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Multiple Model $\mathcal{L}_1$ Adaptive Fault-Tolerant Control of Small Unmanned Aerial Vehicles

Toufik Souanef[1]

[1]Cranfield University, United Kingdom. Email: toufik.souanef@cranfield.ac.uk

ABSTRACT

This paper presents a method for fault-tolerant control of small fixed-wing Unmanned Aerial Vehicles (UAVs). The proposed design is based on multiple-model $\mathcal{L}_1$ adaptive control. The controller is composed of a nominal reference model and a set of suboptimal reference models. The nominal model is the one with desired dynamics that are optimal regarding some specific criteria. In a suboptimal model the performance criteria are reduced, it is designed to ensure system robustness in the presence of critical failures. The controller was tested in simulations and it was shown that the multiple model $\mathcal{L}_1$ adaptive controller stabilizes the system in case of inversion of the control input, while the $\mathcal{L}_1$ adaptive controller with a single nominal model fails.

INTRODUCTION

Unmanned Aerial Vehicles (UAVs), commonly known as drones and referred to as Remotely Piloted Aircraft (RPA) by the International Civil Aviation Organisation (ICAO), are aircraft without a human pilot aboard. According to the assigned missions or to their size, there are many different classes of UAVs (Valavanis and Vachtsevanos, 2015). This work focuses on small fixed-wing UAVs that is, with wingspans less than 2 metres and payload smaller than 2 kg. Small UAVs are gaining growing interest because of their low cost, high manoeuvrability, and simple maintenance. They are used for a wide range of military and civilian tasks (Austin, 2011). The operation of UAVs, especially in urban environments, needs a high degree of safety and reliability. However, small UAVs are generally built with low-cost components and materials, which increases the probability of occurrence of faults and failures. For that reason, the design of fault-tolerant control systems is required (Blanke et al., 2006; Ducard, 2009; Patton, 1997; Zhang and Jiang, 2008).

Fault-tolerant control is defined as a system that possesses the ability to accommodate failures automatically (Zhang and Jiang, 2008). Fault-tolerant control systems are classified as either passive or active (Hwang et al., 2009; Rotondo, 2017). Passive fault-tolerant control is based on robust control while assuming the worst case conditions (Amin and Hasan, 2019; Benosman, 2011; Edwards et al., 2000; Wang, 2010; Yang et al., 2001). Nevertheless, the designed controllers tend to be conservative from performance viewpoint (Jiang and Yu, 2012). Active fault-tolerant controllers are composed of a fault detection scheme and a supervision module. On the basis of the information of the former, the supervision module may decide how to reconfigure the controller (Abbaspour et al., 2020; Amin and Hasan, 2019; Rotondo, 2017). However, applying such advanced control systems for small UAVs is difficult, because of their limited computing resources.

A compromise between the two approaches is adaptive control, which is based on the reconfiguration of the controller parameters without involving an explicit fault detection module (Bodson, 2003; Ma et al., 2020; Nian et al., 2020; Tao, 2004; Xue et al., 2020; Yang et al., 2014).

A crucial aspect in applying adaptive control techniques to real-world systems is the transient response guarantee, in the absence of which, overly poor tracking behaviour can occur before ideal asymptotic convergence takes place (Zang and Bitmead, 1990). Moreover, the transient performance improvement cannot be achieved through high-gain feedback, which will degrade the robustness of the closed-loop system. However, most adaptive control methods focus on the asymptotic performance, providing no transient performance guarantee without resorting to high-gain feedback. One solution to this issue is based on $\mathcal{L}_1$ adaptive control (Hovakimyan and Cao, 2010). The adaptive control architecture decouples the estimation loop from the control loop through the introduction of a low-pass filter. As a result, arbitrarily fast adaptation can be used without sacrificing system robustness. The benefit of $\mathcal{L}_1$ adaptive control is its capacity for fast and robust adaptation that leads to desired transient performance for both system signals, inputs and outputs. These characteristics make it suitable for systems with unknown dynamics and subject to possible faults and external disturbances, such as small UAVs.

Despite the excellent performance of $\mathcal{L}_1$ adaptive control, for fault-tolerant control (Ackerman et al., 2017; Ahmadi et al., 2019; Dobrokhodov et al., 2013; Mühlegg et al., 2015; Patel et al., 2009; Sørensen and Breivik, 2015; Tian et al., 2020; Zhou et al., 2019), it is still true that when the uncertainties induced by disturbances, faults or failures are too large, they may reduce the performance of the controller or even make the system unstable. Actually, if a fault or a failure occurs on the system, the unknown parameters may go outside the predefined sets of the control design, which may lead to poor system performance or more critically to system instability. Furthermore, when a fault affects the system actuators it reduces their capabilities and, if the nominal performance of the system is maintained, the actuators will work beyond their nominal set point, which might lead to severe failures that cannot be compensated by a fault-tolerant controller. Therefore, it is not reasonable to maintain the same desired performance of the system, because after a fault or a failure it is not possible to recover the nominal performance. This is especially true for non-redundant systems such as low-cost UAVs.

The proposed solution is based on the application of the multiple model $\mathcal{L}_1$ adaptive controller (Souanef and Fichter, 2015). The key idea is to design an $\mathcal{L}_1$ adaptive controller with a nominal reference model and a set of suboptimal reference models. The nominal model is the model with desired dynamics that are optimal regarding some specific criteria. A suboptimal model does not necessarily verify these specifications. It is designed to ensure system robustness in the presence of large uncertainties. This multiple-model $\mathcal{L}_1$ adaptive control design can expand the performance of the $\mathcal{L}_1$ adaptive control schemes to effectively deal with plant hard failures such as the inversion of the control direction (a long-standing issue that is difficult for a single-model adaptive controller to deal with) which may be caused by uncertain system structural damage and component (actuator or sensor) failures.

A similar approach for performance degradation based on multiple model control was presented in (Jiang and Zhang, 2006; Zhang and Jiang, 2003). The design was made under the assumption that the model of the plant has no uncertainties, which is not realistic, especially for post-fault systems.

Furthermore, only actuator faults were addressed while structural faults were not considered.

The main contributions of this paper are:

- Development of a method for adaptive fault-tolerant control based on an $\mathcal{L}_1$ adaptive controller with a nominal reference model and a set of suboptimal reference models, so as to avoid system instability in the presence of hard faults/failures.
- Extension of the method proposed in (Souanef and Fichter, 2015) to Multi-Input Multi-Output (MIMO) systems.

NOTATION

Throughout the paper, $\|\cdot\|$ denotes the 2-norm and $\|\cdot\|_\infty$ denotes the infinity norm of a vector or a matrix. The notation $\|\xi\|_{\mathcal{L}_\infty}$ denotes the $\mathcal{L}_\infty$-norm of the vector $\xi(t)$. For a stable proper transfer matrix $G(s)$, $\|G(s)\|_{\mathcal{L}_1}$ denotes its $\mathcal{L}_1$-norm. $\mathbb{R}^n$ denotes the n-dimensional real vector space. $\mathbb{I}$ denotes an identity matrix of appropriate dimensions. Boldface notation is used for matrices, vectors, and tensors; italics are for for all variables and lower-case Greek letters; and Roman for all numerals, upper-case Greek characters, and mathematical operators.

PROBLEM FORMULATION

For control design, the dynamic model of an aircraft can be formulated as the following class of MIMO systems (Lavretsky and Wise, 2013)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_p x(t) + \mathbf{B}_p \mathbf{u}_p(t) + \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{1}$$

where $\mathbf{A}_p = \mathbf{A} + \Delta\mathbf{A} \in \mathbb{R}^{n\times n}$ is an unknown matrix, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a known matrix, $\Delta\mathbf{A} \in \mathbb{R}^{n\times n}$ an unknown matrix of the system dynamics, $\mathbf{B}_p = \mathbf{B}(\mathbb{I}_m + \Delta\mathbf{B}) \in \mathbb{R}^{n\times m}$ is an unknown matrix, $\mathbf{B} \in \mathbb{R}^{n\times m}$ is a known matrix, $\Delta\mathbf{B} \in \mathbb{R}^{m\times m}$ is an unknown matrix of the control input uncertainties, $\mathbf{C} \in \mathbb{R}^{m\times n}$ is a known matrix, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector which is assumed to be available through measurement, $\mathbf{u}_p(t) \in \mathbb{R}^m$ is the control input vector $\mathbf{y}(t) \in \mathbb{R}^m$ is the output vector and $\mathbf{f}(t, \mathbf{x}) \in \mathbb{R}^n$ is a vector of unknown nonlinear functions.

