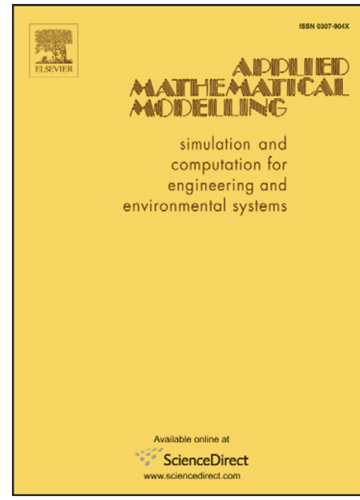

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Reliability analysis for a $k/n(F)$ system with repairable repair-equipment

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Abstract

In this paper, the reliability and replacement policy of a $k/n(F)$ (i.e. $k-out-of-n:F$) system with repairable repair-equipment is analyzed. We assume that both the working and repair times of all components in the system and the repair-equipment follow exponential distributions, and the repairs on the components are perfect whereas that on the repair-equipment is imperfect. Under these assumptions, by using the geometric process, the vector Markov process and the queueing theory, we derive reliability indices for such a system and discuss its properties. We also optimize a replacement policy N under which the repair-equipment is replaced whenever its failure number reaches N. The explicit expression for the expected cost rate (i.e. the expected long-run cost per unit time) of the repair-equipment is derived, and the corresponding optimal replacement policy N^* can be obtained analytically or numerically. Finally, a numerical example for policy N is given.

Key words: Geometric process, supplementary variables, vector Markov process, $M/M/1$ queueing system, repairable repair-equipment.

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1 Introduction

A $k/n(F)$ system consists of n components: it fails if and only if at least k components have failed. The dual of a $k/n(F)$ is $k/n(G)$(i.e. k-out-of-n : G), which consists of n components but it works if and only if at least k components work. Obviously, a $k/n(F)$ system is equivalent to a $(n-k+1)/n(G)$ system. A $1/n(F)$ (or $n/n(F)$) system is a n-component series (or parallel) system. If k components are consecutive in a $k/n(F$ or $G)$ system, the system becomes a $C(k, n : F$ or $G)$ (i.e. consecutive-k-out-of-n: F or G) system. Therefore, a $k/n(F$ or $G)$ system can be seen as an extension of various reliability systems, and it plays an important role in the reliability theory and real applications. This important feature attracts considerable research. For example, Barlow and Proschan [1], Linton and Saw [2], Phillips [3], Gupta and Sharma [4], Kenyon and Newell [5], Nakagawa [6], McGrady [7] and Moustafa [8], etc. They studied such a system with different approaches: they commonly assume that either the system is not repairable or the repair-equipment does not fail. However, in practice, a piece of repair-equipment may experience failures.

The purpose of this paper is to analyze the reliability of a $k/n(F)$ repairable system with repairable repair-equipment. We assume that a failed component in the system can be repaired as good as new, whereas the survival times of the repair-equipment after repair form a geometric process. The geometric process has been applied to optimize maintenance policies in various repairable systems, including simple systems, two-component systems and multi-component systems since it was first introduced by Lam [9, 10]. For more references, the reader is referred to Lam [11], Zhang [12, 13, 14, 15], Lam *et al* [16], Zhang *et al* [17, 18, 19], Lam and Zhang [20, 21, 22], Wu and Clements-Croome [23,24], Wang and Zhang [25, 26], Zhang and Wang [27, 28, 29], and Lam *et al* [30].

Using the queueing theory, the vector Markov process and the geometric process, we not only derive reliability indices of such a $k/n(F)$ repairable system, but also optimize replacement policy N. The replacement policy aims to search an optimum failure number N^* such that the expected cost rate of the repair-equipment is minimized. Finally, a numerical example for policy N is given to illustrate some theoretical results.

This paper is structured as follows. Section 2 introduces the definition of the geometric process and assumptions for the reliability analysis in the paper. Section 3 conducts system analysis using the vector Markov process. Section 4 discusses properties and reliability indices of the $k/n(F)$ system. Section 5 derives replacement policy N^* and provides a numerical example. Section 6 concludes this paper.

2 Definitions and assumptions

Definition 1 Given two random variables ξ and η, ξ is said to be stochastically larger than η or η is stochastically smaller than ξ, if

$$P(\xi > \alpha) \geq P(\eta > \alpha), \quad \text{for all real } \alpha,$$

denoted by $\xi \geq_{st} \eta$ or $\eta \leq_{st} \xi$ (see e.g., Ross[31]). Furthermore, we say that a stochastic process $\{X_n, n = 1, 2, \cdots\}$ is stochastically decreasing if $X_n \geq_{st} X_{n+1}$ and stochastically increasing if $X_n \leq_{st} X_{n+1}$ for all $n = 1, 2, \cdots$.

Definition 2 A stochastic process $\{\xi_n, n = 1, 2, \cdots\}$ is a geometric process, if there exists a real $a > 0$ such that $\{a^{n-1}\xi_n, n = 1, 2, \cdots\}$ forms a renewal process. The real a is called the ratio of the geometric process (see e.g., Lam [10], Zhang [12] for more details).

Obviously, from Definition 2, we have:

(i) If $a > 1$, then $\{\xi_n, n = 1, 2, \cdots\}$ is stochastically decreasing, i.e.

$$\xi_n \geq_{st} \xi_{n+1}, \quad n = 1, 2, \cdots$$

(ii) If $0 < a < 1$, then $\{\xi_n, n = 1, 2, \cdots\}$ is stochastically increasing, i.e.

$$\xi_n \leq_{st} \xi_{n+1}, \quad n = 1, 2, \cdots$$

(iii) If $a = 1$, then the geometric process becomes a renewal process.

(iv) If $E\xi_1 = \frac{1}{\lambda}$, then $E\xi_n = \frac{1}{a^{n-1}\lambda}$.

Suppose the following assumptions hold.

Assumption 1 A system consists of n identical components and repairable repair-equipment. The system fails if and only if at least k components have failed. The n components are repairable, and the order of repair for failed components is with a "first in first out" rule.

Assumption 2 At the beginning, a new $k/n(F)$ system, repairable repair-equipment and one repairman are installed. A failed component is repaired by the repair-equipment and the repair-equipment is repaired by the repairman. Repair for a failed component is perfect whereas repair for the repair-equipment is imperfect. Assume the survival times after repairs for the repair-equipment from a geometric process.

Assumption 3 If the repair-equipment fails while a component is being repaired, the repairman will repair the repair-equipment immediately and the failed component will be waiting for repair. The repair-equipment will be re-started immediately after completion of its repair, and the repair on the failed component will be continued. During the repair for the repair-equipment, the system is shut down and the un-failed components in the system do not fail any more. The repair-equipment does not fail when it is idle. As soon as at least k components are ready for work, the system will be re-started.

Assumption 4 Assume that the successive working times $\xi_n, n = 1, 2, \cdots$ and the consecutive repair times $\eta_n, n = 1, 2, \cdots$ of all n components are respectively i.i.d. random variables, and their survival distributions are

$$F(t) = P(\xi_n \leq t) = 1 - e^{-\lambda t}$$

$$G(t) = P(\eta_n \leq t) = 1 - e^{-\mu t}$$

where $t \geq 0,\ \lambda > 0,\ \mu > 0,\quad n = 1, 2, \cdots$ respectively. Assume that $\mu > \lambda$.

Assumption 5 The time interval between the completions of the $(n-1)$th and nth repairs of the repair-equipment is called the nth cycle of the repair-equipment. Let X_n and Y_n be respectively the working and the repair times of the repair-equipment in the nth cycle, $n = 1, 2, \cdots$. Then $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ form respectively a decreasing geometric process with ratio $a \geq 1$ and a increasing geometric process with ratio $0 < b \leq 1$, and survival distributions of X_n and Y_n are

$$H_n(t) = P(X_n \leq t) = 1 - e^{-a^{n-1}\alpha t}$$

$$K_n(t) = P(Y_n \leq t) = 1 - e^{-b^{n-1}\beta t}$$

where $t \geq 0,\ \alpha > 0,\ \beta > 0,\quad n = 1, 2, \cdots$ respectively.

Assumption 6 $\xi_n,\ \eta_n,\ X_n,\ Y_n,\ n = 1, 2, \cdots$ are all independent random variable sequences.

Remarks

(1) The assumption $\mu > \lambda$ makes the $k/n(F)$ repairable system closer to real situations.

