Cranfield

College of Aeronautics Report No. 9211
September 1992

ARBITRARY-ORDER NUMERICAL SCHEMES FOR LINEAR HYPER BOLIC SYSTEMS

Jian shi

Department of aerospace science
College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL. England

Cranfield

College of Aeronautics Report No. 9211
September 1992

ARBITRARY-ORDER NUMERICAL SCHEMES FOR LINEAR HYPER BOLIC SYSTEMS

Jian shi

Department of aerospace science
College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL. England

ISBN 1 871564492

£8.00

Cranfield

COA REPORT No. 9211
SEPTEMBER, 1992

ARBITRARY-ORDER NUMERICAL SCHEMES FOR LINEAR HYPERBOLIC SYSTEMS

JIAN SHI

DEPARTMENT OF AEROSPACE SCIENCE
COLLEGE OF AERONAUTICS
CRANFIELD INSTITUTE OF TECHNOLOGY
CRANFIELD, BEDS., MK43 OAL

Abstract

This report is an extension of the work carried out in [16].

In [16] we defined arbitrary-order numerical methods for model scalar hyperbolic equation. In this report we extended these methods to linear hyperbolic systems where waves can propagate in both directions.

First, we define a generalized numerical formula which can accommodate arbitrary wave speeds for scalar advection equation. Then to illustrate its application, we derive three, four, and five point generalized numerical schemes.

Finally, according to the theory of linear systems, we extend the generalized schemes to linear hyperbolic systems in a straight forward manner.

Acknowledgements

I would like to express my sincere thanks to Dr. E.F. Toro for his help during this work.

Special thanks go to S. Billett for his corrections and comments of the report, and to all those who helped me in this work.

Contents

LINEAR SYSTEMS

1.1 Introduction

In [16] we defined arbitrary-order numerical methods for model scalar equations. However, these methods are only valid for a single wave propagating in one direction, i.e. either $a > 0$ or $a < 0$ for the scalar model hyperbolic equation.

For linear systems and nonlinear problems where waves can propagate in both directions we need a generalized method suitable for arbitrary wave speeds.

The second order Lax-Wendroff method can be extended to linear system via several different approaches, see [1]-[15]. For a method which is higher than second order, more points are used, and hence more computing cells are involved in the high order numerical flux. The new problem lies in that how can we deal with the multiple cell numerical flux for linear systems.

In this chapter we are therefore going to investegate the issue indicated above, and find a generalized formula to extend arbitrary-order numerical methods to linear systems.

1.2 Generalized Formula of Arbitrary Wave Speeds for Scalar Equation

In this section we will consider the unified formula which is valid for arbitrary wave direction for scalar cases. Before we do so, the following definition is required.

DEFINITION

Two schemes are called **mirror schemes** *if the grid points used in these schemes are symmetrical about the centre point.*

For example, all up-wind schemes are mirror schemes of down-wind schemes. For four points schemes, scheme $U_j^{n+1} = f(U_{j+1}^n, U_j^n, U_{j-1}^n, U_{j-2}^n)$ is mirror scheme of $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n)$, since the points used in these two schemes are symmtrically arranged about the centre point.

Based on the results of three, four, and five points schemes we can conclude that mirror schemes have mirror stable regions. For example, Beam-Warming scheme (2.20), and scheme 2.21 in [16] are mirror schemes. The stable region of B-W scheme is $0 \le c \le 2$, and the mirror region is $-2 \le c \le 0$ for 2.21.

Equiped with this definition we can write arbitrary-order numerical fluxes for $c > 0$ as:

$$F(U^n; j) = a\left(U_j^n + \sum_k D_{j+k+\frac{1}{2}} \Delta U_{j+k+\frac{1}{2}}\right) \tag{1.1}$$

and the mirror numerical fluxes of 1.1 for $c < 0$ are:

$$F^M(U^n; j) = a\left(U_{j+1}^n + \sum_{-k} D_{j+k+\frac{1}{2}} \Delta U_{j+k+\frac{1}{2}}\right) \tag{1.2}$$

here: $-\infty < k < \infty$.

$$\Delta U_{j+k+\frac{1}{2}} = U_{j+k+1} - U_{j+k}$$

and, $D_{j+k+\frac{1}{2}}$ are coefficients which are functions of Courant number, i.e. $D_{j+k+\frac{1}{2}}(c)$. k are integer numbers.