Now consider the control

$$\mathbf{u}_p(t) = \mathbf{u}(t) + \mathbf{K}_l\mathbf{x}(t), \tag{2}$$

where $\mathbf{K}_l \in \mathbb{R}^{m\times n}$ is a gain matrix that defines $\mathbf{A}_m = \mathbf{A} + \mathbf{B}\mathbf{K}_l$, where $\mathbf{A}_m \in \mathbb{R}$ is a Hurwitz matrix that defines the desired dynamics of the system. The resulting system to be controlled by the adaptive control is

$$\dot{\mathbf{x}}(t) = \mathbf{A}_m\mathbf{x}(t) + \mathbf{B}\omega\mathbf{u}(t) + \tilde{\mathbf{f}}(t,\mathbf{x}), \tag{3}$$

where $\omega = \mathbb{I}_m + \Delta\mathbf{B}$ and $\tilde{\mathbf{f}}(t,\mathbf{x}) = \Delta\mathbf{A}\mathbf{x}(t) + (\omega - \mathbb{I}_m)\mathbf{K}_l x(t) + \mathbf{f}(t,\mathbf{x})$. Assuming $\tilde{\mathbf{f}}(t,\mathbf{x}) = \mathbf{B}\big(\theta^\top\mathbf{x}(t) + \eta_m(t)\big) + \eta_u(t)$, the system in (3) can be parametrised as follows

$$\dot{\mathbf{x}}(t) = \mathbf{A}_m\mathbf{x}(t) + \mathbf{B}\big(\omega\mathbf{u}(t) + \theta^\top\mathbf{x}(t) + \eta_m(t)\big) + \eta_u(t), \tag{4}$$

where $\theta^\top \in \mathbb{R}^{m\times n}$ is a matrix of constant unknown parameters representing model uncertainties, $\eta_m(t) \in \mathbb{R}^m$ is an unknown matched disturbance, and $\eta_u(t) \in \mathbb{R}^n$ is an unknown unmatched disturbance.

Assumption 1. The unknown model parameters are bounded, i.e., $\theta \in \Theta$, where Θ is a known compact convex set. The system input gain matrix ω is assumed to be an unknown (non-singular) strictly row-diagonally dominant matrix with $\mathrm{sgn}(\omega_{ii})$ known. Also, it is assumed that there exists a known compact convex set Ω such that $\omega \in \Omega \subset \mathbb{R}^{m\times m}$.

Assumption 2. The non-linear function $\eta_m(t)$ is uniformly bounded, i.e., there exist unknown real constant $L_m > 0$, such that for all $t \geq 0$ the following bound hold:

$$\|\eta_m(t)\| \leq L_m.$$

Assumption 3. There exist unknown real constant $L_u > 0$, such that for all $t \geq 0$ the following bound hold

$$\|\eta_u(t)\| \leq L_u.$$

Remark 1. Assumptions 2 and 3 are acceptable for real systems, given that a superior bound of disturbances, which the system may hold without being broken, is usually known from technical specifications or engineering insights.

The objective is to design a state-feedback controller to ensure that the output of the system tracks a given piecewise continuous bounded reference signal $r(t)$ and consequently maintain the stability of the control system despite the presence of faults and/or external disturbances.

$\mathcal{L}_1$ ADAPTIVE CONTROL

We consider the architecture of the $\mathcal{L}_1$ adaptive controller which is composed of the state predictor, the adaptation law and the control law (Figure 1).

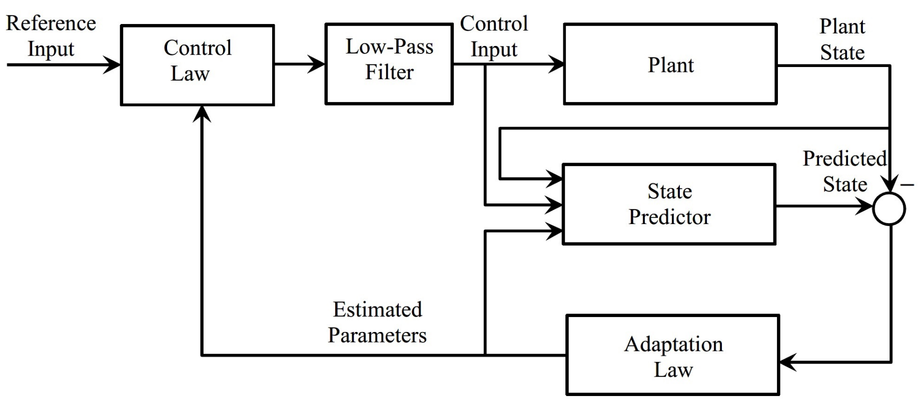


Fig. 1. Block diagram of the $\mathcal{L}_1$ adaptive controller.

The state predictor is defined as

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_m \hat{\mathbf{x}}(t) + \mathbf{B}\big(\hat{\omega}(t)\mathbf{u}(t) + \hat{\theta}^\top(t)\mathbf{x}(t) + \hat{\eta}_m(t)\big) + \hat{\eta}_u(t), \tag{5}$$

where $\hat{\mathbf{x}}(t)$ is the predicted state and $\hat{\theta}(t)$, $\hat{\omega}(t)$, $\hat{\eta}_m(t)$, and $\hat{\eta}_u(t)$ are the estimates of the unknown system parameters and disturbances.

The sliding surface is defined as

$$\sigma(t) = \lambda\tilde{\mathbf{x}}(t), \tag{6}$$

where $\tilde{\mathbf{x}}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$ is the state estimation error and $\lambda \in \mathbb{R}^{m\times n}$ is a constant arbitrary matrix, chosen such that $\lambda\mathbf{B}$ is non-singular and the coefficients $\lambda(i, j) : \ i = 1..n; j = 1..m$ form a stable hyperplane.

The estimation of the matched disturbance $\eta_m(t)$ is defined by

$$\hat{\eta}_m(t) = \begin{cases} -(\lambda\mathbf{B})^{-1}\big(\lambda\mathbf{A}_m\tilde{\mathbf{x}}(t) + \rho\sigma(t)\big) - \hat{L}_m(t)\frac{\mathbf{B}^\top\lambda^\top\sigma(t)}{\|\mathbf{B}^\top\lambda^\top\sigma(t)\|}, & \text{if } \sigma(t) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $\rho > 0$ is arbitrary and the estimated bound $\hat{L}_m(t)$ is given by

$$\dot{\hat{L}}_m(t) = \Gamma\|\sigma^\top(t)\lambda\mathbf{B}\|, \tag{8}$$

where $\Gamma \in \mathbb{R}^+$ is the adaptation rate.

The estimation of the unmatched disturbance $\eta_u(t)$ is defined by

$$\hat{\eta}_u(t) = \begin{cases} -\hat{L}_u(t)\frac{\lambda^\top\sigma(t)}{\|\lambda^\top\sigma(t)\|}, & \text{if } \sigma(t) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where the estimated bound $\hat{L}_u(t)$ is computed by

$$\dot{\hat{L}}_u(t) = \Gamma\|\sigma^\top(t)\lambda\|, \tag{10}$$

The input gain matrix ω and unknown parameters matrix θ are estimated by

$$\begin{aligned}\dot{\hat{\omega}}(t) &= -\Gamma\,\mathrm{Proj}\big(\hat{\omega}(t), \mathbf{u}(t)\,\sigma^\top(t)\lambda\,\mathbf{B}\big)^\top,\\ \dot{\hat{\theta}}(t) &= -\Gamma\,\mathrm{Proj}\big(\hat{\theta}(t), \mathbf{x}(t)\,\sigma^\top(t)\lambda\,\mathbf{B}\big).\end{aligned} \tag{11}$$

The control law is given by

$$\mathbf{u}(s) = -\mathbf{K}\,\mathbf{D}(s)\Big(\hat{\nu}_1(s) + \hat{\nu}_2(s) - \mathbf{K}_g\,\mathbf{r}(s)\Big), \tag{12}$$

where $\mathbf{D}(s)$ is an $m \times m$ proper transfer matrix; $\mathbf{K} \in \mathbb{R}^{m\times m}$; $\mathbf{K}_g = -(\mathbf{C}\mathbf{A}_m^{-1}\mathbf{B})^{-1}$ is the pre-filter of the MIMO control law; $\hat{\nu}_1(s)$ is the Laplace transformation of $\hat{\nu}_1(t) = \hat{\theta}^\top(t)\mathbf{x}(t) + \hat{\omega}(t)\mathbf{u}(t)$; $\mathbf{H}_m(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A}_m)^{-1}\mathbf{B}$; $\mathbf{H}_0(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A}_m)^{-1}$; and $\hat{\nu}_2 = \hat{\eta}_m(t) + \mathbf{H}_m^{-1}(s)\mathbf{H}_0(s)\hat{\eta}_u(s)$.

The design of $\mathbf{D}(s)$ and $\mathbf{K}$ should lead to a strictly proper and stable filter transfer matrix

$$\mathbf{F}(s) = \omega\mathbf{K}\mathbf{D}(s)(\mathbb{I} + \omega\mathbf{K}\mathbf{D}(s))^{-1},$$

with static gain $\mathbf{F}(0) = \mathbb{I}$.