(2) If we regard the three items, the *repair-equipment*, a *failed component* and *a repair* as a *service station*, a *customer* and a *service* respectively, then the three forms a queueing system. Under the above assumptions, the $k/n(F)$ repairable system is equivalent to a repairable $M/M/1$ queueing system with finite customer-source. Hence, the system in this paper can be regarded as a repairable $M/M(M/M)/1/k/n$ queueing system, where symbol (M/M) means that both the working and repair times of the service station (i.e. the repair-equipment) are exponential. The difference between our queueing system and the classical $M/M/1/k/n$ queueing system is that the service station in our queueing system is subject to failure. In this paper, we shall study a $k/n(F)$ repairable system with repairable repair-equipment and regard them as a $M/M(M/M)/1/k/n$ queueing system with repairable service station.

(3) Assumption 3 is reasonable. For example, consider a local area computer network print system with a repairable printer, and there are several workstations connecting to printer. Now, we regard the printer and a print job as a service station and a customer in the print system respectively. Printing jobs submitted from a workstation have to queue up as customers in the print system. If the printer fails to work, it will be repaired, and the jobs have to wait for printing. The printer will be restarted immediately after completion of its repair, and the queueing printing jobs can be conducted. The reader is also referred to Lam *et al* [30] for more detailed discussion. From an application perspective, therefore, the research of this paper is helpful for some maintenance engineers.

3 System analysis

Now, let $N(t)$ be the system state at time t. According to the model assumptions, we have

$$N(t) = \begin{cases} 0, & \text{if at time } t\text{, all components work and the repair-equipment is idle;} \\ & \text{the system works,} \\ 1_w, & \text{if at time } t\text{, a component fails and the repair-equipment is repairing} \\ & \text{the failed component; the system works,} \\ 1_f, & \text{if at time } t\text{, a component fails and the repair-equipment is being repaired,} \\ & \text{and the failed component is waiting for repair; the system is shut down,} \\ \vdots & \vdots \\ m_w, & \text{if at time } t\text{, } m \text{ components fail and the repair-equipment is repairing} \\ & \text{one of the failed components, and the rest } m-1 \text{ failed components are} \\ & \text{waiting for repair; the system is working,} \\ m_f, & \text{if at time } t\text{, } m \text{ components fail, the repair-equipment is being repaired, and} \\ & \text{the } m \text{ failed components are waiting for repair; the system is shut down,} \\ \vdots & \vdots \\ (k-1)_w, & \text{if at time } t\text{, } k-1 \text{ components fail, the repair-equipment is repairing} \\ & \text{one of the failed components, and the rest } k-2 \text{ failed components are} \\ & \text{waiting for repair; the system is working,} \\ (k-1)_f, & \text{if at time } t\text{, } k-1 \text{ components fail, the repair-equipment is being repaired,} \\ & \text{and } k-1 \text{ failed components are waiting for repair; the system is shut down,} \\ k_w, & \text{if at time } t\text{, } k \text{ components fail, the repair-equipment is repairing one of} \\ & \text{the failed components, and the rest } k-1 \text{ failed components are waiting} \\ & \text{for repair; the system fails,} \\ k_f, & \text{if at time } t\text{, } k \text{ components fail, the repair-equipment is being repaired, and} \\ & \text{the } k \text{ failed components are waiting for repair; the system fails.} \end{cases}$$

In fact, the state $N(t)$ of the $k/n(F)$ system as the above-discussed is equivalent to the following state $N(t)$ of a $M/M(M/M)/1/k/n$ queueing system with a repairable service station, i.e.

$$N(t) = \begin{cases} 0, & \text{if at time } t\text{, there is no customer in the queueing system; the service} \\ & \text{station is idle and good,} \\ 1_w, & \text{if at time } t\text{, there is one customer in the queueing system; the service} \\ & \text{station is serving the customer,} \\ 1_f, & \text{if at time } t\text{, there is one customer in the queueing system; the service} \\ & \text{station is being repaired, and the customer is waiting for service, no} \\ & \text{more new customers arrive,} \\ \vdots & \vdots \\ m_w, & \text{if at time } t\text{, there are } m \text{ customers in the queueing system; the service} \\ & \text{station is serving one customer, and the rest } m-1 \text{ customers are} \\ & \text{waiting for service,} \\ m_f, & \text{if at time } t\text{, there are } m \text{ customers in the queueing system; the service} \\ & \text{station is being repaired, and the } m \text{ customers are waiting for service,} \\ & \text{no more new customers arrive,} \\ \vdots & \vdots \\ (k-1)_w, & \text{if at time } t\text{, there are } k-1 \text{ customers in the queueing system;, the} \\ & \text{service station is serving one customer, and the rest } k-2 \text{ customers} \\ & \text{are waiting for service,} \\ (k-1)_f, & \text{if at time } t\text{, there are } k-1 \text{ customers in the queueing system; the} \\ & \text{service station is being repaired, and the } k-1 \text{ customers are waiting} \\ & \text{for service, no more new customers arrive,} \\ k_w, & \text{if at time } t\text{, there are } k \text{ customers in the queueing system; the service} \\ & \text{station is serving one customer, and the rest } k-1 \text{ customers are waiting} \\ & \text{for service; no more new customers arrive,} \\ k_f, & \text{if at time } t\text{, there are } k \text{ customers in the queueing system; the service} \\ & \text{station is repaired, and the } k \text{ customers are waiting for service, no more} \\ & \text{new customers arrive.} \end{cases}$$

Obviously, the state space is $\Omega = \{0, 1_w, 1_f, \cdots, m_w, m_f, \cdots, (k-1)_w, (k-1)_f, k_w, k_f\}$, the set of working states is $W = \{0, 1_w, \cdots, m_w, \cdots, (k-1)_w\}$, and the set of failure states is $F = \{1_f, \cdots, m_f, \cdots, (k-1)_f, k_w, k_f\}$. Although the stochastic process $\{N(t), t \geq 0\}$ is not a Markov process, we can obtain a vector Markov process by introducing a supplementary variable. Let the supplementary variable $S(t) = l_w$ or $l_f, (l = 1, 2, \cdots)$ be the working state or the repair state of the repair-equipment in lth cycle at time t, then $\{N(t), S(t), t \geq 0\}$ forms a vector Markov process.

Denote the state probability of the system by

$$p_{ml_w}(t) = P(N(t) = m_w, S(t) = l_w),\ \ (m = 0, 1, \cdots, k-1; l = 1, 2, \cdots),$$

and

$$p_{ml_f}(t) = P(N(t) = m_f, S(t) = l_f),\ \ (m = 1, 2, \cdots, k-1, k; l = 1, 2, \cdots).$$

With the classical probability theory, it is straightforward to derive the following differential equations:

$$
\begin{aligned}
\frac{d}{dt}p_{0l_w}(t) &= -n\lambda p_{0l_w}(t) + \mu p_{1l_w}(t), \quad (l = 1, 2, \cdots) && (1)\\
\frac{d}{dt}p_{m1_w}(t) &= -((n-m)\lambda + \mu + \alpha)p_{m1_w}(t) + (n-m+1)\lambda p_{(m-1)1_w}(t) + \mu p_{(m+1)1_w}(t),\\
&\quad (m = 1, 2, \cdots, k-1) && (2)\\
\frac{d}{dt}p_{k1_w}(t) &= -(\mu + \alpha)p_{k1_w}(t) + (n-k+1)\lambda p_{(k-1)1_w}(t) && (3)\\
\frac{d}{dt}p_{ml_w}(t) &= -((n-m)\lambda + \mu + a^{l-1}\alpha)p_{ml_w}(t) + (n-m+1)\lambda p_{(m-1)l_w}(t) + \mu p_{(m+1)l_w}(t)\\
&\quad + b^{l-2}\beta p_{m(l-1)_f}(t), \quad (m = 1, 2, \cdots, k-1; \quad l = 2, 3, \cdots) && (4)\\
\frac{d}{dt}p_{kl_w}(t) &= -(\mu + a^{l-1}\alpha)p_{kl_w}(t) + (n-k+1)\lambda p_{(k-1)l_w}(t) + b^{l-2}\beta p_{k(l-1)_f}(t),\\
&\quad (l = 2, 3, \cdots) && (5)\\
\frac{d}{dt}p_{ml_f}(t) &= -b^{l-1}\beta)p_{ml_f}(t) + a^{l-1}\alpha p_{ml_w}(t), (m = 1, 2, \cdots, k-1, k; \quad l = 1, 2, \cdots) && (6)
\end{aligned}
$$