Obtaining a gereralized form for arbitrary wave speeds we can unify 1.1 and 1.2 into a single formula (See [ ]):

$$F^{A-W}(U^n;j) = \frac{1}{2}a(U_j^n + U_{j+1}^n) - \frac{1}{2}|a|\Delta U_{j+\frac{1}{2}} + \sum_{L=-\infty}^{\infty} a_k D_k(c)\Delta U_{j+k+\frac{1}{2}} \quad (1.3)$$

where:

$$\begin{cases} k = L & if\ c > 0 \\ k = -L & if\ c < 0 \end{cases} \quad (1.4)$$

If $D_{j+k+\frac{1}{2}}(c) = D_{j-k+\frac{1}{2}}(c)$, then

$$\begin{cases} a_k = a \\ D_k(c) = D_{j+k+\frac{1}{2}}(c) \end{cases} \quad (1.5)$$

If $D_{j+k+\frac{1}{2}}(c) \neq D_{j-k+\frac{1}{2}}(c)$, then

$$\begin{cases} a_k = |a| \\ D_k(c) = D_{j+k+\frac{1}{2}}(|c|) \end{cases} \quad (1.6)$$

In next section we will use some examples to illustrate the correction of equation 1.3.

1.3 Applications of Generalized Formula

1.3.1 Three Point Schemes

1. $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n)$ **(L-W method)**

$$F^{L-W}(U^n;j) = aU_j^n + \frac{a}{2}(1-c)\Delta U_{j+\frac{1}{2}}$$

From 1.1 we know $k=0$, and $D_{j+0+\frac{1}{2}} = D_{j+\frac{1}{2}} = \frac{1}{2}(1-c)$.

From 1.2, the mirror numerical flux of L-W method is:

$$F^M(U^n;j) = aU_{j+1}^n - \frac{a}{2}(1+c)\Delta U_{j+\frac{1}{2}}$$

here, $-k=-0$, and $D_{j-0+\frac{1}{2}} = D_{j+\frac{1}{2}} = -\frac{1}{2}(1+c)$.

Because $D_{j+0+\frac{1}{2}} \neq D_{j-0+\frac{1}{2}}$, from 1.6 we have:

$$\begin{cases} a_0 = |a| \\ D_0 = \frac{1}{2}(1-|c|) \end{cases}$$

Therefore, from 1.3 the generalized formula of L-W method is:

$$\begin{aligned} F^{A-W}(U^n;j) &= \frac{a}{2}(U_j^n + U_{j+1}^n) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} \\ &\quad + \frac{|a|}{2}(1-|c|)\Delta U_{j+\frac{1}{2}} \end{aligned} \tag{1.7}$$

The stable region of the generalized scheme is:

$$|c| \leq 1 \tag{1.8}$$

2. $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n)$ **(B-W method)**

$$F^{B-W}(U^n;j) = aU_j^n + \frac{a}{2}(1-c)\Delta U_{j-\frac{1}{2}}$$

from 1.1 we know $k=-1$, and $D_{j-1+\frac{1}{2}} = D_{j-\frac{1}{2}} = \frac{1}{2}(1-c)$.