Let

$$\begin{aligned}L &= \max_{\theta\in\Theta}\|\theta\|_1,\ \mathbf{H}(s) = (s\mathbb{I} - \mathbf{A}_m)^{-1}\mathbf{B},\\ \mathbf{G}(s) &= \mathbf{H}(s)\big(\mathbb{I} - \mathbf{F}(s)\big).\end{aligned} \tag{13}$$

The $\mathcal{L}_1$ adaptive controller is subject to the $\mathcal{L}_1$ norm condition

$$\|\mathbf{G}(s)\|_{\mathcal{L}_1} L < 1. \tag{14}$$

Moreover, the design of $\mathbf{F}(s)$ needs to ensure that

$$\mathbf{G}_u(s) = (s\mathbb{I} - \mathbf{A}_m)^{-1} - \mathbf{F}(s)\mathbf{H}(s)\mathbf{H}_m^{-1}(s)\mathbf{H}_0(s), \tag{15}$$

is proper and stable. Furthermore, since the transfer matrix $\mathbf{G}_u(s)$ is proper and stable it has an $\mathcal{L}_1$ norm (Hovakimyan and Cao, 2010).

Remark 2. It has been shown in (Souanef et al., 2015) and (Souanef, 2019) that the adaptation laws of the external disturbances in equations (7) and (9) use the estimated bounds from equations (8) and (10). This relaxes the assumption that the bounds of the external disturbances are known, which is required in $\mathcal{L}_1$ adaptive control based on projection-type adaptive laws (Cao and Hovakimyan, 2008).

Remark 3. If a fault or failure occurs on the system, the unknown parameters may go outside the predefined sets. Therefore, the stability conditions in (14) and (15) may become not satisfied. Hence, it is necessary to maintain system stability and a minimum of good performance, this is done through the design of a set of suboptimal models which become effective when large uncertainties appear on the plant.

MULTIPLE MODEL $\mathcal{L}_1$ ADAPTIVE CONTROL OF MIMO SYSTEMS

Considering probable faults scenario, a set of plant parameterisations, based on multiple models, is arranged, and the objective is that the satisfactory controller is selected automatically to deal with every situation. This means that the model which is the best match of the plant is selected.

The desired performance of each model is made through the design of the pair $(A_{m(i)}, \mathbf{B}_i)$, for $i \in \mathcal{I}$.

The system in (1) can consequently be parameterised as follows

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{m(i)}\mathbf{x}(t) + \mathbf{B}_i\big(\omega_i\mathbf{u}(t) + \theta_i^\top\mathbf{x}(t) + \eta_{m(i)}(t)\big) + \eta_{u(i)}(t), \tag{16}$$

where $\mathbf{A}_{m(i)} \in \mathbb{R}^{n\times n}$ are known Hurwitz matrices that define the desired dynamics of the system $\mathbf{B}_i \in \mathbb{R}^{n\times m}$ are the desired input matrices, $\omega_i \in \mathbb{R}^{m\times m}$ are unknown constant matrices representing the system input gain, $\theta_i^\top \in \mathbb{R}^{m\times n}$ are matrices of constant unknown parameters representing model uncertainties, $\eta_{m(i)}(t) \in \mathbb{R}^m$ are unknown matched disturbances, and $\eta_{u(i)}(t) \in \mathbb{R}^n$ are unknown unmatched disturbances.

Assumption 4. The system input gain matrices ω_i are assumed to be unknown (non-singular) strictly row-diagonally dominant matrices with known signs of diagonals. Also, it is assumed that the unknown parameters are bounded, i.e., $\theta_i \in \Theta_i$, where Θ_i are known compact convex sets. Furthermore, the functions $\eta_{m(i)}$ and $\eta_{u(i)}$ are uniformly bounded, i.e., there exist unknown real constants $L_{m(i)} > 0$ and $L_{u(i)} > 0$, such that for all $t \geq 0$ $|\eta_{m(i)}(t)| \leq L_{m(i)}$ and $\left\|\eta_{u(i)}(t)\right\| \leq L_{u(i)}$.

Controller Design

The multiple model $\mathcal{L}_1$ adaptive controller, as shown in Figure 2, is composed of a set of state predictors, a set of adaptation laws, a set of control laws and a control input selector (switching system). The state predictors are defined by

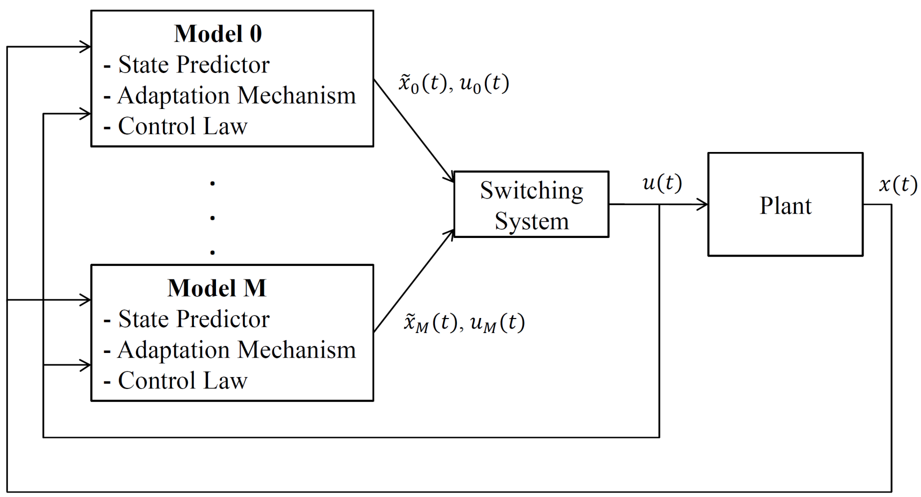


Fig. 2. Block diagram of the multiple model $\mathcal{L}_1$ adaptive controller.

$$\dot{\hat{\mathbf{x}}}_i(t) = \mathbf{A}_{m(i)}\hat{\mathbf{x}}_i(t) + \mathbf{B}_i\left(\hat{\omega}_i(t)\mathbf{u}(t) + \hat{\theta}_i^{\top}(t)\mathbf{x}(t) + \hat{\eta}_{m(i)}(t)\right) + \hat{\eta}_{u(i)}(t), \tag{17}$$

where $\hat{\mathbf{x}}_i(t)$ are the predicted states and, $\hat{\theta}_i(t)$, $\hat{\omega}_i(t)$, $\hat{\eta}_{m(i)}(t)$, and $\hat{\eta}_{u(i)}(t)$ are the estimates of the unknown system parameters and external disturbances. The initial state of the state predictor is equal to the plant state at switching time t_k:

$$\hat{\mathbf{x}}(t_k) = \mathbf{x}(t_k). \tag{18}$$

The sliding surfaces are given by

$$\sigma_i(t) = \lambda_i \tilde{\mathbf{x}}_i(t), \tag{19}$$

where $\tilde{\mathbf{x}}_i(t) = \hat{\mathbf{x}}_i(t) - \mathbf{x}(t)$ are the state estimation errors and $\lambda_i \in \mathbb{R}^{m\times n}$ are constant arbitrary matrices, chosen such that $\lambda_i \mathbf{B}_i$ are non-singular and the coefficients $\lambda_i(k, j) : \; k = 1...n; j = 1...m$ form a stable hyperplane.

The adaptation laws are given by

$$\begin{aligned}
\dot{\hat{\omega}}_i(t) &= -\Gamma_i \operatorname{Proj}\big(\mathbf{u}(t)\, \sigma_i^\top \lambda_i \mathbf{B}_i\big)^\top, \\
\dot{\hat{\theta}}_i(t) &= -\Gamma_i \operatorname{Proj}\big(\mathbf{x}(t)\, \sigma_i^\top \lambda_i \mathbf{B}_i\big), \\
\hat{\eta}_{m(i)}(t) &= \begin{cases} -(\lambda_i \mathbf{B}_i)^{-1}\big(\lambda_i \mathbf{A}_{m(i)} \tilde{\mathbf{x}}_i(t) + \rho_i \sigma_i\big) - \hat{L}_{m(i)}(t) \frac{\mathbf{B}_i^\top \lambda_i^\top \sigma_i}{\|\mathbf{B}_i^\top \lambda_i^\top \sigma_i\|} & \text{if } \sigma_i \neq 0, \\ 0 & \text{if not,} \end{cases} \\
\hat{\eta}_{u(i)}(t) &= \begin{cases} -\hat{L}_{u(i)}(t) \frac{\lambda_i^\top \sigma_i}{\|\lambda_i^\top \sigma_i\|} & \text{if } \sigma_i \neq 0, \\ 0 & \text{if not,} \end{cases} \\
\dot{\hat{L}}_{m(i)}(t) &= \Gamma_i \|\sigma_i^\top \lambda_i \mathbf{B}_i\|, \\
\dot{\hat{L}}_{u(i)}(t) &= \Gamma_i \|\sigma_i^\top \lambda_i\|,
\end{aligned} \tag{20}$$

where $\rho_i > 0$ are arbitrary and $\Gamma_i \in \mathbb{R}^+$ are the adaptation rates.