The initial conditions are:

$$p_{01_w}(0) = 1; \quad p_{0l_w}(0) = 0 \quad (l = 2, 3, \cdots),$$

$$p_{ml_w}(0) = 0 \quad (m = 1, 2, \cdots, k-1, k; \quad l = 1, 2, \cdots),$$

and

$$p_{ml_f}(0) = 0 \quad (m = 0, 1, \cdots, k-1, k; \quad l = 1, 2, \cdots).$$

4 Some characters of the $k/n(F)$ system

It is known there are three important indices in the queueing theory, i.e. queue length, waiting time and busy period and their distributions. This section will derive reliability indices, including system availability, mean waiting time and the idle probability of the repair-equipment on the basis of the queueing theory. Let

$$p^*_{ml_w}(s) = \int_0^\infty e^{-st}p_{ml_w}(t)dt, \quad m = 0, 1, 2, \cdots, k; l = 1, 2, \cdots$$

$$p^*_{ml_f}(s) = \int_0^\infty e^{-st} p_{ml_f}(t)dt, \quad m = 1, 2, \cdots, k; l = 1, 2, \cdots$$

be the Laplace transform of the state probability distribution. Then taking the Laplace transform on the both sides of the differential equations (1)-(6), considering the initial conditions, that the following equations are given

$$\begin{aligned}
(s + n\lambda)p^*_{01_w}(s) &= \mu p^*_{11_w}(s) + 1 && (7)\\
(s + n\lambda)p^*_{0l_w}(s) &= \mu p^*_{1l_w}(s), \quad (l = 2, 3, \cdots) && (8)\\
(s + (n-m)\lambda + \mu + \alpha)p^*_{m1_w}(s) &= (n-m+1)\lambda p^*_{(m-1)1_w}(s) + \mu p^*_{(m+1)1_w}(s), \\
&\quad (m = 1, 2, \cdots, k-1) && (9)\\
(s + \mu + \alpha)p^*_{k1_w}(s) &= (n-k+1)\lambda p^*_{(k-1)1_w}(s) && (10)\\
(s + (n-m)\lambda + \mu + a^{l-1}\alpha)p^*_{ml_w}(s) &= (n-m+1)\lambda p^*_{(m-1)l_w}(s) + \mu p^*_{(m+1)l_w}(s) \\
&\quad + b^{l-2}\beta p^*_{m(l-1)_f}(s), \\
&\quad (m = 1, 2, \cdots, k-1;\ l = 2, 3, \cdots) && (11)\\
(s + \mu + a^{l-1}\alpha)p^*_{kl_w}(s) &= (n-k+1)\lambda p^*_{(k-1)l_w}(s) + b^{l-2}\beta p^*_{k(l-1)_f}(s), \\
&\quad (l = 2, 3, \cdots) && (12)\\
(s + b^{l-1}\beta)p^*_{ml_f}(s) &= a^{l-1}\alpha p^*_{ml_w}(s), \\
&\quad (m = 1, 2, \cdots, k-1;\ l = 1, 2, \cdots) && (13)
\end{aligned}$$

To solve equations (7)-(13), we recall from classical $M/M/1$ queueing system, the busy periods $\{b_1, b_2, \cdots\}$ are i.i.d. with distribution $B(t) = P(b \leq t)$, where b is denoted the busy length.

Lemma 1 The Laplace-Stieltjes tranform of $B(t)$ is given by

$$B^*(s) = \int_0^\infty e^{-st} dB(t) = \frac{s + \lambda + \mu - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\lambda}$$

The proof of Lemma 1 can be find in Takacs[32] or Kleinrock[33].

Lemma 2 The distribution of $\sum\limits_{i=1}^{n} X_i$ is given by

$$H^{(n)}(t) = 1 - \sum_{i=1}^{n} \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{a^{j-1}}{a^{j-1} - a^{i-1}} e^{-a^{i-1}\alpha t}$$

Proof It is known that if n random variables $X_1, X_2, \cdots, X_n$ are independent, and X_i has exponential distribution with the parameter λ_i, i.e.

$$H_i(t) = 1 - e^{-\lambda_i t}, \text{where}, \lambda_i = a^{i-1}\alpha;\ i = 1, 2, \cdots, n$$

with the definition of Laplace-Stieltjes transform of $H^{(n)}(t)$, we have

$$\begin{aligned} H^{*(n)}(s) &= \int_0^\infty e^{-st} dH^{(n)}(t) \\ &= E\{e^{-s(\sum\limits_{i=1}^{n} X_i)}\} = \prod_{i=1}^{n} E\{e^{-sX_i}\} \\ &= \prod_{i=1}^{n} \int_0^\infty e^{-st} dH_i(t) = \prod_{i=1}^{n} \frac{\lambda_i}{s+\lambda_i} = \sum_{i=1}^{n} \frac{c_i \lambda_i}{s+\lambda_i} \end{aligned}$$

where

$$c_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

In reverse, we can obtain

$$H^{(n)}(t) = \sum_{i=1}^{n} c_i H_i(t) = \sum_{i=1}^{n} c_i (1 - e^{-\lambda_i t}) = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} c_i e^{-\lambda_i t}$$

If $t \longrightarrow \infty$ and $H^{(n)}(t) \longrightarrow 1$, $\sum\limits_{i=1}^{n} c_i = 1$. Hence, Lemma 2 holds.

Theorem 1 At time t, the probabilities of all components working in the system are given by

$$\begin{aligned} p_{01_w}(t) &= e^{-\lambda t} + \sum_{n=2}^{\infty} \int_0^t [F^{(n-1)}(t-u) - F^{(n)}(t-u)] e^{-\alpha u} dB^{(n-1)}(u) \\ p_{0l_w}(t) &= \sum_{n=2}^{\infty} \int_0^t [F^{(n-1)} * K^{(l-1)}(t-u) - F^{(n)} * K^{(l-1)}(t-u)] \\ &\quad \cdot [H^{(l-1)}(u) - H^{(l)}(u)] dB^{(n-1)}(u), \quad (l = 2, 3, \cdots) \end{aligned}$$

where

$$F^{(n)}(t) = F(t) * F(t) * \cdots * F(t)$$

$$B^{(n)}(t) = B(t) * B(t) * \cdots * B(t)$$

$$H^{(n)}(t) = H(t) * H(at) * \cdots * H(a^{n-1} t)$$

and

$$K^{(n)}(t) = K(t) * K(bt) * \cdots * K(b^{n-1} t)$$

are respectively the cumulative probability distribution functions of $\sum\limits_{i=1}^{n} \xi_i$, $\sum\limits_{i=1}^{n} b_i$, $\sum\limits_{i=1}^{n} X_i$ and $\sum\limits_{i=1}^{n} Y_i$.

Proof According to the model assumptions and $N(0) = 0$, at $t = 0$, the repair-equipment is idle. Since the idle period v_i and the busy period b_i $(i = 1, 2, \cdots)$ occur alternatively, we have

$$\begin{aligned}
p_{01_w}(t) &= P\{N(t) = 0,\ S(t) = 1_w\} \\
&= P(v_1 > t) + \sum_{n=2}^{\infty} P\{\sum_{i=1}^{n-1}(v_i + b_i) \le t < \sum_{i=1}^{n-1}(v_i + b_i) + v_n; \sum_{i=1}^{n-1} b_i < X_1\} \\
&= 1 - F(t) + \sum_{n=2}^{\infty} \int_0^t P\{\sum_{i=1}^{n-1} v_i \le t - u < \sum_{i=1}^{n} v_i; X_1 > u\} dB^{(n-1)}(u) \\
&= e^{-\lambda t} + \sum_{n=2}^{\infty} \int_0^t [F^{(n-1)}(t-u) - F^{(n)}(t-u)] e^{-\alpha u} dB^{(n-1)}(u)
\end{aligned}$$

$$\begin{aligned}
p_{0l_w}(t) &= P\{N(t) = 0,\ S(t) = l_w\} \\
&= \sum_{n=2}^{\infty} P\{\sum_{i=1}^{n-1}(v_i + b_i) + \sum_{i=1}^{l-1} Y_i \le t < \sum_{i=1}^{n-1}(v_i + b_i) + v_n + \sum_{i=1}^{l-1} Y_i; \\
&\quad \sum_{i=1}^{l-1} X_i \le \sum_{i=1}^{n-1} b_i < \sum_{i=1}^{l} X_i\} \\
&= \sum_{n=2}^{\infty} \int_0^t P\{\sum_{i=1}^{n-1} v_i + \sum_{i=1}^{l-1} Y_i \le t - u < \sum_{i=1}^{n} v_i + \sum_{i=1}^{l-1} Y_i; \sum_{i=1}^{l-1} X_i \le u < \sum_{i=1}^{l} X_i\} \\
&\quad dB^{(n-1)}(u) \\
&= \sum_{n=2}^{\infty} \int_0^t [F^{(n-1)} * K^{(l-1)}(t-u) - F^{(n)} * K^{(l-1)}(t-u)] \\
&\quad \cdot [H^{(l-1)}(u) - H^{(l)}(u)] dB^{(n-1)}(u), \quad (l = 2, 3, \cdots)
\end{aligned}$$