From 1.2, the mirror numerical flux of B-W method is:

$$F^M(Un;j) = aU_{j+1}^n - \frac{a}{2}(1+c)\Delta U_{j+\frac{3}{2}}$$

here: $-k=1$ and $D_{j+1+\frac{1}{2}} = D_{j+\frac{3}{2}} = -\frac{1}{2}(1+c)$

Because $D_{j-1+\frac{1}{2}} \neq D_{j+1+\frac{1}{2}}$, from 1.6 we have:

$$\begin{cases} a_{-1} = |a| \\ D_{-1}(c) = D_{j-\frac{1}{2}}(|c|) = \frac{1}{2}(1-|c|) \end{cases}$$

Therefore according to 1.3, the unified formula of the two schemes is:

$$F^{A-W}(U^n;j) = \frac{a}{2}(U_j^n + U_{j+1}^n) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} + \frac{|a|}{2}(1-|c|)\Delta U_{j+k+\frac{1}{2}} \tag{1.9}$$

where:

$$\begin{cases} k=-1 & if\ c>0 \\ k=1 & if\ c<0 \end{cases}$$

The stable region of the generalized scheme is:

$$|c| \leq 2 \tag{1.10}$$

1.3.2 Four Point Schemes

1. $U_j^{n+1} \;=\; f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$

$$F^{4-p}(U^n;j) \;=\; a\left[U_j^n + \left(\frac{1}{3} - \frac{c}{2} + \frac{c^2}{6}\right)\Delta U_{j+\frac{1}{2}} + \frac{1}{6}(1-c^2)\Delta U_{j-\frac{1}{2}}\right] \qquad (1.11)$$

here, from 1.1 we have

$$\begin{aligned} k_1 &= 0 \\ k_2 &= -1 \\ D_{j+0+\frac{1}{2}} &= D_{j+\frac{1}{2}} = \frac{1}{3} - \frac{c}{2} + \frac{c^2}{6} \\ D_{j-1+\frac{1}{2}} &= D_{j-\frac{1}{2}} = \frac{1}{6}(1-c^2) \end{aligned}$$

The mirror flux of 1.11 according to 1.2 is

$$F^{M}(U^n;j) \;=\; a\left[U_{j+1}^n - \left(\frac{1}{3} + \frac{c}{2} + \frac{c^2}{6}\right)\Delta U_{j+\frac{1}{2}} + \frac{1}{6}(1-c^2)\Delta U_{j+\frac{3}{2}}\right] \qquad (1.12)$$

here,

$$\begin{aligned} -k_1 &= -0 \\ -k_2 &= 1 \\ D_{j-0+\frac{1}{2}} &= D_{j+\frac{1}{2}} = -\left(\frac{1}{3} + \frac{c}{2} + \frac{c^2}{6}\right) \\ D_{j+1+\frac{1}{2}} &= D_{j+\frac{3}{2}} = \frac{1}{6}(1-c^2) \end{aligned}$$

Because

$$\begin{cases} D_{j+0+\frac{1}{2}} \neq D_{j-0+\frac{1}{2}} \\ D_{j-1+\frac{1}{2}} = D_{j+1+\frac{1}{2}} \end{cases}$$

from 1.6 and 1.5 we have

$$\begin{cases} a_0 \;=\; |a| \\ D_0(c) \;=\; \frac{1}{3} - \frac{1}{2}|c| + \frac{1}{6}c^2 \end{cases}$$

and

$$\begin{cases} a_{-1} &= a \\ D_{-1}(c) &= \frac{1}{6}(1-c^2) \end{cases}$$

Therefore, the generalized fomula of the two schemes is:

$$\begin{aligned} F^{A-W}(U^n;j) &= \frac{a}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} \\ &+|a|\left(\frac{1}{3} - \frac{|c|}{2} + \frac{c^2}{6}\right)\Delta U_{j+\frac{1}{2}} \\ &+\frac{a}{6}\left(1-c^2\right)\Delta U_{j+k+\frac{1}{2}} \end{aligned} \tag{1.13}$$

where:

$$\begin{cases} k = -1 & if\ c > 0 \\ k = 1 & if\ c < 0 \end{cases}$$

The stable region of the generalized scheme is:

$$|c| \leq 1 \tag{1.14}$$

If introducing limiteŕs into 1.13, we have:

$$\begin{aligned} F^{A-W}(U^n;j) &= \frac{a}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} \\ &+|a|\left(\frac{1}{3} - \frac{|c|}{2} + \frac{c^2}{6}\right)\phi_j\Delta U_{j+\frac{1}{2}} \\ &+\frac{a}{6}\left(1-c^2\right)\phi_j\Delta U_{j+k+\frac{1}{2}} \end{aligned} \tag{1.15}$$

where:

$$\begin{aligned} \phi_j &= \phi_j(\theta_j) \\ \theta_j &= \frac{\Delta U_{j+k+\frac{1}{2}}}{\Delta U_{j+\frac{1}{2}}} \end{aligned} \qquad \begin{cases} k = -1 & c > 0 \\ k = 1 & c < 0 \end{cases}$$

2. $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n)$

$$F^{4-p}(U^n;j) = a\left[U_j^n + \left(\frac{5}{6} + \frac{c^2}{6} - c\right)\Delta U_{j-\frac{1}{2}} - \left(\frac{1}{3} - \frac{c}{2} + \frac{c^2}{6}\right)\Delta U_{j-\frac{3}{2}}\right]$$

here, from 1.1 we know

$$\begin{aligned} k_1 &= -1 \\ k_2 &= -2 \\ D_{j-1+\frac{1}{2}} &= D_{j-\frac{1}{2}} = \frac{5}{6} + \frac{c^2}{6} - c \\ D_{j-2+\frac{1}{2}} &= D_{j-\frac{3}{2}} = -\left(\frac{1}{3} - \frac{c}{2} + \frac{c^2}{6}\right) \end{aligned}$$

The mirror flux of this scheme is:

$$F^M(U^n;j) = a\left[U_{j+1}^n - \left(\frac{5}{6} + \frac{c^2}{6} + c\right)\Delta U_{j+\frac{3}{2}} + \left(\frac{1}{3} + \frac{c}{2} + \frac{c^2}{6}\right)\Delta U_{j+\frac{5}{2}}\right]$$

here

$$\begin{aligned} -k_1 &= 1 \\ -k_2 &= 2 \\ D_{j+1+\frac{1}{2}} &= D_{j+\frac{3}{2}} = -\left(\frac{5}{6} + \frac{c^2}{6} + c\right) \\ D_{j+2+\frac{1}{2}} &= D_{j+\frac{5}{2}} = \frac{1}{3} + \frac{c}{2} + \frac{c^2}{6} \end{aligned}$$

Because

$$\begin{cases} D_{j-1+\frac{1}{2}} \neq D_{j+1+\frac{1}{2}} \\ D_{j-2+\frac{1}{2}} \neq D_{j+2+\frac{1}{2}} \end{cases}$$

from 1.6 we have

$$\begin{cases} a_{-1} = |a| \\ D_{-1}(c) = \frac{5}{6} + \frac{1}{6}c^2 - |c| \end{cases}$$

and

$$\begin{cases} a_{-2} & = & |a| \\ D_{-2}(c) & = & -\left(\frac{1}{3} - \frac{1}{2}|c| + \frac{1}{6}c^2\right) \end{cases}$$

Therefore, the generalized scheme is:

$$\begin{aligned} F^{A-W}(U^n;j) &= \frac{a}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} \\ &+|a|\left(\frac{5}{6} + \frac{c^2}{6} - |c|\right)\Delta U_{j+k_1+\frac{1}{2}} \\ &-|a|\left(\frac{1}{3} - \frac{|c|}{2} + \frac{c^2}{6}\right)\Delta U_{j+k_2+\frac{1}{2}} \end{aligned} \tag{1.16}$$

where

$$\begin{cases} k_1 = -1, \quad k_2 = -2 & if \;\; c > 0 \\ k_1 = 1, \quad k_2 = 2 & if \;\; c < 0 \end{cases}$$