Let

$$\mathbf{H}_{m(i)}(s) = \mathbf{C}_i \big(s\mathbb{I} - \mathbf{A}_{m(i)}\big)^{-1} \mathbf{B}_i \text{ and } \mathbf{H}_{0(i)}(s) = \mathbf{C}_i \big(s\mathbb{I} - \mathbf{A}_{m(i)}\big)^{-1}.$$

The control laws are given by

$$\mathbf{u}_i(s) = -\mathbf{K}_i\, \mathbf{D}_i(s) \Big(\mathbf{K}_{g(i)}\, \mathbf{r}(s) - \hat{\nu}_{(i)}(s)\Big), \tag{21}$$

where $\hat{\nu}_{(i)}(s) = \hat{\nu}_{1(i)}(s) + \hat{\nu}_{2(i)}(s)\hat{\eta}_{u(i)}(s)$, $\hat{\nu}_{1(i)}(s)$ are the Laplace transformations of $\hat{\nu}_{1(i)}(t) = \hat{\theta}_i^\top(t)\mathbf{x}(t) + \hat{\omega}_i(t)\mathbf{u}_i(t)$, $\hat{\nu}_{2(i)}(s) = \hat{\eta}_{m(i)}(s) + \mathbf{H}_{m(i)}^{-1}(s)\mathbf{H}_{0(i)}(s)\hat{\eta}_{u(i)}(s)$, $\mathbf{K}_{g(i)} = -(\mathbf{C}_i\mathbf{A}_{m(i)}^{-1}\mathbf{B}_i)^{-1}$ are the pre-filters of the MIMO control laws, $\mathbf{D}_i(s)$ are $m \times m$ strictly proper transfer matrices and $\mathbf{K}_i \in \mathbb{R}^{m\times m}$.

Let $\mathbf{B}_i^\dagger = \left(\mathbf{B}_i^\top\mathbf{B}_i\right)^{-1}\mathbf{B}_i^\top$ be the pseudo-inverse of $\mathbf{B}_i$, considering that $\mathbf{B}_i$ has full column rank, then $\hat{\nu}_{2(i)}(t) = \hat{\eta}_{m(i)}(t) + \mathbf{B}_i^\dagger\hat{\eta}_{u(i)}(t)$.

For analysis purposes, without loss of generality, it is assumed that the control laws use the same filter parameters. $\mathbf{D}_i(s)$ are chosen $\mathbf{D}_i(s) = \frac{\mathbf{D}_0(s)}{s}$ and $\mathbf{K}_i = 1$, where $\mathbf{D}_0(s)$ is a proper stable transfer matrix.

Therefore, the control laws can be written as

$$\mathbf{u}_i(s) = \frac{\mathbf{D}_0(s)}{s}\Big(\mathbf{K}_{g(i)}\,\mathbf{r}(s) - \hat{\nu}_{(i)}(s)\Big). \tag{22}$$

The switching logic is defined by

$$\min_{i\in\mathcal{I}}\left\{J_i = c_1\,\|\tilde{\mathbf{x}}_i\|^2 + c_2\int_0^t e^{-c_3(t-\tau)}\|\tilde{\mathbf{x}}_i(\tau)\|^2 d\tau\right\}, \tag{23}$$

where c_1, c_2 and c_3 are arbitrary positive real. The model that minimises the criterion becomes the selected model and its output is applied to the plant.

Remark 4. For practical implementation the discrete-time version of the switching logic in (23) is given by

$$\min_{i\in\mathcal{I}}\left\{J_i = c_1\,\|\tilde{\mathbf{x}}_i(kT)\|^2 + \frac{c_2}{c_3kT+1}\sum_{j=0}^{j}\|\tilde{\mathbf{x}}_i(jT)\|^2\right\}, \tag{24}$$

where T is the sampling period.

Remark 5. It is assumed in this work that the switching is arbitrary, i.e., not dwell time or average dwell time. To prevent arbitrarily fast switching, a non-zero waiting time $T_{min} > 0$ is introduced after every switching. By the end of the waiting period T_{min}, the controller corresponding to the model with the minimum index is chosen (switched) to control the plant Narendra and Balakrishnan

(1997). Note that T_{min} is different from the dwell time τ, and a switching signal has a dwell time $\tau > 0$, used for waiting another controller being activated, if the switching times satisfy $t_{k+1} - t_k \geq \tau,\ \forall k > 0$ (Liberzon, 2003).

Remark 6. It is a common practice to stop the control switching when $J_i(t) \leq \epsilon \forall i = 1, 2, ..., N$, for some pre-chosen, arbitrary and small $\epsilon > 0$ Tan et al. (2017a).

Remark 7. It is quiet understood that both traditional projection-based adaptive law (Cao and Hovakimyan, 2008) and the piecewise-constant adaptive law (Cao and Hovakimyan, 2009) can be applied to the design of the multiple model $\mathcal{L}_1$ adaptive controller. The advantage of the sliding mode adaptation law is that it permits to estimate the upper bounds of the external disturbances which makes the system more robust.

Controller Analysis

In this section, the performance of the $\mathcal{L}_1$ adaptive controller is analysed. More specifically it is shown that:

- The reference models resulting from perfect knowledge of the uncertainties and a corresponding non-adaptive controller are stable, subject to some conditions involving the filter $D_0(s)$.
- The prediction errors, i.e. the errors between the states of the plant and those of the state predictors, are bounded.
- The differences between the states/input of the system and those of the reference systems are proportional to the prediction error

For a switching system, it is not straightforward to compute the $\mathcal{L}_1$ norm condition in (14) and (15). Actually, for Linear Time Invariant (LTI) systems, the $\mathcal{L}_1$ norm is readily computed from the impulse response. However, for a switched system, the impulse response is time dependent (switching signal-dependent), and computing the $\mathcal{L}_1$ norm is not as straightforward as in the LTI case.

A similar approach to (Snyder, 2019) is applied here. It is based on a new method of analysing

$\mathcal{L}_1$ adaptive control. This approach results in necessary and sufficient conditions provided in the form of linear matrix inequalities (LMIs).

Reference Models Analysis

The reference system with the nominal parameters is defined as follows

$$\begin{aligned} \dot{\mathbf{x}}_r(t) =& \mathbf{A}_{m(i)} x_r(t) + \mathbf{B}_i \left(\omega_i \mathbf{u}_r(t) + \theta_i^\top \mathbf{x}_r(t) + \eta_{m(i)}(t)\right) + \eta_{u(i)}(t), \quad \mathbf{x}_r(0) = \mathbf{x}_0 \\ \mathbf{u}_r(s) =& -\frac{\mathbf{D}_0(s)}{s}\left(\nu_{r(i)}(s) - \mathbf{K}_g(i)\mathbf{r}(s)\right) \end{aligned} \tag{25}$$

where $\nu_{r(i)}(s) = \nu_{1(i)}(s) + \nu_{2(i)}(s)$, $\nu_{1(i)}(s)$ are the Laplace transformations of $\nu_{1(i)}(t) = \theta_i^\top \mathbf{x}_r(t) + \omega_i(t)\mathbf{u}_r(t)$ and $\nu_{2(i)}(s) = \eta_{m(i)}(s) + \phi_i(s)\eta_{u(i)}(s)$, with $\phi_i(s) = \mathbf{H}_{m(i)}^{-1}(s)\mathbf{H}_{0(i)}(s)$.

Alternatively we can write $\nu_{2(i)}(t) = \eta_{m(i)}(t) + \mathbf{B}_i^\dagger \eta_{u(i)}(t)$.

Remark 8. It should be noted that the reference control law is not implementable, since it depends on the unknown parameters and it is used only for analysis purposes.

Letting $\left(\mathbf{A}_f, \mathbf{B}_f, \mathbf{C}_f, \mathbf{D}_f\right)$ be a minimal realisation of $\mathbf{D}_0(s)$ with n_f states, the reference system dynamics can be written in state-space form

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_r(t) \\ \dot{\mathbf{x}}_{f_1}(t) \\ \dot{\mathbf{x}}_{I_1}(t) \end{bmatrix} &= \underbrace{\left[\begin{array}{c|cc} \mathbf{A}_{m(i)} + \mathbf{B}_i \theta_i^\top & 0 & -\mathbf{B}_i \omega_i \\ \hline \mathbf{B}_f \theta_i^\top & \mathbf{A}_f & \mathbf{B}_f \omega_i \\ \mathbf{D}_f \theta_i^\top & \mathbf{C}_f & \mathbf{D}_f \omega_i \end{array}\right]}_{\bar{A}_i} \begin{bmatrix} \mathbf{x}_r(t) \\ \mathbf{x}_{f_1}(t) \\ \mathbf{x}_{I_1}(t) \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} \mathbf{B}_i \\ \mathbf{B}_f \\ \mathbf{D}_f \end{bmatrix}}_{\bar{B}_i} \nu_{2(i)}(t) - \begin{bmatrix} 0 \\ \mathbf{B}_f K g_i \\ \mathbf{D}_f K g_i \end{bmatrix} r(t) \\ [u_r(t)] &= \underbrace{\begin{bmatrix} 0 & 0 & -\mathbb{I} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \mathbf{x}_r(t) \\ \mathbf{x}_{f_1}(t) \\ \mathbf{x}_{I_1}(t) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x}_r(0) \\ \mathbf{x}_{f_1}(0) \\ \mathbf{x}_{I_1}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{26}$$

where $\mathbf{x}_{f_1}$, $\mathbf{x}_{I_1}$ are the states of $\mathbf{D}_0(s)$ and the integrator respectively.