Theorem 2

$$\begin{aligned}
p^*_{01_w}(s) &= \frac{1}{s + \lambda - \lambda B^*(s + \alpha)}, && (14) \\
p^*_{0l_w}(s) &= \prod_{j=1}^{l-1} \frac{(ab)^{j-1}\beta}{s + b^{j-1}\beta} \sum_{i=1}^{l} [\prod_{\substack{r=1 \\ r\neq i}}^{l} \frac{1}{a^{r-1} - a^{i-1}}] \frac{\lambda B^*(s + a^{i-1}\alpha)}{(s+\lambda)[s + \lambda - \lambda B^*(s + a^{i-1}\alpha)]}, && (15) \\
&\quad (l = 2, 3, \cdots).
\end{aligned}$$

Proof It follows from Theorem 1 that

$$\begin{aligned}
p^*_{01_w}(s) &= \int_0^{\infty} e^{-st} p_{01_w}(t) dt \\
&= \int_0^{\infty} e^{-st} \{\sum_{n=2}^{\infty} \int_0^t [F^{(n-1)}(t-u) - F^{(n)}(t-u)] e^{-\alpha u} dB^{(n-1)}(u) + e^{-\lambda t}\} dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=2}^{\infty}\int_0^{\infty}\{\int_u^{\infty} e^{-st}[F^{(n-1)}(t-u)-F^{(n)}(t-u)]dt\}e^{-\alpha u}dB^{(n-1)}(u)+\frac{1}{s+\lambda} \\
&= \sum_{n=2}^{\infty}\int_0^{\infty}[\int_0^{\infty} e^{-s(u+v)}(F^{(n-1)}(v)-F^{(n)}(v))dv]e^{-\alpha u}dB^{(n-1)}(u)+\frac{1}{s+\lambda} \\
&= \sum_{n=2}^{\infty}\int_0^{\infty} e^{-su}[\int_0^{\infty} e^{-sv}F^{(n-1)}(v)dv-\int_0^{\infty} e^{-sv}F^{(n)}(v)dv]e^{-\alpha u}dB^{(n-1)}(u)+\frac{1}{s+\lambda} \\
&= \frac{1}{s+\lambda}+\sum_{n=2}^{\infty}\frac{1}{s}[(\frac{\lambda}{s+\lambda})^{n-1}-(\frac{\lambda}{s+\lambda})^{n}]\int_0^{\infty} e^{-(s+\alpha)u}dB^{(n-1)}(u) \\
&= \frac{1}{s+\lambda}+\sum_{n=2}^{\infty}\frac{\lambda^{n-1}}{(s+\lambda)^n}[B^*(s+\alpha)]^{n-1} \\
&= \frac{1}{s+\lambda-\lambda B^*(s+\alpha)}
\end{aligned}
$$

Similarly, from Theorem 1 and Lemma 2, we have

$$
\begin{aligned}
p^*_{0l_w}(s) &= \int_0^{\infty} e^{-st}p_{0l_w}(t)dt \\
&= \int_0^{\infty} e^{-st}\{\sum_{n=2}^{\infty}\int_0^{t}[F^{(n-1)}*K^{(l-1)}(t-u)-F^{(n)}*K^{(l-1)}(t-u)] \\
&\quad [H^{(l-1)}(u)-H^{(l)}(u)]dB^{(n-1)}(u)\}dt \\
&= \sum_{n=2}^{\infty}\int_0^{\infty}\{\int_u^{\infty} e^{-st}[F^{(n-1)}*K^{(l-1)}(t-u)-F^{(n)}*K^{(l-1)}(t-u)]dt\} \\
&\quad [H^{(l-1)}(u)-H^{(l)}(u)]dB^{(n-1)}(u) \\
&= \sum_{n=2}^{\infty}\int_0^{\infty}\{\int_0^{\infty} e^{-s(u+v)}[F^{(n-1)}*K^{(l-1)}(v)-F^{(n)}*K^{(l-1)}(v)]dv\} \\
&\quad [H^{(l-1)}(u)-H^{(l)}(u)]dB^{(n-1)}(u) \\
&= \sum_{n=2}^{\infty}\frac{1}{s}[\frac{\lambda^{n-1}}{(s+\lambda)^{n-1}}-\frac{\lambda^{n}}{(s+\lambda)^{n}}](\prod_{j=1}^{l-1}\frac{b^{j-1}\beta}{s+b^{j-1}\beta})\int_0^{\infty} e^{-su}[H^{(l-1)}(u)-H^{(l)}(u)] \\
&\quad dB^{(n-1)}(u) \\
&= \prod_{j=1}^{l-1}\frac{b^{j-1}\beta}{s+b^{j-1}\beta}\sum_{n=2}^{\infty}\frac{\lambda^{n-1}}{(s+\lambda)^n}\int_0^{\infty} e^{-su}[\sum_{i=1}^{l}\prod_{\substack{r=1\\(r\neq i)}}^{l}\frac{a^{r-1}}{a^{r-1}-a^{i-1}}e^{-a^{i-1}\alpha u}- \\
&\quad \sum_{i=1}^{l-1}\prod_{\substack{r=1\\(r\neq i)}}^{l-1}\frac{a^{r-1}}{a^{r-1}-a^{i-1}}e^{-a^{i-1}\alpha u}]dB^{(n-1)}(u)
\end{aligned}
$$

$$= \prod_{j=1}^{l-1} \frac{b^{j-1}\beta}{s+b^{j-1}\beta}[\sum_{i=1}^{l} \prod_{\substack{r=1 \\ (r\neq i)}}^{l} \frac{a^{r-1}}{a^{r-1}-a^{i-1}} \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{(s+\lambda)^n}(B^*(s+a^{i-1}\alpha))^{n-1} -$$

$$\sum_{i=1}^{l-1} \prod_{\substack{r=1 \\ (r\neq i)}}^{l-1} \frac{a^{r-1}}{a^{r-1}-a^{i-1}} \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{(s+\lambda)^n}(B^*(s+a^{i-1}\alpha))^{n-1}]$$

$$= \prod_{j=1}^{l-1} \frac{b^{j-1}\beta}{s+b^{j-1}\beta}[\sum_{i=1}^{l} \prod_{\substack{r=1 \\ (r\neq i)}}^{l} \frac{a^{r-1}}{a^{r-1}-a^{i-1}} \frac{\lambda B^*(s+a^{i-1}\alpha)}{(s+\lambda)[s+\lambda-\lambda B^*(s+a^{i-1}\alpha)]} -$$

$$\sum_{i=1}^{l-1} \prod_{\substack{r=1 \\ (r\neq i)}}^{l-1} \frac{a^{r-1}}{a^{r-1}-a^{i-1}} \frac{\lambda B^*(s+a^{i-1}\alpha)}{(s+\lambda)[s+\lambda-\lambda B^*(s+a^{i-1}\alpha)]}]$$

$$= \prod_{j=1}^{l-1} \frac{(ab)^{j-1}\beta}{s+b^{j-1}\beta} \sum_{i=1}^{l}[\prod_{\substack{r=1 \\ (r\neq i)}}^{l} \frac{1}{a^{r-1}-a^{i-1}}] \frac{\lambda B^*(s+a^{i-1}\alpha)}{(s+\lambda)[s+\lambda-\lambda B^*(s+a^{i-1}\alpha)]}$$

On the basis of Theorem 2, we can derive the Laplace transform $p^*_{ml_w}(s)$ of $p_{ml_w}(t)$. To do this, first of all it follows from the equation (7) that

$$p^*_{11_w}(s) = \frac{s+n\lambda}{\mu} p^*_{01_w}(s) - \frac{1}{\mu} = \frac{(n-1)\lambda B^*(s+\alpha)}{\mu[s+\lambda-\lambda B^*(s+\alpha)]} \tag{16}$$

Furthermore, we have the following theorem.