The stable region of the generalized scheme is:

$$1 \leq |c| \leq 2 \tag{1.17}$$

1.3.3 Five Point Schemes

1. $U_j^{n+1} \;=\; f(U_{j+2}^n, U_{j+1}^n, U_j^n, U_{j-1}^n, U_{j-2}^n)$

Following the same procedure as above, we can get the generalized formula of this scheme as:

$$\begin{aligned} F^{A-W}(U^n;j) &= \frac{a}{2}\Delta U_{j+\frac{1}{2}} + |a|\left(\frac{1}{12} + \frac{|a|}{24} - \frac{c^2}{12} - \frac{|c|^3}{24}\right)\Delta U_{j-\frac{1}{2}} \\ &+|a|\left(\frac{1}{2} - \frac{7}{12}|c| + \frac{|c|^3}{12}\right)\Delta U_{j+\frac{1}{2}} \end{aligned}$$

$$-|a|\left(\frac{|c|^3}{24}-\frac{c^2}{12}-\frac{|c|}{24}+\frac{1}{12}\right)\Delta U_{j+\frac{3}{2}} \tag{1.18}$$

The stable region of the generalized scheme is:

$$|c| \leq 1 \tag{1.19}$$

2. $U_j^{n+1} \;=\; f(U_{j-4}^n, U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n)$

The generalized formula of this scheme is:

$$\begin{aligned} F^{A-W}(U^n;j) \;=\; & \frac{a}{2}(U_j^n+U_{j+1}^n)-\frac{|c|}{2}\Delta U_{j+\frac{1}{2}} \\ & +|a|\left(\frac{13}{12}-\frac{35}{24}|c|+\frac{5}{12}c^2-\frac{1}{24}|c|^3\right)\Delta U_{j+k_1+\frac{1}{2}} \\ & +|a|\left(\frac{|c|^3}{12}-\frac{2}{3}c^2+\frac{17}{12}|c|-\frac{5}{6}\right)\Delta U_{j+k_2+\frac{1}{2}} \\ & +|a|\left(\frac{1}{4}-\frac{11}{24}|c|+\frac{1}{4}c^2-\frac{1}{24}|c|^3\right)\Delta U_{j+k_3+\frac{1}{2}} \end{aligned} \tag{1.20}$$

where

$$\begin{cases} k_1=-1,\ k_2=-2,\ k_3=-3 & if\ c>0 \\ k_1=1,\ k_2=2,\ k_3=3 & if\ c<0 \end{cases}$$

The stable region of the generalized scheme is:

$$1 \leq |c| \leq 3 \tag{1.21}$$

3. $U_j^{n+1} \;=\; f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$

The generalized formula of the scheme is:

$$F^{A-W}(U^n;j) \;=\; \frac{a}{2}(U_j^n+U_{j+1}^n)-\frac{|a|}{2}\Delta U_{j+\frac{1}{2}}$$

$$
\begin{aligned}
&+|a|\left(\frac{1}{4}-\frac{11}{24}|c|+\frac{1}{4}c^2-\frac{1}{24}|c|^3\right)\Delta U_{j+\frac{1}{2}}\\
&+|a|\left(\frac{1}{12}|c|^3-\frac{1}{3}c^2-\frac{|c|}{12}+\frac{1}{3}\right)\Delta U_{j+k_1+\frac{1}{2}}\\
&-\left(\frac{1}{24}|c|^3-\frac{1}{12}c^2-\frac{1}{24}|c|+\frac{1}{12}\right)\Delta U_{j+k_2+\frac{1}{2}} \qquad (1.22)
\end{aligned}
$$

where

$$
\begin{cases} k_1=-1,\ k_2=-2 & if\ c>0\\ k_1=1,\ k_2=2 & if\ c<0 \end{cases}
$$

The stable region of the scheme is:

$$
|c|\leq 2 \qquad (1.23)
$$

1.4 Generalized Formula of Arbitrary Wave Speed for Linear Systems

In this section we will extend the results of last section into linear hyperbolic systems. In fact, this extension is very natural and straight forward.