The reference control law can be rewritten in more compact form as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \bar{\mathbf{A}}_i \bar{\mathbf{x}}(t) + \bar{\mathbf{B}}_i \nu_{2(i)}(t) + \bar{\mathbf{E}}_i r(t), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0, \\ \mathbf{u}_r(t) &= \bar{\mathbf{C}} \bar{\mathbf{x}}(t), \end{aligned} \tag{27}$$

where $\bar{\mathbf{x}}^\top(t) \triangleq \left[\mathbf{x}_r^\top(t), \mathbf{x}_{f_1}^\top(t), \mathbf{x}_I^\top(t)\right]$.

Lemma 2 Give an arbitrary matrix $\mathbf{Q} = \mathbf{Q}^\top > 0$, if there exists a constant symmetric matrix $\mathbf{P} > 0$ verifying

$$\bar{\mathbf{A}}_i^\top \mathbf{P} + \mathbf{P} \bar{\mathbf{A}}_i \leq -\mathbf{Q}, \ \forall \theta_i \in \Theta_i \text{ and } \forall \omega_i \in \Omega_i,$$

then the Lyapunov function $V = \bar{\mathbf{x}}^\top \bar{\mathbf{P}} \bar{\mathbf{x}}$ guarantees the stability of the switching reference systems in (27).

This fact is straightforward from the converse Lyapunov theorem for LTI systems.

Transient Performance and Steady-State Performance

In the following Lemma, it is stated that the prediction errors $\tilde{\mathbf{x}}_i(t)$ and the estimation errors of the unknown parameters are bounded.

Lemma 3 The following bound holds for the norm of the prediction error $\forall i \in \mathcal{I}$

$$\|\tilde{\mathbf{x}}_i\|_{\mathcal{L}_\infty} \leq \delta, \tag{28}$$

where $\delta > 0$ is an arbitrary small real.

Furthermore, the prediction errors $\tilde{\mathbf{x}}_i(t)$ converge asymptotically to zero, i.e.,

$$\lim_{t \to \infty} \tilde{\mathbf{x}}_i(t) = 0. \tag{29}$$

Proof. The proof is omitted here because of lack os space. Interested readers are referred to

Souanef (2019).

Theorem. There exist positive constants κ_2 and κ_3 such that, for each model i the error between the actual system and the reference system is bounded by

$$\|\mathbf{x}_r(t) - \mathbf{x}(t)\| \leq \kappa_2$$
$$\|\mathbf{u}_r(t) - \mathbf{u}(t)\| \leq \kappa_3.$$

Furthermore, if the closed-loop system is stable then

$$\lim_{t\to\infty} \|\mathbf{x}_r(t) - \mathbf{x}(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\mathbf{u}_r(t) - \mathbf{u}(t)\| \tag{30}$$

Proof. In this section, the dependence of the parameters on (t) is dropped unless it is not clear from the context.

From (22) it can be written

$$\mathbf{u}(s) = -\frac{D_0(s)}{s}\Big(\omega_i \mathbf{u}(s) + \nu_i(s) + \tilde{\nu}_i(s) - \mathbf{K} g_i \mathbf{r}(s)\Big), \tag{31}$$

where $\tilde{\nu}_{(i)}(s) = \tilde{\nu}_{1(i)}(s) + \tilde{\nu}_{2(i)}(s)$, $\tilde{\nu}_{1(i)}(s)$ are the Laplace transformations of $\tilde{\nu}_{1(i)} = \tilde{\theta}_i^\top \mathbf{x}(t) + \tilde{\omega}_i(t)\mathbf{u}(t)$ and $\tilde{\nu}_{2(i)}(s) = \tilde{\eta}_{u(i)}(s) + \phi_i(s)\tilde{\eta}_{u(i)}(s)$.

Consequently, the closed-loop systems (16) and (31) can be written as follows

$$\begin{aligned}\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_{f_1} \\ \dot{\mathbf{x}}_{I_1} \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_{m(i)} + \mathbf{B}_i\theta_i^\top & 0 & -\mathbf{B}_i\omega_i \\ \mathbf{B}_f\theta_i^\top & \mathbf{A}_f & \mathbf{B}_f\omega_i \\ \mathbf{D}_f\theta_i^\top & \mathbf{C}_f & \mathbf{D}_f\omega_i \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_{f_1} \\ \mathbf{x}_{I_1} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_i \\ \mathbf{B}_f \\ \mathbf{D}_f \end{bmatrix} \nu_{2(i)} \\ &+ \begin{bmatrix} 0 \\ \mathbf{B}_f \\ \mathbf{D}_f \end{bmatrix} \tilde{\nu}_i - \begin{bmatrix} 0 \\ \mathbf{B}_f K g_i \\ \mathbf{D}_f K g_i \end{bmatrix} \mathbf{r}.\end{aligned} \tag{32}$$

The error between the state of the reference system and the actual plant, $e = x_r - x$, can be expressed as

$$\begin{bmatrix} \dot{\mathbf{e}} \\ \dot{\mathbf{x}}_{f_1} \\ \dot{\mathbf{x}}_{I_1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{m(i)} + \mathbf{B}_i\theta_i^\top & 0 & -\mathbf{B}_i\omega_i \\ \mathbf{B}_f\theta_i^\top & \mathbf{A}_f & \mathbf{B}_f\omega_i \\ \mathbf{D}_f\theta_i^\top & \mathbf{C}_f & \mathbf{D}_f\omega_i \end{bmatrix} \begin{bmatrix} e \\ x_{f_1} \\ x_{I_1} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_i \\ \mathbf{B}_f \\ \mathbf{D}_f \end{bmatrix} \tilde{\nu}_i. \tag{33}$$

The control error can also be formulated as follows

$$\mathbf{e}_u = \mathbf{u}_r - \mathbf{u} = \begin{bmatrix} 0 & 0 & -\mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{x}_{f_1} \\ \mathbf{x}_{I_1} \end{bmatrix}. \tag{34}$$

From (16) and (17), the prediction error dynamics can be written as

$$\dot{\tilde{\mathbf{x}}}_i = \mathbf{A}_{m(i)}\tilde{\mathbf{x}}_i + \mathbf{B}_i\left(\tilde{\omega}_i\mathbf{u} + \tilde{\theta}_i^\top\mathbf{x} + \tilde{\eta}_{m(i)}\right) + \tilde{\eta}_{u(i)}. \tag{35}$$

Thus

$$\tilde{\nu}_i = \mathbf{B}_i^\dagger \left(\dot{\tilde{\mathbf{x}}} - \mathbf{A}_{m(i)}\tilde{\mathbf{x}}\right). \tag{36}$$

Passing $\mathbf{B}_i^\dagger\dot{\tilde{\mathbf{x}}}$ through the filter $\left(s\mathbb{I} + D_0(s)\omega_i\right)^{-1} D_0(s)$, we can write

$$\begin{bmatrix} \dot{\mathbf{x}}_{f_2} \\ \dot{\mathbf{x}}_{I_2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_f & \mathbf{B}_f\omega_i \\ \mathbf{C}_f & \mathbf{D}_f\omega_i \end{bmatrix} \begin{bmatrix} x_{f_2} \\ x_{I_2} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_f \\ \mathbf{D}_f \end{bmatrix} \mathbf{B}_i^\dagger\tilde{\mathbf{x}}. \tag{37}$$