Theorem 3

$$p^*_{m1_w}(s) = \frac{(n-1)\lambda B^*(s+\alpha)(M^m-N^m)+\mu(MN^m-NM^m)}{\mu(M-N)[s+\lambda-\lambda B^*(s+\alpha)]}, \quad (m=0,1,2,\cdots,k-1)$$

where M and N are two roots of the quadratic equation

$$t^2 - \frac{s+(n-m)\lambda+\mu+\alpha}{\mu} t + \frac{(n-m+1)\lambda}{\mu} = 0 \tag{17}$$

proof According to (9), we have

$$p^*_{(m+1)1_w}(s) = \frac{s+(n-m)\lambda+\mu+\alpha}{\mu} p^*_{m1_w}(s) - \frac{(n-m+1)\lambda}{\mu} p^*_{(m-1)1_w}(s), \tag{18}$$

$$(m=1,2,\cdots,k-1)$$

Because M and N are two roots of the equation (17),then

$$M+N = \frac{s+(n-m)\lambda+\mu+\alpha}{\mu}, \quad MN = \frac{(n-m+1)\lambda}{\mu}$$

and equation (18) becomes

$$p^*_{(m+1)1_w}(s) - Mp^*_{m1_w}(s) = N[p^*_{m1_w}(s) - Mp^*_{(m-1)1_w}(s)]$$

or

$$p^*_{(m+1)1_w}(s) - Np^*_{m1_w}(s) = M[p^*_{m1_w}(s) - Np^*_{(m-1)1_w}(s)]$$

By iteration, it is straightforward that

$$p^*_{m1_w}(s) - Mp^*_{(m-1)1_w}(s) = N^{m-1}[p^*_{11_w}(s) - Mp^*_{01_w}(s)] \tag{19}$$

and

$$p^*_{m1_w}(s) - Np^*_{(m-1)1_w}(s) = M^{m-1}[p^*_{11_w}(s) - Np^*_{01_w}(s)] \tag{20}$$

According to the equations (14), (16), (19) and (20), we can obtain

$$\begin{aligned} p^*_{m1_w}(s) &= \frac{M^m - N^m}{M-N} p^*_{11_w}(s) + \frac{MN^m - NM^m}{M-N} p^*_{01_w}(s) \\ &= \frac{(n-1)\lambda B^*(s+\alpha)(M^m - N^m) + \mu(MN^m - NM^m)}{\mu(M-N)[s+\lambda-\lambda B^*(s+\alpha)]}, \ (m = 0, 1, 2, \cdots, k-1) \end{aligned}$$

According to the equation (13) and Theorem 3, we have

$$\begin{aligned} &p^*_{m1_f}(s) = \frac{\alpha}{s+\beta} p^*_{m1_w}(s) \\ &= \frac{(n-1)\lambda\alpha B^*(s+\alpha)(M^m - N^m) + \mu\alpha(MN^m - NM^m)}{\mu(M-N)(s+\beta)[s+\lambda-\lambda B^*(s+\alpha)]}, \quad (m = 1, 2, \cdots, k) \end{aligned} \tag{21}$$

According to the equation (15) in Theorem 2, we have

$$p^*_{02_w}(s) = \frac{\lambda\beta}{(s+\beta)(a-1)} \frac{B^*(s+\alpha) - B^*(s+a\alpha)}{[s+\lambda-\lambda B^*(s+\alpha)][s+\lambda-\lambda B^*(s+a\alpha)]} \tag{22}$$

According to the equations (8), (11), (12), (21) and (22), we can obtain

$$\begin{aligned} p^*_{12_w}(s) &= \frac{\lambda\beta(s+n\lambda)}{\mu(s+\beta)(a-1)} \cdot \frac{B^*(s+\alpha) - B^*(s+a\alpha)}{[s+\lambda-\lambda B^*(s+\alpha)][s+\lambda-\lambda B^*(s+a\alpha)]} \\ p^*_{22_w}(s) &= \frac{\lambda\beta[(s+n\lambda)(s+(n-1)\lambda+\mu+a\alpha) - n\lambda\mu][B^*(s+\alpha) - B^*(s+a\alpha)]}{\mu^2(s+\beta)(a-1)[s+\lambda-\lambda B^*(s+\alpha)][s+\lambda-\lambda B^*(s+a\alpha)]} \\ &\quad - \frac{(n-1)\lambda\alpha\beta B^*(s+\alpha)}{\mu^2(s+\beta)[s+\lambda-\lambda B^*(s+\alpha)]} \end{aligned}$$

Then we can determine $p^*_{m2_w}(s)$, for $m = 3, 4, \cdots, k-1, k$ by using the equations (11), (12) and the above obtained results again and again. And by using the equation (13), we can get $p^*_{m2_f}(s)$, for $m = 1, 2, \cdots, k-1, k$. In general, on the basis of Theorem 2 and 3, we can also determine the Laplace transform $p^*_{ml_w}(s)$ and $p^*_{ml_f}(s)$ from the equations (7) to (13) recurrently.

4.1 System availability

By the definition, the availability of the system at time t is given by

$$\begin{aligned} A(t) &= P\{N(t) \in W\} = \sum_{l=1}^{\infty}[\sum_{m=0}^{k-1} p_{ml_w}(t)] \\ &= p_{01_w}(t) + \sum_{l=1}^{\infty}[\sum_{m=1}^{k-1} p_{ml_w}(t)] \end{aligned}$$

and the Laplace transform of $A(t)$ is given by

$$A^*(s) = p^*_{01_w}(s) + \sum_{l=1}^{\infty}[\sum_{m=1}^{k-1} p^*_{ml_w}(s)]$$

4.2 Mean of waiting time

A failed component at time t is repaired immediately when the repair-equipment is idle, otherwise it will wait for repair according to the "first in first out" rule. Thus, the mean waiting time of a failed component at time t is an interesting index for the $k/n(F)$ system. Let W_t be the waiting time for repair of a failed component at time t, and let G_m be the total chain-repair time for m failed components in the system and denote the distribution of G_m by $G^{(m)}$. Moreover, let $\hat{X}_i$ be the residual life of X_i. Then we can obtain the following theorem about the distribution of waiting time.

Theorem 4 Let the distribution of W_t be $W_t(x)$, then

$$W_t(x) = \sum_{l=1}^{\infty}\sum_{m=1}^{k-1} p_{ml_w}(t) \sum_{n=0}^{\infty} \int_0^x K_l^{(n)}(x-u)[H_l^{(n)}(u) - H_l^{(n+1)}(u)]dG^{(m)}(u)$$

where n is the failed number of the repair-equipment during the time G_m. And

$$K_l^{(n)}(u) = K_l(u) * K_{l+1}(u) * \cdots * K_{l+n-1}(u)$$

$$H_l^{(n)}(u) = H_l(u) * H_{l+1}(u) * \cdots * H_{l+n-1}(u)$$

$$G^{(m)}(u) = G(u) * G(u) * \cdots * G(u)$$

Proof According to the conditional probability and the formula of total probability, we have

$$\begin{aligned} W_t(x) &= P\{W_t \le x\} \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{k-1} P\{W_t \le x, N(t) = m_w, S(t) = l_w\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=1}^{\infty}\sum_{m=1}^{k-1} P\{N(t)=m_w, S(t)=l_w\}P\{W_t \le x | N(t)=m_w, S(t)=l_w\} \\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{k-1} p_{ml_w}(t)\sum_{n=0}^{\infty} P\{G_m + \sum_{i=l}^{l+n-1} Y_i \le x, \hat{X}_l + \sum_{i=l+1}^{l+n-1} X_i \le G_m < \hat{X}_l + \sum_{i=l+1}^{l+n} X_i\} \\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{k-1} p_{ml_w}(t)\sum_{n=0}^{\infty}\int_0^x P\{\sum_{i=l}^{l+n-1} Y_i \le x-u, \hat{X}_l + \sum_{i=l+1}^{l+n-1} X_i \le u < \hat{X}_l + \sum_{i=l+1}^{l+n} X_i\} \\
&\quad dG^{(m)}(u) \\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{k-1} p_{ml_w}(t)\sum_{n=0}^{\infty}\int_0^x K_l^{(n)}(x-u)[H_l^{(n)}(u) - H_l^{(n+1)}(u)]dG^{(m)}(u)
\end{aligned}$$

Thus, the mean of waiting time to a failed component for repair is given by

$$EW_t = \int_0^{\infty} x dW_t(x)$$

Clearly, if the value of EW_t is larger, we should improve the repair efficiency of the repair-equipment so that the cost of the system is decreased.

4.3 Mean of busy period

It is known that a busy period for the repair-equipment will start when a component in the system fails and the number of failed component in the system is 1, it will end at the time that the number of the failed components in the system reduces to 0. To determine the mean of busy period for the repair-equipment, we study a stochastic process $\{\tilde{N}(t), t \ge 0\}$. The only difference between the processes $\{N(t), t \ge 0\}$ and $\{\tilde{N}(t), t \ge 0\}$ is that the state 0 is an absorbing state in $\{\tilde{N}(t), t \ge 0\}$.

Let $\tilde{B}$ be the length of a busy period, then the distribution function is given by

$$\tilde{B}(t) = P\{\tilde{B} \le t\} = P\{\tilde{N}(t) = 0\}$$

Furthermore, we can obtain the following theorem about the distribution of busy period.

Theorem 5

$$\tilde{B}(t) = \sum_{l=1}^{\infty} B(t)[1 - H_l(t)] + \sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t B(t-u)[H_l^{(n)}(t-u) - H_l^{(n+1)}(t-u)]dK_l^{(n)}(u)$$

where $B(t)$ is the distribution of a busy period in the classical $M/M/1$ queueing system.

