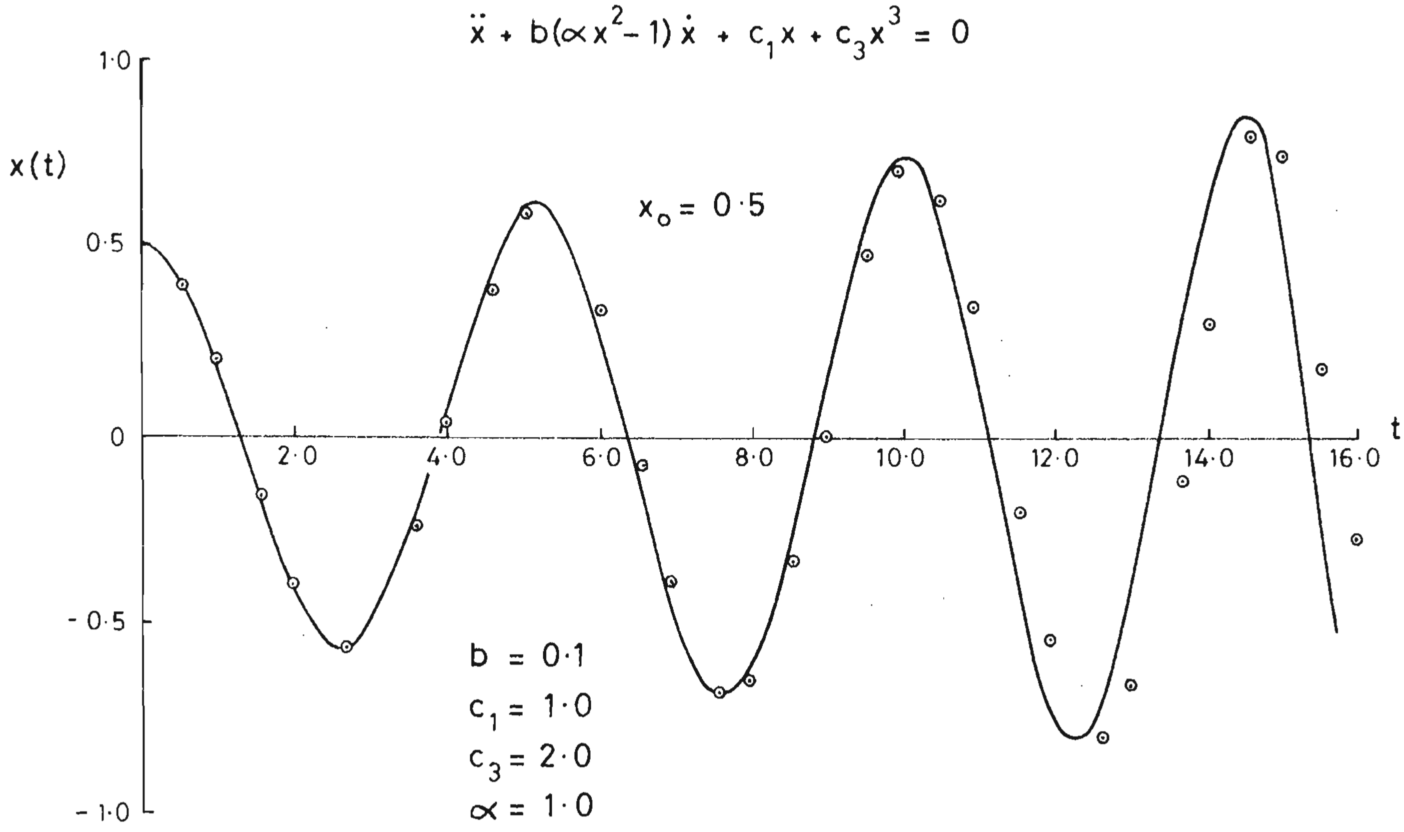


Fig. 11c.

Aero Note 3/74

The Cranfield Institute of Technology

College of Aeronautics

Aerodynamics Division

ON THE USE OF APPROXIMATE SOLUTIONS FOR NON LINEAR OSCILLATIONS

G.D. Padfield

SUMMARY

An approximate solution in terms of Jacobian Elliptic functions is developed for a class of non linear second order differential equations exhibiting oscillatory solutions with weak decay rate (systems with small damping). Using a perturbation method it is found that a straightforward asymptotic expansion leads to secular behaviour in the first order approximation. To correct for this non uniformity the two time scale method is utilized and this yields a zeroth order approximation containing the essential characteristics of the damped non linear oscillation. The value of an approximate technique is an implicit function of its usefulness and with this idea in mind a method is described whereby the parameters of an oscillatory system can be estimated from response measurements with the aid of the approximation. It is hoped that the analysis in this note will serve as a stepping stone to the study of higher order non linear systems.

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August, 1974