Consider the linear system:

$$
\begin{aligned}
u_t+Au_x &= 0 \qquad (1.24)\\
u(x,0) &= u_0(x)
\end{aligned}
$$

where, **u** are vector functions of m conserved variables, and **A** is a m by m constant matrix.

This is a system of conservation laws with the flux function $f(u)=Au$. This system is hyperbolic if A is diagonalizable with real eigenvalues. In this case we can decouple 1.24 by letting

$$A = R\Lambda R^{-1} \tag{1.25}$$

where, Λ = diag $\left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}\right)$ is a diagnal matrix of eigenvalues and $R = \left(r^{(1)}, r^{(2)}, \ldots, r^{(m)}\right)$ is the matrix of right eigenvectors.

From the theory of linear systems, we know

$$\Delta U_{j+k+\frac{1}{2}} = \sum_{p=1}^{m} \alpha^{(p)}_{j+k+\frac{1}{2}} \; r^{(p)}_{j+k+\frac{1}{2}} \tag{1.26}$$

where, $\alpha^{(p)}_{j+k+\frac{1}{2}}$ is called wave strength accross the pth wave traveling at speed of eigenvalue $\lambda^{(p)}_{j+k+\frac{1}{2}}$ in the $\left(j + k + \frac{1}{2}\right)$th computing cell. $r^{(p)}_{j+k+\frac{1}{2}}$ is the pth eigenvector.

Substitution of 1.26 into 1.3 gives arbitrary-order general formula for linear hyperbolic systems:

$$\begin{aligned} F^{A-W}(U^n; j) &= \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}\sum_{p=1}^{m} |\lambda^{(p)}_{j+\frac{1}{2}}| \; \alpha^{(p)}_{j+\frac{1}{2}} \; r^{(p)}_{j+\frac{1}{2}} \\ &+ \sum_{k=-\infty}^{\infty} \sum_{p=1}^{m} |\lambda^{(p)}_{k}| \; D^{(p)}_{k} \; \alpha^{(p)}_{j+k+\frac{1}{2}} \; r^{(p)}_{j+k+\frac{1}{2}} \end{aligned} \tag{1.27}$$

where, F_j and F_{j+1} are physical fluxes evaluated at the data states of j and $j+1$.

Therefore, for examble, we can extend 1.9 into linear systems as:

$$\begin{aligned} F^{A-W}(U^n; j) &= \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}\sum_{p=1}^{m} |\lambda^{(p)}_{j+\frac{1}{2}}| \; \alpha^{(p)}_{j+\frac{1}{2}} \; r^{(p)}_{j+\frac{1}{2}} \\ &+ \frac{1}{2}\sum_{p=1}^{m} \left(1 - |c^{(p)}_{j+k+\frac{1}{2}}|\right) \; |\lambda^{(p)}_{j+k+\frac{1}{2}}| \; \alpha^{(p)}_{j+k+\frac{1}{2}} \; r^{(p)}_{j+k+\frac{1}{2}} \end{aligned} \tag{1.28}$$

where:

$$\begin{cases} k = -1 & if\ c^{(p)}_{j+\frac{1}{2}} > 0 \\ k = 1 & if\ c^{(p)}_{j+\frac{1}{2}} < 0 \end{cases} \tag{1.29}$$

and

$$c^{(p)}_{j+k+\frac{1}{2}} = \frac{\lambda^{(p)}_{j+k+\frac{1}{2}}\ \Delta t}{\Delta x} \tag{1.30}$$