Proof First of all, we introduce a supplementary variable $\tilde{S}(t)$ which is the same as the $S(t)$ in the process $\{N(t), t \ge 0\}$, such that $\tilde{S}(t) = l_w$, if the $(l-1)$th repair has been completed. Thus we can also obtain a vector Markov process $\{\tilde{N}(t), \tilde{S}(t), t \geq 0\}$. Thus

$$\begin{aligned}
\tilde{B}(t) &= P\{\tilde{N}(t) = 0\} \\
&= \sum_{l=1}^{\infty}\sum_{j=l}^{\infty} P\{\tilde{N}(t) = 0, \tilde{S}(t) = j_w, \tilde{S}(0) = l_w\} \\
&= \sum_{l=1}^{\infty} P\{\tilde{N}(t) = 0, \tilde{S}(t) = l_w, \tilde{S}(0) = l_w\} \\
&\quad + \sum_{l=1}^{\infty}\sum_{j=l+1}^{\infty} P\{\tilde{N}(t) = 0, \tilde{S}(t) = j_w, \tilde{S}(0) = l_w\} \\
&= \sum_{l=1}^{\infty} P\{\tilde{N}(t) = 0, \text{the repair-equipment works in}(0,t], \tilde{S}(0) = l_w\} \\
&\quad + \sum_{l=1}^{\infty}\sum_{n \triangleq j-l=1}^{\infty} P\{\tilde{N}(t) = 0, \text{the repair-equipment fails for } n \text{ times in}(0,t], \tilde{S}(0) = l_w\} \\
&= \sum_{l=1}^{\infty} P\{B \leq t, \hat{X}_l > t\} + \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} P\{B + \sum_{i=l}^{l+n-1} Y_i \leq t, \hat{X}_l + \sum_{i=l}^{l+n-1} Y_i + \sum_{i=l+1}^{l+n-1} X_i \leq t \\
&\quad < \hat{X}_l + \sum_{i=l}^{l+n-1} Y_i + \sum_{i=l+1}^{l+n} X_i\} \\
&= \sum_{l=1}^{\infty} B(t)[1 - H_l(t)] \\
&\quad + \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} \int_0^t P\{B \leq t-u, \hat{X}_l + \sum_{i=l+1}^{l+n-1} X_i \leq t-u < \hat{X}_l + \sum_{i=l+1}^{l+n} X_i\} dK_l^{(n)}(u) \\
&= \sum_{l=1}^{\infty} B(t)[1 - H_l(t)] + \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} \int_0^t B(t-u)[H_l^{(n)}(t-u) - H_l^{(n+1)}(t-u)] dK_l^{(n)}(u)
\end{aligned}$$

Thus, the mean of busy period for the repair-equipment is given by

$$E\tilde{B} = \int_0^{\infty} t d\tilde{B}(t)$$

4.4 The idle probability of the repair-equipment

Clearly, the repair-equipment will be idle when all components are working at time t.Thus, according to Theorem 1, the idle probability of the repair-equipment at time t is given by

$$I(t) = P\{N(t) = 0\} = p_{01_w}(t) + p_{0l_w}(t)$$

$$
\begin{aligned}
&= e^{-\lambda t} + \sum_{n=2}^{\infty} \int_0^t [F^{(n-1)}(t-u) - F^{(n)}(t-u)] e^{-\alpha u} dB^{(n-1)}(u) \\
&\quad + \sum_{n=2}^{\infty} \int_0^t [F^{(n-1)} * K^{(l-1)}(t-u) - F^{(n)} * K^{(l-1)}(t-u)] \\
&\quad \cdot [H^{(l-1)}(u) - H^{(l)}(u)] dB^{(n-1)}(u), \quad (l = 1, 2, \cdots)
\end{aligned}
$$

4.5 Repair-equipment MTTFF

To determine the mean time to first failure(MTTFF) of the repair-equipment, we derive the distribution of the time to first failure of the repair-equipment. Given that there is no failed component in the system at the beginning, let T_f be the time to the first failure of the repair-equipment, and let the distribution of T_f be

$$\Psi_0(t) = P\{T_f \le t | N(0) = 0\} \tag{23}$$

and denote the Laplace-Stieltjes tranform of $\Psi_0(t)$ by $\Psi_0^*(s) = \int_0^\infty e^{-st} d\Psi_0(t)$. Then we have the following theorem.

Theorem 6

$$\Psi_0^*(s) = \frac{\lambda\alpha[1 - B^*(s+\alpha)]}{(s+\alpha)[s + \lambda - \lambda B^*(s+\alpha)]}$$

Proof As before, let v_i be the ith idle period, then it follows from the equation (23) that

$$
\begin{aligned}
\Psi_0(t) &= P\{T_f \le t | N(0) = 0\} \\
&= \sum_{n=1}^{\infty} P\{\sum_{i=1}^{n} v_i + X_1 \le t, \sum_{i=1}^{n-1} b_i < X_1 \le \sum_{i=1}^{n} b_i\} \\
&= \sum_{n=1}^{\infty} \int_0^t P\{\sum_{i=1}^{n} v_i \le t - u, \sum_{i=1}^{n-1} b_i < u \le \sum_{i=1}^{n} b_i\} dH_1(u) \\
&= \sum_{n=1}^{\infty} \int_0^t F^{(n)}(t-u)[B^{(n-1)}(u) - B^{(n)}(u)] dH_1(u)
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\Psi_0^*(s) &= \int_0^\infty e^{-st} d\Psi_0(t) \\
&= \sum_{n=1}^{\infty} (\frac{\lambda}{s+\lambda})^n \int_0^\infty e^{-st} [B^{(n-1)}(t) - B^{(n)}(t)] dH_1(t) \\
&= \sum_{n=1}^{\infty} (\frac{\lambda}{s+\lambda})^n \int_0^\infty \alpha e^{-(s+\alpha)t} [B^{(n-1)}(t) - B^{(n)}(t)] dH_1(t)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} (\frac{\lambda}{s+\lambda})^n [\frac{\alpha}{s+\alpha}(B^*(s+\alpha))^{n-1} - \frac{\alpha}{s+\alpha}(B^*(s+\alpha))^n] \\
&= \frac{\alpha}{s+\alpha} \sum_{n=1}^{\infty} [\frac{\lambda}{s+\lambda}(\frac{\lambda B^*(s+\alpha)}{s+\lambda})^{n-1} - (\frac{\lambda B^*(s+\alpha)}{s+\lambda})^n] \\
&= \frac{\lambda\alpha[1-B^*(s+\alpha)]}{(s+\alpha)[s+\lambda-\lambda B^*(s+\alpha)]}
\end{aligned}
$$

Thus, the mean time to first failure (MTTFF) of the repair-equipment is given by

$$ET_f = -\frac{d}{ds}\Psi_0^*(s)|_{s=0} = \frac{1}{\alpha} + \frac{1}{\lambda[1-B^*(\alpha)]}$$

4.6 Availability of the repair-equipment

Let the availability of the repair-equipment at time t be

$$A_f(t) = P\{\text{the repair-equipment works at time } t|N(0)=0\}.$$

Then

$$\bar{A}_f(t) = P\{\text{the repair-equipment fails at time } t|N(0)=0\}$$

is the probability that the repair-equipment fails at time t. Now, denote the Laplace transforms of $A_f(t)$ and $\bar{A}_f(t)$ by $A_f^*(s)$ and $\bar{A}_f^*(s)$ respectively. Then, the following theorem follows directly.

Theorem 7

$$A_f^*(s) = \sum_{m=0}^{k-1}\sum_{l=1}^{\infty} p_{ml_w}^*(s)$$

Proof It is clear that

$$
\begin{aligned}
A_f(t) &= P\{\text{the repair-equipment works at time } t|N(0)=0\} \\
&= \sum_{m=0}^{k-1}\sum_{l=1}^{\infty} P\{N(t)=m_w, S(t)=l_w|N(0)=0\} \\
&= \sum_{m=0}^{k-1}\sum_{l=1}^{\infty} p_{ml_w}(t)
\end{aligned}
$$

Therefore

$$A_f^*(s) = \sum_{m=0}^{k-1}\sum_{l=1}^{\infty} p_{ml_w}^*(s)$$

Furthermore, due to the fact $A_f(t) + \bar{A}_f(t) = 1$, we have

$$A_f^*(s) + \bar{A}_f^*(s) = \frac{1}{s}$$

Consequently, $\bar{A}_f^*(s)$ can also be determined.