Applying this to the error dynamics in (33) we have

$$\begin{bmatrix} \dot{\mathbf{e}} \\ \dot{\mathbf{x}}_{f_1} \\ \dot{\mathbf{x}}_{I_1} \\ \dot{\mathbf{x}}_{f_2} \\ \dot{\mathbf{x}}_{I_2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{m(i)} + \mathbf{B}_i\theta_i^\top & 0 & -\mathbf{B}_i\omega_i & -\mathbf{B}_i\mathbf{C}_f & -\mathbf{B}_i\mathbf{D}_f\omega_i \\ \mathbf{B}_f\theta_i^\top & \mathbf{A}_f & \mathbf{B}_f\omega_i & 0 & 0 \\ \mathbf{D}_f\theta_i^\top & \mathbf{C}_f & \mathbf{D}_f\omega_i & 0 & 0 \\ 0 & 0 & 0 & \mathbf{A}_f & \mathbf{B}_f\omega_i \\ 0 & 0 & 0 & \mathbf{C}_f & \mathbf{D}_f\omega_i \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{x}_{f_1} \\ \mathbf{x}_{I_1} \\ \mathbf{x}_{f_2} \\ \mathbf{x}_{I_2} \end{bmatrix} . \tag{38}$$
$$+ \begin{bmatrix} -\mathbf{D}_f\mathbf{B}_i^\dagger \\ -\mathbf{B}_f\mathbf{B}_i^\dagger\mathbf{A}_{m(i)} \\ -\mathbf{D}_f\mathbf{B}_i^\dagger\mathbf{A}_{m(i)} \\ -\mathbf{B}_f\mathbf{B}_i^\dagger \\ -\mathbf{D}_f\mathbf{B}_i^\dagger \end{bmatrix} \tilde{\mathbf{x}},$$

and

$$\mathbf{e}_u = \begin{bmatrix} 0 & 0 & -\mathbb{I} & -\mathbf{C}_f & -\mathbf{D}_f\omega_i \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{x}_{f1} \\ \mathbf{x}_{I_1} \\ \mathbf{x}_{f_2} \\ \mathbf{x}_{I_2} \end{bmatrix} + \begin{bmatrix} -\mathbf{D}_f\mathbf{B}_i^\dagger \end{bmatrix} \tilde{\mathbf{x}}. \tag{39}$$

Letting

$$\bar{H}_i = \begin{bmatrix} -\mathbf{B}_i\mathbf{C}_f & -\mathbf{B}_i\mathbf{D}_f\omega_i \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{J}_i = \begin{bmatrix} -\mathbf{D}_f\mathbf{B}_i^\dagger \\ -\mathbf{B}_f\mathbf{B}_i^\dagger\mathbf{A}_{m(i)} \\ -\mathbf{D}_f\mathbf{B}_i^\dagger\mathbf{A}_{m(i)} \end{bmatrix},$$
$$\bar{G}_i = \begin{bmatrix} -\mathbf{B}_f\mathbf{B}_i^\dagger\mathbf{A}_{m(i)} \\ -\mathbf{D}_f\mathbf{B}_i^\dagger\mathbf{A}_{m(i)} \end{bmatrix}, \quad \bar{L}_i = \begin{bmatrix} 0 & \mathbf{C}_f & \mathbf{D}_f\omega_i \end{bmatrix},$$

it follows from (38) and (39) that

$$\begin{bmatrix} \dot{\bar{\mathbf{e}}} \\ \dot{\bar{\mathbf{x}}}_{f_2} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_i & \bar{\mathbf{H}}_i \\ 0 & \bar{\mathbf{F}}_i \end{bmatrix} \begin{bmatrix} \bar{\mathbf{e}} \\ \bar{\mathbf{x}}_{f_2} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{J}}_i \\ \bar{\mathbf{G}}_i \end{bmatrix} \tilde{\mathbf{x}}, \tag{40}$$

and

$$\mathbf{e}_u = \begin{bmatrix} \bar{\mathbf{C}} & \bar{\mathbf{L}}_i \end{bmatrix} \begin{bmatrix} \bar{\mathbf{e}} \\ \bar{\mathbf{x}}_{f_2} \end{bmatrix} + \begin{bmatrix} -\mathbf{D}_f \mathbf{B}_i^{\dagger} \end{bmatrix} \tilde{\mathbf{x}}, \tag{41}$$

where $\bar{\mathbf{e}} = \left[\mathbf{e}^\top, \mathbf{x}_{f_1}^\top, \mathbf{x}_{I_1}^\top\right]^\top$ and $\bar{\mathbf{x}}_{f_2} = \left[\mathbf{x}_{f_2}^\top, \mathbf{x}_{I_2}^\top\right]^\top$.

Note that the reference system is stable and the filter represented by $\bar{\mathbf{F}}_i$ is a subsystem of the reference system when $\theta = 0$. Therefore, from Lemma 1, there exists positive definite matrices $\mathbf{Q}_i(\omega_i) > 0$ such that for all $\omega_i \in \Omega$,

$$\bar{\mathbf{F}}_i^\top \bar{\mathbf{Q}}_i + \bar{\mathbf{Q}}_i \bar{\mathbf{F}}_i \leq -\mathbb{I}. \tag{42}$$

Let $\bar{V}_i(t) = \bar{\mathbf{x}}_{f_2}^\top \bar{\mathbf{Q}}_i \bar{\mathbf{x}}_{f_2}$, where $V_i(0) = 0$. Differentiating along the system trajectories it follows that

$$\begin{aligned} \dot{V}_i =& \bar{\mathbf{x}}_{f_2}^\top \left(\bar{\mathbf{F}}_i^\top \bar{\mathbf{Q}}_i + \bar{\mathbf{Q}}_i \bar{\mathbf{F}}_i\right) \bar{\mathbf{x}}_{f_2} + 2\bar{\mathbf{x}}_{f_2}^\top \bar{\mathbf{Q}}_i \bar{\mathbf{G}}_i \tilde{\mathbf{x}} \\ \leq& -\left\|\bar{\mathbf{x}}_{f_2}\right\|^2 + 2\|\bar{\mathbf{x}}_{f_2}\|\beta_F \|\tilde{\mathbf{x}}\|_{\mathcal{L}_\infty} \\ \leq& -\left\|\bar{\mathbf{x}}_{f_2}\right\|^2 + \beta_F^2 \|\tilde{\mathbf{x}}\|_{\mathcal{L}_\infty}^2 \end{aligned} \tag{43}$$

where the last line follows from square completion and $\beta_F = \sqrt{n} \max_{i \in \mathcal{I}} \|\bar{\mathbf{Q}}_i \bar{\mathbf{G}}_i\|$.

By integrating it is straightforward to show that the following bound holds for $\bar{\mathbf{x}}_{f_2}$

$$\|\bar{\mathbf{x}}_{f_2}\|_{\mathcal{L}_\infty} \leq \kappa_1, \tag{44}$$

where $\kappa_1 = \sqrt{n} \max_{i \in \mathcal{I}} \|\bar{\mathbf{Q}}_i \bar{\mathbf{G}}_i\| \delta$ and δ is the upper bound of $\tilde{\mathbf{x}}_i$ defined in Lemma 2.

We now define the Lyapunov functions $\bar{W}_i = \bar{\mathbf{e}}^\top \bar{\mathbf{P}}_i \bar{\mathbf{e}}$. Differentiating along the system trajectories it follows that

$$
\begin{aligned}
\dot{W}_i &= \bar{\mathbf{e}}^\top \left(\bar{\mathbf{A}}_i^\top \bar{\mathbf{P}}_i + \bar{\mathbf{P}}_i \bar{\mathbf{A}}_i\right) \bar{\mathbf{e}} + 2\bar{\mathbf{e}}^\top \bar{\mathbf{P}}_i \bar{\mathbf{H}}_i \bar{\mathbf{x}}_{f_2} + 2\bar{\mathbf{e}}^\top \bar{\mathbf{P}}_i \bar{\mathbf{J}}_i \tilde{\mathbf{x}} \\
&\leq -\|\bar{\mathbf{e}}\|^2 + 2\|\bar{\mathbf{e}}\|\beta_{\bar{e}}\|\tilde{\mathbf{x}}\|_{\mathcal{L}_\infty} \\
&\leq -\|\bar{\mathbf{e}}\|^2 + \beta_{\bar{e}}^2\|\tilde{\mathbf{x}}\|_{\mathcal{L}_\infty}^2,
\end{aligned}
\tag{45}
$$

where $\beta_e = \left(\kappa_1 \max_{i\in\mathcal{I}} \left\|\bar{\mathbf{P}}_i\bar{\mathbf{H}}_i\right\| + \sqrt{n}\max_{i\in\mathcal{I}} \left\|\bar{\mathbf{P}}_i\bar{\mathbf{J}}_i\right\|\right)$.

Therefore, the following bound holds

$$
\|\bar{\mathbf{e}}\|_{\mathcal{L}_\infty} \leq \kappa_2, \tag{46}
$$

where $\kappa_2 = \left(\kappa_1 \max_{i\in\mathcal{I}} \left\|\bar{\mathbf{P}}_i\bar{\mathbf{H}}_i\right\| + \sqrt{n}\max_{i\in\mathcal{I}} \left\|\bar{\mathbf{P}}_i\bar{\mathbf{J}}_i\right\|\right)\delta$.