And 1.13 can be extended into linear systems as:

$$\begin{aligned} F^{A-W}(U^n; j) &= \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}\sum_{p=1}^{m} |\lambda^{(p)}_{j+\frac{1}{2}}|\ \alpha^{(p)}_{j+\frac{1}{2}}\ r^{(p)}_{j+\frac{1}{2}} \\ &+ \sum_{p=1}^{m} \left(\frac{1}{3} - \frac{1}{2}|c^{(p)}_{j+\frac{1}{2}}| + \frac{1}{6}(c^{(p)}_{j+\frac{1}{2}})^2\right)\ |\lambda^{(p)}_{j+\frac{1}{2}}|\ \alpha^{(p)}_{j+\frac{1}{2}}\ r^{(p)}_{j+\frac{1}{2}} \\ &+ \sum_{p=1}^{m} \frac{1}{6}\left(1 - (c^{(p)}_{j+k+\frac{1}{2}})^2\right)\ \lambda^{(p)}_{j+k+\frac{1}{2}}\ \alpha^{(p)}_{j+k+\frac{1}{2}}\ r^{(p)}_{j+k+\frac{1}{2}} \end{aligned} \tag{1.31}$$

where:

$$\begin{cases} k = -1 & if\ c^{(p)}_{j+\frac{1}{2}} > 0 \\ k = 1 & if\ c^{(p)}_{j+\frac{1}{2}} < 0 \end{cases} \tag{1.32}$$

Note, if $c^{(p)}_{j+\frac{1}{2}}\ c^{(p)}_{j+k+\frac{1}{2}} < 0$, this means that the waves travel in different directions in cells $j + \frac{1}{2}$ and $j + k + \frac{1}{2}$. In this case, let $\lambda^{(p)}_{j+k+\frac{1}{2}} = 0$.

From these two examples above we know that all three, four, and five point scalar generalized schemes obtained in last section can be extended to linear systems in the same way.

Bibliography

[1] **SOD, G.**, A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws, J. Comput. Phy., 27(1978), pp. 1-31.

[2] **HARTEN, A.**, High Resolution Schemes for Hyperbolic Conservation Laws, J. Comput. Phys., 49(1983), pp.

[3] **ROE, P.L.**, Numerical Algorithms for the Linear Wave Equation, Technical Report 81047, Cranfield Institute of Technology.

[4] **ROE, P.L.**, Generalized Formulation of TVD Lax-Wendroff Schemes, NASA Contractor Report No. 84-53, 1984. 357-393.

[5] **OSHER, S. and CHAKRAVARTHY, S.**, High Resolution Schemes and the Entropy Condition, SIAM J. Num. Anal., 21(1984), pp. 995-984.

[6] **SWEBY, P.K.**, High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws, SIAM J. Num. Anal., 21(1984), pp 995-1011.

[7] **HARTEN, A. and OSHER, S.**, Uniformly High-order Accurate Nonoscillatory Schemes 1, SIAM J. Num. Anal., 24(1987), pp. 279-309.

[8] **SHU, C.**, TVB Uniformly High-order Schemes for Conservation Laws, Math. Comp., 49(1987), pp. 105-121.

[9] **GEODMAN, J.B. and LEVEQUE, P.J.**, A Geometric Approach to High Resolution TVD Schemes, SIAM J. Num. Anal., 25(1988), pp. 268-284.

[10] **SHU, S. and OSHER, S.**, Efficient Implementation of Essentially Nonoscillatory Shock Capturing Schemes, J. Comput. Phys., 77(1988), pp. 439-471.

[11] **HARTEN, A.**, ENO Schemes with Subcell Resolution, J. Comput. Phys., 83(1989).

[12] **RICHTMYER, R.D. and MORTON, K.W.**, Difference Methods for Initial-value Problems, Wiley-Interscience, 1967.

[13] **STRIKWERDA, J.C.**, Finite Difference Schemes and Partial Differential Equations, Wadsworth and Brooks/cole, 1989.

[14] **LEVEQUE, R.J.**, Numerical Methods for Conservation Laws, Birkhauser, 1990.

[15] **TORO, E.F.**, Shock-capturing Methods, Leture Note in Computational Fluid Dynamics, College of Aeronautics, Cranfield Institute of Technology, 1992.

[16] **SHI, J.**, Arbitrary-order High Resolution Schemes for Model Hyperbolic Conservation Laws, COA Report No. 9290, College of Aeronautics, Cranfield Institute of Technology, 1992.