4.7 Repair-equipment ROCOF

The ROCOF is one of important indices in reliability theory. Let $M_f(t)$ be the mean failure number of the repair-equipment in $(0, t]$, then its derivative $m_f(t) = \frac{d}{dt}M_f(t)$ is called the rate of occurrence of failure (ROCOF). According to Lam [34], the ROCOF can be evaluated in the following way:

$$m_f(t) = \sum_{m\in W, n\in F} \sum_{l=1}^{\infty} p_{ml_w}(t) r_{mnl}$$

where r_{mnl} is transition rate from state m to state n in lth cycle. Thus, we can obtain

$$m_f^*(s) = \sum_{m=1}^{k} \sum_{l=1}^{\infty} a^{l-1} \alpha p_{ml_w}^*(s)$$

where $m_f^*(s)$ is the Laplace transform of $m_f(t)$. Since $p_{ml_w}^*(s)$ has been determined in Section 4, we can then evaluate $m_f^*(s)$.

5 Replacement policy for the repair-equipment

5.1 Expected cost rate under policy N

In this section, we consider a replacement policy N based on the number of failures of the repair-equipment. The repair-equipment will be replaced by a new and identical one whenever the failure number of the repair-equipment reaches N. Our objective is to search an optimal replacement policy N^* such that the expected cost rate of the repair-equipment is minimized. To do this, we add the following assumptions.

Assumption 7 A replacement policy N based on the number of failures of the repair-equipment is used. The repair-equipment will be replaced sometime by a new and identical one, and the replacement time is negligible.

Assumption 8 The repair cost rate of the repair-equipment is c_r, the working reward rate of the repair-equipment is c_w, and the fixed replacement cost of the repair-equipment is C.

Let τ_1 be the first replacement time of the repair-equipment after installation, and $\tau_n (n \geq 2)$ be the time between the $(n-1)$th and the nth replacements of the repair-equipment under policy N. Clearly, $\{\tau_1, \tau_2, \cdots\}$ forms a renewal process, and the time between two consecutive replacements is called a renewal cycle.

Let $C(N)$ be the expected cost rate of the repair-equipment under policy N. Thus, according to the model assumptions and the renewal reward theorem (see, for example

Ross [31]), we have

$$
\begin{aligned}
C(N) &= \frac{\text{the expected cost incurred in a renewal cycle}}{\text{the expected length of a renewal cycle}} \\
&= \frac{E(c_r \sum_{j=1}^{N-1} Y_j + C - c_w \sum_{j=1}^{N} X_j)}{E(\sum_{j=1}^{N-1} Y_j + \sum_{j=1}^{N} X_j)} \\
&= \frac{c_r \sum_{j=1}^{N-1} \frac{1}{b^{j-1}\beta} + C - c_w \sum_{j=1}^{N} \frac{1}{a^{j-1}\alpha}}{\sum_{j=1}^{N-1} \frac{1}{b^{j-1}\beta} + \sum_{j=1}^{N} \frac{1}{a^{j-1}\alpha}} \qquad (24)
\end{aligned}
$$

Obviously, we can determine an optimal replacement policy N^* by analytical or numerical methods such that $C(N)$ is minimized.

5.2 Optimal replacement policy N^*

In order to determine the optimal replacement policy N^* for minimizing $C(N)$ explicitly, we rewrite the equation (24) as

$$C(N) = A(N) - c_w,$$

where

$$A(N) = \frac{(c_r + c_w) \sum_{j=1}^{N-1} \frac{1}{b^{j-1}\beta} + C}{\sum_{j=1}^{N} \frac{1}{a^{j-1}\alpha} + \sum_{j=1}^{N-1} \frac{1}{b^{j-1}\beta}}.$$

Thus, to minimize $C(N)$ is equivalent to minimize $A(N)$. The difference of $A(N+1)$ and $A(N)$ is given as:

$$
\begin{aligned}
A(N+1) - A(N) &= \frac{(c_r + c_w) \sum_{j=1}^{N} \frac{1}{b^{j-1}\beta} + C}{\sum_{j=1}^{N+1} \frac{1}{a^{j-1}\alpha} + \sum_{j=1}^{N} \frac{1}{b^{j-1}\beta}} - \frac{(c_r + c_w) \sum_{j=1}^{N-1} \frac{1}{b^{j-1}\beta} + C}{\sum_{j=1}^{N} \frac{1}{a^{j-1}\alpha} + \sum_{j=1}^{N-1} \frac{1}{b^{i-1}\beta}} \\
&= \frac{\frac{c_r + c_w}{b^{N-1}\beta} \sum_{j=1}^{N} b^{j-1} + C}{\frac{1}{a^N\alpha} \sum_{j=1}^{N+1} a^{j-1} + \frac{1}{b^{N-1}\beta} \sum_{j=1}^{N} b^{j-1}} - \frac{\frac{c_r + c_w}{b^{N-2}\beta} \sum_{j=1}^{N-1} b^{j-1} + C}{\frac{1}{a^{N-1}\alpha} \sum_{j=1}^{N} a^{j-1} + \frac{1}{b^{N-2}\beta} \sum_{j=1}^{N-1} b^{j-1}} \\
&= \frac{(c_r + c_w)h(N) - C(a^N\alpha + b^{N-1}\beta)}{a^N b^{N-1}\alpha\beta[\frac{1}{a^N\alpha} \sum_{j=1}^{N+1} a^{j-1} + \frac{1}{b^{N-1}\beta} \sum_{j=1}^{N} b^{j-1}][\frac{1}{a^{N-1}\alpha} \sum_{j=1}^{N} a^{j-1} + \frac{1}{b^{N-2}\beta} \sum_{j=1}^{N-1} b^{j-1}]},
\end{aligned}
$$

where $h(N) = \sum_{j=1}^{N} a^j - \sum_{j=1}^{N-1} b^j$.

We now structure an auxiliary function

$$B(N) = \frac{(c_r + c_w)h(N)}{C(a^N \alpha + b^{N-1}\beta)} \tag{25}$$

Because the denominator of $A(N+1) - A(N)$ is always positive, the sign of $A(N+1) - A(N)$ is the same as the sign of its numerator. Thus, the following lemma is straightforward.

Lemma 3

$$A(N+1) \begin{matrix} > \\ = \\ < \end{matrix} A(N) \iff B(N) \begin{matrix} > \\ = \\ < \end{matrix} 1.$$

Lemma 1 shows that the monotonicity of $A(N)$ is determined by the value of $B(N)$. We can simplify the difference of $B(N+1)$ and $B(N)$ as follows.

$$\begin{aligned} B(N+1) - B(N) &= \frac{(c_r + c_w)h(N+1)}{C(a^{N+1}\alpha + b^N \beta)} - \frac{(c_r + c_w)h(N)}{C(a^N \alpha + b^{N-1}\beta)} \\ &= \frac{c_r + c_w}{C}\left(\frac{h(N+1)(a^N\alpha + b^{N-1}\beta) - h(N)(a^{N+1}\alpha + b^N\beta)}{(a^{N+1}\alpha + b^N\beta)(a^N\alpha + b^{N-1}\beta)}\right) \\ &= \frac{c_r + c_w}{C}\left(\frac{a^N\alpha[h(N+1) - ah(N)] + b^{N-1}\beta[h(N+1) - bh(N)]}{(a^{N+1}\alpha + b^N\beta)(a^N\alpha + b^{N-1}\beta)}\right), \end{aligned}$$

where

$$h(N+1) - ah(N) = \left(\sum_{j=1}^{N+1} a^j - \sum_{j=1}^{N} b^j\right) - a\left(\sum_{j=1}^{N} a^j - \sum_{j=1}^{N-1} b^j\right) = (a - b^N) + (a-1)\sum_{j=1}^{N-1} b^j \geq 0,$$

$$h(N+1) - bh(N) = \left(\sum_{j=1}^{N+1} a^j - \sum_{j=1}^{N} b^j\right) - b\left(\sum_{j=1}^{N} a^j - \sum_{j=1}^{N-1} b^j\right) = (1-b)\sum_{j=1}^{N} a^j + (a^{N+1} - b) \geq 0.$$

Thus, $B(N+1) - B(N) \geq 0$, this implies:

Lemma 4 $B(N)$ is nondecreasing in N.

According to Lemmas 3 and 4, an analytic expression for an optimal policy for minimizing $A(N)$ can be obtained. The following theorem can be obtained.

Theorem 8 The optimal replacement policy N^* can be determined by

$$N^* = \min\{N \mid B(N) \geq 1\} \tag{26}$$

Furthermore, if $B(N^*) > 1$, then the optimal policy N^* is unique.

Because $B(N)$ is nondecreasing in N, there exists an integer N^* such that

$$B(N) \geq 1 \iff N \geq N^*$$

and

$$B(N) < 1 \iff N < N^*.$$

Note that N^* is the minimum satisfying (26), and the policy N^* is an optimal replacement policy. Furthermore, it is easy to see that if $B(N^*) > 1$, then the optimal policy is also uniquely existent.