Furthermore

$$
\begin{aligned}
\|e_u\|_{\mathcal{L}_\infty \leq} \leq &\|\bar{\mathbf{C}}\|\|\bar{\mathbf{e}}\|_{\mathcal{L}_\infty} + \left\|\bar{\mathbf{L}}_i\right\|\left\|\bar{\mathbf{x}}_{f_2}\right\|_{\mathcal{L}_\infty} + \left\|\mathbf{D}_f\mathbf{B}_i^\dagger\right\|\|\tilde{\mathbf{x}}\|_{\mathcal{L}_\infty}, \\
\leq &\kappa_3,
\end{aligned}
\tag{47}
$$

where $\kappa_3 = \|\bar{\mathbf{C}}\|\kappa_2 + \left(\max_{i\in\mathcal{I}} \left\|\bar{\mathbf{L}}_i\right\| + \max_{i\in\mathcal{I}} \left\|\mathbf{D}_f\mathbf{B}_i^\dagger\right\|\right)\delta$.

This completes the proof. □

UAV LATERAL-DIRECTIONAL CONTROL IN CASE OF INVERSION OF THE COMMANDS

A critical situation in flight control systems is that in case of structural damage of the aircraft, the control direction can be inverted. In fact, if an aircraft suffers damage, a control surface may generate a totally opposite angular acceleration, which means the actuation signs will be changed (Liu et al., 2010; Tan et al., 2017b). The inversion of the sign of the control direction can also result from actuators or software faults. This situation cannot be handled by $\mathcal{L}_1$ adaptive controller with a single model. Actually, a conservative condition in adaptive control is that the sign of input vector must be known and should not change (Ioannou and Sun, 2012).

Controller Design

The lateral-directional equations of motion of a fixed-wing aircraft are described by the set of states (β, p, r, ϕ), where β is the sideslip angle, p is the roll rate, r is the yaw rate and ϕ is the roll angle. The control inputs are the aileron deflection δ_a and the rudder deflection δ_r.

The objective is to design a control input $u = [\delta_a, \delta_r]^\top$ to enable tracking of the roll command ϕ_c and the sideslip angle command β_c.

From (Stevens and Lewis, 2003), the linearised model of the lateral-directional dynamics of a fixed-wing aircraft can be written in matrix form as follows

$$\underbrace{\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \frac{Y_\beta}{V_a} & \frac{Y_p}{V_a} & \frac{Y_r}{V_a} - 1 & \frac{g}{V_a} \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}_p} \underbrace{\begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{Y_{\delta_a}}{V_a} & \frac{Y_{\delta_r}}{V_a} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}_p} \underbrace{\begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}}_{\mathbf{u}}, \tag{48}$$

where $(Y_\beta, Y_p, Y_r, Y_{\delta_a}, Y_{\delta_r})$, $(L_\beta, L_p, L_r, L_{\delta_a}, L_{\delta_r})$ and $(N_\beta, N_p, N_r, N_{\delta_a}, N_{\delta_r})$ are the lateral-directional stability derivatives, V_a is the trimmed airspeed and g is the gravity. It should be recalled that the stability derivatives cannot be measured, and they vary depending on flight conditions.

Taking the external disturbances and the model uncertainties into account, the system in (48) can be extended as follows

$$\dot{\mathbf{x}} = \mathbf{A}_p \mathbf{x} + \mathbf{B}_p \mathbf{u} + \mathbf{f}(t, \mathbf{x}). \tag{49}$$

The system with its nominal desired dynamics can be parameterised to become similar to the class of MIMO systems in (16) defined by

$$\dot{\mathbf{x}} = \mathbf{A}_{m(0)} x + \mathbf{B}_0 \left(\omega_0 \mathbf{u} + \theta_0^\top \mathbf{x} + \eta_{m(0)} \right) + \eta_{u(0)}(t).$$

A second model for the case of inversion of the sign of the aileron command is given by

$$\dot{\mathbf{x}} = \mathbf{A}_{m(1)} x + \mathbf{B}_0 \beta_1 \left(\omega_1 \mathbf{u} + \theta_1^\top \mathbf{x} + \eta_{m(1)} \right) + \eta_{u(1)}(t),$$

where $\beta_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

A third model for the case of inversion of the sign of the rudder command is given by

$$\dot{\mathbf{x}} = \mathbf{A}_{m(2)} x + \mathbf{B}_0 \beta_2 \left(\omega_2 \mathbf{u} + \theta_2^\top \mathbf{x} + \eta_{m(2)} \right) + \eta_{u(2)}(t),$$

where $\beta_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

A fourth model for the case of inversion of both the signs of the aileron and the rudder commands is given by

$$\dot{\mathbf{x}} = \mathbf{A}_{m(3)} x + \mathbf{B}_0 \beta_3 \left(\omega_3 \mathbf{u} + \theta_3^\top \mathbf{x} + \eta_{m(3)} \right) + \eta_{u(3)}(t),$$

where $\beta_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

The input matrix B_0 was taken to be the same for both models.

Simulation Results

In order to show the efficiency of the multiple model controller, simulations were first made using only the nominal controller, i.e., the $\mathcal{L}_1$ adaptive controller with one model. Two situations were considered in this case:

1. Control inputs loss of effectiveness of 50% without the inversion of the commands;
2. Control inputs loss of effectiveness of 50% with the inversion of the sign of the aileron command.

Furthermore, the following uncertainties were added to the plant at simulation time $t = 7\ s$:

- Linear-in-state unknown parameters;
- Matched disturbance $d_m = \sin(2\pi t)$ deg.

Simulation results for the nominal $\mathcal{L}_1$ adaptive controller, without inversion of actuation signs, are shown in Figure 3. As expected, the system has good performance in the presence of uncertainties. The aileron command δ_a and the rudder command δ_r are within acceptable limits.

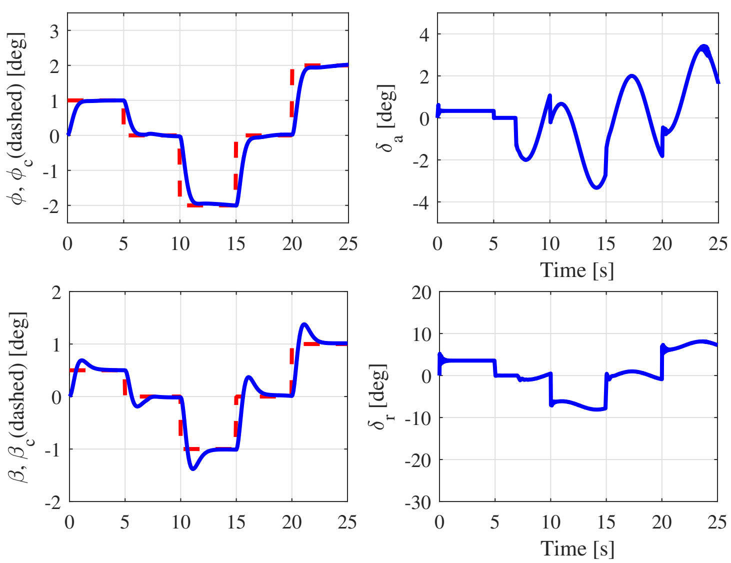


Fig. 3. Closed-loop tracking performance of the nominal controller without inversion of the sign of the commands.

In the second scenario of loss of effectiveness of 50% with the inversion of the sign of the aileron command, the system with only the nominal controller has become unstable as it can be observed in Figure 4. This is a direct consequence of the fact the adaptation laws (7)-(11) cannot match the correct control direction when it is inverted.

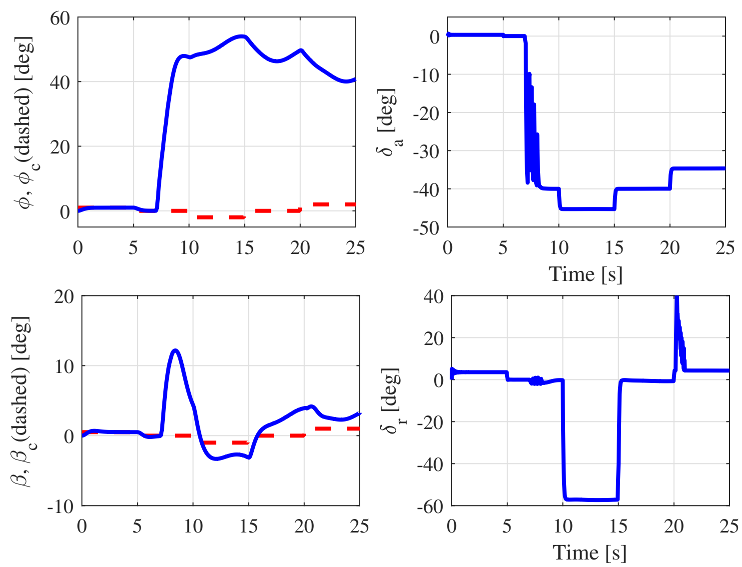


Fig. 4. Closed-loop tracking performance of the nominal controller with inversion of the sign of the aileron.