5.3 A numerical example for policy N

In this section, we provide a numerical example to illustrate the optimal replacement policy N^* for minimizing $C(N)$. Now, let

$$l_1 = \sum_{j=1}^{N} \frac{1}{a^{j-1}},\ l_2 = \sum_{j=1}^{N-1} \frac{1}{b^{j-1}},\ l_3 = \sum_{j=1}^{N} a^j,\ l_4 = \sum_{j=1}^{N-1} b^j,$$

then equations (24) and (25) become respectively

$$C(N) = \frac{\frac{c_r}{\beta} l_2 + C - \frac{c_w}{\alpha} l_1}{\frac{l_1}{\alpha} + \frac{l_2}{\beta}} \tag{27}$$

and

$$B(N) = \frac{(c_r + c_w)(l_3 - l_4)}{C(a^N \alpha + b^{N-1}\beta)}. \tag{28}$$

Further let $a = 1.15, b = 0.85, \alpha = 0.02, \beta = 1, c_r = 15, c_w = 60$ and $C = 4000$. Substituting the above values into equations (27) and (28), we can respectively obtain the results presented in Figure 1 and Table 1.

It is easy to find that $C(10) = -42.3998$ is the minimum of the expected cost rate of the repair-equipment. In other words, the optimal policy is $N^* = 10$ and we should replace the repair-equipment at the time of the 10th failure. And the optimal policy $N = 10$ is unique from Figure 1, Table 1 or the conclusion of Theorem 8 because $B(10) = 1.1396 > 1$.

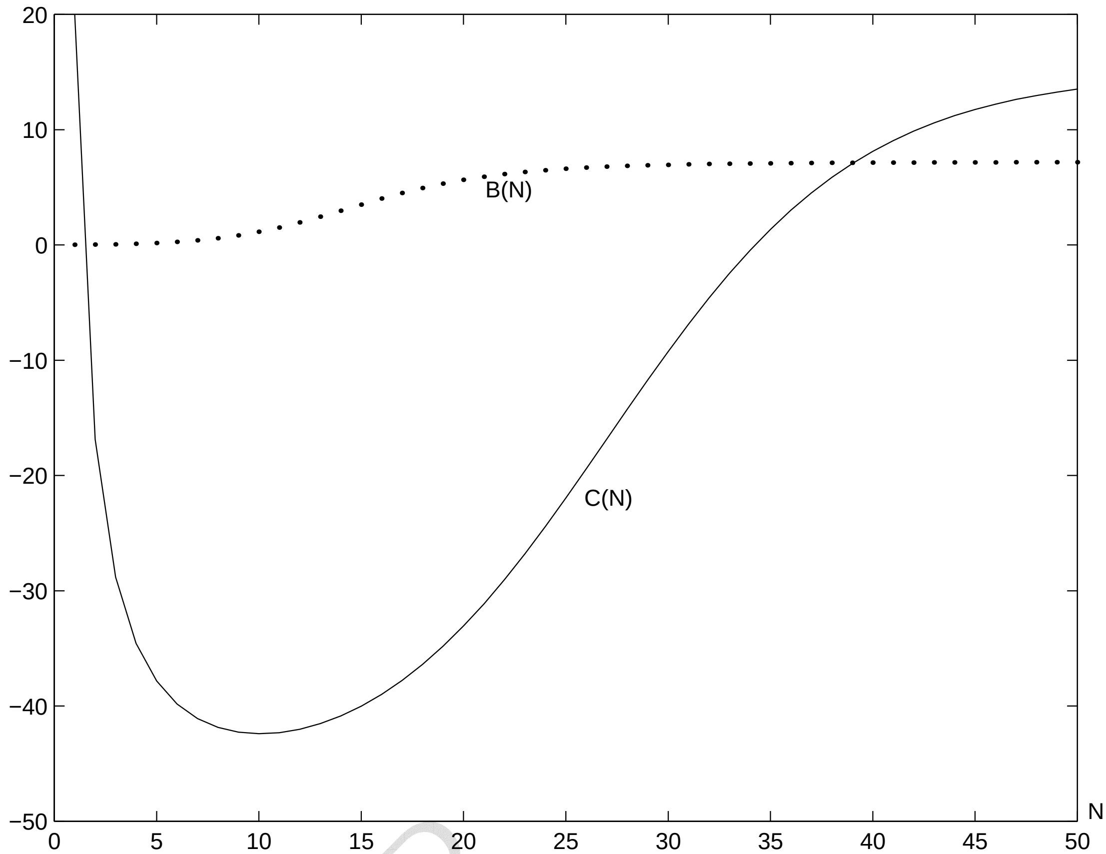

Fig.1 The plots of C(N) and B(N) against failure number N

Table 1: Results obtained from Equations (27) and (28)

N	C(N)	B(N)	N	C(N)	B(N)	N	C(N)	B(N)
1	20.0000	0.0211	11	-42.3063	1.5166	21	-31.1221	5.9243
2	-16.8684	0.0347	12	-42.0093	1.9569	22	-29.0294	6.1500
3	-28.8058	0.0603	13	-41.5236	2.4485	23	-26.7891	6.3349
4	-34.5588	0.1027	14	-40.8560	2.9723	24	-24.4224	6.4856
5	-37.8255	0.1682	15	-40.0090	3.5050	25	-21.9552	6.6080
6	-39.8282	0.2646	16	-38.9820	4.0234	26	-19.4178	6.7074
7	-41.0853	0.4007	17	-37.7733	4.5075	27	-16.8432	6.7882
8	-41.8512	0.5862	18	-36.3815	4.9439	28	-14.2662	6.8541
9	-42.2633	0.8302	19	-34.8067	5.3255	29	-11.7212	6.9080
10	**-42.3998**	**1.1396**	20	-33.0515	5.6513	30	-9.2408	6.9523

6 Concluding remarks

In this paper, the reliability and replacement policy of a $k/n(F)$ system with repairable repair-equipment are analyzed. We assume that the working time distributions and the repair time distributions of all components, and the repair-equipment in the system are exponential, repair on a failed component is perfect, and repair on the repair-equipment is imperfect, i.e. the survival times of the repair-equipment after repair form a geometric process. We then derive properties and reliability indices for such a system, as well as replacement policy for the repair-equipment, using theories from the geometric process, the vector Markov process and the queueing theory. The results in this paper are interesting from a theoretical perspective and useful from a practical application perspective. The following findings are also achieved.

(1) According to the assumptions in Section 2, we have indicated that the $k/n(F)$ repairable system with repairable repair-equipment is equivalent to a $M/M(M/M)/1/k/n$ queueing system with repairable service station. By using the queueing theory, we derived properties and reliability indices of the $k/n(F)$ repairable system on the basis of the concept of busy period for classical $M/M/1$ queueing system. It is a generalization of the existing work. For example, if we let $a = b = 1$, the $M/M(M/M)/1/k/n$ queueing system with repairable service station in which repair is imperfect will reduce to a $M/M(M/M)/1/k/n$ queueing system with repairable service station in which repair is perfect. Therefore, the method introduced in this paper is new for analyzing the reliability of the $k/n(F)$ repairable system with repairable repair-equipment.

(2) Although the geometric process has been wildly applied to the maintenance optimization for the simple repairable system and the multi-component series, parallel and cold standby repairable systems, this is the first work to apply the geometric process to a $k/n(F)$ system with repairable repair-equipment.

(3) Let $N(t)$ be the state of the $k/n(F)$ system with repairable repair-equipment at time t. It is clear from model assumptions that $\{N(t),\ t \geq 0\}$ is not a Markov process. However, it can be extended to be a vector Markov process (i.e. a two-dimensional Markov process) by introducing a supplementary variable. To obtain properties and reliability indices of the system, we need to determine the state probabilities of the system at time t. Accordingly, we can derive the system of differential equations about $p_{ml_w}(t),\ (m = 0, 1, \cdots, k-1; l = 1, 2, \cdots)$ and $p_{ml_f}(t),\ (m = 1, 2, \cdots, k-1, k; l = 1, 2, \cdots)$. Finally, the Laplace transform results of reliability indices of the system are obtained. In general, conducting an inverse Laplace transformation to obtain transient results of reliability indices is not easy, and results from Laplace transformation of reliability in-

dices of the system are hard to obtain for practical application. Thus, for practical use, a numerical method based on the Runge-Kutta method is often adopted (see, e.g., Zhang and Wang [29]).

(4) In this paper, we consider a replacement policy N based on the number of failures of the repairable repair-equipment. An optimal replacement policy N^* for minimizing $C(N)$ is determined. The uniqueness of the optimal replacement policy N^* is proved. And a given numerical example can also illustrate the theoretical result. Theorem 8 can be used in practice, as based on this theorem one can stop searching the optimum whenever $B(N)$ crosses over 1.

(5) Our future work will be to conduct research for the situation where all components in the system are not "as good as new" or there are $r(r > 1)$ repairable repair-equipment.

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