Next, the multiple model controller was applied. The tuning parameters and the desired dynamics of the suboptimal controller were the same as the nominal controller. Three situations were considered in simulations:

1. Control inputs loss of effectiveness of 50% with the inversion of the sign of the aileron command;
2. Control inputs loss of effectiveness of 50% with the inversion of the sign of the rudder command;
3. Control input loss of effectiveness of 50% with the of both the signs of the aileron and the rudder commands.

Moreover, the same uncertainties as in previous simulations were added to the plant. The failures were introduced at simulation time $t = 7\ s$.

The simulation results in the case of inversion of the sign of aileron command, the rudder command, and both the ailerons and the rudder commands, are shown in Figure 5, Figure 6 and Figure 7, respectively. In each situation, the system has stayed stable and has shown a good tracking performance. The aileron command δ_a and the rudder command δ_r were within acceptable limits.

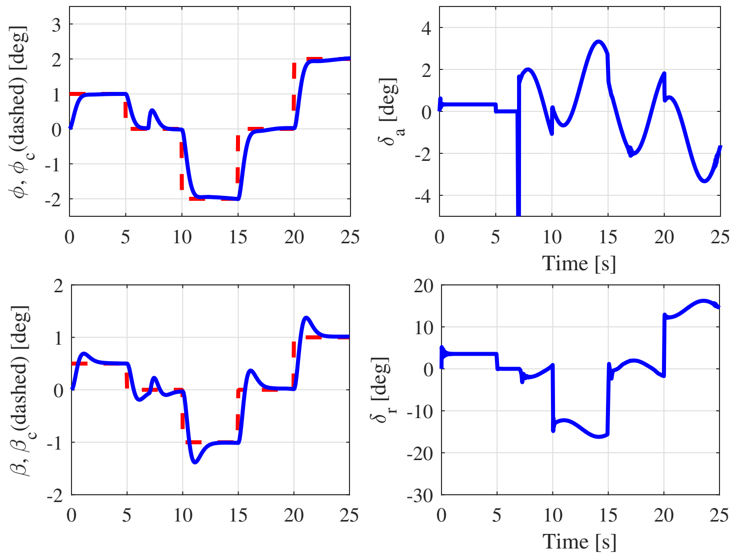


Fig. 5. Closed-loop tracking performance of the multiple model controller with inversion of the sign of the aileron.

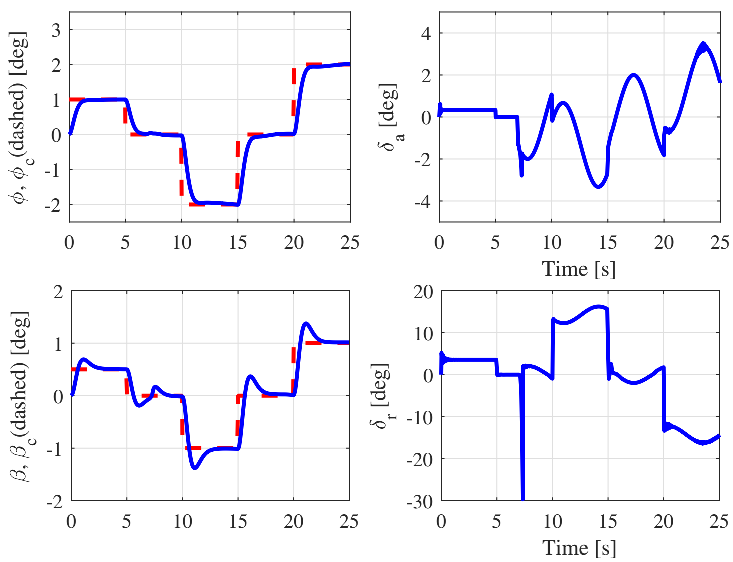


Fig. 6. Closed-loop tracking performance of the multiple model controller with inversion of the sign of the rudder.

Furthermore, as it is shown on Figure 8, the matching model to each failure case corresponds to the minimum cost function defined in (23).

These simulations conclude that the application of the multiple model $\mathcal{L}_1$ adaptive controller is justified in case of structural damages or faults that lead to inversion of the sign of the control input of flight systems.

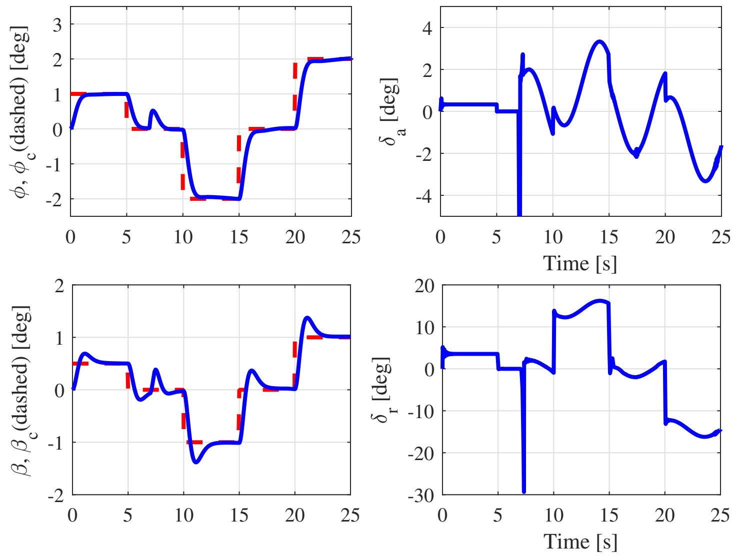
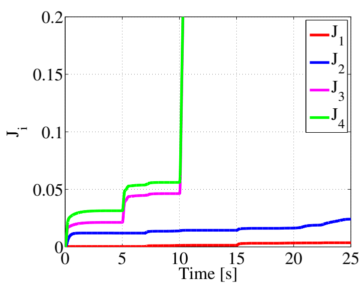
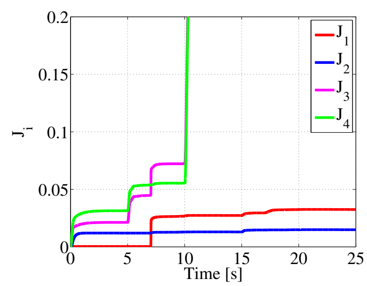


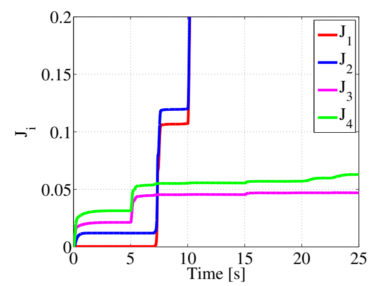
Fig. 7. Closed-loop tracking performance of the multiple model controller with inversion of the sign of both the aileron and the rudder.



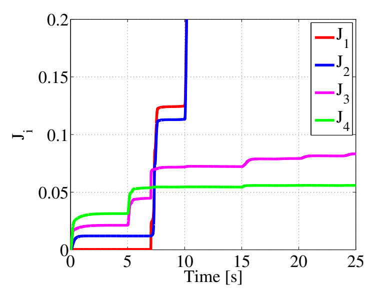
(a) 1st failure case.



(b) 2nd failure case.



(c) 3rd failure case.



(d) 4th failure case.

Fig. 8. Cost Functions

SUMMARY

In this paper, an approach for $\mathcal{L}_1$ adaptive fault-tolerant control MIMO systems is proposed. The aim is the fault-tolerant control of small fixed-wing UAVs in the presence of critical failures.

The design is based on a nominal model for a fault-free plant and a set of suboptimal models for the plant under failures. The switching between the models is based on a simple quadratic criterion.

The main advantage of this approach is that it allows a larger class of uncertainties and faults to be considered and can achieve better accommodation and preserve system integrity. Simulations have shown that the multiple model $\mathcal{L}_1$ adaptive has stabilised the system in case of inversion of the control input, while the controller with a single model has failed.

PRACTICAL APPLICATIONS

Small drones or Unmanned Aerial Vehicles (UAVs) that is, with wingspans less than 2 metres and payload smaller than 2 kg, are generally built based on commercial Radio Controlled (RC) airplane. Small UAVs are gaining growing interest because of their low cost, high manoeuvrability and simple maintenance. Autonomy, although relative because they are still operated under human supervision, is the main feature of small UAVs compared to RC airplane. Autonomy has been made possible through the development of advanced autopilot (flight control) systems. They are used for a wide range of military and civilian tasks, such as: inspection, detection, transportation, monitoring, search and rescue, photography, imaging, mapping, intelligence, surveillance, reconnaissance, agriculture, entertainment etc. However, small UAVs are generally built with low-cost components and materials, which increases the probability of occurrence of faults and failures. The proposed flight control solution permits to maintain the UAVs in flight in the presence of hard failure, while other approaches fail. Therefore, the mission can be safely terminated, either automatically, or manually. Alternatively, it it can also be decided to complete the operation in degraded (limited) mode. Therefore, this solution is recommended for the operation of drones, especially in urban environments that needs a high degree of safety and reliability.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the author upon reasonable request.

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Souanef, Toufik

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